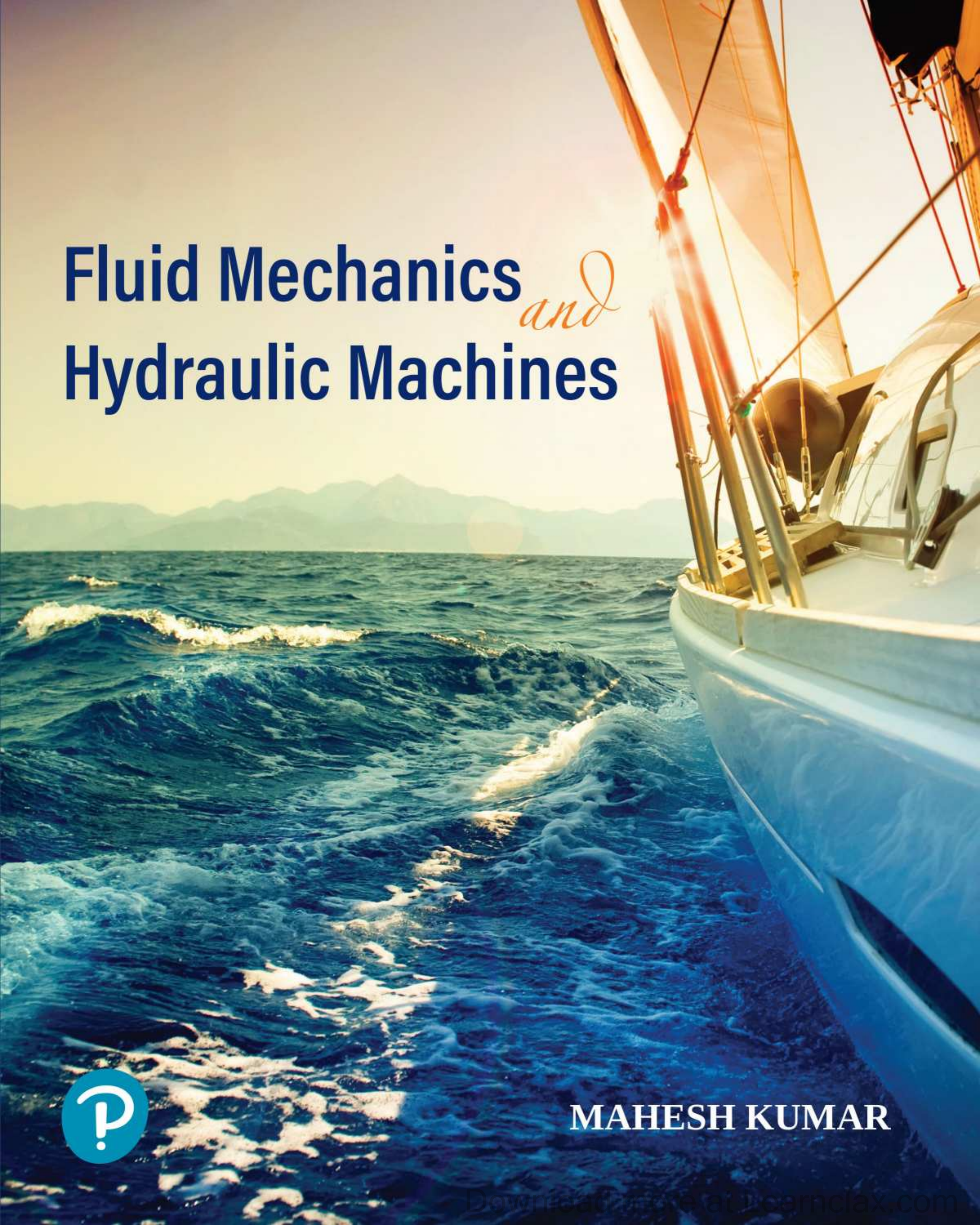

Fluid Mechanics and Hydraulic Machines

MAHESH KUMAR

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Mahesh Kumar
Assistant Professor
Department of Mechanical Engineering
Guru Jambheshwar University of Science and Technology
Hisar

Pearson

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Registered Office: 4th Floor, Software Block, Elnet Software City, TS-140, Block 2 & 9, Rajiv Gandhi Salai, Taramani, Chennai 600 113, Tamil Nadu, India.
Fax: 080-30461003, Phone: 080-30461060
website: in.pearson.com, Email: companysecretary.india@pearson.com

This book is dedicated to
My loving children
Reena
and
Tushar

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Brief Contents

Contents

Preface

This text book '*Fluid Mechanics and Hydraulic Machines*' is meticulously prepared to assist the candidates who partake in the curricula of undergraduate engineering courses (B.E./B.Tech.) in mechanical, civil and agricultural streams offered by many technological universities and engineering institutes. The only intent to prepare this book is to emphasize the theoretical concepts in a clear, concise and simplified manner for the students to grasp the subject without any hurdles. It also aims to cater the needs of students who prepare for various competitive and professional examinations.

This book wholly consists of 27 chapters following the SI system of units. Each and every chapter is enhanced with a number of figure illustrations and solved examples. Various applications to real time problems have also been comprehensively emphasized. All the basic principles and theories have been discussed in a simple and lucid language which can be easily grasped. The summary, objective type questions, review questions and unsolved problems have also been provided at the end of each chapter.

Chapter 1 covers the basic concepts of fluid flows and engineering properties of fluids. **Chapter 2** describes the relation for variation of pressure in a fluid under static conditions and manometers and mechanical gauges for measuring fluid pressure and aerostatics. **Chapter 3** gives the hydrostatic equations and methods to determine the resultant force acting on a submerged surface under static fluid conditions. In **Chapter 4**, liquid in a container subjected to uniform acceleration and constant rotation under relative equilibrium conditions is described. **Chapter 5** deals with the equilibrium of floating and submerged bodies. In **Chapter 6**, the basic concepts related to fluid kinematics and the methods for determining velocity and acceleration are described. In **Chapter 7**, the derivation of energy and momentum equations along with their applications for solving a wide variety of fluid flow problems have been described. **Chapter 8** describes the characteristics of vortex flow, equation of vortex motion and rotation of liquid in a closed cylindrical vessel. **Chapter 9** provides information on the important cases of potential flow with the help of potential and stream functions. **Chapter 10** discusses the orifice and mouthpieces for measuring the rate of flow of fluid. **Chapter 11** deals with concepts regarding notches and weirs. **Chapter 12** presents expressions relating to shear stress and pressure gradients in laminar flow, power absorbed in bearings and various viscometers. **Chapter 13** discusses some semi-empirical theories developed for turbulent flow and provides information on turbulence and turbulent flow in pipes. **Chapter 14** deals with various problems of pipe flow based on major and minor energy losses, pipes in series, parallel and branches, flow through syphon and nozzles, water hammer and power transmission through pipes. **Chapter 15** explains the boundary layer thickness parameters, shear stress and the associated drag on the flat plate surface. **Chapter 16** describes the simple approach of analysing the drag and lift forces acting on the submerged moving bodies (plates, circular cylinders, spheres and airfoils). In **Chapter 17**, the basic equations for compressible flow and its analysis for flow through nozzles and venturimeter is introduced. **Chapter 18** describes flow in open channels pertaining to steady flow under uniform and non-uniform flow conditions. In **Chapter 19**, Rayleigh method, Buckingham pi method, types of similarities and various model laws are presented. In **Chapter 20**, different cases of force exerted by free water jet on stationary and moving vanes of different shapes, propulsion of ship, and basics of fluid machines are described. **Chapter 21** deals with construction, working, governing, work done, efficiency and design aspects of Pelton turbine (impulse turbine). In **Chapter 22**, the general features, various working proportions and design aspects of

Francis turbine (radial flow reaction turbines) are described. **Chapter 23** describes the propeller and Kaplan turbines (axial flow reactions turbines), cavitation, draft tube and new turbines, namely Deriaz, tubular and bulb turbines. **Chapter 24** explains the unit quantities, characteristic curves, specific speeds, model relationship and testing of impulse and reaction turbines. **Chapter 25** presents the main components, theoretical analysis for determining power requirements and other associated problems with centrifugal pumps. **Chapter 26** deals with reciprocating pumps covering air vessels, effects of acceleration, friction and its characteristic curves. **Chapter 27** provides information on hydraulic devices, such as press, accumulator, intensifier, ram, lift, crane, coupling, torque converter, air lift pump, jet pump and gear pump.

Utmost care has been taken at every stage of proof reading and checking but it is possible that some unintentional errors and misprints might have crept in. I shall be very grateful to the readers for pointing out the errors. Valuable suggestions for the improvement in future editions of this text book are warmly welcome. I hope that this book will be of great use to the students and teachers.

Complete solutions manual and lecture PPTs are also available for the instructors at www.pearsoned.co.in/maheshkumar.

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I wish to express my sincere appreciation to my parents Sh. Jai Bhagwan Sharma and Smt. Sona Devi and teachers at all levels of my studies. I would like to express my appreciation to my wife Promila Devi and loving children Reena and Tushar for their wholehearted support and patience throughout the preparation of this text book. I extend gratitude to my departmental colleagues who were a source of encouragement through the course of this writing work. I would like to thank my B.Tech., M.Tech., and Ph.D. students who provided feedback from their perspectives. I express sincere thanks to Dr Himanshu Manchanda for his valuable suggestions and discussion. I am thankful to Rakesh Bisla for his help in preparing few drawings. I am also thankful to Dr Gulshan Sachdeva (NIT Kurukshetra) for his valuable suggestions.

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Ajay Pratap Singh
Ajay Kumar Garg Engineering College
Ghaziabad, Uttar Pradesh

Basudeb Bhattacharyya
Indian Institute of Engineering Science
and Technology, Shibpur, West Bengal

Darshana Amrutbhai Patel
Government Engineering College
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Dr Kanchan Chatterjee
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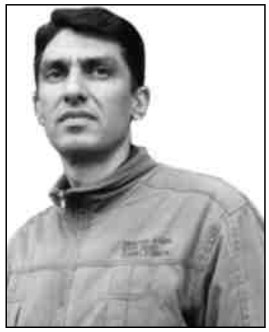
Durgapur, West Bengal

G. S. S. Adithya
Manipal University, Jaipur, Rajasthan

Deepesh Singh
Harcourt Butler Technical University
Kanpur, Uttar Pradesh

Mahesh Kumar

About the Author



Mahesh Kumar is an assistant professor at the Department of Mechanical Engineering, Guru Jambheshwar University of Science and Technology, Hisar. He has both B.Tech. and M.E. degrees in Mechanical Engineering from Delhi College of Engineering, Delhi (now Delhi Technological University, Delhi). He was conferred with a Ph.D. in Mechanical Engineering from NIT Kurukshetra. He has published over 55 research articles in both national and international journals and also presented over 25 research papers, both at national and international conferences. He has also authored a text book named '*Elements of Mechanical Engineering*'. Owing to his expertise in teaching for 15 years, Dr. Kumar has shimmered as a guidance of knowledge to a number of M.Tech. and Ph.D. students in his career. His current fields of research are heat transfer, thermal engineering, energy conversion, energy conservation and solar energy.

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Chapter 1

Basic Concepts and Properties of Fluids

1.1 ❒ INTRODUCTION

It is evident that matter exists in two forms, namely fluid and non-fluid (or solid). A fluid is a substance which deforms continuously when subjected to shear forces. The deformation occurs at a finite rate and it can be determined by the applied shear force and fluid properties. The act of continuous deformation is called the flow. Thus, fluid may be defined as a substance which is capable of flowing and it includes liquids and gases.

A fluid does not have its own shape but it conforms to the shape of the container. The mass of a fluid has definite volume at particular temperature and pressure. A liquid is practically considered as incompressible (i.e., density remains constant), but it takes the shape of the container. If the container is of a larger volume, then it forms a free surface. Gases are highly compressible and it takes the shape of the containing vessel but it occupies the whole volume of the container without any free surface.

The fluid flow analysis is carried out at macro-level by considering the fluid as a continuum. In this chapter, the basic concepts used in the analysis of fluid flow and the various engineering properties of fluids that are essential in the study of their behaviour are described.

1.2 ❒ FLUID MECHANICS AND ITS APPLICATIONS

The subject of fluid mechanics deals with the behaviour of the fluids at rest or in motion, as well as its interaction with solids or other fluids at the boundaries. Broadly, this subject is classified into statics, kinematics and dynamics of fluids. Fluid mechanics can also be divided into several categories, such as hydrodynamics, hydraulics, gas dynamics and aerodynamics.

1. **Fluid statics:** It deals with the behaviour of fluids at rest.
2. **Fluid kinematics:** It deals with motion of fluids without considering the forces causing flow.
3. **Fluid dynamics:** It deals with fluid flow subjected to forces.
4. **Hydrodynamics:** It deals with the motion of incompressible fluids, such as liquids (mainly water) and gases at low speeds.
5. **Hydraulics:** It deals with liquid flows, for example, flows in pipes and open channel.
6. **Gas dynamics:** It deals with the fluid flows that undergo significant density changes, for example, flows of gases through nozzles at high speeds.
7. **Aerodynamics:** It mainly deals with the flow of air over bodies, like automobiles, aircrafts, spacecrafts and rockets.

1.2.1 Application Areas of Fluid Mechanics

Fluid mechanics is widely used in a variety of applications, such as pumps, turbines, airplanes, ships, submarines, fans, blowers, windmills, pipes, engines, jets, rockets, sprinklers, rivers, designing of buildings and bridges, hydraulic systems, pneumatic systems, artificial hearts, breathing machines, automobiles like components related with transportation of the fuel, hydraulic brakes, lubrication systems and cooling systems, pumping of blood in human body, naturally occurring flows such as in meteorology, oceanography, hydrology and many more.

1.3 ❒ UNITS AND DIMENSIONS

Physical quantities are characterized quantitatively by dimensions. The magnitudes assigned to the dimensions are called units which are accepted as standards. The four basic dimensions, namely mass (m), length (L), time (T) and temperature (θ) are primary or fundamental dimensions. The other dimensions such as velocity (V), density (ρ), force (F), etc., are secondary or derived units. The most commonly used dimensions and units in SI (System International) system are given in Table 1.1.

The SI system of units is based on a decimal relationship and thus, it requires prefix before the unit. Some of the commonly used prefixes are listed in Table 1.2.

In the CGS system of units, the unit of force is dyne which is equal to 1 g cm/s^2 and the gravitational units kg_f and g_f are also used as force units and it needs conversion to SI units. Some of the important conversions to SI units are given in Table 1.3.

Table 1.1 *SI system of units*

Quantity	Dimensions	SI unit with symbol
Mass	$[M]$	Kilogram (kg)
Length	$[L]$	Metre (m)
Time	$[T]$	Second (s)
Temperature	$[\theta]$	Degree Celsius (°C) or Kelvin (K)
Force	$[MLT^{-2}]$	Newton (N)
Pressure	$[ML^{-1}T^{-2}]$	Pascal (Pa)
Work	$[ML^2T^{-2}]$	Joule (J)
Power	$[ML^2T^{-3}]$	Watt (W)
Frequency	$[T^{-1}]$	Hertz (Hz)

Table 1.2 *Standard prefixes in SI units with symbols*

Multiple	Prefix	Multiple	Prefix
10^{15}	Peta (P)	10^{-1}	Deci (d)
10^{12}	Tera (T)	10^{-2}	Centi (c)
10^{9}	Giga (G)	10^{-3}	Milli (m)
10^{6}	Mega (M)	10^{-6}	Micro (μ)
10^{3}	Kilo (k)	10^{-9}	Nano (n)
10^{2}	Hecto (h)	10^{-12}	Pico (p)
10^{1}	Deca (da)	10^{-15}	Femto (f)

Table 1.3 *Important conversions of units*

1 kg_f	g_f	1 N	1 J	1 W	1 Pa	1 HP	°C
9.81 N	981 dynes	10^5 dynes	1 Nm	1 J/s	1 N/m^2	736 W (Metric) 746 W (British)	(°C + 273.15) K (1.8°C + 32)°F

1.4 ❐ PRESSURE IN FLUIDS

Pressure is a general characteristic of a fluid which is exerted normal to a solid boundary or any plane drawn through the fluid. Consider a small area (δA) on the surface of a body (Figure 1.1). Let δF be the force acting on the elementary area δA. This force can be resolved into two components, (i) δF_n along the normal to the small area δA, also called normal force or pressure force and (ii) δF_t along the plane of the small area δA, also called tangential force or shear force.

The normal force exerted by a fluid per unit area is known as pressure or intensity of pressure (p) and in case of solids it is termed as normal stress. The tangential force per unit area is called shear stress or tangential stress (τ). Thus, the expression for intensity of pressure is given as follows.

$$p = \underset{\delta A \to 0}{Lt} \frac{\delta F_n}{\delta A}$$

Shear stress is given by,

$$\tau = \underset{\delta A \to 0}{Lt} \frac{\delta F_t}{\delta A}$$

In fluids at rest, there is no shear force and consequently, the force exerted is normal to the surface of the containing vessel. This normal force per unit area is termed as pressure which is measured in N/m^2 or Pascal.

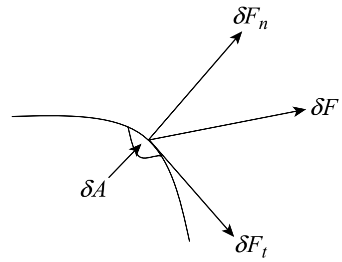

Figure 1.1 *Normal and shear forces*

1.5 ❐ FLUID CONTINUUM

Fluid is made up of atoms or molecules which are widely spaced for gas and closely spaced for liquid. In fluids, the distance between molecules is very large when compared to its molecular diameter. It is convenient to ignore the atomic or molecular nature of a substance and view it as a continuous and homogeneous matter with no voids or holes and therefore, the fluid is called a continuum.

The continuum idealization allows us to consider the properties of fluids as continuous function of space variables. In other words, the variation in properties is so smooth that the differential calculus can be used to analyse the fluid behaviour. Thus, according to the mathematical idealization of continuum any property function P defined at a point (x, y, z) is a continuous and differential function of space variables x, y and z. The continuum idealization is valid as long the size of any system is large relative to the space between the molecules which practically exists in all problems being studied. However, the continuum concept is not useful in high vacuum and very high elevation problems where rarefied gas flow theory is applicable.

1.6 ❐ FLUID PROPERTIES

The characteristics of a fluid by which its physical condition may be described are called properties of fluid. It helps in the formulation of general laws which govern fluid motion. Some of the important properties of fluids are density, specific weight, specific volume, specific gravity, viscosity, surface tension, capillarity, compressibility and vapour pressure. The properties that are independent of the mass of a system are called intensive properties, for example, temperature, pressure and density. The properties that depend on the size (or extent) of the system are called extensive properties. Thus, in dividing a system into two equal parts if the properties become half of the original system then the properties are said to be extensive properties, for example, mass and volume.

1.7 ❐ MASS DENSITY OR DENSITY

The mass density or density (ρ) is the ratio of mass (m) of a fluid to its volume (v). The mass density of a fluid is mathematically expressed as given below.

$$\rho = \frac{\text{Mass}}{\text{Volume}} = \frac{m}{v} \tag{1.1}$$

Its units are kg/m^3 and dimensions are $[ML^{-3}]$. The density of water is denoted by ρ_w and its typical value is $1000\ kg/m^3$ (at 4°C and 1 atm). The typical value of density for air is $1.3\ kg/m^3$ (at 0°C and 1 atm).

1.8 ❒ SPECIFIC WEIGHT OR WEIGHT DENSITY

The specific weight or weight density (w) is the ratio of the weight (W) of a fluid to its volume (v). The specific weight of a fluid is mathematically expressed as given below.

$$w = \frac{\text{Weight}}{\text{Volume}} = \frac{W}{v} = \frac{mg}{v} = \frac{m}{v} \times g = \rho \times g \tag{1.2}$$

The units of specific weight or weight density is N/m^3 and the dimensions are $[ML^{-2}T^{-2}]$. The value of specific weight for water is given below.

$$\boxed{w = \rho_w g = 1000 \times 9.81 = 9810\ N/m^3}$$

1.9 ❒ SPECIFIC VOLUME

The volume (v) of a fluid per unit mass (m) is called its specific volume (v_s). It is the reciprocal of mass density and it is mathematically expressed as given below.

$$v_s = \frac{v}{m} = \frac{1}{(m/v)} = \frac{1}{\rho} \tag{1.3}$$

Its units are m^3/kg and dimensions are $[M^{-1}L^3]$. The concept of specific volume is used in the study of flow of gases (i.e., compressible fluids).

1.10 ❒ SPECIFIC GRAVITY OR RELATIVE DENSITY

Specific gravity (S) is defined as the ratio of the density (or weight density) of a fluid to the density (or weight density) of a standard fluid. For liquids, the standard fluid is assumed as water (at 4°C), whereas for gases it is air (at 0°C). The specific gravity is also known as relative density and it has no units. The specific gravity of a fluid is mathematically expressed as given below.

$$S = \frac{\rho_{\text{liquid}}}{\rho_{\text{water}}} = \frac{w_{\text{liquid}}}{w_{\text{water}}} \qquad \text{(For liquids)} \tag{1.4}$$

$$S = \frac{\rho_{\text{gas}}}{\rho_{\text{air}}} = \frac{w_{\text{gas}}}{w_{\text{air}}} \qquad \text{(For gases)} \tag{1.4a}$$

The specific gravities of some common substances are given in Table 1.4.

Table 1.4 *Specific gravities of some substances at 0°C*

Substance	Water	Mercury	Seawater	Petrol	Ice
S	1	13.6	1.025	0.7	0.92

Example 1.1 Determine (i) specific weight, (ii) density and (iii) specific gravity of 3 litres of a liquid that weighs 24 N.

Solution

Let $v = 3$ litres $= 3\times10^{-3}\ \text{m}^3$ and $W = 24$ N.

(i) $w = \dfrac{W}{v} = \dfrac{24}{3\times10^{-3}} = \mathbf{8000\ N/m^3}$

(ii) $\rho = \dfrac{w}{g} = \dfrac{8000}{9.81} = \mathbf{815.5\ kg/m^3}$

(iii) $S = \dfrac{\rho}{\rho_w} = \dfrac{815.5}{1000} = \mathbf{0.8155}$

Example 1.2 A liquid has specific gravity of 0.76, determine its (i) density, (ii) specific volume and (iii) specific weight.

Solution

Let $S = 0.76$.

(i) $\rho = S\times\rho_w = 0.76\times1000 = \mathbf{760\ kg/m^3}$

(ii) $v_s = \dfrac{1}{\rho} = \dfrac{1}{760} = \mathbf{1.316\times10^{-3}\ m^3/kg}$

(iii) $w = \rho\times g = 760\times 9.81 = \mathbf{7455.6\ N/m^3}$

1.11 ❒ VISCOSITY OR DYNAMIC VISCOSITY

Viscosity is the property of a fluid which offers resistance to the movement of one layer of fluid over an adjacent layer. In other words, it is a measure of the internal fluid friction which causes resistance to flow. The viscosity is due to cohesion and molecular momentum transfer between fluid layers. If there is any fluid flow, then these factors add up and appear as shearing stresses. A typical velocity profile developed during the flow of a fluid over a stationary solid flat surface is illustrated in Figure 1.2(a).

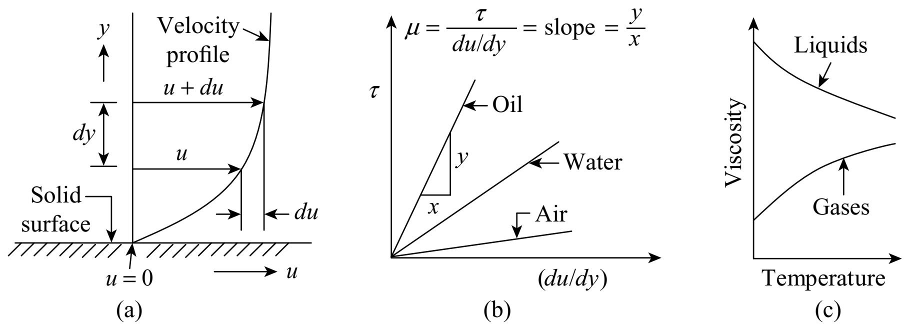

Figure 1.2 *(a) velocity profile (b) shear stress versus velocity gradient (c) viscosity versus temperature*

Here, no slip occurs at the point of contact between the fluid and the solid surface where the fluid motion completely stops and has a zero velocity $(u = 0)$. The layer that sticks to the solid surface slows down the next adjacent fluid layer due to the presence of viscous forces and it slows down the next layer and so on. Thereby, the velocity of each successive layer increases. Thus, the stream velocity in the fluid layers which is far away from the solid surface attains the free stream velocity. In fact, the velocity variation between each layer is due to viscosity.

1.11.1 Newton's Law of Viscosity

Consider two adjacent layers of a fluid at a distance apart and denoted as dy. The lower layer moves with a velocity u and the upper layer moves with a velocity $(u + du)$ which is higher than the lower one as shown in Figure 1.2(a). The upper layer drags the lower layer along with it by means of force F acting over an area of contact A. However, the lower layer tries to retard the upper one with an equal and opposite force F. These two equal and opposite forces causes shear stress τ and it is mathematically expressed as given below.

$$\tau = \frac{F}{A} \tag{1.5}$$

The Newton's law of viscosity states that the shear stress on a fluid layer is directly proportional to the velocity gradient or the rate of shear strain (du/dy) as shown in Figure 1.2(b) and it is mathematically expressed as given below.

$$\tau \, \alpha \, \frac{du}{dy}$$

Thus

$$\boxed{\tau = \mu \frac{du}{dy}} \tag{1.6}$$

Here, μ is the constant of proportionality known as coefficient of viscosity or the dynamic viscosity or simply the viscosity.

Fluids which obey Newton's law of viscosity are known as Newtonian fluid, for example, water, air and molten metals. Fluids which do not obey this law are known as non-Newtonian fluids, for example, human blood and thick lubricating oils. The variation of shear stress (τ) with velocity gradient (du/dy) for Newtonian fluid is a straight line whose slope (y/x) is the viscosity of the fluid (Figure 1.2(b)).

It can be noticed that the Newton's law of viscosity for fluids is analogous to the Hooke's law of elasticity for solids.

1.11.2 Units of Viscosity

From Equation (1.6), we get:

$$\mu = \frac{\tau}{\left(\frac{du}{dy}\right)} = \frac{\text{N/m}^2}{\left(\frac{\text{m}}{\text{s}} \times \frac{1}{\text{m}}\right)} = \frac{\text{Ns}}{\text{m}^2} = \frac{\text{N}}{\text{m}^2} \cdot \text{s} = \text{Pa} \cdot \text{s}$$

In SI units, viscosity is measured in Ns/m^2 or $\text{Pa} \cdot \text{s}$ or kg/ms and its dimensions are $[ML^{-1}T^{-1}]$. Commonly, viscosity is measured in poise or centipoise (cP).

$$\boxed{1 \text{ poise} = 0.1 \text{ Ns/m}^2 \text{ and } 1 \text{ cP} = 0.01 \text{ poise}}$$

For water at 20°C: $\mu = 1 \text{ cP} = 10^{-3} \text{ Ns/m}^2$
For air at 20°C: $\mu = 0.0181 \text{ cP} = 0.0181 \times 10^{-3} \text{ Ns/m}^2$

1.11.3 Variation of Viscosity with Temperature

The viscosity of fluids varies greatly with temperature. With an increase in temperature, the viscosity of liquids decreases while for gases it increases (Figure 1.2(c)). This can be explained by the fact that the property of viscosity is due to intermolecular forces of cohesion and the momentum transfer due to exchange of molecules between adjacent layers of fluid under shear. In liquids due to closely packed molecules, the cohesive forces predominates the molecular momentum transfer. With increase in temperature, the molecular cohesion decreases due to increase in distance between the molecules and as a result, the viscosity of liquids decreases. In case of gases, the molecular cohesive forces are very small and the viscosity is mainly due to molecular momentum transfer. With increase in temperature, the molecular activity increases with increase in momentum transfer and also the viscosity.

