FINITE DIFFERENCE METHODS

The basic idea of these methods is the following: we substitute the derivatives in the (ordinary or partial) differential equation and in the initial or (and) boundary conditions by expressions of approximating derivation formulas. In this way we can get an "approximating system of equations" for the required function values. We illustrate the possibilities on two well-known problems.

Solution of the Second Order Differential Equations using Finite Difference Method

The most general linear second order differential equation is in the form:

$$y''(x) + p(x)y'(x) + q(x)y(x) = r(x), \quad a \le x \le b.$$

We need to specify the value of the solution at two distinct points

$$y(a) = A \text{ and } y(b) = B.$$

These are typically called boundary conditions.

We can divide the interval I=[a,b] into a chosen number **n** of subintervals of equal width.

$$x_i = a + ih \quad (i = 0, 1, ..., n)$$

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x_0 x_1 x_2 x_3 x_{n-1} x_n

Thus, the step size, h, of each of the **n** subintervals is given by $h = \frac{b-a}{n}$.

We will use the notation to denote y_i the value of the function at the i-th node of the computational grid. , $y_0 = A \ and \ \ y_n = B$.

Finite Difference Approximation:

First derivative, forward FD:

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h)$$

First derivative, backward FD:

$$f'(x) = \frac{f(x) - f(x-h)}{h} + O(h)$$

First derivative, central FD:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

Second derivative, central FD:

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2)$$

y_1, y_2,..., y_{n-1} can be calculated as follow:

At any mesh point $x = x_i$, the finite-difference representation of the differential equation can be written as follows (based on central FD)

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + p_i \frac{y_{i+1} - y_{i-1}}{2h} + q_i y_i = r_i, \quad (i = 1,2,.., n-1)$$

$$2(y_{i+1} - 2y_i + y_{i-1}) + hp_i(y_{i+1} - y_{i-1}) + 2h^2 q_i y_i = 2h^2 r_i, \quad (i = 1,2,.., n-1)$$

and arranging the equations with respect to $y_1, \ldots y_n$, we can obtain the system of linear equations

$$(2 + hp_i)y_{i+1} + (2h^2 q_i - 4)y_i + (2 - hp_i)y_{i-1} = 2h^2 r_i, \quad (i = 1,2,.., n-1)$$

The boundary conditions provide of the solution at the two ends of the grid: $y_0 = A$ and

$y_n = B$.

We can interpret **y** as a vector and write the equation formally as an algebraic matrix equation:

$$A_h Y_h = R_h$$

where

$$A_h = \begin{bmatrix} (2h^2 q_1 - 4) & (2 + hp_1) & 0 & \cdots & \cdots & 0 \\ (2 - hp_2) & (2h^2 q_2 - 4) & (2 + hp_3) & 0 & \cdots & 0 \\ 0 & (2 - hp_3) & (2h^2 q_3 - 4) & (2 + hp_3) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & (2 - hp_{n-2}) & (2h^2 q_{n-2} - 4) & (2 + hp_{n-2}) \\ 0 & \cdots & \cdots & 0 & (2 - hp_{n-1}) & (2h^2 q_{n-1} - 4) \end{bmatrix},$$

$$R_h = \begin{bmatrix} 2h^2 r_1 - (2 - hp_1)A \\ 2h^2 r_2 \\ 2h^2 r_3 \\ \vdots \\ 2h^2 r_{n-2} \\ 2h^2 r_{n-1} - (2 + hp_{n-1})B \end{bmatrix},$$

$$Y_h = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{bmatrix}.$$

Example 1: Approximate the solution of the initial-value problem

$$y'' - (1 - \frac{x}{5})y = x, \text{ y(1) = 2 and y(3) = -1}$$

on the interval $1 \le x \le 3$.

Solution:

Put $x_n = x_0 + nh$, $x_0 = 1, h = 0.5$ and let y_n be the calculated approximation for $y_n = y(x_n)$.