The relationship between viscosity and temperature for liquids is expressed as follows.

$$\mu = \frac{\mu_o}{1 + aT + bT^2} \tag{1.7}$$

Here, μ is the viscosity of liquid at T°C in poise, μ_o is the viscosity at 0°C in poise, and a and b are the constants depending on the liquid.

For water:

$$\mu_o = 0.0179 \text{ poise}, \quad a = 0.03368 \text{ and } b = 0.000221$$

From Equation (1.7), it can also be observed that viscosity of liquids decrease with increase in temperature.

The relationship between viscosity and temperature for gases is expressed as follows.

$$\mu = \mu_o + aT - bT^2 \tag{1.8}$$

Here, μ is the viscosity of gas at T°C, μ_o is the viscosity at 0°C, and a and b are the constants depending on the gas.

For air:

$$\mu_o = 1.7 \times 10^{-5} \text{ Ns/m}^2, \; a = 0.56 \times 10^{-7} \text{ and } b = 0.1189 \times 10^{-9}$$

From Equation (1.8), it can also be observed that viscosity of gases increases with increase in temperature.

The dynamic viscosity of liquids and gases does not change appreciably with pressure values generally encountered in practice.

1.12 ❒ KINEMATIC VISCOSITY

The kinematic viscosity (ν) is the ratio of dynamic viscosity (μ) to the density (ρ) of a fluid. It represents the momentum diffusivity and it is mathematically expressed as given below.

$$\nu = \frac{\text{Viscosity}}{\text{Density}} = \frac{\mu}{\rho} \tag{1.9}$$

In SI units, kinematic viscosity is measured in m^2/s and its dimensions are $[L^2T^{-1}]$. Commonly, kinematic viscosity is measured in stoke or centistoke.

$$\boxed{1 \text{ stoke} = 10^{-4} \text{ m}^2/\text{s and } 1 \text{ centistoke} = 0.01 \text{ stoke}}$$

In case of liquids, the kinematic viscosity decreases with the increase in temperature, whereas in case of gases it increases. The kinematic viscosity of gases changes with pressure due to change in density.

1.13 ❒ TYPES OF FLUIDS

The fluids may be classified as (i) ideal and real fluids, (ii) Newtonian fluid, (iii) non-Newtonian fluid, (iv) ideal plastic fluid and (v) thixotropic fluids as shown in Figure 1.3.

1. **Ideal and real fluids:** An imaginary fluid which is incompressible and has zero viscosity is called an ideal fluid. Practically, all the fluids have some viscosity and are called real fluids. In Figure 1.3(a), an ideal fluid is represented by the horizontal axis for which shear stress $\tau = 0$.

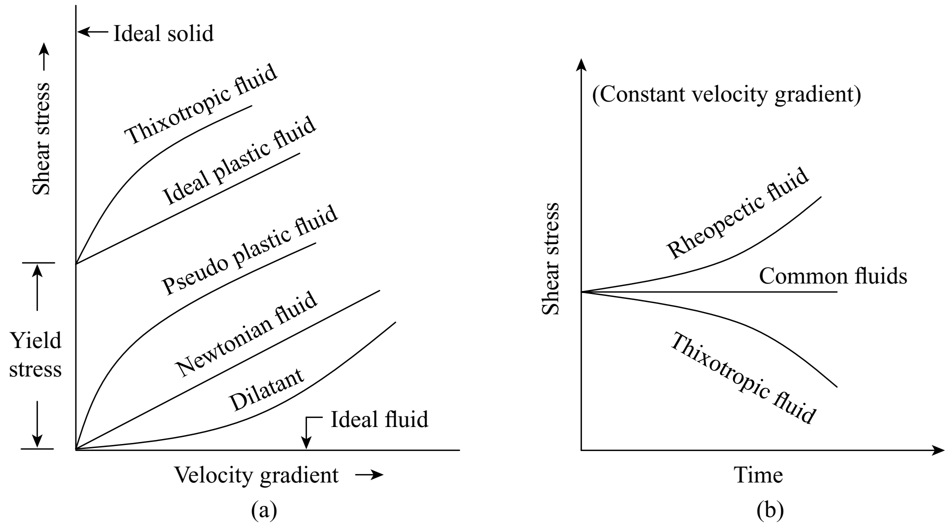

Figure 1.3 *Types of fluids*

2. **Newtonian fluid:** Fluids which obey the Newton's law of viscosity are known as Newtonian fluids. These are real fluids in which there is a linear relation between the magnitude of shear stress and the resulting velocity gradient (Figure 1.3(a)). Some of the Newtonian fluids are air, water, glycerine, kerosene and molten metals. It is important to note that this book deals with Newtonian fluids only.
3. **Non-Newtonian fluid:** Fluids which do not obey the Newton's law of viscosity are known as non-Newtonian fluids. These are real fluids in which there is a non-linear relation between the magnitude of shear stress and velocity gradient as shown in Figure 1.3(a). The behaviour of non-Newtonian fluid may be given by power law as expressed below.

$$\boxed{\tau = k\left(\frac{du}{dy}\right)^n} \tag{1.10}$$

Here, k is a consistency index and n is a flow behaviour index. For a Newtonian fluid $k = \mu$ and $n = 1$.

Some of the non-Newtonian fluids are slurries, polymer solutions, suspensions, grouts, human blood, thick lubricating oil, toothpaste and gels. The study of non-Newtonian fluids is known as '*Rheology*' and it comes under the following groups.

(a) **Pseudo-plastic fluids:** Non-Newtonian fluids for which $n < 1$ are called pseudo-plastic fluids (or shear thinning fluids) and its dynamic viscosity decreases as the rate of shear increases. Some examples of this type of fluids are milk, blood, colloidal solutions, clay and liquid cement.

(b) **Dilatant:** Non-Newtonian fluids for which $n > 1$ are called dilatant (or shear thickening fluids) and its dynamic viscosity increases as the rate of shear increases. Some examples of this type of fluids are concentrated sugar solution, aqueous suspension of rice starch and quicksand.

(c) **Ideal plastic fluid:** Non-Newtonian fluids in which shear stress is more than the yield value and there is a linear relation between shear stress and the velocity gradient is known as ideal plastic fluid or Bingham plastic. For example, sewage sludge and toothpaste does not flow out of the tube until a finite stress is applied by squeezing.

(d) **Thixotropic fluid:** Non-Newtonian fluids which have a non-linear relationship between the shear stress and the velocity gradient beyond an initial yield stress are called thixotropic fluids, for example, printer's ink. These fluids thin out with time and require decreasing stress to maintain a constant velocity gradient, whereas fluids which require increasing shear stress to maintain a constant velocity gradient are called rheopectic (Figure 1.3(b)).

Example 1.3 A liquid has kinematic viscosity of 5 stokes and specific gravity of 1.6, determine its dynamic viscosity in poise.

Solution

Let $\nu = 5 \text{ stokes} = 5\times10^{-4}\ \text{m}^2/\text{s}$ and $S = 1.6$.

$$\rho = S\times\rho_w = 1.6\times1000 = 1600\ \text{kg/m}^3$$

$$\mu = \rho\times\nu = 1600\times5\times10^{-4} = 0.8\ \text{Ns/m}^2 = 0.8\times10 = \mathbf{8\ poise}$$

Example 1.4 The shear stress at a point in oil of density 800 kg/m^3 is 0.25 N/m^2 and the rate of shear strain at that point is 0.15 per second, determine its kinematic viscosity in stoke.

Solution

Let $\rho = 800\ \text{kg/m}^3$, $\tau = 0.25\ \text{N/m}^2$ and $(du/dy) = 0.15$ per second.

From Equation (1.6), we get:

$$\mu = \frac{\tau}{(du/dy)} = \frac{0.25}{0.15} = 1.67\ \text{Ns/m}^2$$

$$\nu = \frac{\mu}{\rho} = \frac{1.67}{800} = 2.0875\times10^{-3}\ \text{m}^2/\text{s}$$

$$\therefore\ \nu = 2.0875\times10^{-3}\times10^4 = \mathbf{20.875\ stokes}$$

Example 1.5 A horizontal flat plate which is at a distance of 0.04 mm from another fixed flat plate moves with a velocity of 1 m/s. It requires a force of 1.8 N/m^2 to maintain its speed in the oil placed between the plates. Find the viscosity of oil in poise.

Solution

Let $dy = 0.04\ \text{mm} = 0.04\times10^{-3}\ \text{m}$, $u = 1\ \text{m/s}$ and $\tau = 1.8\ \text{N/m}^2$.

$$du = (u-0) = 1\ \text{m/s}$$

From Equation (1.6), we get:

$$\mu = \tau\frac{dy}{du} = 1.8\times\frac{0.04\times10^{-3}}{1} = 7.2\times10^{-5}\ \text{Ns/m}^2$$

$$\therefore\ \mu = 7.2\times10^{-5}\times10 = \mathbf{7.2\times10^{-4}\ poise}$$

Example 1.6 Two horizontal flat plates are placed 0.12 mm apart and the space between them is filled with an oil of viscosity 1.2 poise. The upper plate of area 1.6 m^2 is pulled with a speed of 0.45 m/s relative to the lower plate. Evaluate the shear stress, shear force, and power required to maintain the given speed.

Solution

Let $dy = 0.12\ \text{mm} = 0.12\times10^{-3}\ \text{m}$, $\mu = 1.2\ \text{poise} = 0.12\ \text{Ns/m}^2$, $A = 1.6\ \text{m}^2$ and $u = 0.45\ \text{m/s}$.

Let F be the shear force and P be the power required.

$$du = (u - 0) = 0.45 \text{ m/s}$$

$$\tau = \mu \frac{du}{dy} = 0.12 \times \frac{0.45}{0.12 \times 10^{-3}} = \mathbf{450\ N/m^2}$$

$$F = \tau \times A = 450 \times 1.6 = \mathbf{720\ N}$$

$$P = F \times u = 720 \times 0.45 = \mathbf{324\ W}$$

Example 1.7 A flat plate having an area of 0.64 m^2 slides down the inclined plane at an angle of 30° to the horizontal with a speed of 0.35 m/s. A lubricant layer of 1.6 mm thickness is placed between the plane and the plate. Determine the viscosity of the lubricant used if the weight of the plate is 250 N.

Solution

Refer Figure 1.4. Let $A = 0.64$ m^2, $\alpha = 30°$, $u = 0.35$ m/s, $dy = 1.6$ mm = 0.0016 m and $W = 250$ N.

The load along the plate is equal to the shear force on the bottom of the plate and it is given as follows.

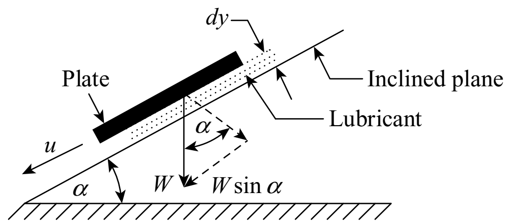

Figure 1.4

$$F = W \sin\alpha = 250 \sin 30° = 125 \text{ N}$$

$$\tau = \frac{F}{A} = \frac{125}{0.64} = 195.31 \text{ N/m}^2$$

$$du = (u - 0) = 0.35 \text{ m/s}$$

From Equation (1.6), we get:

$$\mu = \tau \frac{dy}{du} = 195.31 \times \frac{0.0016}{0.35} = \mathbf{0.8928\ Ns/m^2}$$

Example 1.8 The velocity distribution for flow over a plate is given by $u = 3y - y^2$, where u is the velocity in m/s at a distance y metre above the plate. Find the velocity gradient and shear stress at the boundary and 0.2 m from it when $\mu = 0.86$ Ns/m^2.

Solution

Let $u = 3y - y^2$, $y = 0$ m and 0.2 m and $\mu = 0.86$ Ns/m^2.

$$\frac{du}{dy} = 3 - 2y$$

Velocity gradient at the boundary, i.e., at $y = 0$ is given by,

$$\left(\frac{du}{dy}\right)_{y=0} = 3 - 2(0) = \mathbf{3\ per\ s}$$

Velocity gradient at $y = 0.2$ m is given by,

$$\left(\frac{du}{dy}\right)_{y=0.2} = 3 - 2(0.2) = \mathbf{2.6\ per\ s}$$

Shear stress at the boundary, i.e., at $y = 0$ is given by,

$$(\tau)_{y=0} = \mu\left(\frac{du}{dy}\right)_{y=0} = 0.86\times 3 = \mathbf{2.58\ N/m^2}$$

Shear stress at $y = 0.2$ m is given by,

$$(\tau)_{y=0.2} = \mu\left(\frac{du}{dy}\right)_{y=0.2} = 0.86\times 2.6 = \mathbf{2.236\ N/m^2}$$

Example 1.9 The space between two horizontal square flat plates of sides 0.7 m each is filled with a lubricant film of thickness 1 cm. The upper plate requires a force of 100 N to maintain its speed of 2 m/s while the lower plate is fixed. Evaluate the dynamic viscosity in poise and the kinematic viscosity in stokes of the lubricant if its specific gravity is 0.96.

Solution

Let $x = 0.7$ m, $dy = 1\text{ cm} = 0.01$ m, $F = 100$ N, $u = 2$ m/s and $S = 0.96$.

$$A = x^2 = 0.7^2 = 0.49\text{ m}^2$$

$$du = (u - 0) = 2\text{ m/s}$$

$$\tau = \frac{F}{A} = \frac{100}{0.49} = 204.08\text{ N/m}^2$$

From Equation (1.6), we get:

$$\mu = \tau\frac{dy}{du} = 204.08\times\frac{0.01}{2} = 1.0204\text{ Ns/m}^2$$

$$\therefore\ \mu = 1.0204\times 10 = \mathbf{10.204\ poise}$$

$$\rho = S\times\rho_w = 0.96\times 1000 = 960\text{ kg/m}^3$$

$$\nu = \frac{\mu}{\rho} = \frac{1.0204}{960} = 1.0629\times 10^{-3}\text{ m}^2/\text{s}$$

$$\therefore\ \nu = 1.0629\times 10^{-3}\times 10^4 = \mathbf{10.629\ stokes}$$

Example 1.10 The parabolic velocity profile of a fluid over a flat plate with vortex 25 cm from the plate is given by $u = ay^2 + by + c$, where the vortex velocity is 150 cm/s. If the dynamic viscosity of the fluid is 8.9 poise, then determine (i) velocity gradients and (ii) shear stresses at distances of 0 cm and 12.5 cm from the plate.

Solution

Refer Figure 1.5. Let $y_{\text{vortex}} = 25$ cm, $u = ay^2 + by + c$, $u_{\text{vortex}} = 150$ cm/s, $\mu = 8.9\text{ poise} = 0.89\text{ Ns/m}^2$ and $y = 0$ cm and 12.5 cm.

The boundary conditions becomes,

(a) at $y = 0$, $u = 0$,

(b) at $y = 25$ cm, $u = 150$ cm/s and $(du/dy) = 0$

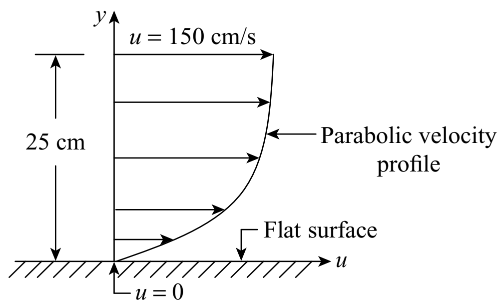

Figure 1.5

$$u = ay^2 + by + c \quad \text{(i)}$$

$$\frac{du}{dy} = 2ay + b \quad \text{(ii)}$$

Substituting boundary condition (a) in Equation (i), we get:

$$0 = a(0)^2 + b(0) + c \Rightarrow c = 0$$

Substituting boundary condition (b) and value of $c = 0$ in Equation (i), we get:

$$150 = a(25)^2 + b(25) + (0) \Rightarrow 625a + 25b = 150 \quad \text{(iii)}$$

Substituting boundary condition (b) in Equation (ii), we get:

$$0 = 2a(25) + b \Rightarrow 50a + b = 0 \quad \text{(iv)}$$

Solving equations (iii) and (iv), we get:

$$a = -0.24 \text{ and } b = 12$$

Substituting the values of a, b and c in Equation (i), we get:

$$u = -0.24y^2 + 12y$$

(i) Velocity gradients:

$$\frac{du}{dy} = -0.48y + 12$$

Velocity gradient at $y = 0$ cm is given by,

$$\left(\frac{du}{dy}\right)_{y=0} = -0.48(0) + 12 = \mathbf{12\ per\ s}$$

Velocity gradient at $y = 12.5$ cm is given by,

$$\left(\frac{du}{dy}\right)_{y=12.5} = -0.48(12.5) + 12 = \mathbf{6\ per\ s}$$

(ii) Shear stress at $y = 0$ is given by,

$$(\tau)_{y=0} = \mu\left(\frac{du}{dy}\right)_{y=0} = 0.89 \times 12 = \mathbf{10.68\ N/m^2}$$

Shear stress at $y = 12.5$ cm is given by,

$$(\tau)_{y=12.5} = \mu\left(\frac{du}{dy}\right)_{y=12.5} = 0.89 \times 6 = \mathbf{5.34\ N/m^2}$$

Example 1.11 A vertical cylinder of diameter 16 cm rotates concentrically inside another cylinder of diameter 16.1 cm. The clearance space between the cylinders is filled with a liquid of unknown viscosity which has a linear viscosity profile and both cylinders are 24 cm high. Find the viscosity of the liquid if a torque of 10 Nm is required to rotate the inner cylinder at a speed of 50 rpm.

Solution

Let $D = 16$ cm $= 0.16$ m, $D_1 = 16.1$ cm $= 0.161$ m, $l = 24$ cm $= 0.24$ m, $T = 10$ Nm and $N = 50$ rpm.

Let u be the speed of the cylinder and t be the fluid film thickness.

$$t = \frac{D_1 - D}{2} = \frac{0.161 - 0.16}{2} = 0.0005 \text{ m}$$

$$u = \frac{\pi DN}{60} = \frac{\pi \times 0.16 \times 50}{60} = 0.4189 \text{ m/s}$$

$$F = \tau \times A = \mu \frac{du}{dy} \times A = \mu \frac{u}{t} \times \pi Dl \qquad \text{[Linear velocity profile]}$$

Since
$$T = F \times R = \mu \frac{u}{t} \times \pi Dl \times \frac{D}{2}$$

$$10 = \mu \times \frac{0.4189}{0.0005} \times \pi \times 0.16 \times 0.24 \times \frac{0.16}{2}$$

$$\therefore \ \mu = \frac{10 \times 0.0005 \times 2}{0.4189 \times \pi \times 0.16 \times 0.24 \times 0.16} = \mathbf{1.237 \ Ns/m^2}$$

Example 1.12 A dash pot is 0.2 m in diameter and 0.25 m long and it slides down in a vertical cylinder of diameter 0.21 m. The lubricating oil filled in the annular space has a viscosity of 0.5 poise and has a linear velocity profile. When the load on the piston is 25 N, find the speed with which the piston slides down.

Solution

Refer Figure 1.6. Let $D = 0.2$ m, $l = 0.25$ m, $D_1 = 0.21$ m, $\mu = 0.5$ poise $= 0.05 \text{ Ns/m}^2$ and $F = 25$ N.

Let u be the speed of dash pot and t be the thickness of fluid film.

$$t = \frac{D_1 - D}{2} = \frac{0.21 - 0.2}{2} = 0.005 \text{ m}$$

$$F = \tau \times A = \mu \frac{u}{t} \times \pi Dl \qquad \text{[Linear velocity profile]}$$

Thus
$$25 = 0.05 \times \frac{u}{0.005} \times \pi \times 0.2 \times 0.25$$

$$\therefore \ u = \frac{25 \times 0.005}{0.05 \times \pi \times 0.2 \times 0.25} = \mathbf{15.915 \ m/s}$$

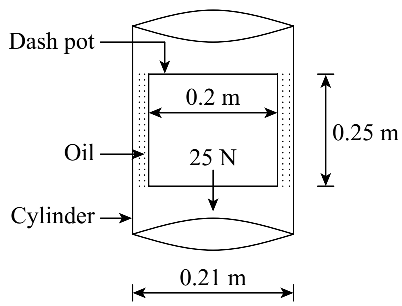

Figure 1.6

Example 1.13 In a 75 mm long horizontal journal bearing arrangement, a shaft of diameter 25 mm rotates at 1500 rpm. The clearance space between the two at concentric condition is 0.12 mm in which a Newtonian lubricant of viscosity 0.2 poise is filled. Find the frictional torque and the corresponding power loss if the velocity variation in the lubricant is linear.

Solution

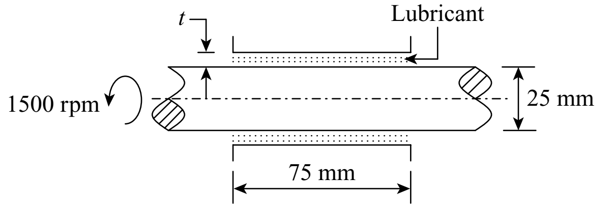

Figure 1.7

Refer Figure 1.7. Let $l = 75$ mm $= 0.075$ m, $D = 25$ mm $= 0.025$ m, $N = 1500$ rpm, $t = 0.12$ mm $= 0.12 \times 10^{-3}$ m and $\mu = 0.2$ poise $=$ 0.02 Ns/m^2.

Let u be the speed, t be the fluid film thickness, T be the frictional torque and P be the power loss.

$$u = \frac{\pi DN}{60} = \frac{\pi \times 0.025 \times 1500}{60} = 1.96 \text{ m/s}$$

$$T = F \times R = \mu \frac{u}{t} \times \pi Dl \times \frac{D}{2} \qquad \text{[Linear velocity profile]}$$

$$\therefore T = 0.02 \times \frac{1.96}{0.12 \times 10^{-3}} \times \pi \times 0.025 \times 0.075 \times \frac{0.025}{2} = \mathbf{0.024\ Nm}$$

$$P = \frac{2\pi NT}{60} = \frac{2\pi \times 1500 \times 0.024}{60} = \mathbf{3.77\ W}$$

Example 1.14 A central thin plate of area 3 m^2 equidistant from both the fixed planes 16 mm apart is being pulled with a force of 100 N. Find the velocity at which the central thin plate moves when the viscosities of the two fluids are in the ratio of 1 : 4 and the viscosity of top fluid is 0.15 Ns/m^2.

Solution

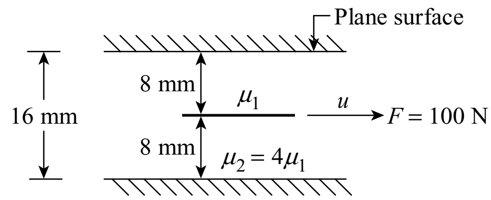

Figure 1.8

Refer Figure 1.8. Let $A = 3\text{ m}^2$, $y = 16$ mm, $F = 100$ N, $\mu_2 = 4\mu_1$ and $\mu_1 = 0.15\text{ Ns/m}^2$.

Let F_1 be the shear force on the upper side of thin plate, F_2 be the shear force on the lower side of thin plate, F be the total force required to drag the plate, u be the thin plate speed and t be the fluid film thickness between the plates and the planes. Assume linear velocity profile.

$$t_1 = t_2 = t = \frac{y}{2} = \frac{16}{2} = 8 \text{ mm} = 8 \times 10^{-3} \text{ m}$$

Since
$$F = F_1 + F_2 = \tau_1 A + \tau_2 A = \mu_1 \left(\frac{u}{t}\right) A + \mu_2 \left(\frac{u}{t}\right) A$$

$$F = \mu_1 \left(\frac{u}{t}\right) A + 4\mu_1 \left(\frac{u}{t}\right) A = 5\mu_1 \left(\frac{u}{t}\right) A$$

$$100 = 5 \times 0.15 \times \left(\frac{u}{8 \times 10^{-3}}\right) \times 3$$

$$\therefore\ u = \frac{100 \times 8 \times 10^{-3}}{5 \times 0.15 \times 3} = 0.355 \text{ m/s}$$

Example 1.15 A 20 mm wide gap between two vertical plane surfaces is filled with a lubricating fluid of specific gravity 0.9 and dynamic viscosity 20 poise. A metal plate of thickness 1 m × 1 m × 0.002 m and weight 50 N is placed midway in the gap. Determine the force required when the plate is lifted up with a constant velocity of 0.1 m/s.

Solution

Refer Figure 1.9. Let $d = 20$ mm = 0.02 m, $S = 0.9$, $\mu = 20$ poise = 2 Ns/m^2, $l = 1$ m, $b = 1$ m, $h = 0.002$ m, $W = 50$ N and $u = 0.1$ m/s.

Let F_1 be the shear force on the left side of the metal plate, F_2 be the shear force on the right side of the metal plate, F be the drag force against the plate, u be the speed of metal plate and t be the fluid thickness between the plate and the plane surface. Assume linear velocity profile.

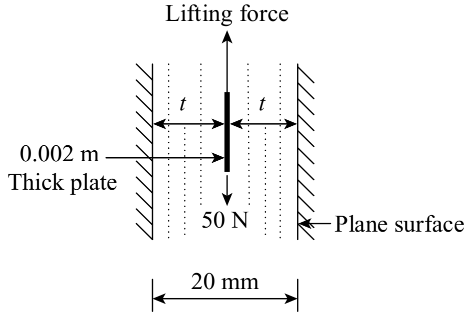

Figure 1.9

$$t_1 = t_2 = t = \frac{d-h}{2} = \frac{0.02-0.002}{2} = 0.009 \text{ m}$$

Since $$F = F_1 + F_2 = \tau_1 A + \tau_2 A = \mu\left(\frac{u}{t}\right)A + \mu\left(\frac{u}{t}\right)A = 2\mu\left(\frac{u}{t}\right)A$$

$$\therefore\ F = 2\times 2\times\left(\frac{0.1}{0.009}\right)\times(1\times 1) = 44.44 \text{ N}$$

Buoyant force on the plate is given by,

$$F_b = w\times v = S\rho_w g\times lbh = 0.9\times 1000\times 9.81\times 1\times 1\times 0.002 = 17.66 \text{ N}$$

Effective weight of the plate is given by,

$$F_e = W - F_b = 50 - 17.66 = 32.34 \text{ N}$$

Therefore, the total force required to lift the metal plate is given by,

$$F_t = F + F_e = 44.44 + 32.34 = \mathbf{76.78\ N}$$

Example 1.16 A skater weighing 500 N skates at 10 m/s and is supported by an average skating area of 10 cm^2. If the viscosity of water is 1 centipoise and the coefficient of friction between skates and ice is 0.02, then find the thickness of thin film of water existing between the skates and the ice.

Solution

Let $W = 500$ N, $u = 10$ m/s, $A = 10 \text{ cm}^2 = 10\times 10^{-4} \text{ m}^2$, $\mu = 1$ cP = 0.001 Ns/m^2 and $\mu_f = 0.02$.