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$x_0 = 1$ $x_1 = 1.5$ $x_2 = 2.0$ $x_3 = 2.5$ $x_4 = 3.0$

We now use the central finite-difference approximation:

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} - (1 - \frac{x_i}{5})y_i = x_i, \quad i = 1,2,3.$$

This gives

$$y_{i+1} - \left[2 + (1 - \frac{x_i}{5})\Delta x^2\right]y_i + y_{i+1} = x_i \Delta x^2, \quad i = 1,2,3.$$

for i=1: $$y_0 - \left[2 + (1 - \frac{1.5}{5})(0.5)^2\right]y_1 + y_2 = 1.5(0.5)^2$$

for i=2: $$y_1 - \left[2 + (1 - \frac{2}{5})(0.5)^2\right]y_2 + y_3 = 2(0.5)^2$$

for i=3: $$y_2 - \left[2 + (1 - \frac{2.5}{5})(0.5)^2\right]y_3 + y_4 = 2.5(0.5)^2$$

We set $y_0 = 2$, $y_4 = -1$ and get the system of equations with the three unknowns y_1, y_2, y_3:

The linear system:

$$\begin{bmatrix} -2.175 & 1 & 0 \\ 1 & -2.150 & 1 \\ 0 & 1 & -2.125 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -1.625 \\ 0.5 \\ 1.625 \end{pmatrix}$$

Solution:

$$y_1 = 0.5520, y_2 = -0.4244, y_3 = -0.9644.$$

Example 2: Approximate the solution of the **nonlinear** ordinary differential equation

$$8y'' + yy' = 2x^3 + 32 \quad y(1) = 17, y(4) = 45.$$

on the interval $1 \le x \le 4$, with h = 1.

Solution:

We have h = 1, this means that our interval [1; 4] is divided into 3 equal subintervals.

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$x_0 = 1$ $x_1 = 2$ $x_2 = 3$ $x_3 = 4.0$

We now use the central finite-difference approximation:

$$8\frac{y_{i+1}-2y_i+y_{i-1}}{\Delta x^2}+\frac{y_{i+1}-y_{i-1}}{2\Delta x}y_i = 2x_i^3+32, \quad i=1,2.$$

This gives

$$16(y_{i+1}-2y_i+y_{i-1})+\Delta x(y_{i+1}-y_{i-1})y_i=(2x_i^3+32)2\Delta x^2, \quad i=1,2.$$

Rearranging the terms, we have:

$$16y_{i+1}-32y_i+16y_{i-1}+y_{i+1}y_i\Delta x-y_{i-1}y_i\Delta x=(2x_i^3+32)2\Delta x^2, \quad i=1,2.$$

for i=1: we have: $16y_2-32y_1+16y_0+y_2y_1-y_0y_1=(2*2^3+32)2,$

for i=2: we have: $16y_3-32y_2+16y_1+y_3y_2-y_1y_2=(2*3^3+32)2.$

We set $y_0=17,\ y_3=45$ and get the system of equations with the two unknowns y_1, y_2:

The nonlinear system:

$$16y_2-32y_1+272+y_2y_1-17y_1=96,$$
$$720-32y_2+16y_1+45y_2-y_1y_2=172.$$

We can express the nonlinear system as a matrix with a corresponding vector.

$$F(Y)=\begin{bmatrix} f_1(y_1,y_2) \\ f_2(y_1,y_2)\end{bmatrix}=\begin{bmatrix} y_1y_2-49y_1+16y_2+176 \\ -y_1y_2+16y_1+13y_2+548\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

To approximate the solution to this nonlinear algebraic system, we use the Newton's iterative method:

We can do this by taking an initial approximation Y^0:

$$Y^{n+1}=Y^n-(J(Y^n))^{-1}F(Y^n), \quad n\ge 0.$$

where $(J(Y^n))^{-1}$ is the inverse of the Jacobian matrix.

$$J(Y)=\begin{bmatrix} y_2-49 & y_1+16 \\ -y_2+16 & -y_1+13\end{bmatrix}.$$

The first iteration:

If we take $Y^0=\begin{bmatrix}0\\0\end{bmatrix}\Rightarrow F_0=F(Y^0)=\begin{bmatrix}176\\548\end{bmatrix},$

$$\mathbf{J}(Y^0)=\mathbf{J}_0=\begin{bmatrix}-49 & 16\\ 16 & 13\end{bmatrix} \quad \Rightarrow \mathbf{J}_0^{-1}=\begin{bmatrix}-0.0146 & 0.0179\\ 0.0179 & 0.0549\end{bmatrix}$$

$$Y^1=Y^0-\mathbf{J}_0^{-1}F^0=\begin{bmatrix}0\\0\end{bmatrix}-\begin{bmatrix}-0.0146 & 0.0179\\ 0.0179 & 0.0549\end{bmatrix}\begin{bmatrix}176\\548\end{bmatrix}=\begin{bmatrix}-7.2564\\-33.2228\end{bmatrix}$$