Let F be the frictional force which equals viscous shear force, u be the skater speed and t be the water film thickness between the skates and the ice.

$$F = \mu_f \times W = 0.02\times 500 = 10 \text{ N}$$

$$F = \tau\times A = \mu\frac{u}{t}\times A \qquad \text{[Linear velocity profile]}$$

$$10 = 0.001 \times \left(\frac{10}{t}\right) \times (10 \times 10^{-4})$$

$$\therefore\ t = \frac{0.001 \times 10 \times 10 \times 10^{-4}}{10} = \mathbf{1 \times 10^{-6}\ m}$$

Example 1.17 A hydraulic lift used for lifting cars has a ram of 40 cm diameter and it slides in a 40.02 cm diameter cylinder. The annular space is filled with oil whose kinematic viscosity is 5 cm²/s and specific gravity is 0.85. Determine the viscous resistance when the ram of 2 m length in the cylinder travels with a velocity of 15 cm/s.

Solution

Let $D = 40$ cm $= 0.4$ m, $D_1 = 40.02$ cm $= 0.4002$ m, $v = 5$ cm²/s $= 5 \times 10^{-4}$ m²/s, $S = 0.85$, $l = 2$ m and $u = 15$ cm/s $= 0.15$ m/s.

Let u be the speed of ram, t be the oil film thickness and F be the viscous force.

$$t = \frac{D_1 - D}{2} = \frac{0.4002 - 0.4}{2} = 0.0001\ \text{m}$$

$$\rho = S\rho_w = 0.85 \times 1000 = 850\ \text{kg/m}^3$$

$$\mu = \rho v = 850 \times 5 \times 10^{-4} = 0.425\ \text{Ns/m}^2$$

Since

$$F = \tau \times A = \mu \frac{u}{t} \times \pi D l$$

$$\therefore\ F = 0.425 \times \frac{0.15}{0.0001} \times \pi \times 0.4 \times 2 = \mathbf{1602.21\ N}$$

Example 1.18 A disc of diameter 80 mm is rotated on a spindle and is enclosed in a small chamber filled with oil of viscosity 5 poise. Determine the torque required to rotate the disc at 60 rpm when the clearance at the top and the bottom of the disc is 1 mm. Assume linear velocity profile in the oil film.

Solution

Refer Figure 1.10. Let $D = 80\ \text{mm} = 0.08\ \text{m}$, $\mu = 5\ \text{poise} = 0.5\ \text{Ns/m}^2$, $N = 60$ rpm and $t = 1\ \text{mm} = 0.001$ m.

Let u be the tangential velocity of disc, t be the oil film thickness, F be the viscous force and T be the total torque acting on the disc.

Consider an element of width dr at a radial distance r. Shear force acting on this element after assuming linear velocity profile can be given as follows.

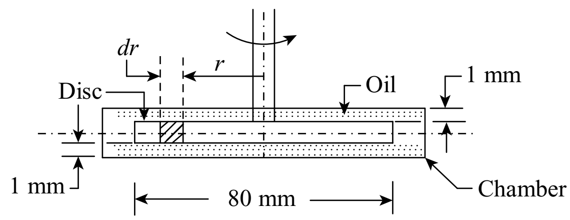

Figure 1.10

$$dF = \tau \times A = \mu \frac{u}{t} \times (2\pi r dr) = \mu\left(\frac{\omega r}{t}\right) \times (2\pi r dr)$$

$[\because u = \omega r]$

Viscous torque on one face of the element is given by,

$$dT = dF \times r = \mu \frac{\omega r}{t} \times 2\pi r dr \times r$$

Therefore, the torque acting on both faces is obtained by integrating the above expression from $r = 0$ to $r = R$ and by multiplying two as given below.

$$T = 2\left[\frac{2\pi\mu\omega}{t}\int_0^R r^3 dr\right] = \frac{\pi\mu\omega}{t}R^4 = \frac{\pi\mu}{t}\times\frac{2\pi N}{60}\times\left(\frac{D}{2}\right)^4 \qquad [\because \omega = 2\pi N/60]$$

$$\therefore T = \frac{\pi\times 0.5}{0.001}\times\frac{2\pi\times 60}{60}\times\left(\frac{0.08}{2}\right)^4 = \mathbf{0.0253\ Nm}$$

Example 1.19 Two large fixed parallel planes are 20 mm apart. The space between the surfaces is filled with a lubricating oil of viscosity 0.85 Ns/m^2. A flat thin plate 0.4 m^2 area moves through the oil at a velocity of 0.5 m/s. Determine the drag force when the thin plate is at a distance of 6 mm from one of the plane surfaces.

Solution

Refer Figure 1.11. Let y = 20 mm = 0.02 m, μ = 0.85 Ns/m^2, A = 0.4 m^2, u = 0.5 m/s, t_1 = 6 mm = 0.006 m and t_2 = 20 − 6 = 14 mm = 0.014 m.

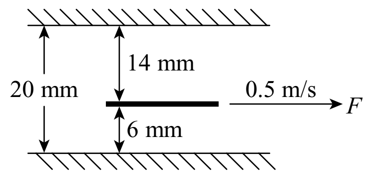

Figure 1.11

Since $$F = \mu\left(\frac{u}{t_1}\right)A + \mu\left(\frac{u}{t_2}\right)A$$

$$\therefore\ F = 0.85\times\left(\frac{0.5}{0.006}\right)\times 0.4 + 0.85\times\left(\frac{0.5}{0.014}\right)\times 0.4 = \mathbf{40.4762\ N}$$

1.14 ❐ THERMODYNAMIC PROPERTIES

1.14.1 Perfect Gas Law

The gases are compressible and its behaviour is different from liquids. All the gases at high temperature and low pressures (relative to their critical point) obey the perfect gas law (or characteristic equation for gases) as given in the below expression.

$$pv = mRT \quad \text{or} \quad pv_s = RT \tag{1.11}$$

or

$$\frac{p}{\rho} = RT \qquad [\because v_s = 1/\rho] \tag{1.11a}$$

Here, p is the absolute pressure in N/m^2, v is the volume of m kg of gas in m^3, v_s is the specific volume in m^3/kg, ρ is the mass density in kg/m^3, T is the absolute temperature in K and R is the gas constant.

The units of gas constant can be given from Equation (1.11a) and we get the following expression.

$$R = \frac{p}{\rho T} = \frac{(\text{N/m}^2)}{(\text{kg/m}^3)\times\text{K}} = \frac{\text{Nm}}{\text{kgK}} = \frac{\text{J}}{\text{kgK}}$$

1.14.2 Universal Gas Constant

The characteristic equation given by Equation (1.11) can also be expressed in mole basis which is universally applicable.

Let for a gas, m be the mass in kg, M be its molecular weight in kg/mol and n be the number of moles in mol, then we get the following expression.

$$m = nM \tag{1.12}$$

From Equation (1.11), we get:

$$pv = nMRT$$

$$MR = \frac{pv}{nT} \tag{1.13}$$

According to Avogadro's hypothesis, the molal volume (v/n) is same for all the gases at the same values of p and T.

From Equation (1.13), we get:

$$MR = \frac{pv}{nT} = R_o \tag{1.13a}$$

$$R = \frac{R_o}{M} \tag{1.14}$$

Here, R_o is the universal gas constant whose value is given below.

The volume of one mole of any perfect gas at N.T.P., $p = 1\ \text{bar} = 1.01325\times10^5\ \text{N/m}^2$ and $T = 0\ °\text{C} = 273.15\ \text{K}$ is approximately equal to 22.4136 m^3.

From Equation (1.13a), we get:

$$R_o = \frac{pv}{nT} = \frac{1.01325\times10^5\times22.4136}{1\times273.15} = 8314.3\ \text{Nm/mol K}$$

From Equation (1.14), we get:

$$\boxed{R = \frac{8314.3}{M}\ \text{Nm/kg K}} \tag{1.14a}$$

Therefore, the value of gas constant of a gas can be calculated from Equation (1.14a) if its molecular weight is given.

1.14.3 Isothermal Process (Constant Temperature Process)

When the change in density of a fluid system occurs at constant temperature, then it is called isothermal process.

From Equation (1.11), we get:

$$pv = \text{Constant} \qquad \text{(Boyle's law)}$$

$$pv_s = \frac{p}{\rho} = C \qquad [\because v_s = 1/\rho] \tag{1.15}$$

1.14.4 Isobaric Process (Constant Pressure Process)

When the change in density of a fluid system occurs at constant pressure, then the process is called isobaric process.

From Equation (1.11), we get:

$$\frac{v}{T} = \text{Constant} \qquad \text{(Charle's law)}$$

$$\frac{v_s}{T} = \frac{1}{\rho T} = C \qquad [\because v_s = 1/\rho] \tag{1.16}$$

1.14.5 Reversible Adiabatic Process (Isentropic Process)

When the change in density of a fluid system occurs without involving any heat transfer and irreversibility (such as friction), then the process is called reversible adiabatic process. A reversible adiabatic process (or frictionless adiabatic process) is also known as isentropic process. The relation for reversible adiabatic process is given in the following expression.

$$pv^{\gamma} = \text{Constant}$$

$$p{v_s}^{\gamma} = \frac{p}{\rho^{\gamma}} = C \qquad [\because v_s = 1/\rho] \tag{1.17}$$

Here, $\gamma = (c_p/c_v)$, c_p is the specific heat at constant pressure, c_v is the specific heat at constant volume and C is a constant. For air, $\gamma = 1.4$ and it may also be shown that $R = (c_p - c_v)$.

Example 1.20 Determine the gas constant, density, specific volume and specific weight of CO_2 containing in a vessel at a temperature of 25°C and absolute pressure of 5 bar.

Solution

Let $T = 25°\text{C} = 298.15$ K and $p = 5$ bar $= 5\times10^5$ N/m^2.

Molecular weight of CO_2:

$$M = 12 + 2\times16 = 44$$

$$R = \frac{8314.3}{M} = \frac{8314.3}{44} = \mathbf{188.96\ Nm/kgK}$$

$$\rho = \frac{p}{RT} = \frac{5\times10^5}{188.96\times298.15} = \mathbf{8.875\ kg/m^3}$$

$$v_s = \frac{1}{\rho} = \frac{1}{8.875} = \mathbf{0.1127\ m^3/kg}$$

$$w = \rho g = 8.875\times9.81 = \mathbf{87.064\ N/m^3}$$

Example 1.21 Determine the density and gas constant of a gas containing in a vessel at a temperature of 30°C and absolute pressure of 2 bar when it weighs 10 N/m^3.

Solution

Let $T = 30°\text{C} = 303.15$ K, $p = 2$ bar $= 2\times10^5$ N/m^2 and $w = 10$ N/m^3.

$$\rho = \frac{w}{g} = \frac{10}{9.81} = \mathbf{1.019\ kg/m^3}$$

$$R = \frac{p}{\rho T} = \frac{2\times10^5}{1.019\times303.15} = \mathbf{647.44\ Nm/kgK}$$

1.15 ❒ SURFACE TENSION

The surface of the liquids behaves like a stretched elastic membrane under tension. Due to intermolecular attraction (i.e., cohesion) between molecules, a pulling force acts parallel to the surface. The magnitude of this force per unit length is called surface tension which is denoted by σ and it is usually expressed in N/m. Its dimensions are $[MT^{-2}]$.

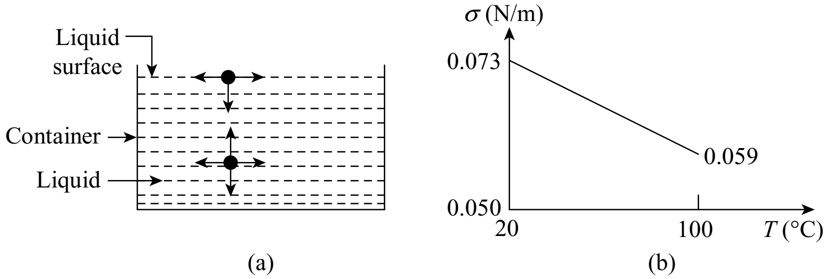

Figure 1.12 *(a) Surface tension (b) Surface tension of water versus temperature*

This phenomenon also appears as surface energy which may be defined as the work done against the pulling force for the formation of a surface and it is expressed in $\mathrm{N \cdot m/m^2}$ or $\mathrm{J/m^2}$.

A liquid molecule inside a fluid mass is equally attracted on all the sides and thus, the forces of attraction are in equilibrium (Figure 1.12(a)). However, a molecule at the surface of the liquid does not have any liquid molecule above it and consequently, there is a net downward force on it due to the attraction by the molecules below it. Thus, a film or an elastic membrane seems to form on the liquid surface which remains in tension and it can support small loads like a small steel needle that can float on it and water strider (an insect) can walk on the water surface in a pond. The surface tension phenomenon can be observed in a mercury drop which forms a sphere, a soap bubble, a liquid fuel injected into an engine which forms a mist of spherical droplets, water droplets and capillarity.

The surface tension (σ) of liquids decreases with increase in temperature (*T*). It depends on the cohesive forces and the fluid in contact with the liquid surface. The magnitude of surface tension of water in contact with air at 20°C is 0.073 N/m and it decreases to 0.059 N/m when the temperature is increased to 100°C (Figure 1.12(b)). The surface tension of some fluids in air at 20°C is given in Table 1.5. The addition of soaps and detergents lower the surface tension of water.

The effect of surface tension is to minimize the surface of the liquid and thus, the drops of liquid tend to take a spherical shape. For a droplet, surface tension increases the internal pressure *p* to balance the surface force. In the following sections, the pressure inside a liquid droplet, soap bubble and a liquid jet is described.

Table 1.5 *Surface tension of some fluids in air at 20°C*

Liquid	SAE 30 oil	Petrol	Soap solution	Glycerine	Ethyl alcohol	NH_3	Hg	Kerosene
σ(N/m)	0.035	0.022	0.025	0.063	0.023	0.021	0.44	0.028

1.15.1 Pressure Inside a Liquid Droplet

Consider a small spherical droplet of liquid of diameter *d* as illustrated in Figure 1.13.

Let σ be the surface tension of the liquid and *p* be the pressure intensity inside the liquid droplet which is above the atmospheric pressure.

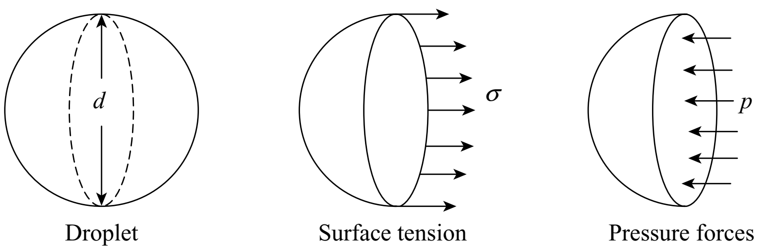

Figure 1.13 *Pressure inside the liquid droplet*

The forces acting on one half of the droplet are as follows.

1. Tensile force due to surface tension which acts around the circumference is given below.

$$F_\sigma = \sigma \times \text{circumference} = \sigma \times \pi d$$

2. Pressure force is given by,

$$F_p = p \times \frac{\pi}{4} d^2$$

$$F_p = F_\sigma \qquad \text{[For equilibrium]}$$

$$p \times \frac{\pi}{4} d^2 = \sigma \times \pi d$$

$$\therefore \; p = \frac{\sigma \times \pi d}{(\pi/4)\, d^2} = \frac{4\sigma}{d} \tag{1.18}$$

From Equation (1.18), it can be observed that pressure intensity inside the liquid droplet decreases with increase in its diameter or size.

1.15.2 Pressure Inside a Soap Bubble

A soap bubble has two surfaces in contact with air, one inside the bubble and the other outside it as schematically shown in Figure 1.14. Thus, the surface tension force will act on both the surfaces.

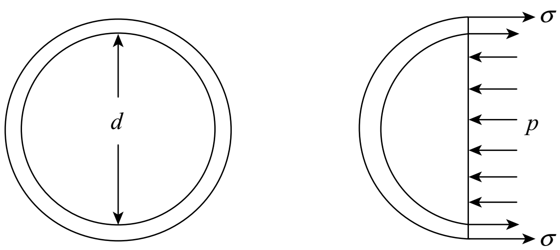

Figure 1.14 *Pressure inside the soap bubble*

For equilibrium of the soap bubble, the required condition is as follows.

$$p \times \frac{\pi}{4} d^2 = 2(\sigma \times \pi d)$$

$$\therefore p = \frac{2\sigma \times \pi d}{(\pi/4)\, d^2} = \frac{8\sigma}{d} \tag{1.19}$$

1.15.3 Pressure Inside a Liquid Jet

A schematic view of a cylindrical liquid jet of diameter d and length l is shown in Figure 1.15.

The following forces act on the liquid jet.

$$\text{Pressure force} = p \times l \times d$$

$$\text{Surface tension force} = \sigma \times 2 \times l$$

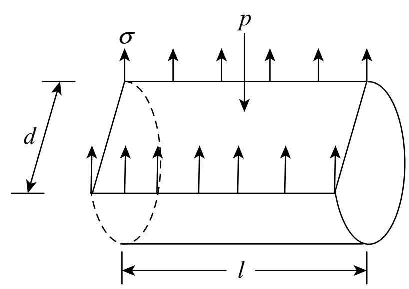

Figure 1.15 *Pressure inside a liquid jet*

For equilibrium of the liquid jet, equating the above forces, we get:

$$p \times l \times d = \sigma \times 2l$$

$$\therefore \ p = \frac{\sigma \times 2l}{l \times d} = \frac{2\sigma}{d} \tag{1.20}$$

Example 1.22 The pressure inside a soap bubble of 25 mm diameter is 2 N/m^2 above the atmosphere. Determine the surface tension of the soap film.

Solution

Let $d = 25 \text{ mm} = 0.025 \text{ m}$ and $p = 2 \text{ N/m}^2$.

$$\sigma = \frac{pd}{8} = \frac{2 \times 0.025}{8} = \mathbf{6.25 \times 10^{-3} \ N/m}$$

Example 1.23 Determine the power required to convert 1.2 litres of water per minute at a temperature of 20°C into a mist having an average drop size of 3.5×10^{-6} m. Also determine the pressure intensity inside the mist droplets and neglect any thermal effects.

Solution

Let $v = 1.2 \ l/\min = 1.2 \times 10^{-3} \ \text{m}^3/\text{min}$, $T = 20°\text{C}$ and $d = 3.5 \times 10^{-6}$ m.

Take $\sigma = 0.073$ N/m

Number of drops produced per second (n) can be calculated by dividing the total volume per sec by volume of single drop as given below.

$$n = \frac{(v/60)}{(\pi/6)d^3} = \frac{1.2 \times 10^{-3} \times 6}{60\pi \times (3.5 \times 10^{-6})^3} = 8.91 \times 10^{11}$$

Power is equal to the product of surface tension and the rate of surface area produced.

$$P = \sigma(n \times 4\pi r^2)$$

$$\therefore \ P = 0.073 \times 8.91 \times 10^{11} \times 4\pi \times \left(\frac{3.5 \times 10^{-6}}{2}\right)^2 = \mathbf{2.503 \ Watts}$$

$$p = \frac{4\sigma}{d} = \frac{4 \times 0.073}{3.5 \times 10^{-6}} = \mathbf{83428.57 \ N/m^2}$$

Example 1.24 Determine the surface tension of a liquid of specific gravity 0.82 when a tube of diameter 3.2 mm is immersed in the liquid to a depth of 0.055 m. The maximum pressure of air applied through the tube is 500 N/m^2 above atmospheric. Neglect small differences in depth due to bubble formation.

Solution

Let $S = 0.82$, $d = 3.2 \text{ mm} = 0.0032 \text{ m}$, $h = 0.055$ m and $p = 500 \text{ N/m}^2$.

$$\rho = S\rho_w = 0.82 \times 1000 = 820 \text{ kg/m}^3$$

The net pressure acting on the film is given by,

$$p_{net} = p - \rho g h = 500 - 820 \times 9.81 \times 0.055 = 57.569 \text{ N/m}^2$$

Therefore, pressure force is given by,

$$F_p = p_{\text{net}} \times \text{area} = p_{\text{net}} \times \frac{\pi}{4} d^2 = 57.569 \times \frac{\pi}{4} \times 0.0032^2 = 4.63 \times 10^{-4} \text{ N}$$

Surface tension force is given by,

$$F_\sigma = \sigma \times \pi d = \sigma \times \pi \times 0.0032$$

$$F_\sigma = F_p \qquad \text{[Under equilibrium]}$$

$$\sigma \times \pi \times 0.0032 = 4.63 \times 10^{-4}$$

$$\therefore \ \sigma = \frac{4.63 \times 10^{-4}}{\pi \times 0.0032} = \mathbf{0.046 \ N/m}$$

1.16 ❒ CAPILLARITY (CAPILLARY EFFECT)

Capillarity means the rise or fall of a liquid surface in a small diameter tube relative to the adjacent general level of liquid when the tube is inserted vertically into the liquid. The small diameter tube is called the capillary tube and the curved free surface of the liquid in this tube is called the meniscus. The rise of kerosene through the wick dipped into the sump of a kerosene lamp is a good example of capillary effect. The rise of liquid surface is called capillary rise and the lowering of liquid surface is called capillary depression and it is expressed in terms of mm of liquid or m of liquid.

The strength of capillary effect can be known by the contact angle α, which is the angle that the tangent to the liquid surface makes with the solid surface at the point of contact. The liquid wet the surface when $\alpha < 90°$ and the liquid does not wet the surface when $\alpha > 90°$. The contact angle of water with clean glass tube is about zero, i.e., $\alpha \approx 0°$. Therefore, the surface tension force acts upward on water in a glass tube and consequently, water rises in the tube until the weight of the liquid in the tube above the liquid level of the reservoir balances the surface tension force.

Capillarity is due to both cohesion (forces between like molecules) and adhesion (forces between unlike molecules). Adhesion between glass and water molecule is greater than cohesion between water molecules. Thus, water rises in the tube and forms a concave meniscus with very small angle of contact. In the case of mercury (Hg), the cohesion force between the molecules is more than the adhesion force between the mercury molecules and the glass surface. Thus, the mercury in the clean glass tube goes down relative to the free surface in the container and forms a convex meniscus with the angle of contact for about 130°.

1.16.1 Expression for the Capillary Rise or Fall

Let σ be the surface tension of the liquid, ρ be the density of the liquid, h be the height of the liquid in the tube and α be the contact angle as shown in Figure 1.16.

The magnitude of the capillary rise or depression (fall) in a circular glass tube can be determined from a force balance given below.

Weight of the liquid raised or lowered in the capillary tube is given by,

$$\Rightarrow \text{Area of tube} \times \text{rise or fall} \times \rho g = \frac{\pi}{4} d^2 \times h \times \rho g \qquad \text{(i)}$$

Vertical component of the surface tension force is given by,

$$\Rightarrow \sigma \cos\alpha \times \text{circumference} = \sigma \cos\alpha \times \pi d \qquad \text{(ii)}$$

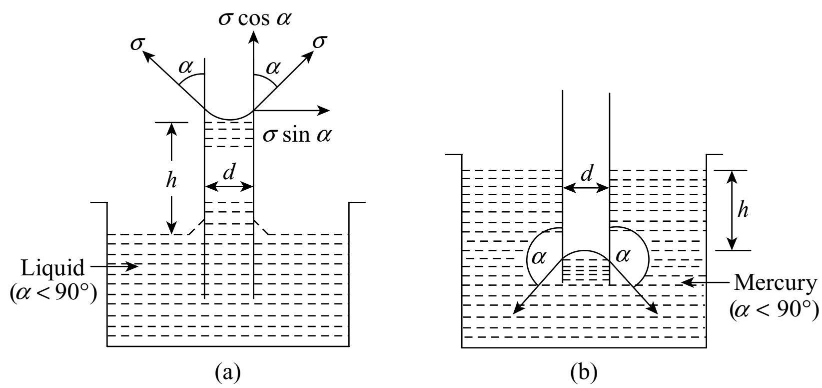

Figure 1.16 *Capillary rise and depression*

For equilibrium, equating the above forces given by the expressions (i) and (ii), we get:

$$\frac{\pi}{4}d^2 \times h \times \rho g = \sigma \cos\alpha \times \pi d$$

$$\therefore \quad \boxed{h = \frac{4\sigma\cos\alpha}{\rho g d}} \tag{1.21}$$

It can be observed from Equation (1.21) that for $0° < \alpha < 90°$, h is positive, i.e., capillary rise and that for $90° < \alpha < 180°$, h is negative, i.e., capillary depression. The capillary rise or fall is inversely proportional to the tube diameter. For minimizing the error in reading liquid levels in fine gauge tubes, the minimum tube diameter for mercury and water should be 6 mm.

Example 1.25 Determine the capillary rise in a glass tube of 3 mm diameter when inserted vertically in water and mercury (Hg). The values of surface tensions for water and Hg in contact with air are 0.073 N/m and 0.44 N/m, respectively. Assume the values of specific gravity of mercury as 13.6 and the angle of contact for Hg and water as 130° and 0°, respectively.

Solution

Let $d = 3$ mm $= 0.003$ m, $\sigma_w = 0.073$ N/m, $\sigma_{Hg} = 0.44$ N/m, S_{Hg} or $S_m = 13.6$, for mercury $\alpha = 130°$ and for water $\alpha = 0°$.

$$\rho_{Hg} = S_{Hg}\rho_w = 13.6 \times 1000 = 13600 \text{ kg/m}^3$$

(i) $h = \dfrac{4\sigma_w \cos\alpha}{\rho_w g d} = \dfrac{4 \times 0.073 \cos 0°}{1000 \times 9.81 \times 0.003} =$ **0.0099 m or 9.9 mm**

(ii) $h = \dfrac{4\sigma_{Hg} \cos\alpha}{\rho_{Hg} g d} = \dfrac{4 \times 0.44 \cos 130°}{13600 \times 9.81 \times 0.003} =$ **−0.0028 m or − 2.8 mm**

For mercury, negative sign shows the capillary depression.

Example 1.26 Calculate the minimum size of a glass tube that can be used to measure water level when capillary rise in the tube is not to exceed 0.4 mm and surface tension for water in contact with air is 0.0735 N/m. Take angle of contact $\alpha = 0°$.

Solution

Let $h = 0.4$ mm $= 0.4 \times 10^{-3}$ m, $\sigma = 0.0735$ N/m and $\alpha = 0°$.

From Equation (1.21), we get:

$$d = \frac{4\sigma\cos\alpha}{\rho_w gh} = \frac{4\times 0.0735\cos 0°}{1000\times 9.81\times 0.4\times 10^{-3}} = \mathbf{0.075\ m\ or\ 7.5\ cm}$$

Example 1.27 A U-tube is made of two capillaries of bore 1.2 mm and 2.4 mm, respectively. The vertical tube is partially filled with a liquid of surface tension 0.055 N/m and zero contact angle. If the estimated difference in the level of the two menisci is 1 cm, then determine the density of the liquid.

Solution

Let $d_1 = 1.2$ mm = 0.0012 m, $d_2 = 2.4$ mm = 0.0024 m, $\sigma = 0.055$ N/m, $\alpha = 0°$ and $(h_1 - h_2) = 1$ cm = 0.01 m.

Let h_1 and h_2 be the heights of liquid columns and ρ be the density of the liquid.