The second iteration:

We have $Y^1=\begin{bmatrix}-7.2564\\-33.2228\end{bmatrix}\Rightarrow F^1=F(Y^1)=\begin{bmatrix}241.0795\\-241.0795\end{bmatrix}$

$$\mathbf{J}_1=\begin{bmatrix}-82.2228 & 8.7436\\ 49.2228 & 20.2564\end{bmatrix} \quad \Rightarrow \mathbf{J}_1^{-1}=\begin{bmatrix}-0.0097 & 0.0042\\ 0.0235 & 0.0392\end{bmatrix}$$

$$Y^2 = \begin{bmatrix} 7.2564 \\ 33.2228 \end{bmatrix} - \mathbf{J}_1^{-1} \begin{bmatrix} 241.0795 \\ -241.0795 \end{bmatrix} = \begin{bmatrix} -3.9208 \\ -29.4671 \end{bmatrix}.$$

Solution of the Partial Differential Equations

using Finite Difference Method

Partial differential equations (PDE) have a huge application in mathematics physics, hydrodynamics, acoustics and other scientific and application-oriented working areas. In most cases, however, they cannot be solved in an obvious way and that is why their approximate solution is widespread.

We will consider the grid method for solving linear differential equations of the second order. The building of different diagrams by means of the grid method depends on the type of the PDE and the kind of boundary conditions that they satisfy.

The most widely-spread partial solutions of linear PDE of the second order are:

1) Poisson's equation, which is an elliptic equation

$$\frac{\partial^2 u}{\partial x} + \frac{\partial^2 u}{\partial y^2} = f(x, y),$$

2) a parabolic equation, which is a heat-conductivity equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x, y),$$

3) a hyperbolic equation, which is a wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x, y).$$

The first example

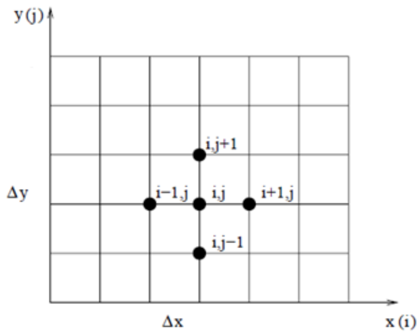
We consider the elliptic partial differential equation, knows as Poisson’s equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f \quad \text{where } (x, y) \in \Omega = [a, b] \times [c, d].$$

The solution is known on the boundary:

$$u(x, y) = g(x, y).$$

To solve Poisson’s equation by difference method, the region Ω is partitioned into a grid consisting of n x m rectangles with sides h and k.



The mesh point are given by

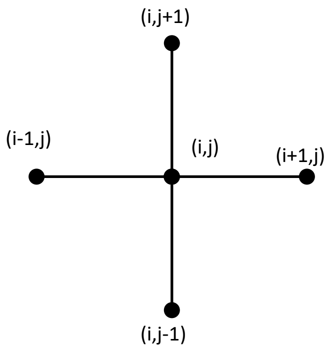
$$x_i = a + i\Delta x\,,\ y_j = c + j\Delta y\,,\quad i = 0,1,\ldots,n\,,\ j = 0,1,\ldots,m\,.$$

At this time, it is convenient to introduce an abbreviated notation: $\begin{cases} u_{ij} = u(x_i, y_j) \\ f_{ij} = f(x_i, y_j) \\ g_{ij} = g(x_i, y_j) \end{cases}$

By using central difference approximations, the finite difference equation at the mesh point (i, j) is

$$\frac{u_{(i+1)j} - 2u_{ij} + u_{(i-1)j}}{\Delta x^2} + \frac{u_{i(j+1)} - 2u_{ij} + u_{i(j-1)}}{\Delta y^2} = f_{ij}\,.$$

The FD equation at the grid point (x_i, y_j) involves five grid points in a five-point stencil, (x_{i-1}, y_j), (x_{i+1}, y_j), (x_i, y_{j-1}), (x_i, y_{j+1}) and (x_i, y_j).