Since
$$h_1 - h_2 = \frac{4\sigma\cos\alpha}{\rho g d_1} - \frac{4\sigma\cos\alpha}{\rho g d_2} = \frac{4\sigma\cos\alpha}{\rho g}\left[\frac{1}{d_1} - \frac{1}{d_2}\right]$$

$$\rho = \frac{4\sigma\cos\alpha}{(h_1 - h_2)g}\left[\frac{1}{d_1} - \frac{1}{d_2}\right]$$

$$\therefore \rho = \frac{4\times 0.055\cos 0°}{0.01\times 9.81}\times\left[\frac{1}{0.0012} - \frac{1}{0.0024}\right] = \mathbf{934.42\ kg/m^3}$$

Example 1.28 A single column U-tube manometer made of glass has a nominal inside diameter of 2.6 mm, which is used to measure air pressure in a vessel. If the limb opened to atmosphere is 10% oversize, then find the error in m of Hg in the measurement of air pressure due to surface tension when $\sigma_{Hg} = 0.52$ N/m, $S_{Hg} = 13.6$ and angle of contact $\alpha = 128°$.

Solution

Let $d_1 = 2.6$ mm = 0.0026 m, $d_2 = 0.0026\times 1.1 = 0.00286$ m, $\sigma_{Hg} = 0.52$ N/m, S_{Hg} or $S_m = 13.6$ and $\alpha = 128°$.

Let $\Delta h = (h_1 - h_2)$ be the error in measurement.

$$\rho_{Hg} = S_{Hg}\rho_w = 13.6\times 1000 = 13600\ \text{kg/m}^3$$

Since
$$\Delta h = \frac{4\sigma_{Hg}\cos\alpha}{\rho_{Hg} g d_1} - \frac{4\sigma_{Hg}\cos\alpha}{\rho_{Hg} g d_2} = \frac{4\sigma_{Hg}\cos\alpha}{\rho_{Hg} g}\left[\frac{1}{d_1} - \frac{1}{d_2}\right]$$

$$\therefore \Delta h = \frac{4\times 0.52\cos 128°}{13600\times 9.81}\times\left[\frac{1}{0.0026} - \frac{1}{0.00286}\right] = \mathbf{-3.356\times 10^{-4}\ m}$$

1.17 ❐ COMPRESSIBILITY AND THE BULK MODULUS

The compressibility of a fluid is the measure of volumetric strain caused by unit change in pressure. In other words, compressibility or coefficient of compressibility (β) is the reciprocal of bulk modulus of elasticity of the fluid. The bulk modulus of elasticity (K) is the ratio of compressive stress to volumetric strain. Mathematically, the bulk modulus of elasticity is given in the following expression.

$$K = \frac{\text{Change in pressure}}{\left(\frac{\text{Change in volume}}{\text{Original volume}}\right)} = \frac{dp}{\left(\frac{-dv}{v}\right)} \tag{1.22}$$

Always increase in pressure causes a decrease in volume and thus, minus sign is included in the above equation to give a positive value of K.

Thus, the value of compressibility is given as follows.

$$\beta = \frac{1}{K} = -\frac{(dv/v)}{dp} \tag{1.23}$$

$$m = \rho v \tag{i}$$

$$dm = \rho dv + v d\rho \qquad \text{[Differentiating (i)]}$$

$$\rho dv + v d\rho = 0 \qquad [\because m = \text{Constant}]$$

$$\frac{d\rho}{\rho} = -\frac{dv}{v}$$

From Equation (1.22), we get:

$$\boxed{K = -\frac{dp}{(dv/v)} = -\frac{dp}{(d\rho/\rho)}} \tag{1.22a}$$

The bulk modulus of elasticity is expressed in N/m^2 and its dimensions are $[ML^{-1}T^{-2}]$. The values for the bulk modulus of elasticity for water and air at normal temperature and pressure are about 2.06×10^9 N/m^2 and 1.03×10^5 N/m^2, respectively. This indicates that air is about 20000 times more compressible than water. A large value of the bulk modulus of elasticity indicates that a large pressure will be required to cause a small change in volume. Thus, a fluid with large value of K will be incompressible, for example, liquids. However, in some problems where the changes in pressure of liquid is either very large or very rapid, such as water hammer or rapid closure of valve, the effect of compressibility is to be considered.

On the other hand, gases are highly compressible. Thus, the value of K for gases is not constant. It is proportional to pressure and changes very rapidly. It may be noted that compressibility of air is considered only at high velocities, i.e., nearing the local speed of sound in that medium and otherwise the flow of air is considered as incompressible. In Chapter 17, it is explained that velocity of sound in an ideal gas is given by the following expression.

$$\boxed{C = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{K}{\rho}} = \sqrt{\gamma RT}} \tag{1.24}$$

Equation (1.24) is for an isentropic process, since there is negligible heat transfer and the disturbance is small.

The relationships between the bulk modulus of elasticity and pressure for an ideal gas for isothermal and adiabatic process are discussed in the following sections.

1.17.1 Bulk Modulus for an Isothermal Process

The relation for isothermal process is given by the following expression.

$$pv = \text{Constant} \tag{ii}$$

$$pdv + vdp = 0 \qquad \text{[Differentiating (ii)]}$$

$$p = -\frac{dp}{(dv/v)}$$

$$\therefore \boxed{K = p} \tag{1.25}$$

Thus, for a perfect gas, the bulk modulus of elasticity equals the pressure for an isothermal process.

1.17.2 Bulk Modulus for Reversible Adiabatic Process (or Isentropic Process)

The relation for reversible adiabatic process is given by the following expression.

$$pv^{\gamma} = \text{Constant} \qquad \text{(iii)}$$

$$p\gamma v^{\gamma-1}dv + v^{\gamma}dp = 0 \qquad \text{[Differentiating (iii)]}$$

$$p\gamma = -\frac{dp}{(dv/v)}$$

$$\therefore \quad \boxed{K = p\gamma} \qquad (1.26)$$

Thus, for an ideal gas, the bulk modulus of elasticity equals γ times the pressure for an adiabatic process (or isentropic process).

Example 1.29 Calculate the increase in pressure for water necessary to produce (i) 1.2% reduction in volume at the same temperature and (ii) 1.2% reduction in volume of air undergoing adiabatic compression when standard atmospheric conditions are considered and the bulk modulus of elasticity of water is given as 2.06×10^6 kN/m^2. Also comment on the results.

Solution

Let $-(dv/v) = 1.2\% = 0.012$ and $K_w = 2.06 \times 10^6$ kN/m^2.

(i) $dp = \left(\frac{-dv}{v}\right)K_w = 0.012 \times 2.06 \times 10^6 = \mathbf{24720\ kN/m^2}$

(ii) $K_{\text{air}} = p\gamma = 101.325 \times 1.4 = 141.855$ kN/m^2

$$dp = \left(\frac{-dv}{v}\right)K_{\text{air}} = 0.012 \times 141.855 = \mathbf{1.7023\ kN/m^2}$$

The pressure required to bring a reduction in the volume of water is about 14521.53 times the pressure required for the same percentage reduction in the volume of air.

Example 1.30 Calculate the bulk modulus of elasticity and the coefficient of compressibility of the liquid when increase in its pressure from 6000 kN/m^2 to 12000 kN/m^2 causes 0.15 per cent decrease in volume.

Solution

Let $p_1 = 6000$ kN/m^2, $p_2 = 12000$ kN/m^2 and $-(dv/v) = 0.15\% = 0.0015$.

$$K = -\frac{dp}{(dv/v)} = \frac{12000 - 6000}{0.0015} = \mathbf{4 \times 10^6\ N/m^2}$$

$$\beta = \frac{1}{K} = \frac{1}{4 \times 10^6} = \mathbf{2.5 \times 10^{-7}\ m^2/N}$$

Example 1.31 Calculate the velocity of sound through water when the bulk modulus of water is given as 2.06×10^6 kN/m^2.

Solution

Let $K_w = 2.06 \times 10^6$ kN/m^2 and taking $\rho_w = 1000$ kg/m^3.

$$C = \sqrt{\frac{K_w}{\rho_w}} = \sqrt{\frac{2.06 \times 10^6 \times 10^3}{1000}} = \mathbf{1435.27\ m/s}$$

Example 1.32 Calculate the bulk modulus of air kept in a cylinder of volume 0.2 m^3 at 100 kPa when compressed to 0.05 m^3 by (i) isothermal compression and (ii) adiabatic compression.

Solution

Let $v_1 = 0.2$ m^3, $p_1 = 100$ kPa and $v_2 = 0.05$ m^3.

(i) $$p_2 = \frac{p_1 v_1}{v_2} = \frac{100 \times 10^3 \times 0.2}{0.05} = 4 \times 10^5 \text{ N/m}^2 \qquad [\because p_1 v_1 = p_2 v_2]$$

$$K = p = \mathbf{4 \times 10^5\ N/m^2}$$

(ii) $$p_1 v_1^{\gamma} = p_2 v_2^{\gamma}$$

$$p_2 = p_1 \left(\frac{v_1}{v_2}\right)^{\gamma} = 100 \times 10^3 \times \left(\frac{0.2}{0.05}\right)^{1.4} = 6.96 \times 10^5 \text{ N/m}^2$$

$$K = \gamma p = 1.4 \times 6.96 \times 10^5 = \mathbf{9.74 \times 10^5\ N/m^2}$$

1.18 ❐ VAPOUR PRESSURE

Atmospheric air is a mixture of dry air and water vapour. Thus, atmospheric pressure is the sum of the partial pressure of dry air and the partial pressure of water vapour. The partial pressure of water vapour constitutes only about 3% of the atmospheric pressure. The vapour pressure of water (liquid) can be defined as the pressure exerted by its vapour in phase equilibrium with water at a given temperature. When both vapour and water are present and the system is in phase equilibrium, then the partial pressure of vapour must be equal to the vapour pressure, and the system is said to be saturated. A liquid changes into vapour when exposed to atmosphere and its rate of evaporation is controlled by the difference between the vapour pressure and the partial pressure. The vapour pressure of air at a given temperature is equal to the saturation pressure of water at that temperature. Consider a large tub of water at 30°C in a room with dry air at one atmosphere. The evaporation of water starts but it stops when the partial pressure of water vapour in the room rises to 4.25 kPa at which phase equilibrium is attained.

At a given pressure, the temperature at which a pure substance changes its phase is termed as saturation temperature. Similarly, at a given temperature, the pressure at which a pure substance changes phase is called the saturation pressure. When the liquid is confined in a closed vessel, the ejected vapour molecules get accumulated in the space between the free liquid surface and the top of the vessel. These accumulated vapours of the liquid exert a partial pressure on the liquid surface which is known as vapour pressure of the liquid. There may be an interchange of vapour molecules between the liquid and the gaseous space above it. The vapour pressure will have a constant value when the vapour molecules leave and enter the liquid at the same rate (i.e., equilibrium state). The constant vapour pressure is called the saturated vapour pressure (or saturation pressure) and it greatly depends on the temperature. The vapour pressure and saturation pressure are equivalent for phase changes processes and it increases with temperature. For example, water at 0°C has a saturation pressure (or vapour pressure) of 0.611 kPa, whereas it changes to 2.34 kPa, 4.25 kPa and 12.35 kPa at 20°C, 30°C and 50°C, respectively. The saturated pressure of water at 100°C is equal to 101.325 kPa. When the pressure above the liquid is equal to its saturation pressure, the liquid starts to boil. Thus, if the pressure above the liquid surface is reduced by some means to such an extent that it becomes equal to or less than the saturation pressure of the liquid, then boiling of the liquid starts irrespective of the temperature. The vapour pressure of mercury is very low and hence, it is an excellent fluid to be used in a barometer. On the other hand, the vapour pressure of volatile liquids, like benzene, petrol, etc., is very high.

1.19 ❐ CAVITATION

In a liquid flow system, the pressure at any location in the liquid may drop below its vapour pressure. This causes vaporization of the liquid and results in the formation of small cavities of vapour bubbles and dissolved gases. The vapour bubbles so formed are carried by the flowing liquid from low pressure region to a high pressure region where they collapse suddenly and generate very high pressure waves. The pressure developed due to collapsing of bubbles may cause pitting, erosion and fatigue failure of the adjoining solid surfaces. This destructive phenomenon is called cavitation which results in noise, vibration, loss of efficiency and damage to machines. Cavitation phenomenon may occur in hydraulic machines, such as turbines, pumps and propellers and it can be sensed by its characteristic tumbling sound. To avoid cavitation, the pressure at any point in the fluid flow should not be allowed to drop below the vapour pressure at the local temperature. To avoid problems related to flow of water, the pressure should not be permitted to fall below 2.5 m of water.

Summary

1. Fluid is a substance which is capable of flowing, for example, liquids and gases.
2. Fluid mechanics deals with the behaviour of the fluids at rest or in motion.
3. Fluid statics deals with the behaviour of fluids at rest.
4. Fluid kinematics deals with motion of fluids without considering the forces.
5. Fluid dynamics deals with fluid flow subjected to forces.
6. Pressure: Normal force per unit area which is measured in N/m^2 or Pascal.
7. Mass density (ρ): Ratio of mass (m) of a fluid to its volume (v).
8. Weight density (w): Ratio of weight of a fluid to its volume.
9. Specific volume (v_s): Reciprocal of mass density.
10. Specific gravity (S): Ratio of density of a fluid to density of a standard fluid.
11. Viscosity (μ): Measure of internal fluid friction which causes resistance to flow.
12. Newton's law of viscosity: $\tau = \mu(du/dy)$, here (du/dy) is velocity gradient.
13. Unit of viscosity (μ): Ns/m^2 or $Pa \cdot s$ or kg/ms, 1 poise = 0.1 Ns/m^2.
14. Kinematic viscosity: $\nu = \mu/\rho$. It is measured in m^2/s, 1 stoke = 10^{-4} m^2/s.
15. Perfect gas law: $pv = mRT$ or $(p/\rho) = RT$, here R is the gas constant.
16. Units of gas constant R: J/kgK.
17. Universal gas constant: $R_o = MR = 8314.3$ Nm/molK, here M is molecular weight.
18. Isothermal process: $pv =$ constant, Isobaric process: $v/T =$ constant.
19. Adiabatic process: $pv^\gamma =$ constant, here $\gamma = (c_p/c_v)$ is the specific heat ratio.
20. Surface tension (σ) is usually expressed in N/m.
21. Pressure inside a liquid droplet: $p = 4\sigma/d$, here d is the diameter of droplet.
22. Pressure inside a soap bubble: $p = 8\sigma/d$; Pressure inside a liquid jet: $p = 2\sigma/d$.
23. Capillarity is the rise or fall (h) of a liquid surface in a small diameter tube (d) that is expressed as $h = (4\sigma\cos\alpha)/(\rho g d)$, here α is the contact angle, $\alpha = 0°$ for water and $\alpha = 130°$ for mercury.
24. Bulk modulus of elasticity (K): Ratio of compressive stress to volumetric strain. For isothermal process, $K = p$ and for isentropic process, $K = \gamma p$.
25. Compressibility: Reciprocal of the bulk modulus of elasticity of the fluid.

Multiple-choice Questions

1. Newton's law of viscosity relates to
(a) Shear stress, temperature, velocity and viscosity.
(b) Pressure, viscosity and rate of angular deformation in a fluid.
(c) Pressure, velocity and viscosity.
(d) Shear stress and rate of angular deformation in a fluid.

2. Poise is the unit of
(a) Kinematic viscosity. (b) Dynamic viscosity.
(c) Density. (d) None of these.

3. The unit of kinematic viscosity is
(a) m/kg s (b) Ns/m^2
(c) kg/m^2 s (d) m^2/s

4. Multiplying factor for converting one stoke into m^2/s is
(a) 10^{-4} (b) 10^4
(c) 10^{-2} (d) 10^2

5. With increase in temperature, the viscosity of liquids
(a) Increase.
(b) Decrease.
(c) First decreases and then increases.
(d) First increases and then decreases.

6. In the phenomenon of cavitation, the characteristic fluid property is
(a) Viscosity. (b) Surface tension.
(c) Vapour pressure. (d) None of these.

7. The density of a fluid is sensitive to changes and pressure, the fluid is
(a) Real fluid. (b) Newtonian fluid.
(c) Compressible fluid. (d) None of these.

8. Surface tension has the units of
(a) N/m^2 (b) N/m
(c) N/m^3 (d) None of these

9. The ratio of specific weight of the liquid to the specific weight of a standard fluid is called
(a) Specific gravity. (b) Mass density.
(c) Viscosity. (d) None of these.

10. Printer's ink is an example of
(a) Newtonian fluid. (b) Non-Newtonian fluid.
(c) Elastic solid. (d) Thixotropic fluid.

11. With rise in pressure bulk modulus of liquid
(a) Remains constant. (b) Decreases.
(c) Increases. (d) None of these.

12. Bulk modulus of elasticity is the ratio of
(a) Compressive stress to compressive strain.
(b) Tensile stress to tensile strain.
(c) Compressive stress to volumetric strain.
(d) None of the above.

13. The rain drop is spherical due to
(a) Incompressibility. (b) Capillarity.
(c) Surface tension. (d) None of these.

14. The height of liquid in a capillary tube
(a) Increases with increase in diameter.
(b) Increases with decrease in diameter.
(c) Increases with increase in specific weight.
(d) Increases with decrease in surface tension.

15. Paper pulp is an example of
(a) Newtonian fluid.
(b) Non-Newtonian fluid.
(c) Pseudoplastic fluid.
(d) Bingham plastic.

16. Soap helps in cleaning clothes because
(a) Solution is more viscous.
(b) Dirt is absorbed.
(c) Chemical constituents of soap changed.
(d) Surface tension of solution is decreased.

17. Ball pen works on the principle of
(a) Density. (b) Viscosity.
(c) Capillarity. (d) Surface tension.

18. Oil in the wick of an oil pump rises due to
(a) Density. (b) Viscosity.
(c) Capillarity. (d) Surface tension.

Review Questions

1. Define fluid mechanics and give its application areas. What is fluid continuum?

2. Define the following properties of fluid: (i) mass density, (ii) specific weight, (iii) specific volume and (iv) specific gravity.

3. What do you mean by viscosity? State and explain Newton's law of viscosity. Also discuss how does viscosity of liquids vary with temperature?

4. Define a fluid and also discuss the various types of fluids.

5. Define surface tension and explain its cause.

6. Derive expressions for internal pressure inside a droplet and soap bubble.

7. Explain the phenomenon of capillarity and also obtain an expression for capillary rise or fall of a liquid in a very small diameter glass tube.

8. Explain why in a capillary tube, the meniscus of water is concave upwards while the meniscus of mercury is convex upwards.

9. Define compressibility. How is it related to bulk modulus of elasticity?

10. Explain why some insects can walk on water and soap bubbles rise up in air?

11. Explain why a water column in a thin glass tube be lifted up while a mercury column be depressed?

12. Define cavitation, evaporation, vapour pressure and boiling.

Problems

1. Find the mass density, specific weight and weight of one litre of a liquid of specific gravity of 0.72.
[**Ans.** 720 kg/m^3, 7063.2 N/m^3, 7.0632 N]

2. Two horizontal plates are placed 12.5 mm apart, the space between them is filled with an oil of viscosity of 12 poises. Determine shear stress in oil if upper plate is moving with a velocity of 2 m/s.
[**Ans.** 192 N/m^2]

3. Calculate the intensity of shear of an oil of viscosity of 1.2 poise. The oil is filled in the clearance of 1.2 mm between a journal bearing and shaft of diameter 100 mm when it rotates at 120 rpm.
[**Ans.** 62.8 N/m^2]

4. The velocity distribution u in m/s over a plate is given in terms of distance y m above the plate by $u = (2/3)y - y^2$. Find shear stress at $y = 0$ and $y = 0.15$ m when viscosity of the fluid is given as 9 poise.
[**Ans.** 0.6003 N/m^2, 0.3303 N/m^2]

5. Determine the viscosity of a lubricant film of thickness 1.4 mm used between a square plate of size 0.8 m × 0.8 m and an inclined plane with angle of inclination 30°. The weight of the square plate is 200 N and it slides down the inclined plane with a uniform velocity of 0.25 m/s.
[**Ans.** 8.75 poise]

6. The space between two square plates of sides of 0.5 m is filled with an oil film of thickness 1.2 cm. The upper plate moving with a velocity of 2.4 m/s needs a force of 98 N to maintain its speed. Calculate the dynamic viscosity and kinematic viscosity of the oil if its specific gravity is 0.85.
[**Ans.** 19.6 poise, 23.06 stokes]

7. A plate 0.06 mm distant from a fixed plate moves at 1.2 m/s and needs a force of 2 N/m^2 to maintain its speed. Calculate the viscosity of the fluid used between the plates.
[**Ans.** 1×10^{-3} poise]

8. A 100 mm diameter vertical cylinder rotates concentrically inside another cylinder of diameter 101 mm. The space between the cylinders is filled with a liquid of unknown viscosity and both cylinders are 200 mm high. Determine the viscosity of the liquid when 15 Nm torque is required to rotate the inner cylinder at a speed of 125 rpm.
[**Ans.** 36.5 poise]

9. An oil layer thickness of 1.4 mm having viscosity of one poise is being used between a shaft of diameter 0.5 m and a sleeve of length 10 cm. Determine the power lost in the bearing when the shaft is revolving at a speed of 250 rpm.
[**Ans.** 480.27 Watts]

10. A circular disc of diameter d is slowly rotates in a liquid of large viscosity μ at a small distance h from a fixed surface. Derive an expression for torque T required to maintain an angular speed of ω.
[**Ans.** $T = (\pi\mu\omega d^4)/(32h)$]

11. A fluid has an absolute viscosity of 0.048 Pa·s and a specific gravity of 0.913. For flow of such a fluid over a flat surface, the velocity at a point 75 mm away from the surface is 1.125 m/s. Calculate the shear stresses at the solid boundary and also at points 25 mm, 50 mm and 75 mm away from the surface in normal direction, if the velocity distribution across the surface is (i) linear and (ii) parabolic with vertex at the points 75 mm away from the surface.
[**Ans.** (i) 0.72 N/m^2, (ii) 1.44 N/m^2, 0.96 N/m^2, 0.48 N/m^2, 0]

12. A square plate 0.5 m × 0.5 m weighing 100 N is allowed to slide down an inclined plane which is laid a slope of 1 vertical to 2.5 horizontal. If 0.03 mm thick oil film of viscosity 2×10^{-3} Ns/m^2 is maintained between the inclined plane and the plate. Determine the terminal velocity attained by the plate.
[**Ans.** 2.23 m/s]

13. In a 60 mm long journal bearing arrangement, a shaft of 30 mm diameter rotates at 500 rpm. The clearance between the journal and bearing at concentric condition is filled with a Newtonian fluid of thickness 0.12 mm having viscosity of 0.5 Ns/m^2. If velocity variation in the fluid is linear, then determine frictional torque overcome by the journal and the corresponding power loss.
[**Ans.** 0.277 Nm, 14.5 W]

14. A hydraulic lift used for lifting cars has 200 mm diameter ram sliding in a 200.16 mm diameter cylinder. The clearance space between the cylinder and ram is filled with a lubricating oil of kinematic viscosity 2 stokes and specific gravity of 0.8. If the travel of 3 m long ram has a uniform rate of 0.1 m/s, then determine the frictional resistance experienced by the ram.
[**Ans.** 376.99 N]

15. A 6 cm disc rotates on a table separated by an oil film of 1.5 mm thickness. Determine the viscosity of oil if the torque required to rotate the disc at 30 rpm is 3×10^{-4} Nm.
[**Ans.** 1.126 poise]

16. A thin plate of very large area is placed in a gap of height h with oils of viscosities μ_1 and μ_2 on two sides of the plate. The thin plate is pulled at a constant velocity u and y is the distance of the plate from one of the surfaces of the gap. Find the position of plate so that (i) shear force on the two sides of the plate is equal and (ii) force required to drag the plate is minimum. Assume viscous flow and neglect all end effects.

[**Ans.** (i) $y = (\mu_2 h)/(\mu_1 + \mu_2)$, (ii) $y = h/(1+\sqrt{\mu_1/\mu_2})$]

17. In the bubble method of measuring surface tension, a tube of diameter 3 mm is immersed in the liquid of specific gravity 0.85 to a depth of 6 cm. The maximum pressure of air that could be applied through the tube was 540 Pa above atmosphere. Determine the surface tension.

[**Ans.** 0.02977 N/m]

18. Find the power required to convert one litre of water per minute at a temperature of 20°C into a mist having an average drop size of 5×10^{-3} mm. Also determine the pressure intensity inside the mist droplets if thermal effects are neglected.

[**Ans.** 1.46 Watts, 58.4 kN/m^2]

19. A capillary tube of internal diameter 2 mm is immersed into a pool of water to a depth of 12 mm. If the air bubbles are intended to have a diameter of 2 mm, then determine the pressure required to form the air bubble if the surface tension of water is 0.073 N/m.

[**Ans.** 263.72 N/m^2]

20. A 1.5 mm diameter glass tube is immersed in mercury. Determine the depression if the surface tension for mercury is 0.46 N/m and the contact angle is 130°.

[**Ans.** 5.91 mm]

21. Derive an expression for the capillary rise of a liquid having surface tension σ and contact angle α between two vertical parallel plates at a distance of x apart.

[**Ans.** $h = (2\sigma\cos\alpha)/(\rho g x)$]

22. In the bubble method, a tube of 1.5 mm internal diameter is immersed to a depth of 12.5 mm in a mineral oil of relative density 0.85. Air is forced through the tube forming a bubble at the lower end. Determine the unit surface energy indicated when the maximum bubble pressure intensity is 150 N/m^2.

[**Ans.** 0.0172 N/m]

23. Determine the percentage reduction in volume of water if there is an increase in pressure by 10^4 kN/m^2 over the atmospheric pressure of 101.3 kN/m^2. The value of bulk modulus for water is given as 206×10^4 kN/m^2. If the same reduction in volume of air is to be obtained by an isothermal process, then determine the required increase in pressure. Also comment on your results.

[**Ans.** 0.4854%, 0.492 kN/m^2, air is 20325.2 times more compressible than water]

24. Gas A at 125 kPa (abs) is compressed isothermally and gas B at 100 kPa (abs) is compressed isentropically with compression index of 1.4. State which gas is more compressible?

[**Ans.** Gas A is more compressible]

25. Determine (i) density, (ii) weight density and (iii) specific volume at a depth of 8000 m from the surface of ocean where the pressure is found to be 82.5×10^3 kN/m^2. The density at the surface is 1020 kg/m^3 and bulk modulus is 2.4×10^6 kN/m^2 for the given pressure.

[**Ans.** 1055.06 kg/m^3, 10350.14 N/m^3, 9.478×10^{-4} m^3/kg]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (d)	**2.** (b)	**3.** (d)	**4.** (a)	**5.** (b)
6. (c)	**7.** (c)	**8.** (b)	**9.** (a)	**10.** (d)
11. (c)	**12.** (c)	**13.** (c)	**14.** (b)	**15.** (c)
16. (d)	**17.** (d)	**18.** (c)		

Chapter **2**

Fluid Pressure and Its Measurement

2.1 ❐ INTRODUCTION

A fluid remains in contact with any surface and it needs a container to store it. It always exerts a normal force on the surfaces. In case of liquids, this force is mainly due to specific weight of the liquid, whereas in gases it is mainly because of molecular activity. This normal force exerted by a fluid per unit area is called fluid pressure which is also termed as static or hydrostatic pressure.

Fluid statics deals with the study of fluid at absolute rest or relative rest. If both the fluid and its container are at rest or both are moving in the same direction at the same speed, then it is said to be in absolute rest. If the fluid is rotated or moved as a solid mass (i.e., there is no shear stress) in a stationary container, then the fluid is said to be in relative rest or relative equilibrium. The weight (W) of a liquid element can be resolved into two components, namely tangential component ($W \sin\alpha$) and perpendicular component ($W \cos\alpha$) as shown in Figure 2.1. The angle of the water surface to the horizontal (α) causes motion of the liquid due to shear. In a static liquid, $W \sin\alpha = 0$, i.e., the surface is horizontal.

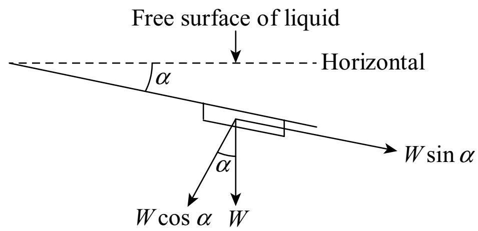

Figure 2.1 *Free surface of liquid*

This chapter describes the relation for variation of pressure along a vertical depth in a fluid under static conditions. The relation has been applied to the measurement of pressure at a point with manometers. The mechanical gauges which measure fluid pressure are briefly explained. The relation for variation of pressure in a compressible fluid (atmosphere) is also described.

2.2 ❐ FLUID PRESSURE

Fluid pressure (or pressure) is defined as the normal force exerted by a fluid per unit area. Let F be the total force uniformly distributed over an area (A), then pressure (p) at any point is given below.

$$p = \frac{F}{A}$$

However, if the force is not uniformly distributed and the pressure varies from point to point on the given area, then pressure at any point is given below.

$$p = \frac{dF}{dA}$$

Here, dF is the force acting on an infinitesimal area dA in a large mass of fluid. In SI units, the pressure is measured in N/m^2 and it is called pascal (Pa), i.e., 1 Pa = 1 N/m^2. Pascal is a very small unit and so commonly, pressure is given in kPa (1 kPa = 10^3 Pa) and MPa (1 MPa = 10^6 Pa) units. The other pressure units which are commonly used are as follows.

- 1 bar = 10^5 Pa = 10^5 N/m^2 = 0.1 MPa = 100 kPa
- 1 atm = 101325 Pa = 101.325 kpa = 1.01325 bar

2.3 ❒ PASCAL'S LAW

The Pascal's law states that pressure or intensity of pressure at a point in a fluid at rest is equal in all directions. This law was established by B. Pascal, a French Mathematician in 1653.

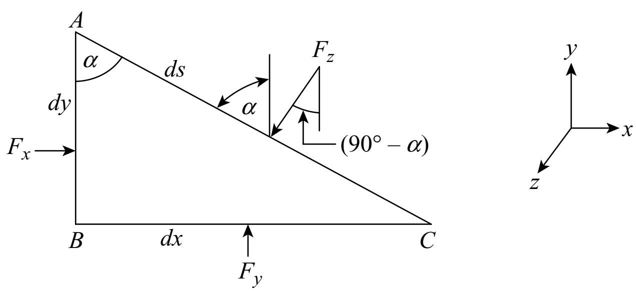

Figure 2.2 *Forces on a static fluid element*

Consider an infinitesimal wedge shaped element of stationary fluid as a free body. Assume the dimensions of this element as dx, dy, ds and it has a unit depth perpendicular to the plane of the paper as shown in Figure 2.2. Let w be the weight density of the liquid and W be the weight of the fluid element ABC. Since the fluid is at rest there is no shear force. The two forces which act on the element are normal pressure force and the vertical force due to the weight of the element. Let F_x, F_y and F_z be the pressure forces acting on the faces AB, BC and CA, respectively and p_x, p_y and p_z be the corresponding pressures.

Let $\angle BAC = \alpha$

$$\therefore ds = \frac{dy}{\cos\alpha} = \frac{dx}{\sin\alpha}$$

The pressure force acting on a surface is given by the product of the pressure intensity and the surface area perpendicular to the direction of pressure. The pressure forces are given by the following expressions.

$$F_x = p_x \times (dy \times 1) = p_x\, dy;\ F_y = p_y \times (dx \times 1) = p_y\, dx;\ F_z = p_z \times (ds \times 1) = p_z\, ds$$

The vertical force due to weight of the element is given by the product of volume and weight density as follows.

$$W = \left(\frac{1}{2} \times dx \times dy \times 1\right) \times w = \frac{1}{2} wdxdy$$

Since the element ABC is in equilibrium, the forces must be in equilibrium. Resolving the forces in x-direction, we get:

$$F_x - F_z \sin(90° - \alpha) = 0$$

$$F_x - F_z \cos\alpha = 0$$

$$p_x dy - p_z ds \cos\alpha = 0$$

$$p_x dy - p_z \times \frac{dy}{\cos\alpha} \times \cos\alpha = 0$$

$$p_x\, dy - p_z\, dy = 0$$

$$\therefore p_x = p_z \tag{i}$$

Resolving the forces in y-direction, we get:

$$F_y - F_z \cos(90° - \alpha) - \frac{1}{2} w dx\, dy = 0$$

$$p_y dx - p_z ds \sin\alpha - \frac{1}{2} w\,dx\,dy = 0$$

$$p_y dx - p_z \times \frac{dx}{\sin\alpha} \times \sin\alpha - \frac{1}{2} w\,dx\,dy = 0$$

$$p_y dx - p_z dx - \frac{1}{2} w\,dx\,dy = 0$$

Since the term $\left(\frac{1}{2}\right) w\,dx\,dy$ involves a product of infinitesimal quantities, it may be neglected.

Thus $$p_y\,dx - p_z\,dx = 0$$

$$\therefore p_y = p_z \quad \text{(ii)}$$

Therefore, from the expressions (i) and (ii), we get:

$$p_x = p_y = p_z \quad (2.1)$$

From Equation (2.1), it can be seen that the pressure at any point in the stationary liquid is same in all the directions.

Example 2.1 In a hydraulic press, the diameters of ram and plunger are 100 mm and 15 mm, respectively. Determine the weight lifted by the press when the force applied on the plunger is 300 N.

Solution

Let D = 100 mm = 0.1 m, d = 15 mm = 0.015 m and F = 300 N. Let W be the weight lifted by the hydraulic press.

Since $$p = \frac{F}{a} = \frac{W}{A} \quad \text{[Pascal's law]}$$

$$\therefore W = \frac{FA}{a} = \frac{F \times (\pi/4) D^2}{(\pi/4) d^2} = \frac{300 \times 0.1^2}{0.015^2} = \mathbf{13333.33\ N}$$

2.4 ❐ HYDROSTATIC LAW (PRESSURE VARIATION IN A STATIC FLUID)

The hydrostatic law states that the rate of increase of pressure in a vertically downward direction must be equal to specific weight of the fluid at that point.

In a stationary fluid, consider a small fluid element $ABCD$ of cross sectional area ΔA and height Δh. Let p be the intensity of pressure (above atmospheric pressure) on face BC and h be its distance from free surface as shown in Figure 2.3. Let w be the weight density of the fluid and W be the weight of the fluid element $ABCD$. The following forces act on this small element.

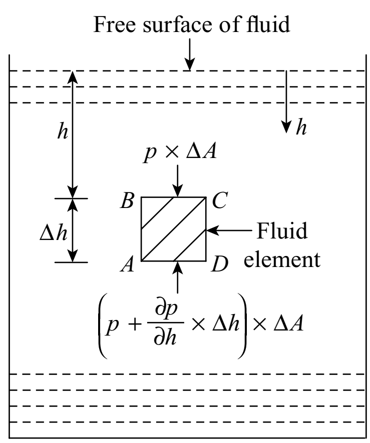

Figure 2.3 *Forces acting on a static fluid element*

1. Pressure force on the face BC acting downward is given by,

$$= p \times \Delta A$$

2. Pressure force on the face AD acting upward is given by,

$$= \left(p + \frac{\partial p}{\partial h} \Delta h\right) \times \Delta A$$

3. Weight of the fluid element acting downward is given by,

$$W = \text{Weight density} \times \text{Volume} = w \times \Delta A \Delta h$$

4. Pressure forces on surfaces *AB* and *CD* are equal and opposite, it cancels each other.

Since the element *ABCD* is in equilibrium, we get:

$$p\Delta A - \left(p + \frac{\partial p}{\partial h}\Delta h\right)\Delta A + w\Delta A\Delta h = 0$$

$$p\Delta A - p\Delta A - \frac{\partial p}{\partial h}\Delta h\Delta A + w\Delta A\Delta h = 0$$

$$-\frac{\partial p}{\partial h}\Delta h\Delta A + w\Delta A\Delta h = 0$$

$$\frac{\partial p}{\partial h}\Delta h\Delta A = w\Delta A\Delta h$$

$$\frac{\partial p}{\partial h} = w = \rho g \tag{2.2}$$

Since the pressure function (p) depends on a single variable distance (h) only, the partial derivative can be replaced by ordinary (or exact) derivative.

$$\therefore \frac{dp}{dh} = w = \rho g \tag{2.2a}$$

It is pertinent to mention here that in the above equation, h is measured vertically downward. But when h is measured vertically up then Equation (2.2a) becomes,

$$\frac{dp}{dh} = -w = -\rho g \tag{2.2b}$$

From Equation (2.2), it can be seen that rate of pressure increase in a vertical direction is equal to weight density of the stationary fluid at that point.

Now integrating Equation (2.2), we get:

$$\int \partial p = \rho g \int \partial h$$

$$p = \rho g h = wh \tag{2.3}$$

$$\boxed{h = \frac{p}{\rho g} = \frac{p}{w}} \tag{2.4}$$

In Equation (2.4), 'h' denoted as the vertical height of the free surface above any point in a fluid at rest is known as pressure head. In case the pressure at a point in a fluid at rest is p_a, then the pressure at any point below it at a depth h will be given by the following expression.

$$p = p_a + \rho g h = p_a + wh \tag{2.5}$$

From Equation (2.5), it is clear that the pressure at any point in a static fluid depends on the vertical depth of the point below the free surface and the specific weight of the fluid. It does not depend on the shape and size of the container. The containers of different shapes are interconnected in Figure 2.4. Thus, the weight of the liquid in each part differs but the pressures at points *A*, *B*, *C*, *D* and *E* lying on the same horizontal level are same. This is due to the same vertical height of the fluid column at the given points below the free surface and it is known as hydrostatic paradox.

If h_1 and h_2 be the heights of the columns of liquids of weight densities w_1 and w_2, respectively which develop the same pressure p at any point, then from Equation (2.3), we get the following expression.

$$p = w_1 h_1 = w_2 h_2 \tag{2.6}$$

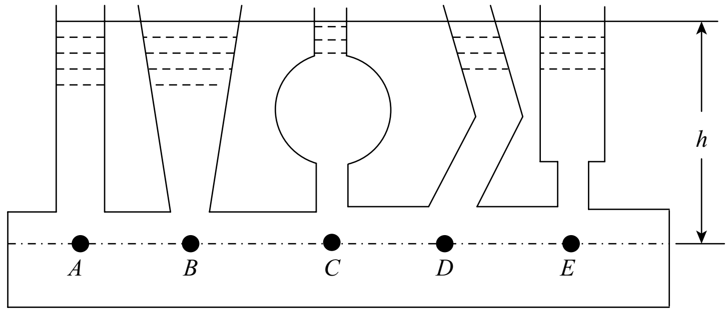

Figure 2.4 *Fluid pressure is same at all points on a horizontal plane*

If S_1 and S_2 be the specific gravities of the two liquids, then Equation (2.6) is as follows.

$$p = S_1 h_1 = S_2 h_2 \tag{2.7}$$

2.5 ❒ ATMOSPHERIC, ABSOLUTE, GAUGE AND VACUUM PRESSURES

1. **Atmospheric pressure:** The atmospheric pressure (p_{atm}) may be defined as a normal pressure exerted by atmospheric air on all surfaces with which it is in contact. It varies with altitude and measured by a barometer and it is also called barometric pressure. At sea level, under normal conditions, its value is 1.01325×10^5 N/m^2 or 1.01325 bar or 10.336 m of water or 760 mm of Hg (760 mmHg). The unit mmHg is also known as Torr in the honour of Torricelli (1608–1647). Thus, 1 atm = 760 torr and 1 torr = 133.3223 Pa. The atmospheric pressure can be measured with a mercury barometer that consists of a glass tube filled with mercury. The open end of the tube is immersed in a mercury container that is open to the atmosphere as shown in Figure 2.5(a). The column of mercury reaches to an equilibrium position where its weight balances the force due to the atmospheric pressure at point A. The pressure at A equals to that at B, since both lie on the same horizontal and therefore, we get the following expression.

$$\boxed{p_{atm} = \rho g h}$$

   Here, ρ is the density of mercury, g is the local acceleration due to gravity and h is the height of the mercury column above the free surface. It is to be noted that vapour pressure of mercury (p_v) is very low relative to atmospheric pressure and thus, it has been neglected. This almost vacuum condition above the mercury in the barometer is known as Torricellian vacuum. At sea level, h = 76 cm and it varies place to place as per the height above sea level.

   Generally, fluid pressures are measured by taking either absolute zero pressure (or complete vacuum) as datum or local atmospheric pressure as datum.

2. **Absolute pressure:** When pressure is measured above absolute zero, it is called absolute pressure (abs) and all values of absolute pressure are positive.
3. **Gauge pressure:** When pressure is measured by taking atmospheric pressure as datum, it is called gauge pressure and it is measured by pressure gauges. All pressure gauges show zero value when open to the atmosphere and it indicates only the difference between the fluid pressure and the atmospheric pressure.
4. **Vacuum pressure:** When the pressure of a fluid is below atmospheric pressure, it is called vacuum pressure (or negative gauge pressure) and it is measured by a vacuum gauge.

   Thus, the mathematical relation between the above pressures is given by,

   Absolute pressure = Atmospheric pressure + Gauge pressure

   Absolute pressure = Atmospheric pressure − Vacuum pressure

   The above relationships between pressures are shown in Figure 2.5(b).

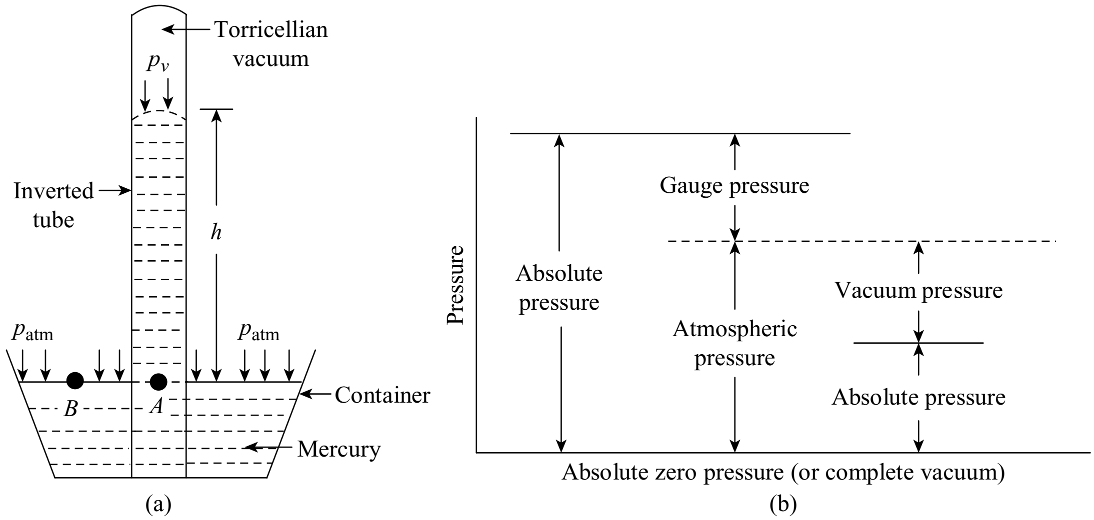

Figure 2.5 *(a) A barometer (b) The scale of pressure*

Example 2.2 Determine the depth of a point below sea water surface where pressure intensity is 1005 kPa. Take specific gravity of sea water as 1.03.

Solution

Let p = 1005 kPa = 1005×10^3 Pa and S = 1.03. Let h be the depth to be found out.

$$\rho = S\rho_w = 1.03 \times 1000 = 1030 \text{ kg/m}^3$$

$$h = \frac{p}{\rho g} = \frac{1005 \times 10^3}{1030 \times 9.81} = \mathbf{99.463\ m}$$

Example 2.3 Convert water pressure head of 100 m to kerosene of specific gravity 0.82.

Solution

Let h_1 = 100 m, S_2 = 0.82 and S_1 = 1 (specific gravity of water). Let h_2 be the kerosene pressure head.

Since
$$S_1 h_1 = S_2 h_2$$
$$1 \times 100 = 0.82 \times h_2$$
$$\therefore h_2 = \frac{100}{0.82} = \mathbf{121.95\ m}$$

Example 2.4 The diameters of small and large pistons in a hydraulic jack are 0.03 m and 0.15 m, respectively. If the force applied on small piston is 500 N, then determine the load lifted by the large piston when smaller piston is 0.25 m above the large piston. Take specific gravity of oil as 0.82.

Solution

Let d = 0.03 m, D = 0.15 m, F = 500 N, h = 0.25 m and S = 0.82. Let W be the load lifted.

$$\rho = S\rho_w = 0.82 \times 1000 = 820 \text{ kg/m}^3$$

Since
$$p = \left(\frac{F}{a} + \rho g h\right) = \frac{W}{A} \qquad \text{[Pascal's law]}$$

$$W = \left(\frac{F}{a} + \rho gh\right)A = \left[\frac{F}{(\pi/4)d^2} + \rho gh\right] \times \frac{\pi}{4}D^2$$

$$\therefore W = \left[\frac{500}{(\pi/4)\times 0.03^2} + 820 \times 9.81 \times 0.25\right] \times \frac{\pi}{4} \times 0.15^2 = \mathbf{12535.54\ N}$$

Example 2.5 An open tank contains water in its bottom up to 2 m depth and then oil of specific gravity 0.8 up to a depth of 1.5 m. Determine the pressure at the bottom of the tank and at the interface of water and oil.

Solution

Let $h_1 = 2$ m, $S_2 = 0.8$ and $h_2 = 1.5$ m. Let p_b be the pressure at the bottom of the tank and p_i be the pressure at the interface of water and oil.

Since
$$p_b = \rho_w g h_1 + S_2 \rho_w g h_2$$

$$\therefore p_b = 1000 \times 9.81 \times 2 + 0.8 \times 1000 \times 9.81 \times 1.5 = \mathbf{31392\ N/m^2}$$

$$p_i = S_2 \rho_w g h_2 = 0.8 \times 1000 \times 9.81 \times 1.5 = \mathbf{11772\ N/m^2}$$

Example 2.6 Determine the pressure due to a column of 0.5 m of an oil of specific gravity 0.85 and mercury having specific gravity of 13.6.

Solution

Let $h = 0.5$ m, $S_{oil} = 0.85$ and S_{Hg} or $S_m = 13.6$.

(i) Pressure due to oil is given by,

$$p = S_{oil}\rho_w gh = 0.85 \times 1000 \times 9.81 \times 0.5 = \mathbf{4169.25\ N/m^2}$$

(ii) Pressure due to mercury is given by,

$$p = S_{Hg}\rho_w gh = 13.6 \times 1000 \times 9.81 \times 0.5 = \mathbf{66708\ N/m^2}$$

Example 2.7 A cylindrical container of 2.5 m diameter and 5 m high is fully filled with an oil of specific gravity 0.8. Determine (i) intensity of pressure and the total force on the bottom of the tank and (ii) total force on the vertical surface.

Solution

Let $d = 2.5$ m, $h = 5$ m and $S = 0.8$.

(i) Pressure intensity at the bottom is given by,

$$p = S\rho_w gh = 0.8 \times 1000 \times 9.81 \times 5 = \mathbf{39240\ N/m^2}$$

Total pressure force at the bottom is given by,

$$F = p \times \frac{\pi}{4}d^2 = 39240 \times \frac{\pi}{4} \times 2.5^2 = \mathbf{192618.9\ N}$$

(ii) Minimum pressure intensity (p_{min}) at the top is given by,

$$p_{min} = 0$$

Maximum pressure intensity (p_{max}) at the bottom and it is given by,

$$p_{max} = 39240\ \text{N/m}^2$$

$$\therefore p_{av} = \frac{p_{min} + p_{max}}{2} = \frac{0 + 39240}{2} = 19620\ \text{N/m}^2$$

Total force on the wall is given by,

$$F = p_{av} \times \pi dh = 19620 \times \pi \times 2.5 \times 5 = \mathbf{770475.6\ N}$$

Example 2.8 Determine the gauge and absolute pressures at a point 2.5 m from the free surface of the liquid whose density is 1525 kg/m^3 when barometer reads 740 mm of Hg. Assume standard values for density of water and specific gravity of mercury.

Solution

Let $h = 2.5$ m, $\rho = 1525$ kg/m^3 and $h_{atm} = 740$ mm Hg = 0.74 mHg.

Gauge pressure (p) is given by,

$$p = \rho gh = 1525 \times 9.81 \times 2.5 = \mathbf{37400.625\ N/m^2}$$

Atmospheric pressure (p_{atm}) is given by,

$$p_{atm} = S_{Hg}\rho_w gh_{atm} = 13.6 \times 1000 \times 9.81 \times 0.74 = 98727.84\ \text{N/m}^2$$

Absolute pressure (p_{abs}) is given by,

$$p_{abs} = p_{atm} + p = 98727.84 + 37400.625 = \mathbf{136128.465\ N/m^2}$$

Example 2.9 Determine the height of the mountain if the density of air is 1.23 kg/m^3 and the pressure at the base and the top of the mountain are 0.76 mHg and 0.70 mHg respectively. Take specific gravity of mercury = 13.6 and density of water = 1000 kg/m^3.

Solution

Let $\rho = 1.23$ kg/m^3, h_{base} = 0.76 mHg, h_{top} = 0.70 mHg, S_{Hg} = 13.6 and $\rho_w = 1000$ kg/m^3. Let h be the height of the mountain.

Since

$$\rho gh = p_{base} - p_{top}$$

$$\rho gh = S_{Hg}\rho_w gh_{base} - S_{Hg}\rho_w gh_{top}$$

$$h = \frac{S_{Hg}\rho_w g \times (h_{base} - h_{top})}{\rho g}$$

$$\therefore h = \frac{13.6 \times 1000 \times 9.81 \times (0.76 - 0.70)}{1.23 \times 9.81} = \mathbf{663.415\ m}$$

Example 2.10 For a 30 cm deep gasoline tank of a car, the fuel gauge fitted at the bottom of the tank indicates some value. The fuel tank accidentally contains 1.8 cm height of water column in addition to gasoline (specific gravity = 0.68). Determine the height of the air $(\rho = 1.21\ \text{kg/m}^3)$ remaining at the top when the gauge erroneously reads full.

Solution

Let h = 30 cm = 0.3 m, h_3 = 1.8 cm = 0.018 m, S = 0.68 and $\rho_{air} = 1.21$ kg/m^3. Let h_1 be the height of the air column, h_3 be the height of water column and $h_2 = (0.3 – 0.018 – h_1) = (0.282 – h_1)$ m be the gasoline column.

When the tank is full of gasoline, we get:

$$p = S\rho_w gh = 0.68 \times 1000 \times 9.81 \times 0.3 = 2001.24\ \text{N/m}^2$$

Since

$$p = \rho_w gh_3 + S\rho_w gh_2 + \rho_{air} gh_1$$

$$2001.24 = 1000 \times 9.81 \times 0.018 + 0.68 \times 1000 \times 981 \times (0.282 - h_1) + 1.21 \times 9.81 h_1$$

$$2001.24 = 176.58 + 1881.166 - 6670.8 h_1 + 11.87\ h_1$$

$$6658.93\, h_1 = 56.506$$

$$\therefore h_1 = \frac{56.506}{6658.93} = \mathbf{8.486 \times 10^{-3}\ m\ or\ 0.8486\ cm}$$

2.6 ❒ MEASUREMENT OF PRESSURE

The fluid pressure is measured by the manometers and mechanical gauges which are described below.

2.6.1 Manometers

These devices measure pressure at a point in a liquid by balancing the column of liquid by the same or another column of liquid. The manometers may be classified as

1. **Simple manometers:** (a) Piezometer, (b) U-tube manometer (or Double column manometer) and (c) Single column manometer
2. **Differential manometers:** (a) U-tube differential manometer, (b) Inverted U-tube differential manometer and (c) Micromanometer.

2.6.2 Mechanical Gauges

In these pressure measuring devices, an elastic element like spring is used against the liquid pressure to be measured. Some of the commonly used mechanical gauges are (a) Bourdon tube pressure gauge, (b) Diaphragm pressure gauge, (c) Bellow pressure gauge and (d) Dead weight pressure gauge.

2.7 ❒ SIMPLE MANOMETERS (OPEN TYPE MANOMETERS)

Simple manometers measure pressure at a point in a fluid contained in a pipe or a vessel. Generally, it consists of a glass tube with one end connected to the point where pressure is to be measured and the other end remains open to atmosphere. These manometers are also known as open type manometers and some of the simple manometers are discussed below.

2.7.1 Piezometer

A piezometer is the simplest form of manometer which is used to measure the moderate pressure of a liquid. It is essentially a single column manometer which consists of a glass tube whose one end is connected to the point where pressure is to be measured and the other end remains open to atmosphere. The usual types of these manometers are shown in Figure 2.6.

Since the surface of the liquid in the tube is open to atmospheric pressure and therefore, a piezometer measures gauge pressure only. If ρ is the density of a liquid and h is its height in piezometer tube, then pressure at point 'A' in Figure 2.6 is given by the expression $p = \rho g h$.

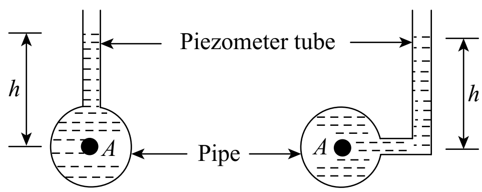

Figure 2.6 *Piezometers*

Limitations

1. A piezometer cannot be used when large pressure in a lighter liquid is to be measured, as it requires very long tubes and it cannot be handled easily.
2. It cannot measure gas pressure, as gas does not form any free surface with the atmosphere.

Example 2.11 An absolute pressure of 2.4×10^5 N/m^2 acts on the free surface of water closed in a container. A piezometer is fitted to the container at a depth of 2 m from the free surface. Determine the height of the water rising in the piezometer if atmospheric pressure is 1.01325×10^5 N/m^2.

Solution

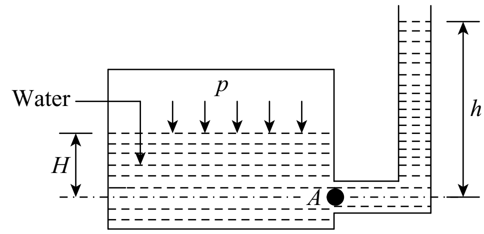

Figure 2.7

Refer Figure 2.7. Let $p = 2.4 \times 10^5$ N/m^2, $H = 2$ m, $p_{atm} = 1.01325 \times 10^5$ N/m^2. Let h be height of the water rising in the piezometer tube.

The manometric equation for point 'A' is given by,

$$p + \rho_w gH = p_{atm} + \rho_w gh$$

$$h = \frac{p + \rho_w gH - p_{atm}}{\rho_w g}$$

$$\therefore h = \frac{2.4 \times 10^5 + 1000 \times 9.81 \times 2 - 1.01325 \times 10^5}{1000 \times 9.81} = \mathbf{16.136\ m}$$

Example 2.12 Determine the least diameter of a piezometer tube so that error due to capillary action in measuring the air gauge pressure of 120 N/m^2 lies within 2%. Take surface tension for water as 0.0725 N/m and the angle of contact as 0°.