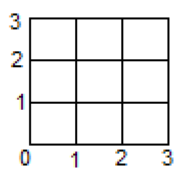
The following recurrence relationship is established by re-arranging the terms in the above difference equation and we set $\Delta x = \Delta y = h$:

$$-4u_{ij} + u_{i(j-1)} + u_{i(j+1)} + u_{(i-1)j} + u_{(i+1)j} = h^2 f_{ij}\,,\ i = 1,\ldots,n-1,\ j = 1,\ldots,m-1.$$

The approximate solution values u_{ij} at the grid points can be obtained by solving the previous linear system of algebraic equations.

Application: $\Omega = [0,1]\times[0,1]$, $\Delta x = \Delta y = h = \frac{1}{3}$, $f(x, y) = 2xy$ and $g(x, y) = 40.$

We have: $f_{ij} = 2(ih)(jh) = 2ijh^2$ and $g_{ij} = 40.$



For i = 1 j = 1: we have: $-4u_{11} + u_{10} + u_{12} + u_{01} + u_{21} = h^2 f_{11}$

For i = 2 j = 1: we have: $-4u_{21} + u_{20} + u_{22} + u_{11} + u_{31} = h^2 f_{21}$

For i = 1 j = 2: we have: $-4u_{12} + u_{11} + u_{13} + u_{02} + u_{22} = h^2 f_{12}$

For i = 2 j = 2: we have: $-4u_{22} + u_{21} + u_{23} + u_{12} + u_{32} = h^2 f_{22}$

The boundary conditions provide the solution on the boundary:

$-4u_{11} + g_{10} + u_{12} + g_{01} + u_{21} = h^2 f_{11} \Rightarrow -4u_{11} + u_{12} + u_{21} = h^2 2h^2 - 2*40$

$-4u_{21} + g_{20} + u_{22} + u_{11} + g_{31} = h^2 f_{21} \Rightarrow -4u_{21} + u_{22} + u_{11} = h^2 2*2h^2 - 2*40$

$-4u_{12} + u_{11} + g_{13} + g_{02} + u_{22} = h^2 f_{12} \Rightarrow -4u_{12} + u_{11} + u_{22} = h^2 2*2h^2 - 2*40$

$-4u_{22} + u_{21} + g_{23} + u_{12} + g_{32} = h^2 f_{22} \Rightarrow -4u_{22} + u_{21} + u_{12} = h^2 2*4h^2 f_{22} - 2*40$

We can interpret the solution as a vector and write the equation formally as an algebraic matrix equation:

$$\begin{pmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{pmatrix} = \begin{pmatrix} 2(1/3)^4 - 80 \\ 4(1/3)^4 - 80 \\ 4(1/3)^4 - 80 \\ 8(1/3)^4 - 80 \end{pmatrix}$$

Solution:

$u_{11} = 39.980452674897123$,
$u_{21} = 39.973251028806587$,
$u_{12} = 39.973251028806580$,
$u_{22} = 39.961934156378597$.

The second example

We now begin to study finite difference methods for time-dependent partial differential equations, where variations in space are related to variations in time. We begin with the heat equation

$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2}, \quad x \in [0,1], \quad t \geq 0.$$

Along with this equation we need initial conditions at some time $t = 0$, which we typically take to be

$$T(x,0) = f(x)$$

and also boundary conditions

$$\begin{cases} T(0,t) = g(t) \\ T(1,t) = l(t) \end{cases}, \qquad t \geq 0.$$

To solve the heat equation numerically, we divide the interval [0, 1] into N equal subintervals:

$$x_i = i\Delta x, \quad i = 0,1,2...$$

Each time step is given an integer index i so that the time can be written as

$$t_j = j\Delta t, \quad j = 0,1,2...$$

where Δt is the time step.

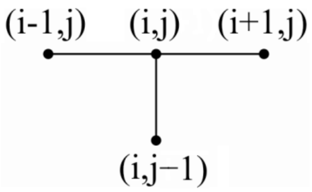
At this time, it is convenient to introduce an abbreviated notation: $T_{ij} = T(x_i, t_j)$.