Solution

Let $p = 120$ N/m^2, error = 2%, $\sigma = 0.0725$ N/m and $\alpha = 0^o$. Let d be the least diameter of piezometer tube.

The height of the water in the tube is given by,

$$h = \frac{4\sigma \cos\alpha}{\rho_w gd} \tag{i}$$

The rise of water in the piezometer tube is also given by,

$$h = \frac{p}{\rho_w g} \tag{ii}$$

Error can be obtained by dividing expression (i) by expression (ii) as given below.

$$\frac{(4\sigma \cos\alpha)/(\rho_w gd)}{p/(\rho_w g)} = 0.02$$

$$\therefore d = \frac{4\sigma \cos\alpha}{0.02p} = \frac{4 \times 0.0725 \cos 0°}{0.02 \times 120} = \mathbf{0.1208\ m\ or\ 12.08\ cm}$$

2.7.2 U-tube Manometer (Double Column Manometer)

A U-tube manometer consists of a glass tube bent in U-shape, where one end is connected to a point at which pressure is to be measured and the other end remains open to the atmosphere as shown in Figure 2.8.

The U-tube contains a liquid of specific gravity greater than that of the fluid whose pressure is to be measured. For measuring high pressure, generally mercury (specific gravity = 13.6) is used as manometric liquid and for measuring low pressure, liquids of lower specific gravities, such as carbon tetrachloride (specific gravity = 1.59) and acetylene tetrabromide (specific gravity = 2.59) are used. A U-tube manometer can be used to measure both the gauge pressure and vacuum pressure.

1. **U-tube manometer for gauge pressure:** Let p be the pressure which is to be measured at point 'A' and Z–Z be the horizontal datum line as schematically shown in Figure 2.8(a). Due to the application of pressure, manometric liquid moves downward in the left limb and correspondingly, it rise in the right limb of the manometer.

   Let h_1 and h_2 be the heights of light and heavy liquids from datum line, respectively, S_1 and S_2 be the specific gravities of light and heavy liquids, respectively and $\rho_1 = S_1\rho_w$ and $\rho_2 = S_2\rho_w$ be the densities of light and heavy liquids, respectively, where ρ_w is the density of water and p_{atm} is the atmospheric pressure.

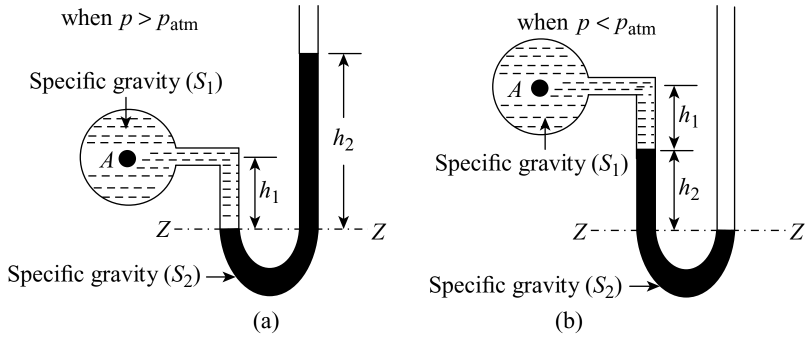

Figure 2.8 *U-tube manometers*

Section *Z–Z* is the surface of manometric liquid in the left and right limbs. Thus, liquid has continuous connection at the same horizontal. Therefore, pressure above the horizontal datum line *Z–Z* in the left column will be equal to the pressure above the horizontal datum line *Z–Z* in the right column of the manometer. The corresponding expressions are as follows.

$$p + \rho_1 g h_1 = \rho_2 g h_2$$

$$\therefore\ p = \rho_2 g h_2 - \rho_1 g h_1 \qquad (2.8)$$

2. **U-tube manometer for vacuum pressure (or negative pressure):** Here, Figure 2.8(b) illustrates the balancing of the light and heavy liquids in the two limbs of the manometer. Due to negative pressure, the manometric liquid sucks upwards in the left limb and correspondingly falls in the right limb of the manometer. Thus, we get the following expression.

$$p + \rho_1 g h_1 + \rho_2 g h_2 = 0$$

$$\therefore\ p = -(\rho_1 g h_1 + \rho_2 g h_2) \qquad (2.9)$$

Example 2.13 The right limb of a U-tube manometer filled with mercury (specific gravity = 13.6) is open to the atmosphere while the left limb is connected to a pipe containing a fluid of specific gravity 0.8. The centre of the pipe lies 15 cm below the level of mercury in the right limb. Determine the pressure of fluid in the pipe when the difference of mercury level in the two limbs is 30 cm.

Solution

Refer Figure 2.9. Let $S_2 = 13.6$, $S_1 = 0.8$, $h_2 = 30$ cm = 0.3 m and $h_1 = h_2 - 15 =$ 15 cm = 0.15 m.

Let p be the fluid pressure in the pipe at point '*A*'. Since the pressures in the left and right limbs at the plane *Z–Z* are equal, we get the following expression.

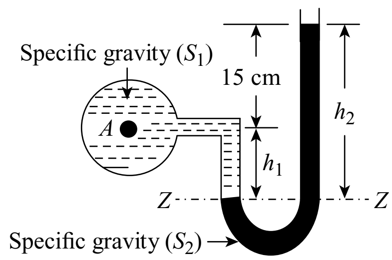

Figure 2.9

$$p + \rho_1 g h_1 = \rho_2 g h_2$$

$$p = \rho_2 g h_2 - \rho_1 g h_1 = S_2 \rho_w g h_2 - S_1 \rho_w g h_1$$

$$\therefore\ p = 13.6 \times 1000 \times 9.81 \times 0.3 - 0.8 \times 1000 \times 9.81 \times 0.15 = \mathbf{38847.6\ N/m^2}$$

Example 2.14 The right limb of a U-tube manometer filled with mercury (specific gravity = 13.6) is open to the atmosphere while the left limb is connected to a pipe containing a fluid of specific gravity 0.85. The level of mercury in the left limb is 10 cm below the centre of the pipe. Determine the vacuum pressure of fluid in the pipe when the difference of mercury level in the two limbs is 25 cm.

Solution

Refer Figure 2.10. Let $S_2 = 13.6$, $S_1 = 0.85$, $h_1 = 10$ cm = 0.1 m and $h_2 = 25$ cm = 0.25 m. Let p be the fluid pressure in the pipe at point 'A'. Since the pressures in the left and right limbs at the plane Z–Z are equal, we derive following expression.

$$p + \rho_1 g h_1 + \rho_2 g h_2 = 0$$

$$p = -(\rho_2 g h_2 + \rho_1 g h_1) = -(S_2 \rho_w g h_2 + S_1 \rho_w g h_1)$$

$$P = -(13.6 \times 1000 \times 9.81 \times 0.25 + 0.85 \times 1000 \times 9.81 \times 0.1)$$

$$\therefore \; p = \mathbf{-34187.85\ N/m^2}$$

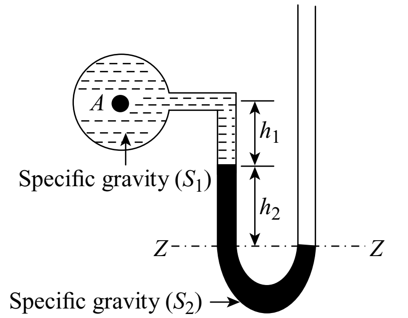

Figure 2.10

Example 2.15 The left limb of a U-tube mercury manometer is connected to a pipe line conveying water, the level of mercury (specific gravity = 13.6) in the leg being 0.5 m below the centre of pipe line and the right limb is open to atmosphere. The level of mercury in the right limb is 0.4 m above that in the left limb and the space above mercury in the right limb contains benzene (specific gravity = 0.88) to a height of 0.2 m. Determine the pressure of water in the pipe.

Solution

Refer Figure 2.11. Let $S_2 = 13.6$, $h_1 = 0.5$ m, $h_2 = 0.4$ m, $S_3 = 0.88$ and $h_3 = 0.2$ m.

Let p be the fluid pressure in the pipe at point 'A'. Since the pressures in the left and right limbs at the plane Z–Z are equal, we get the following expression.

$$p + \rho_w g h_1 = \rho_2 g h_2 + \rho_3 g h_3$$

$$p = \rho_2 g h_2 + \rho_3 g h_3 - \rho_w g h_1 = S_2 \rho_w g h_2 + S_3 \rho_w g h_3 - \rho_w g h_1$$

$$P = 13.6 \times 1000 \times 9.81 \times 0.4 + 0.88 \times 1000 \times 9.81 \times 0.2 - 1000 \times 9.81 \times 0.5$$

$$\therefore \; P = \mathbf{50187.96\ N/m^2}$$

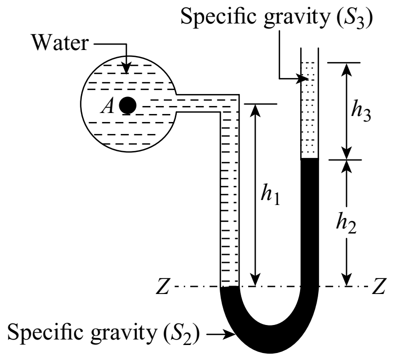

Figure 2.11

Example 2.16 Determine the pressure of water in the pipeline when the difference in the mercury level in the limbs of a U-tube manometer connected to the water pipe is 20 cm and the free surface of mercury (specific gravity = 13.6) is in level with the centre of pipe. Also determine the new difference in the level of mercury when the pressure of water in the pipeline is reduced to 9.95×10^3 N/m^2.

Solution

(i) Refer Figure 2.12(a). Let $h_1 = 20$ cm = 0.2 m and $S_2 = 13.6$. Let p be the fluid pressure in the pipe at point 'A'. Since the pressures in the left and right limbs at the plane Z–Z are equal, we get the following expression.

$$p + \rho_w g h_1 = \rho_2 g h_1$$

$$p = \rho_2 g h_1 - \rho_w g h_1 = S_2 \rho_w g h_1 - \rho_w g h_1$$

$$\therefore \; P = 13.6 \times 1000 \times 9.81 \times 0.2 - 1000 \times 9.81 \times 0.2 = \mathbf{24721.2\ N/m^2}$$

(ii) Refer Figure 2.12(b). With reduction in the pressure of water in the pipe, mercury level in the left limb will rise and there will be corresponding fall in the right limb. Let x cm be the new difference in the level of mercury. $h_1 = (0.2 - 0.01x)$ m, $h_2 = (0.2 - 0.02x)$ m, $S_2 = 13.6$ m, $p = 9.95 \times 10^3$ N/m^2 be the fluid pressure in the pipe at point 'A'. Since the pressures in the left and right limbs at the plane Z_1–Z_1 are equal, we get the following expression.

$$p + \rho_w g h_1 = \rho_2 g h_2$$

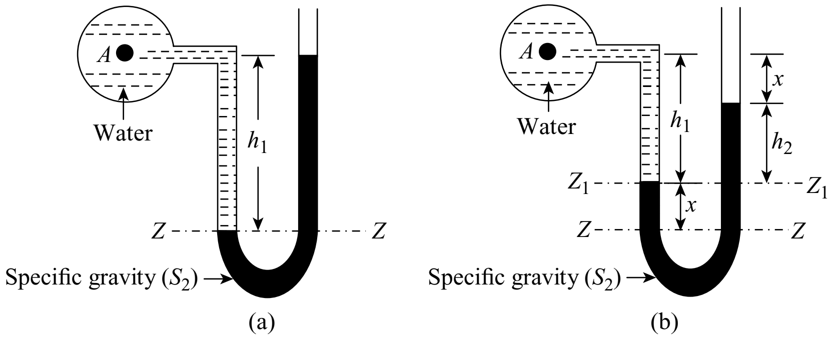

Figure 2.12

or
$$p + \rho_w g h_1 = S_2 \rho_w g h_2$$

$$9.95 \times 10^3 + 1000 \times 9.81 \times (0.2 - 0.01x) = 13.6 \times 1000 \times 9.81 \times (0.2 - 0.02x)$$

$$9950 + 1962 - 98.1x = 26683.2 - 2668.32x$$

$$2570.22x = 14771.2$$

$$\therefore x = \frac{14771.2}{2570.22} = 5.747 \text{ cm}$$

Thus, new difference of mercury is given as follows.

$$h_2 = 20 - 2x = 20 - 2 \times 5.747 = \mathbf{8.506\ cm}$$

Example 2.17 A U-tube manometer containing mercury (specific gravity = 13.6) is connected to the outlet of a conical vessel as shown in Figure 2.13(a). The reading of the manometer given in the figure shows when the vessel is empty. Find the reading of the manometer when the vessel is completely filled with water.

Solution

Refer Figure 2.13(a). Let $S_2 = 13.6$ and $h_2 = 0.15$ m.

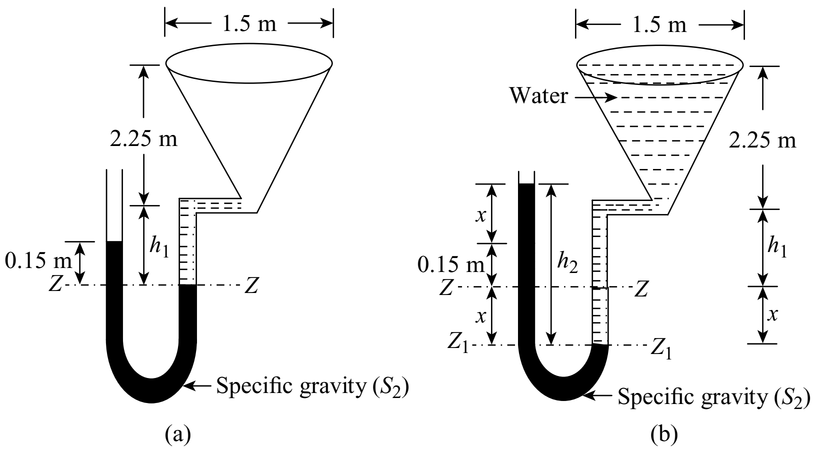

Figure 2.13

Let h_1 be the height of the water above the plane $Z–Z$. Since the pressures in the left and right limbs at the plane $Z–Z$ are equal, we get the following expression.

$$\rho_w g h_1 = \rho_2 g h_2$$

or

$$\rho_w g h_1 = S_2 \rho_w g h_2$$

$$1000 \times 9.81 \times h_1 = 13.6 \times 1000 \times 981 \times 0.15$$

$$\therefore h_1 = \mathbf{2.04\ m}$$

Refer Figure 2.13(b). Let x m be the fall in mercury level in the right limb with a corresponding rise of x m in the left limb. Thus, the difference of mercury level in two limbs becomes $h_2 = (0.15 + 2x)$ m and $S_2 = 13.6$. Since the pressures in the left and right limbs at the plane $Z_1–Z_1$ are equal, we get the following expression.

$$\rho_w g \times (2.25 + h_1 + x) = \rho_2 g \times (0.15 + 2x)$$

or

$$\rho_w g \times (2.25 + h_1 + x) = S_2 \rho_w g \times (0.15 + 2x)$$

$$1000 \times 9.81 \times (2.25 + 2.04 + x) = 13.6 \times 1000 \times 9.81 \times (0.15 + 2x)$$

$$22072.5 + 20012.4 + 9810x = 20012.4 + 266832x$$

$$257022x = 22072.5$$

$$\therefore x = \frac{22072.5}{257022} = 0.0859 \text{ m}$$

Thus, the difference of mercury level in the two limbs is given by,

$$h_2 = 0.15 + 2 \times 0.0859 = \mathbf{0.3218\ m\ or\ 32.18\ cm}$$

2.7.3 Single Column Manometer

The U-tube manometers require readings of liquid levels at two points since a change in pressure causes a rise of liquid in one limb of the manometer and a drop in the other. This difficulty can be overcome by using single column manometer.

A single column manometer is a modified form of a U-tube manometer in which one of the two limbs is made a large reservoir having a cross-sectional area of about 100 times to that of area of the narrow tube in the other limb (Figure 2.14). The change in liquid level in the reservoir due to any variation in pressure will be so small that it may be neglected and the pressure can be measured by the height of the liquid in the narrow tube. Thus, only one reading in the narrow tube of the manometer is to be recorded for pressure measurement. Based on the position of the narrow tube, single column manometers are classified into two types, namely (i) vertical single column manometer and (ii) inclined single column manometer.

1. **Vertical single column manometer:** A vertical single column manometer is shown in Figure 2.14(a).

   The surface of the manometric liquid will stand at level $Z–Z$ when the manometer is not connected to the vessel or pipe. When the reservoir limb of the manometer is connected to the vessel containing liquid at pressure p (greater than atmospheric pressure), the manometric liquid level in the reservoir falls down by Δh and there is a corresponding level rise h_2 in the narrow right limb. Thus, the manometric liquid will stand at level $Z_1–Z_1$ as shown in Figure 2.14(a).

   Let h_1 be the height of the centre of the pipe, S_1 and ρ_1 be the specific gravity and density of liquid in the pipe, S_2 and ρ_2 be the specific gravity and density of liquid in the reservoir and right limb of the manometer, A be the cross-sectional area of the reservoir, and a be the cross-sectional area of the right limb. Fall of manometric liquid in the reservoir causes a rise in liquid in the right limb.

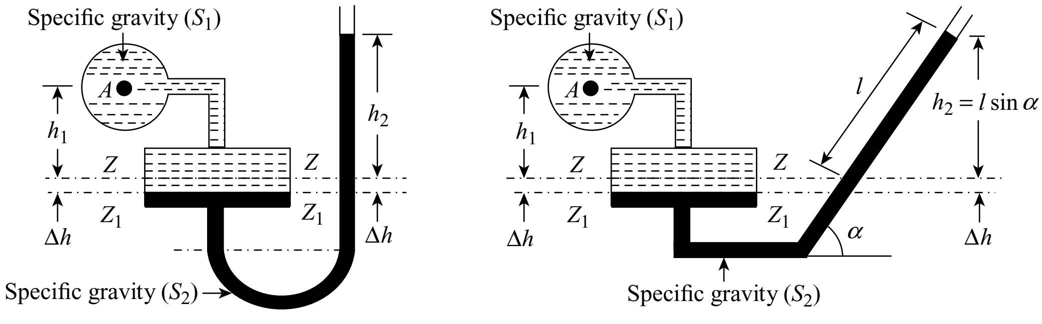

Figure 2.14 *(a) vertical single column manometer (b) inclined single column manometer*

Thus
$$A \times \Delta h = a \times h_2$$

$$\Delta h = \frac{a}{A} h_2$$

Now considering the datum line Z_1–Z_1, the pressure in the left limb will be equal to the pressure in the right limb. Thus, we get the following expression.

$$p + \rho_1 g(h_1 + \Delta h) = \rho_2 g(\Delta h + h_2)$$

$$p = \rho_2 g(\Delta h + h_2) - \rho_1 g(h_1 + \Delta h) = \Delta h(\rho_2 g - \rho_1 g) + \rho_2 g h_2 - \rho_1 g h_1$$

$$p = \frac{a}{A} h_2 \times (\rho_2 g - \rho_1 g) + \rho_2 g h_2 - \rho_1 g h_1 \qquad (2.10)$$

The reservoir area (A) is very large in comparison to cross-sectional area a and thus, the ratio (a/A) is very small and it can be neglected. Therefore, Equation (2.10) is expressed as given below.

$$p = \rho_2 g h_2 - \rho_1 g h_1 \qquad (2.10a)$$

2. **Inclined single column manometer:** An inclined single column manometer is shown in Figure 2.14(b). Due to inclined tube the distance moved by the liquid in the narrow right limb will be more. Thus, the inclined tube type single manometers are more sensitive and are preferred for the measurement of small pressures. These are also used for determining the draft in steam generator settings.

   Let α be the inclination of the right limb to the horizontal axis, l be the rise of manometric liquid in the right limb and $h_2 = l \sin \alpha$ be the corresponding vertical rise in the tube. Sensitivity of inclined single column manometer increases by a factor of $1/\sin \alpha$ when compared to vertical single column manometer. Here, Equation (2.10a) can also be used for an inclined single column manometer after substituting the value of h_2. Thus, we get the following expression.

$$p = \rho_2 g \times l \sin \alpha - \rho_1 g h_1 \qquad [\because h_2 = l \sin \alpha] \qquad (2.10b)$$

The fall of liquid in the reservoir of the manometer is given by,

$$\Delta h = l \times \frac{a}{A} \qquad (2.11)$$

Example 2.18 A single column mercury manometer (specific gravity = 13.6) is connected to the centre of the pipe which carries a liquid of specific gravity 0.85. The area of narrow tube limb is one hundredths of the area of the reservoir of the manometer. Determine the pressure of the liquid flowing through the pipe when the centre of the pipe and the free mercury surface in the narrow tube are 25 cm and 45 cm, respectively above the datum in the reservoir of the manometer.

Solution

Refer Figure 2.15. Let $S_2 = 13.6$, $S_1 = 0.85$, $a/A = 1/100$, $h_1 = 25$ cm = 0.25 m and $h_2 = 45$ cm = 0.45 m. Let p be the pressure of the liquid in the pipe.

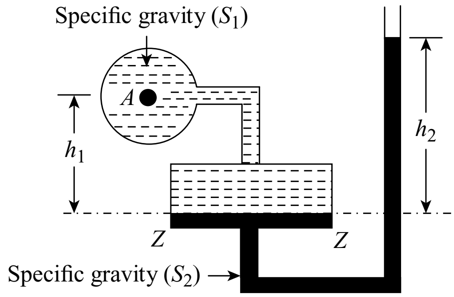

Figure 2.15

$$\rho_2 = S_2\rho_w = 13.6\times 1000 = 13600 \text{ kg/m}^3$$

$$\rho_1 = S_1\rho_w = 0.85\times 1000 = 850 \text{ kg/m}^3$$

Since $$p = \frac{a}{A}h_2(\rho_2 g - \rho_1 g) + \rho_2 g h_2 - \rho_1 g h_1 \quad \text{[Equation (2.10)]}$$

$$p = \frac{0.45}{100}\times(13600\times 9.81 - 850\times 9.81) + 13600\times 9.81\times 0.45 - 850\times 9.81\times 0.25$$

$$\therefore p = 562.849 + 60037.2 - 2084.625 = \mathbf{58515.424\ N/m^2}$$

Example 2.19 The diameter of the reservoir of an inclined mercury manometer is 50 mm. The diameter of the manometer tube is 2.5 mm and the inclination angle to the horizontal axis is 30°. Determine (i) change in pressure when the length of the mercury in the manometer tube changes by 5 mm and (ii) percentage error when neglecting the change in level in the container.

Solution

Refer Figure 2.16. Let $D = 50$ mm, $d = 2.5$ mm, $\alpha = 30^o$, $l =$ 5 mm. Let p be the change in pressure, A and a be the areas of reservoir and manometer tube, respectively. Here, h be the total change in head and Δh be the fall of liquid in the reservoir.

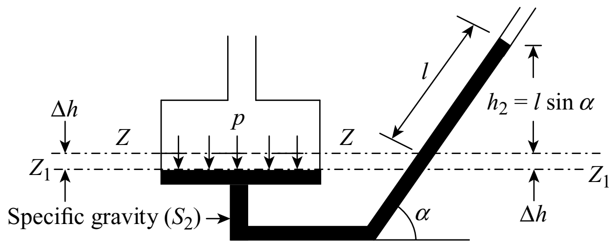

Figure 2.16

(i) $$\frac{A}{a} = \frac{(\pi/4)D^2}{(\pi/4)d^2} = \frac{50^2}{2.5^2} = 400$$

$$\Delta h = l\times\frac{a}{A} = \frac{5}{400} = 0.0125 \text{ mm}$$

$$h_2 = l\sin\alpha = 5\sin 30° = 2.5 \text{ mm}$$

$$h = h_2 + \Delta h = 2.5 + 0.0125 = 2.5125 \text{ mm}$$

$$\rho = 13.6\rho_w = 13.6\times 1000 = 13600 \text{ kg/m}^3$$

$$p = \rho g h = 13600\times 9.81\times 2.5125\times 10^{-3} = \mathbf{335.21\ N/m^2}$$

(ii) $$\% \text{ Error} = \left(\frac{h - h_2}{h}\right)\times 100 = \left(\frac{2.5125 - 2.5}{2.5125}\right)\times 100 \approx \mathbf{0.5\%}$$

2.7.4 Double U-tube Manometer (Compound Manometer)

If the pressure of a fluid to be measured is very high, then a very long U-tube will be required which is difficult to handle. In order to have a U-tube of reasonable size, a number of U-tubes, having the same or different manometric fluids may be used. A double U-tube manometer also known as compound manometer is shown in Figure 2.17.

Let p be the pressure which is to be measured at point 'A', Z–Z and Z_1–Z_1 be the horizontal datum lines, h_1, h_2, h_3 and h_4 be the heights of the liquids from datum lines, S_1, S_2, S_3 and S_4 be the specific gravities and ρ_1, ρ_2, ρ_3 and ρ_4 be the corresponding densities of the liquids as schematically shown in Figure 2.17.

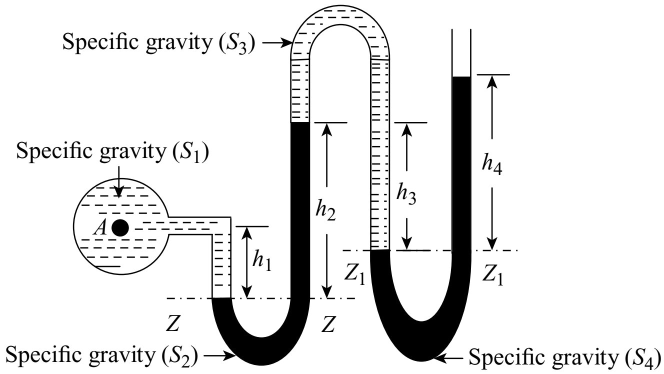

Figure 2.17 *Compound manometer*

$$p + \rho_1 g h_1 - \rho_2 g h_2 + \rho_3 g h_3 - \rho_4 g h_4 = 0$$

$$\therefore p = \rho_2 g h_2 - \rho_1 g h_1 + \rho_4 g h_4 - \rho_3 g h_3 \quad (2.12)$$

Example 2.20 Determine the gauge pressure at point '*A*' by a compound manometer as shown in Figure 2.18. Take specific gravity of mercury as 13.6.

Solution

Refer Figure 2.18. Let $h_1 = 0.15$ m, $h_2 = 0.35$ m, $h_3 = 0.3$ m, $h_4 = 0.4$ m, $\rho_1 = \rho_3 = 1000\ \text{kg/m}^3$, $\rho_2 = \rho_4 = 13.6 \times 1000 = 13600\ \text{kg/m}^3$. Let p be the pressure at point '*A*' which is to be determined.

Since $\quad p = \rho_2 g h_2 - \rho_1 g h_1 + \rho_4 g h_4 - \rho_3 g h_3$

$$p = 13600 \times 9.81 \times 0.35 - 1000 \times 9.81 \times 0.15 + 13600 \times 9.81 \times 0.4 - 1000 \times 9.81 \times 0.3$$

$$\therefore p = 46695.6 - 1471.5 + 53366.4 - 2943 = \mathbf{95647.5\ N/m^2}$$

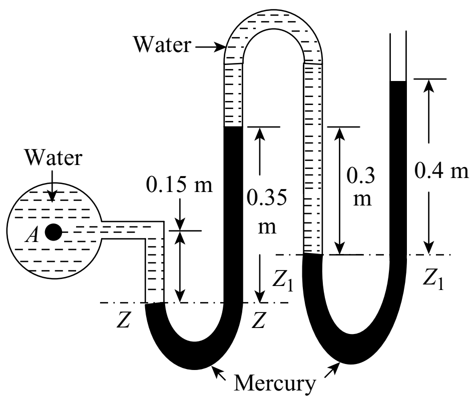

Figure 2.18

Example 2.21 Determine the gauge pressure at point '*A*' by a compound manometer as shown in Figure 2.19. Take specific gravity of mercury as 13.6 and density of air as 1.2 kg/m^3.

Solution

Refer Figure 2.19. Let $S_{Hg} = 13.6$ and $\rho_a = 1.23\ \text{kg/m}^3$. Let p be the pressure at point '*A*', $\rho_{Hg} = 13.6 \times 1000 = 13600\ \text{kg/m}^3$, $h_1 = 0.4$ m and $h_2 = 0.6$ m.

Since $\quad p + \rho_a g(h_1 + h_2) - \rho_{Hg} g h_2 + \rho_w g h_2 - \rho_{Hg} g h_2 + \rho_w g h_2 - \rho_{Hg} g h_2 = 0$

or $\quad p = 3\rho_{Hg} g h_2 - 2\rho_w g h_2 - \rho_a g(h_1 + h_2)$

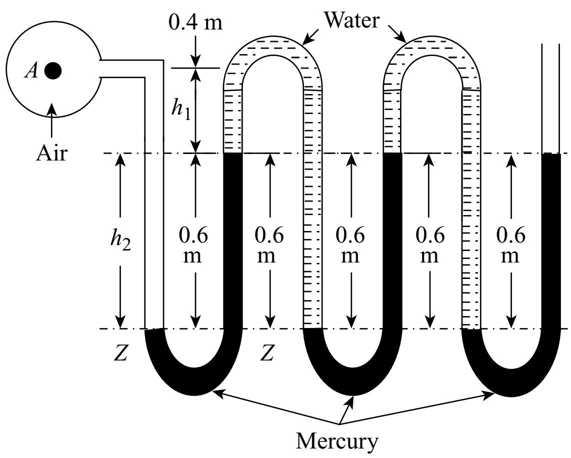

Figure 2.19

$$p = 3 \times 13600 \times 9.81 \times 0.6 - 2 \times 1000 \times 9.81 \times 0.6 - 1.23 \times 9.81 \times (0.4 + 0.6)$$

$$\therefore p = 240148.8 - 11772 - 12.07 = \mathbf{228364.73\ N/m^2}$$

2.8 ❐ DIFFERENTIAL MANOMETERS

A differential manometer is used to measure the difference of pressures in two pipes or between two points in a pipeline. Generally, a differential manometer consists of a bent glass tube (U-tube) containing a heavy liquid (usually mercury). The two ends of the U-tube are connected to the two gauge points between which the pressure difference is to be measured. Thus, none of the ends of the limbs of a differential manometer are exposed to the atmosphere. These manometers are used in venturi meters and orifice or nozzle flow meters. The two most common types of differential manometers, namely U-tube differential manometer and inverted U-tube differential manometer are described in this section.

2.8.1 U-tube Differential Manometer (or Upright U-tube Differential Manometer)

The pipes connected to the differential manometers may be at same level or it may be at different levels as shown in Figure 2.20.

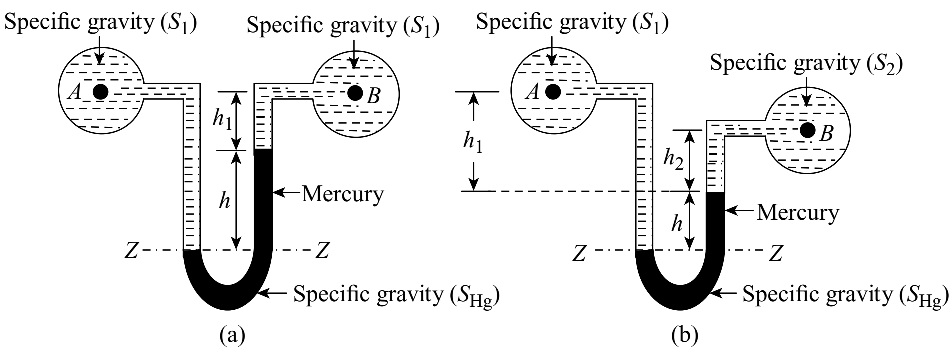

Figure 2.20 *U-tube differential manometers*

Case I: When the pipes are at same level as shown in Figure 2.20(a).

Let h be the difference of mercury level in the U-tube, h_1 be the distance of centres of pipes A and B from the mercury level in the right limb, S_1 be the specific gravity of the liquid in the two pipes at A and B. Here, $\rho_1 = S_1\rho_w$ be the density of the liquid in the two pipes at A and B, and S_{Hg} be the specific gravity of the mercury, and $\rho_{Hg} = S_{Hg}\rho_w$ be the density of mercury.

For obtaining the difference in pressure heads at A and B, equating the pressures in the left limb and right limb of the manometer at datum Z–Z, we get the following expression.

$$p_A + \rho_1 g(h_1 + h) = p_B + \rho_1 g h_1 + \rho_{Hg} g h$$

$$p_A - p_B = \rho_1 g h_1 + \rho_{Hg} g h - \rho_1 g h_1 - \rho_1 g h = gh(\rho_{Hg} - \rho_1) \tag{2.13}$$

Case II: When the pipes are at different levels as shown in Figure 2.20(b).

Let h be the difference of mercury level in the U-tube, h_1 be the distance of centre of pipe A from the mercury level in the right limb, h_2 be the distance of centre of pipe B from the mercury level in the right limb, S_1 be the specific gravity of the liquid in the pipe at A, S_2 be the specific gravity of the liquid in the pipe at B, $\rho_1 = S_1\rho_w$ be the density of the liquid in the pipe at A, $\rho_2 = S_2\rho_w$ be the density of the liquid in the pipe at B and S_{Hg} be the specific gravity of the mercury and $\rho_{Hg} = S_{Hg}\rho_w$ be the density of mercury.

For obtaining the difference in pressure heads at *A* and *B*, equating the pressures in the left limb and right limb of the manometer at datum *Z–Z*, we get the following expression.

$$p_A + \rho_1 g(h_1 + h) = p_B + \rho_2 gh_2 + \rho_{Hg} gh$$

$$p_A - p_B = \rho_2 gh_2 + \rho_{Hg} gh - \rho_1 gh_1 - \rho_1 gh$$

$$\therefore (p_A - p_B) = gh(\rho_{Hg} - \rho_1) + \rho_2 gh_2 - \rho_1 gh_1 \tag{2.14}$$

Example 2.22 A differential mercury manometer connected at the two points *A* and *B* in a pipeline containing an oil of specific gravity 0.85 indicates a difference in mercury levels as 20 cm. Determine the difference in pressures at the two points. Take specific gravity of mercury as 13.6.

Solution

Refer Figure 2.20(a). Let $S_1 = 0.85$, $h = 20$ cm = 0.2 m and $S_{Hg} = 13.6$.

Let p_A be the pressure at point '*A*' and p_B be the pressure at point '*B*'.

$$\rho_1 = S_1 \rho_w = 0.85 \times 1000 = 850 \text{ kg/m}^3$$

$$\rho_{Hg} = S_{Hg} \rho_w = 13.6 \times 1000 = 13600 \text{ kg/m}^3$$

For obtaining the difference in pressure heads at *A* and *B*, equating the pressures in the left limb and right limb of the manometer at datum *Z–Z*, we get the following expression.

$$p_A + \rho_1 g(h_1 + h) = p_B + \rho_1 gh_1 + \rho_{Hg} gh$$

$$(p_A - p_B) = gh(\rho_{Hg} - \rho_1)$$

$$\therefore (p_A - p_B) = 9.81 \times 0.2 \times (13600 - 850) = \mathbf{25015.5 \text{ N/m}^2}$$

Example 2.23 The centres of two pipes are connected to a differential manometer. The pipe at higher level carries a liquid (specific gravity = 0.95) at a pressure of 1.5 bar while the pipe at lower level carries a liquid (specific gravity = 1.3) at a pressure of 1 bar. The centres of pipes having high pressure liquid and low pressure liquid are 2 m and 1 m above the higher mercury level in the manometer respectively. Determine the difference in mercury level in the manometer. Take specific gravity of mercury as 13.6.

Solution

Refer Figure 2.20(b). Let $S_1 = 0.95$, $p_A = 1.5$ bar, $S_2 = 1.3$, $p_B = 1$ bar, $h_1 = 2$ m, $h_2 = 1$ m and $S_{Hg} = 13.6$. Let h be the difference of mercury level in the U-tube.

$$\rho_1 = S_1 \rho_w = 0.95 \times 1000 = 950 \text{ kg/m}^3$$

$$\rho_2 = S_2 \rho_w = 1.3 \times 1000 = 1300 \text{ kg/m}^3$$

$$\rho_{Hg} = S_{Hg} \rho_w = 13.6 \times 1000 = 13600 \text{ kg/m}^3$$

For obtaining the difference in pressure heads at *A* and *B*, equating the pressures in the left limb and right limb of the manometer at datum *Z–Z*, we get the following expression.

$$p_A + \rho_1 g(h_1 + h) = p_B + \rho_2 gh_2 + \rho_{Hg} gh$$

$$1.5 \times 10^5 + 950 \times 9.81 \times (2 + h) = 1 \times 10^5 + 1300 \times 9.81 \times 1 + 13600 \times 9.81\ h$$

$$168639 + 9319.5\ h = 112753 + 133416\ h$$

$$133416\,h - 9319.5\,h = 168639 - 112753$$

$$124096.5\,h = 55886$$

$$\therefore h = \frac{55886}{124096.5} = \mathbf{0.45\ m}$$

Example 2.24 The difference in height of the centres of two pipes connected to a differential mercury manometer is 2.5 m. The pipe at higher level carries a liquid (specific gravity = 0.9) at a pressure of 1.75 bar while the pipe at lower level carries a liquid (specific gravity = 1.45) at a pressure of 1 bar. The centre of pipe having low pressure liquid is 1.5 m above the higher mercury level in the manometer. Determine the difference in mercury level in the manometer. Take specific gravity of mercury as 13.6.

Solution

Refer Figure 2.21. Let h_3 = 2.5 m, S_1 = 0.9, p_A = 1.75 bar, S_2 = 1.45, p_B = 1 bar, h_2 = 1.5 m, $h_1 = h_3 + h_2$ = 2.5 + 1.5 = 4 m and S_{Hg} = 13.6.

Let h be the difference of mercury level in the U-tube.

$$\rho_1 = S_1\rho_w = 0.9\times1000 = 900 \text{ kg/m}^3$$

$$\rho_2 = S_2\rho_w = 1.45\times1000 = 1450 \text{ kg/m}^3$$

$$\rho_{Hg} = S_{Hg}\rho_w = 13.6\times1000 = 13600 \text{ kg/m}^3$$

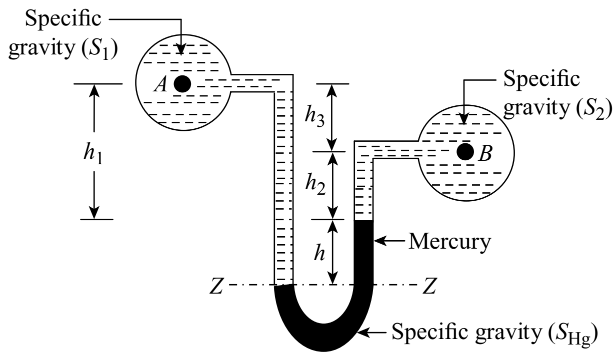

Figure 2.21

For obtaining the difference in pressure heads at A and B, equating the pressures in the left limb and right limb of the manometer at datum Z–Z, we get the following expression.

$$p_A + \rho_1 g(h_1 + h) = p_B + \rho_2 g h_2 + \rho_{Hg} g h$$

$$1.75\times10^5 + 900\times9.81\times(4+h) = 1\times10^5 + 1450\times9.81\times1.5 + 13600\times9.81h$$

$$210316 + 8829\,h = 121336.75 + 133416\,h$$

$$133416\,h - 8829\,h = 210316 - 121336.75$$

$$124587\,h = 88979.25$$

$$\therefore h = \frac{88979.25}{124587} = \mathbf{0.7142\ m\ or\ 71.42\ cm}$$

Example 2.25 A differential manometer is connected at two gauge points A and B. At point A, in the pipe having water connected to left limb of the manometer, the air pressure is one bar. The centre of the pipe having water and air is 0.75 m above the lower mercury level in the manometer while the centre of the pipe having liquid (specific gravity = 0.95) is 0.25 m above the higher mercury level in the manometer. The difference in mercury level in the manometer is 0.12 m. The specific gravity of mercury is 13.6. Determine the gauge pressure at point B.

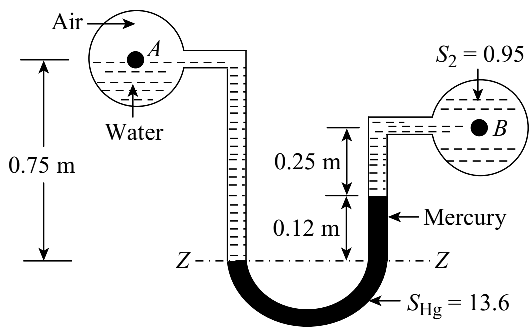

Figure 2.22

Solution

Refer Figure 2.22. Let p_A = 1 bar, h_1 = 0.75 m, S_2 = 0.95, h_2 = 0.25 m, h = 0.12 m and S_{Hg} = 13.6. Let p_B be the gauge pressure at point B.

$$\rho_{Hg} = S_{Hg}\rho_w = 13.6\times1000 = 13600 \text{ kg/m}^3$$

$$\rho_2 = S_2\rho_w = 0.95\times1000 = 950 \text{ kg/m}^3$$

For obtaining the pressure p_B at point B, equating the pressures in the left limb and right limb of the manometer at datum Z–Z, we get the following expression.

$$p_A + \rho_w g h_1 = p_B + \rho_2 g h_2 + \rho_{Hg} g h$$

$$1\times10^5 + 1000\times9.81\times0.75 = p_B + 950\times9.81\times0.25 + 13600\times9.81\times0.12$$

$$107357.5 = p_B + 18339.79$$

$$\therefore p_B = 107357.5 - 18339.79 = \mathbf{89017.71\ N/m^2}$$

Example 2.26 Two pipes containing water are connected to a compound differential manometer as shown in Figure 2.23. Determine the pressure difference between the points at A and B when the specific gravities of mercury and oil are 13.6 and 0.9, respectively.

Solution

Refer Figure 2.23. Let $h_1 = 1$ m, $h_2 = 0.5$ m, $h_3 = 0.3$ m, $h_4 = 0.4$ m, $h_5 = 0.15$ m and $S_{Hg} = 13.6$. Here, p_A and p_B be the pressure at points A and B, respectively.

$$\rho_{Hg} = S_{Hg}\rho_w = 13.6\times1000 = 13600 \text{ kg/m}^3$$

$$\rho_2 = S_2\rho_w = 0.9\times1000 = 900 \text{ kg/m}^3$$

For obtaining the pressure difference between the points A and B, equating the pressures, we get the following governing manometric equation.

$$p_A + \rho_w g h_1 - \rho_{Hg} g h_2 + \rho_2 g h_3 - \rho_{Hg} g h_4 - \rho_w g h_5 = p_B$$

$$p_A - p_B = \rho_{Hg} g(h_2 + h_4) - \rho_w g(h_1 - h_5) - \rho_2 g h_3$$

$$(p_A - p_B) = 13600\times9.81\times(0.5+0.4) - 1000\times9.81 \times(1-0.15) - 900\times9.81\times0.3$$

$$\therefore (p_A - p_B) = \mathbf{109087.2\ N/m^2}$$

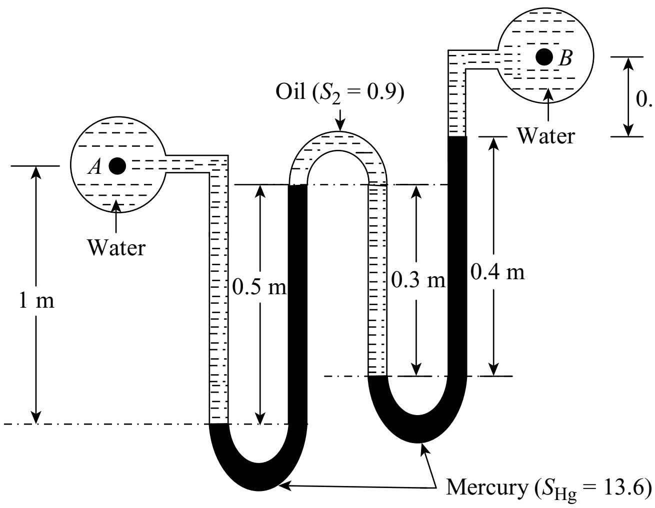

Figure 2.23

Example 2.27 Determine the pressure difference between the points at A and B shown in Figure 2.24. Take the specific gravity of mercury as 13.6 and density of air as 1.23 kg/m^3.

Solution

Refer Figure 2.24. Let $h_1 = 0.25$ m, $h_2 = 0.4$ m, $l_3 = l_4 = 0.2$ m, $S_{Hg} = 13.6$, $S_1 = 1.2$, $\rho_{air} = 1.23$ kg/m^3 and $\alpha = 30°$. Here, p_A and p_B be the pressure at points A and B, respectively.

$$\rho_{Hg} = S_{Hg}\rho_w = 13.6\times1000 = 13600 \text{ kg/m}^3$$

$$\rho_1 = S_1\rho_w = 1.2\times1000 = 1200 \text{ kg/m}^3$$

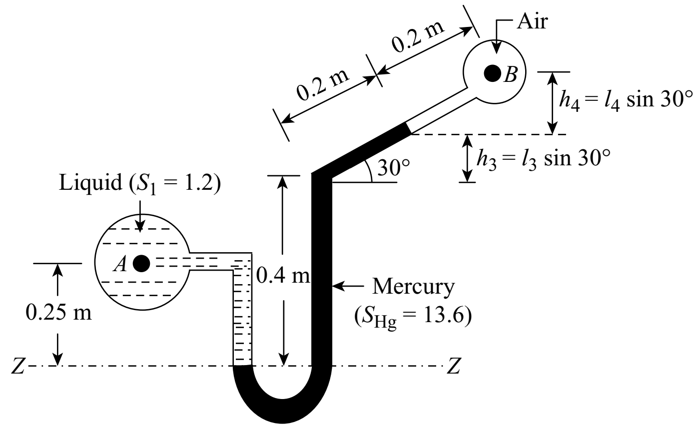

Figure 2.24

For obtaining the difference in pressure heads at A and B, equating the pressures in the left limb and right limb of the manometer at datum Z–Z, we get the following expression.

$$p_A + \rho_1 g h_1 = \rho_{Hg} g h_2 + \rho_{Hg} g l_3 \sin 30° + \rho_{air}\, g l_4 \sin 30° + p_B$$

$$p_A - p_B = g(\rho_{Hg} h_2 + \rho_{Hg} l_3 \sin 30° + \rho_{air}\, l_4 \sin 30° - \rho_1 h_1)$$

$$(p_A - p_B) = 9.81 \times (13600 \times 0.4 + 13600 \times 0.2 \times 0.5 + 1.23 \times 0.2 \times 0.5 - 1200 \times 0.25)$$

$$\therefore\ (p_A - p_B) = \mathbf{63766.207\ N/m^2}$$

2.8.2 Inverted U-tube Manometer

The inverted U-tube manometer is used when the difference of pressure to be measured between two points in a pipeline or between two pipes is small. This manometer contains lighter fluid than the working fluid in the pipeline. Here, Figure 2.25 illustrates an inverted U-tube manometer connected at two gauge points in which the pressure at A is more than the pressure at B.

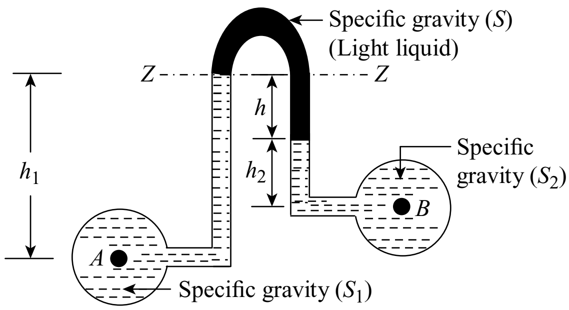

Figure 2.25 *Inverted U-tube differential manometer*

Let h be the difference of lighter liquid level in the U-tube, h_1 be the distance of centre of pipe A from the lighter liquid level in the left limb, h_2 be the distance of centre of pipe B from the lighter liquid level in the right limb, S_1 be the specific gravity of the liquid in the pipe at A, S_2 be the specific gravity of the liquid in the pipe at B, $\rho_1 = S_1 \rho_w$ be the density of the liquid in the pipe at A, $\rho_2 = S_2 \rho_w$ be the density of the liquid in the pipe at B and S be the specific gravity of the lighter liquid in U-tube and $\rho = S \rho_w$ be the density of lighter liquid.

For obtaining the difference in pressure heads at A and B, equating the pressures in the left limb and right limb of the manometer at datum Z–Z, we get the following expression.

$$p_A - \rho_1 g h_1 = p_B - \rho_2 g h_2 - \rho g h$$

$$\therefore (p_A - p_B) = \rho_1 g h_1 - \rho_2 g h_2 - \rho g h \tag{2.15}$$

When both the pipes are at same level, then $h_1 = h_2$ and Equation (2.15) becomes,

$$\therefore (p_A - p_B) = gh_1(\rho_1 - \rho_2) - \rho gh \tag{2.16}$$

Example 2.28 An inverted tube manometer is connected to two horizontal pipes at the points *A* and *B*. Pipe *A* carries water, pipe *B* carries oil (specific gravity = 0.94) and the manometer contains a light fluid (specific gravity = 0.85). Pipe *A* is at lower level when compared to pipe *B*. The common level is at 0.9 m above the centre of pipe *A* and 0.8 m above the centre of pipe *B*. The 0.8 m is consisted of 0.2 m light liquid and the remaining oil. Determine (i) pressure difference between the points at *A* and *B* and (ii) absolute pressure in pipe '*A*' in m of water if pressure at point *B* is 5.2×10^4 N/m^2 and the atmospheric pressure is 750 mmHg.

Solution

Refer Figure 2.26. Let $S_2 = 0.94$, $S = 0.85$, $h_1 = 0.9$ m, $h + h_2 = 0.8$ m, $h = 0.2$ m and $h_2 = 0.6$ m. Here, p_A and p_B be the pressure at points *A* and *B*, respectively, $p_B = 5.2 \times 10^4$ N/m^2 and $p_{atm} = 750$ mmHg = 0.75 mHg.

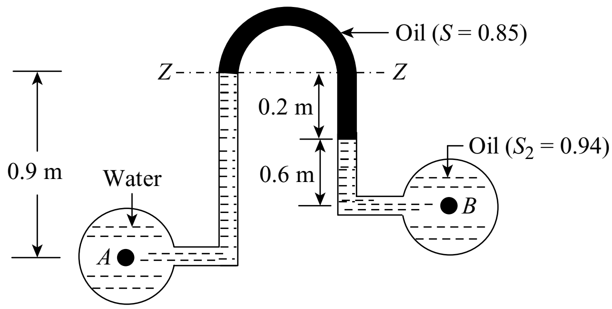

Figure 2.26

(i) $\rho = S\rho_w = 0.85 \times 1000 = 850$ kg/m^3

$\rho_2 = S_2\rho_w = 0.94 \times 1000 = 940$ kg/m^3

For obtaining the difference in pressure heads at *A* and *B*, equating the pressures in the left limb and right limb of the manometer at datum *Z–Z*, we get the following expression.

$$p_A - \rho_w gh_1 = p_B - \rho_2 gh_2 - \rho gh$$

$$p_A - p_B = g(\rho_w h_1 - \rho_2 h_2 - \rho h)$$

$$\therefore (p_A - p_B) = 9.81 \times (1000 \times 0.9 - 940 \times 0.6 - 850 \times 0.2) = \mathbf{1628.46\ N/m^2}$$

(ii) Pressure in pipe *A* is given by,

$$p_A = 1628.46 + p_B = 1628.46 + 5.2 \times 10^4 = 53628.46 \text{ N/m}^2$$

Atmospheric pressure is given by,

$$p_{atm} = 1000 \times 9.81 \times 13.6 \times 0.75 = 100062 \text{ N/m}^2$$

Absolute pressure is given by,

$$p_A \text{ (abs)} = p_A + p_{atm} = 53628.46 + 100062 = 153690.46 \text{ N/m}^2$$

$$\therefore p_A\text{(abs)} = \frac{153690.46}{1000 \times 9.81} = \mathbf{15.67\ m\ of\ water}$$

Example 2.29 An inverted U-tube manometer containing a manometric light fluid (specific gravity = 0.7) is connected to two pipes at points *A* and *B*. Pipe *A* carries liquid of specific gravity 1.2 and pipe *B* carries water. The pipes are at the same level. The height of the liquid of specific gravity 1.2 from the centre of the pipe is 30 cm. If all liquids are immiscible and the pressure in pipe *B* is 0.2 kPa above the pressure in the pipe *A*, determine the differential reading of the manometer.

Solution

Refer Figure 2.27. Let $S = 0.7$, $S_1 = 1.2$, $h_1 = 30$ cm = 0.3 m, p_A and p_B be the pressure at points *A* and *B*, respectively. Thus, $p_B = p_A + 0.2$ kPa = $(p_A + 200)$ N/m^2. Let *h* be the differential reading of the given manometer.

$$\rho_1 = S_1\rho_w = 1.2\times1000 = 1200 \text{ kg/m}^3$$

$$\rho = S\rho_w = 0.7\times1000 = 700 \text{ kg/m}^3$$

$$\rho_2 = \rho_w = 1000 \text{ kg/m}^3$$

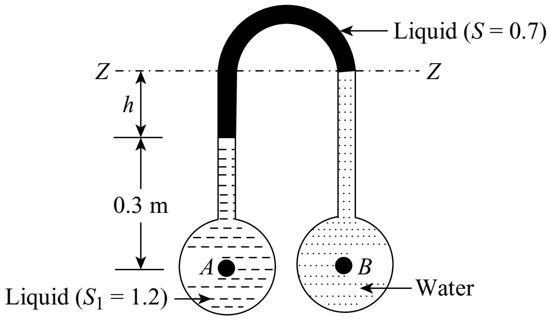

Figure 2.27

Equating the pressures in the left limb and right limb of the manometer at datum *Z*–*Z*, we get the following expression.

$$p_A - \rho_1 g h_1 - \rho g h = p_B - \rho_2 g(0.3+h)$$

$$p_A - 1200\times9.81\times0.3 - 700\times9.81\times h = (p_A+200) - 1000 \times 9.81\times(0.3+h)$$

$$-3531.6 - 6867\,h = 200 - 2943 - 9810\,h$$

$$2943\,h = 788.6$$

$$\therefore h = \frac{788.6}{2943} = \mathbf{0.268\ m}$$

Example 2.30 An inverted U-tube manometer containing oil as a manometric liquid (specific gravity = 0.75) is connected to the two horizontal water pipes at the points *A* and *B*. Pipe *B* is at lower level when compared to pipe *A*. The common level is at 0.4 m above pipe *A* and 0.75 m above pipe *B*. The 0.75 m height consists of *h* m of manometric liquid and the remaining is water. If the pressure in pipe *B* is 2.5 kPa above the pressure in the pipe *A*, then determine the value of differential reading *h* of the manometer.