We approximate the time derivative as: (here we use the backward finite difference)

$$\frac{\partial T(x_i, t_j)}{\partial t} = \frac{T_{ij} - T_{i(j-1)}}{\Delta t}$$

The spatial term is typically approximated as: (here we use the central finite difference)

$$\frac{\partial^2 T(x_i, t_j)}{\partial x^2} = \frac{T_{(i+1)j} - 2T_{ij} + T_{(i-1)j}}{\Delta x^2}$$



Grid points used in difference equation

Combining these two expressions we obtain:

$$\frac{T_{ij} - T_{i(j-1)}}{\Delta t} = \frac{T_{(i+1)j} - 2T_{ij} + T_{(i-1)j}}{\Delta x^2}, \; i = 1, \ldots, n-1, \; j = 1, \ldots, m-1.$$

Re-arranging the terms in the above difference equation, the solution at time t_j can be obtained by solving the following linear system of algebraic equations:

$$-\frac{T_{(i+1)j}}{\Delta x^2} + \left(\frac{1}{\Delta t} + \frac{2}{\Delta x^2}\right)T_{ij} - \frac{T_{(i-1)j}}{\Delta x^2} = \frac{T_{i(j-1)}}{\Delta t}, \; i = 1, \ldots, n-1.$$

Application: N = 4, $\Delta x = \frac{1}{4}, \Delta t = \frac{1}{10},$

$f(x) = 25 \Rightarrow \; T_{i0} = f_{i0} = 25$

$g(t) = 25 \Rightarrow \; T_{0j} = g_{0j} = 25$

and $l(t) = 40 \Rightarrow \; T_{4j} = l_{4j} = 40$.

The solution at time $t = \Delta t$:

For i = 1 j = 1: we have: $-\frac{T_{21}}{\Delta x^2} + \left(\frac{1}{\Delta t} + \frac{2}{\Delta x^2}\right)T_{11} - \frac{T_{01}}{\Delta x^2} = \frac{T_{10}}{\Delta t}$

For i = 2 j = 1: we have: $-\frac{T_{31}}{\Delta x^2} + \left(\frac{1}{\Delta t} + \frac{2}{\Delta x^2}\right)T_{21} - \frac{T_{11}}{\Delta x^2} = \frac{T_{20}}{\Delta t}$

For i = 3 j = 1: we have: $-\frac{T_{41}}{\Delta x^2} + \left(\frac{1}{\Delta t} + \frac{2}{\Delta x^2}\right)T_{31} - \frac{T_{21}}{\Delta x^2} = \frac{T_{30}}{\Delta t}$

Using the boundary conditions and the initial condition we have:

$$-\frac{T_{21}}{\Delta x^2}+(\frac{1}{\Delta t}+\frac{2}{\Delta x^2})T_{11}-\frac{25}{\Delta x^2}=\frac{25}{\Delta t} \Rightarrow -16T_{21}+(10+32)T_{11}=25*10+25*16$$
$$-\frac{T_{31}}{\Delta x^2}+(\frac{1}{\Delta t}+\frac{2}{\Delta x^2})T_{21}-\frac{T_{11}}{\Delta x^2}=\frac{25}{\Delta t} \Rightarrow -16T_{31}+(10+32)T_{21}-16T_{11}=25*10$$
$$-\frac{40}{\Delta x^2}+(\frac{1}{\Delta t}+\frac{2}{\Delta x^2})T_{31}-\frac{T_{21}}{\Delta x^2}=\frac{25}{\Delta t} \Rightarrow (10+32)T_{31}-16T_{21}=25*10+40*16$$

We can interpret the solution as a vector and write the equation formally as an algebraic matrix equation:

$$\begin{pmatrix} 42 & -16 & 0 \\ -16 & 42 & -16 \\ 0 & -16 & 42 \end{pmatrix}\begin{pmatrix} T_{11} \\ T_{21} \\ T_{31} \end{pmatrix}=\begin{pmatrix} 650 \\ 250 \\ 890 \end{pmatrix}.$$

The solution at time $t=\frac{1}{10}$ is:

$T_{01}=25.0,$

$T_{11}=26.1684,$

$T_{21}=28.0671,$

$T_{31}=31.8827,$

$T_{41}=40.0.$