Solution

Refer Figure 2.28. Let S = 0.75 and h be the differential reading of the manometer, h_1 = 0.4 m, $h + h_2$ = 0.75 m. Here, p_A and p_B be the pressure at points *A* and *B*, respectively. Thus, $p_B = p_A + 2.5\text{ kPa} = (p_A + 2500)\text{ N/m}^2$.

$$\rho = S\rho_w = 0.75\times1000 = 750 \text{ kg/m}^3$$

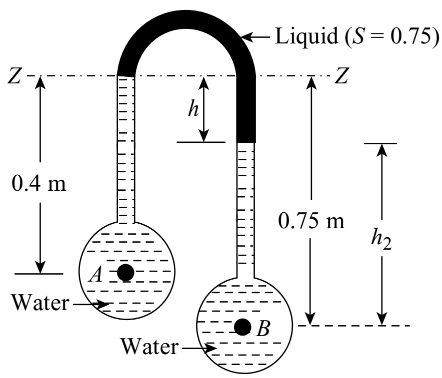

Figure 2.28

Equating the pressures in the left limb and right limb of the manometer at datum *Z*–*Z*, we get the following expression.

$$p_A - \rho_w g\times0.4 = p_B - \rho_w g\times(0.75-h) - \rho g h$$

$$p_A - 1000\times9.81\times0.4 = (p_A+2500) - 1000\times9.81\times(0.75-h) - 750\times9.81\times h$$

$$-3924 = 2500 - 7357.5 + 9810\,h - 7357.5\,h$$

$$2452.5\,h = 933.5$$

$$\therefore h = \frac{933.5}{2452.5} = \mathbf{0.3806\ m\ or\ 38.06\ cm}$$

2.9 ❐ ADVANTAGES AND LIMITATIONS OF MANOMETERS

The following are the advantages of manometers.

1. These are relatively inexpensive, easy to fabricate and have good accuracy.
2. These have high sensitivity (i.e., respond rapidly to pressure changes).
3. These require little maintenance and do not get affected by vibrations.

4. It is suited to low pressure and low differential pressures.
5. Sensitivity can be changed by changing the quantity of manometric liquid.

The following are the limitations of manometers.

1. These are bulky, large and fragile.
2. The readings of the manometers are affected by change in temperature and altitude.
3. Surface tension of manometric fluid causes capillary effect.
4. Meniscus is to be measured carefully to obtain accurate readings.
5. The differential head is limited to low pressure differences in usual type of differential manometers. However, high differential heads can also be measured easily by using mercury as manometric liquid.

2.10 ❐ MICROMANOMETERS

The micromanometer consists of a U-tube with two reservoirs of wider cross section at the top of both limbs as shown in Figure 2.29.

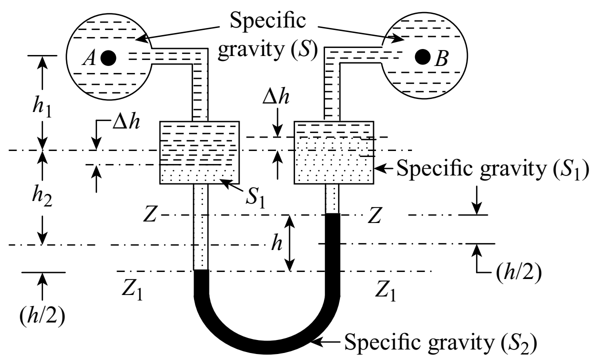

Figure 2.29 *Micromanometer*

It measures very small pressure differences with very high precision. The micromanometer uses two manometric liquids of different specific gravities. The manometric liquids are immiscible with each other and with the fluid for which the pressure difference is to be measured. When the manometer is connected to the water pipes 'A' and 'B' subjected to pressure p_A and p_B such that $p_A > p_B$, then the level of the lighter liquid falls in the left reservoir and rises in the right reservoir by the amount Δh. Similarly, the level of the heavier manometric liquid falls in the left limb and correspondingly rises in the right limb to the datum Z–Z.

Let A and a be the cross-sectional areas of the reservoir and the tube, respectively. The volume of the liquid displaced in each reservoir is equal to the volume of the liquid displaced in each limb.

$$A\Delta h = a \times \frac{h}{2}$$

$$\Delta h = \frac{a}{A} \times \frac{h}{2} \tag{2.17}$$

Let w and S be the weight density and specific gravity of the water in the pipe A and B, S_1 be the specific gravity of the lighter liquid in the reservoirs and S_2 be the specific gravity of the manometric heavier liquid. The other readings are shown in Figure 2.29. Equating the pressures at the datum Z_1–Z_1, we get the following expression.

$$\frac{p_A}{w} + (h_1 + \Delta h)S + \left(h_2 - \Delta h + \frac{h}{2}\right)S_1 = \frac{p_B}{w} + (h_1 - \Delta h)S + \left(h_2 + \Delta h - \frac{h}{2}\right)S_1 + hS_2$$

$$\frac{p_A}{w} - \frac{p_B}{w} = 2\Delta h S_1 - 2\Delta h S - hS_1 + hS_2$$

Substituting the value of Δh from Equation (2.17), we get:

$$\frac{p_A}{w} - \frac{p_B}{w} = 2\times\frac{ah}{2A}S_1 - 2\times\frac{ah}{2A}S - hS_1 + hS_2$$

$$\frac{p_A}{w} - \frac{p_B}{w} = h\left[S_2 - S_1\left(1-\frac{a}{A}\right) - S\frac{a}{A}\right] \tag{2.18}$$

When the cross-sectional area of the reservoir is large as compared with the cross-sectional area of the tube, then the ratio (a/A) becomes very small and thus, from Equation (2.18), we get the following expression.

$$\frac{p_A}{w} - \frac{p_B}{w} = h(S_2 - S_1) \tag{2.19}$$

$$p_A - p_B = wh(S_2 - S_1) = \rho_w gh(S_2 - S_1) \tag{2.19a}$$

Example 2.31 In a micromanometer, the specific gravities of the liquids used at the top and bottom portions of the U-tube are 0.9 and 1, respectively. The ratio of areas of the reservoir to the tube is 25. When the micromanometer is connected to two pipes A and B carrying oil of specific gravity 0.8, the common surface displaces 20 cm. If the pressure is large in pipe A than pipe B, then determine the difference in pressure between A and B.

Solution

Refer Figure 2.29. $S_1 = 0.9$, $S_2 = 1$, $A/a = 25$, $S = 0.8$, $h = 20$ cm $= 0.2$ m. Let p_A and p_B be the pressure at points A and B, respectively and $p_A > p_B$.

From Equation (2.18), we get:

$$p_A - p_B = \rho_w gh\left[S_2 - S_1\left(1-\frac{a}{A}\right) - S\frac{a}{A}\right]$$

$$(p_A - p_B) = 1000\times 9.81\times 0.2\times\left[1 - 0.9\left(1-\frac{1}{25}\right) - 0.8\times\frac{1}{25}\right]$$

$$\therefore\ (p_A - p_B) = \mathbf{204.048\ N/m^2}$$

2.11 ❒ MECHANICAL GAUGES

2.11.1 Bourdon Tube Pressure Gauge

A schematic view of Bourdon tube pressure gauge is shown in Figure 2.30(a). It consists of an elastic metallic tube called Bourdon tube which senses the pressure. The Bourdon tube is made of phosphor bronze and it is of elliptical cross section bent in the form of circular arc. One end of the tube is fixed which is connected to the gauge point of the source of pressure through a syphon filled with water. The other closed end of tube is connected to a toothed sector wheel through a link. The toothed sector meshes with a pinion fixed on a spindle which carries the pointer to read the pressure on a dial gauge.

The fluid pressure forces the water inside the elliptical tube. The tube tries to become straight and thus, the closed end of Bourdon tube moves outward. This slight movement of the tube actuates the sector and the pinion rotates on which pointer is mounted. Hence, the slight movement of the Bourdon tube is considerably magnified and the pointer moves on the graduated scale which directly gives the fluid pressure in the container. A Bourdon tube pressure gauge may be used to measure high as well as low pressures. This can also be used to measure vacuum pressure. A compound Bourdon tube pressure gauge measures both above and below the atmospheric pressure.

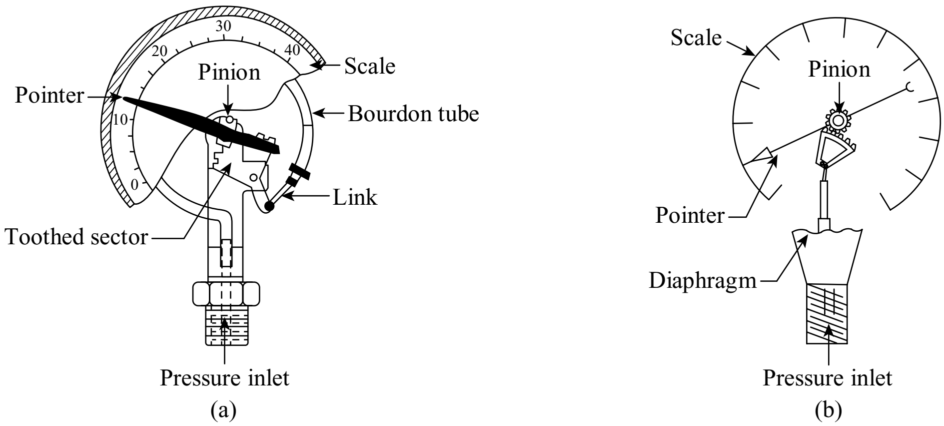

Figure 2.30 *(a) Bourdon tube pressure gauge (b) Diaphragm pressure gauge*

2.11.2 Diaphragm Pressure Gauge

A schematic view of diaphragm pressure gauge is shown in Figure 2.30(b). The difference between diaphragm pressure gauge and the Bourdon tube pressure gauge is that it uses a corrugated metallic diaphragm for transmission of pressure instead of Bourdon tube. The fluid pressure acts on the diaphragm which gets deformed and the deformation is transmitted to the needle point that moves on the calibrated scale with the help of a pinion arrangement. The diaphragm pressure gauge measures relatively low pressure.

2.11.3 Bellows Pressure Gauge

A schematic view of bellows pressure gauge is shown in Figure 2.31(a). In this gauge, the pressure is sensed by a thin metallic tube which has deep circumferential corrugations. When this pressure gauge is connected to the source of pressure the elastic element deforms and thereby, the pointer moves on the graduated scale. This pressure can be directly read from the scale.

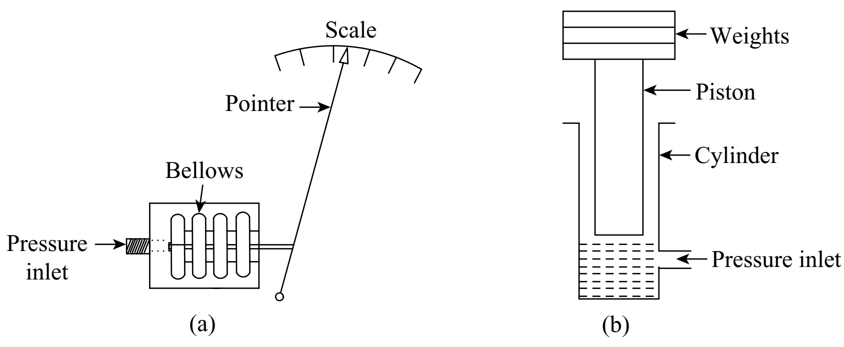

Figure 2.31 *(a) Bellows pressure gauge (b) Dead weight pressure gauge*

2.11.4 Dead Weight Pressure Gauge

A schematic view of dead weight pressure gauge is shown in Figure 2.31(b). This pressure gauge consists of a cylinder and piston arrangement of known area which is connected to fluid by a tube. The pressure exerted by the fluid on the piston is counter-balanced by known dead weights placed on the top of the vertical pistons. Let p be the fluid pressure, $A = [(\pi/4)d^2]$ be the area of the piston, d be the diameter of the piston and W be dead weight, then pressure is given by, $p = W/A$. Both W and A are known quantities and thus, the value of p can be determined. The dead weight pressure gauge is the most accurate pressure gauge. It is used for precision work and for the calibration of other pressure measuring devices.

2.12 ❒ PRESSURE VARIATION IN COMPRESSIBLE FLUID (AEROSTATICS)

Atmosphere is a body of compressible fluid in which the pressure and temperature both vary with altitude. Depending on the changes in temperature, troposphere and stratosphere are two major layers in atmosphere. The troposphere is the lowest layer which extends up to about 11000 m above sea level. In this layer, the temperature of air decreases at an average rate of 0.0065°C per metre of height. The stratosphere extends from 11000 m to 32000 m. In this region, the temperature remains constant at about −56.5°C, i.e., isothermal condition may be assumed. However, beyond 32000 m the temperature increases again. The density varies with both the pressure and temperature. For variable density, Equation (2.2) cannot be integrated, unless the relation between p and ρ is known. The relation between density, pressure and temperature for a perfect gas is given by Equation (1.11a) and it is expressed below.

$$\rho = \frac{p}{RT}$$

Now from Equation (2.2a), we get:

$$\frac{dp}{dh} = \rho g = \frac{pg}{RT}$$

$\because$ h is measured vertically upward and we get the following expression.

$$\therefore \frac{dp}{p} = -\frac{g}{RT} dh \tag{2.20}$$

Pressure variation can be measured by assuming (i) isothermal process (temperature remains constant) and (ii) adiabatic process (no heat exchange).

2.12.1 Isothermal Process

In isothermal process, the temperature remains constant and by integrating Equation (2.20), we get the following expression.

$$\int_{p_o}^{p} \frac{dp}{p} = -\frac{g}{RT}\int_{h_o}^{h} dh$$

$$\ln\left(\frac{p}{p_o}\right) = -\frac{g}{RT}(h - h_o)$$

As h_o is considered as ground level, $h_o = 0$.

$$\therefore \frac{p}{p_o} = \exp\left(-\frac{gh}{RT}\right)$$

Thus, pressure at a height h is given by,

$$\boxed{p = p_o \exp\left(-\frac{gh}{RT}\right)} \tag{2.21}$$

2.12.2 Adiabatic Process

The relation between p and ρ for an adiabatic process is given from Equation (1.17) and the expression is given below.

$$\frac{p}{\rho^\gamma} = \frac{p_o}{\rho_o{}^\gamma} \qquad [\because p/\rho^\gamma = C]$$

$$\rho = \rho_o p^{1/\gamma} p_o{}^{-1/\gamma}$$

From Equation (2.2a) by taking h in vertically upward direction, we get:

$$\frac{dp}{dh} = -\rho g = -\rho_o p^{1/\gamma} p_o^{-1/\gamma} g$$

$$\frac{dp}{p^{1/\gamma}} = -g\rho_o p_o^{-1/\gamma} dh$$

Integrating the above equation, we get:

$$\int_{p_o}^{p} \frac{dp}{p^{1/\gamma}} = -g\rho_o p_o^{-1/\gamma} \int_{h_o}^{h} dh$$

$$\frac{\gamma}{\gamma - 1}\left[p^{\frac{\gamma-1}{\gamma}} - p_o^{\frac{\gamma-1}{\gamma}}\right] = -g\rho_o p_o^{-1/\gamma}(h - h_o)$$

Using $\rho_o = p_o/(RT_o)$ and taking h_o as ground level, i.e., $h_o = 0$, we get:

$$\frac{\gamma}{\gamma - 1}\left[p^{\frac{\gamma-1}{\gamma}} - p_o^{\frac{\gamma-1}{\gamma}}\right] = -g\frac{p_o}{RT_o} p_o^{-1/\gamma}(h - 0) = -p_o^{\frac{\gamma-1}{\gamma}} \frac{gh}{RT_o}$$

$$\left(\frac{p}{p_o}\right)^{\frac{\gamma-1}{\gamma}} = 1 - \frac{\gamma - 1}{\gamma}\frac{gh}{RT_o}$$

$$\boxed{\therefore p = p_o\left[1 - \frac{\gamma - 1}{\gamma}\frac{gh}{RT_o}\right]^{\frac{\gamma}{\gamma-1}}} \tag{2.22}$$

Using the following relations for an adiabatic process, we get:

$$\frac{\rho}{\rho_o} = \left(\frac{p}{p_o}\right)^{\frac{1}{\gamma}} \text{ and } \frac{T}{T_o} = \left(\frac{p}{p_o}\right)^{\frac{\gamma-1}{\gamma}}$$

We obtain the following expressions for temperature and density with altitude,

$$\boxed{T = T_o\left[1 - \frac{\gamma - 1}{\gamma}\frac{gh}{RT_o}\right]} \tag{2.23}$$

and

$$\boxed{\rho = \rho_o\left[1 - \frac{\gamma - 1}{\gamma}\frac{gh}{RT_o}\right]^{\frac{1}{\gamma-1}}} \tag{2.24}$$

Temperature lapse rate It is defined as the rate of decrease of temperature with altitude. It is denoted by λ and it can be obtained by differentiating Equation (2.23), we get the following expression.

$$\lambda = \frac{dT}{dh} = \frac{d}{dh}\left[T_o - T_o\frac{\gamma - 1}{\gamma}\frac{gh}{RT_o}\right] = -\frac{g}{R}\left(\frac{\gamma - 1}{\gamma}\right) \tag{2.25}$$

When $\gamma = 1$ then $\lambda = 0$ and it indicates that temperature does not vary with altitude and the process is isothermal. When $\gamma > 1$, $\lambda = -\text{ve}$ and it indicates that temperature decreases with increase in altitude.

Example 2.32 If the atmospheric pressure at sea level is 1.01325 bar and the temperature is 27°C, then determine the pressure, temperature and density at an elevation 4500 m above sea level, assuming (i) constant density (ii) isothermal process and (iii) adiabatic process.

Solution

Let p_o = 1.01325 bar = 101325 N/m^2, T = 27°C = 300.15 K and h = 4500 m.

(i) $\rho = \dfrac{p}{RT} = \dfrac{101325}{287 \times 300.15} = \mathbf{1.1762\ kg/m^3}$

For constant density, pressure is given by integrating Equation (2.2a), we get:

$$\int_{p_o}^{p} dp = -\rho g \int dh$$

$$p - p_o = -\rho g(h - h_o) = -\rho_o gh \qquad [\because h_o = 0]$$

$$p = p_o - \rho_o gh$$

$$\therefore p = 101325 - 1.1762 \times 9.81 \times 4500 = \mathbf{49401.651\ N/m^2}$$

$$T = \frac{p}{\rho R} = \frac{49401.651}{1.1762 \times 287} = \mathbf{146.345\ K}$$

(ii) For isothermal assumption pressure at a height h is given by,

$$p = p_o \exp\left(-\frac{gh}{RT}\right) = p_o \exp\left(-\frac{gh\rho_o}{p_o}\right)$$

$$\therefore p = 101325 \exp\left(-\frac{9.81 \times 4500 \times 1.1762}{101325}\right) = \mathbf{60696.714\ N/m^2}$$

Since it is isothermal process, we get the following value.

$$\therefore T = \mathbf{300.15\ K}$$

$$\rho = \frac{p}{RT} = \frac{60696.714}{287 \times 300.15} = \mathbf{0.7046\ kg/m^3}$$

(iii) For adiabatic assumption, pressure at a height h is given by,

$$p = p_o \left[1 - \frac{\gamma - 1}{\gamma} \frac{gh}{RT_o}\right]^{\frac{\gamma}{\gamma - 1}}$$

$$\therefore p = 101325 \times \left[1 - \frac{1.4 - 1}{1.4} \times \frac{9.81 \times 4500}{287 \times 300.15}\right]^{\frac{1.4}{1.4-1}} = \mathbf{58220.47\ N/m^2}$$

Since $$T = T_o \left[1 - \frac{\gamma - 1}{\gamma} \frac{gh}{RT_o}\right]$$

$$\therefore T = 300.15\times\left[1-\frac{1.4-1}{1.4}\times\frac{9.81\times 4500}{287\times 300.15}\right] = \mathbf{256.2\ K}$$

$$\rho = \frac{p}{RT} = \frac{58220.47}{287\times 256.2} = \mathbf{0.7918\ kg/m^3}$$

Example 2.33 Determine the pressure, temperature and density of air at an elevation of 5000 m above sea level where pressure and temperature of the air are 1.01325 bar and 25°C, respectively. Assume the temperature lapse rate as 0.0065°C per metre and the density of air at sea level as 1.23 kg/m^3

Solution

Let h = 5000 m, p_o = 1.01325 bar = 101325 N/m^2, $T = 25°\text{C} = 298.15\ \text{K}$, $\lambda = -0.0065°\text{C/m}$ and $\rho_o = 1.23\ \text{kg/m}^3$.

$$R = \frac{p_o}{\rho_o T_o} = \frac{101325}{1.23\times 298.15} = 276.3\ \text{J / kg K}$$

$$\lambda = \frac{dT}{dh} = -\frac{g}{R}\left(\frac{\gamma-1}{\gamma}\right)$$

$$-0.0065 = -\frac{9.81}{276.3}\times\left(\frac{\gamma-1}{\gamma}\right)$$

$$\frac{\gamma-1}{\gamma} = 0.1831 \Rightarrow \gamma-1 = 0.1831\gamma$$

$$\therefore \gamma = \frac{1}{0.8169} = 1.224$$

(i) For adiabatic assumption, pressure at a height h is given by,

$$p = p_o\left[1-\frac{\gamma-1}{\gamma}\frac{gh}{RT_o}\right]^{\frac{\gamma}{\gamma-1}}$$

$$\therefore p = 101325\times\left[1-\frac{1.224-1}{1.224}\times\frac{9.81\times 5000}{276.3\times 298.15}\right]^{\frac{1.224}{1.224-1}} = \mathbf{53941.79\ N/m^2}$$

(ii) $T = T_o\left[1-\frac{\gamma-1}{\gamma}\frac{gh}{RT_o}\right]$

$$\therefore T = 298.15\times\left[1-\frac{1.224-1}{1.224}\times\frac{9.81\times 5000}{276.3\times 298.15}\right] = \mathbf{265.66\ K}$$

(iii) $\rho = \frac{p}{RT} = \frac{53941.79}{276.3\times 265.66} = \mathbf{0.7349\ kg/m^3}$

Summary

1. Pressure is the normal force exerted by a fluid per unit area.
2. **Pascal's law:** Pressure at a point in a static fluid is equal in all directions.
3. **Hydrostatic law:** Rate of increase of pressure in a vertically downward direction is equal to specific weight of the fluid at that point, i.e., $dp/dh = w = \rho g$.
4. **Absolute pressure:** Pressure measured above absolute zero.
5. **Gauge pressure:** Pressure measured by taking atmospheric pressure as datum.
6. **Vacuum pressure:** Pressure of a fluid below atmospheric pressure.
7. **Manometer:** It measures pressure at a point in a liquid by balancing the column of liquid by the same or another column of liquid. Manometers are of two types, namely simple manometers and differential manometers.
8. Bourdon tube pressure gauge is a mechanical gauge that measures pressure by using an elastic element against the liquid pressure to be measured.
9. **Piezometer:** It is a single column manometer which consists of a glass tube whose one end is connected to the gauge point and other end remains open to atmosphere.
10. **U-tube manometer:** It consists of a glass tube bent in U-shape, where one end is connected to gauge point and the other end remains open to the atmosphere.
11. **Single column manometer:** It is a modified form of a U-tube manometer in which one of the two limbs is made a large reservoir having a cross-sectional area of about 100 times to that of area of the narrow tube in the other limb.
12. **Inclined tube type single manometer:** It is more sensitive and preferred for the measurement of small pressures.
13. A double U-tube manometer also known as compound manometer.
14. **Differential manometer:** It is used to measure the difference of pressures in two pipes or between two points in a pipeline.
15. Inverted U-tube differential manometer is used when the difference of pressure to be measured between two points in a pipeline or between two pipes is small.
16. Micromanometer consists of a U-tube with two reservoirs at the top. It measures very small pressure differences with very high precision.
17. **Atmosphere:** A body of compressible fluid in which the pressure and temperature both vary with altitude. It has two layers, such as troposphere and stratosphere.
18. Troposphere extends up to about 11000 m above sea level in which the temperature of air decreases at an average rate of 0.0065°C per metre of height.
19. Stratosphere extends from 11000 m to 32000 m in which temperature remains constant at about −56.5°C, i.e., isothermal condition may be assumed.
20. Pressure at a height h from ground level in a static compressible fluid under isothermal condition: $p = p_o \exp[-(gh)/RT]$.
21. The pressure, temperature and density at a height h from ground level in a static compressible fluid under adiabatic condition is respectively given as follows.

$$p = p_o\left[1-\frac{\gamma-1}{\gamma}\frac{gh}{RT_o}\right]^{\frac{\gamma}{\gamma-1}};\quad T = T_o\left[1-\frac{\gamma-1}{\gamma}\frac{gh}{RT_o}\right];$$

$$\rho = \rho_o\left[1-\frac{\gamma-1}{\gamma}\frac{gh}{RT_o}\right]^{\frac{1}{\gamma-1}}$$

Here, p_o, T_o and ρ_o are the pressure, temperature and density, respectively at ground level.

22. Temperature lapse rate (λ) is the rate of decrease of temperature with altitude. It is given by $\lambda = \frac{dT}{dh} = -\frac{g}{R}\left(\frac{\gamma-1}{\gamma}\right)$.

If $\gamma = 1$, then $\lambda = 0$ and the process is isothermal. If $\gamma > 1$, $\lambda = -\text{ve}$ and temperature decreases with increase in altitude.

Multiple-choice Questions

1. Pressure at a point in a static mass of liquid depends on
   (a) The depth below the free liquid surface.
   (b) The shape and size of the container.
   (c) The specific weight of liquid, depth below the free liquid surface and the shape and size of the container.
   (d) The specific weight of liquid and depth below the free liquid surface.
2. Piezometer measures only
   (a) Gauge pressure. (b) Atmospheric pressure.
   (c) Absolute pressure. (d) All of these.
3. Inclined single column manometer is useful in measuring
   (a) High pressure. (b) Medium pressure.
   (c) Low pressure. (d) All of these.

4. Which one of the followings statement is correct?
(a) A barometer indicates the difference between local and standard atmospheric pressure.
(b) Local atmospheric pressure depends only upon the height of locality above mean sea level.
(c) Standard atmospheric pressure is the mean local atmospheric pressure at sea level.
(d) None of the above.

5. Differential manometer is used to measure
(a) Pressure at a point in a fluid.
(b) Velocity difference between two points in a fluid.
(c) Velocity at a point in a fluid.
(d) Pressure difference between two points in a fluid.

6. Manometers are comparatively suitable for measuring
(a) Very high pressure. (b) High pressure.
(c) Low pressure. (d) All of these.

7. The differential equation for pressure variation in a static fluid when h is measured vertically up may be written as
(a) $dp = \rho g dh$
(b) $dp = -\rho g dh$
(c) $dp = g d\rho$
(d) None of the above.

8. Inverted U-tube manometer is used when the difference of pressure to be measured between two points in a pipeline is
(a) High. (b) Small.
(c) Very high. (d) All of these.

9. The micromanometer is used to measure
(a) Very small pressure differences with very high precision.
(b) Very high pressure differences with very high precision.
(c) Very small pressure differences with very low precision.
(d) None of the above.

Review Questions

1. Define fluid pressure and give an expression for it at a point in a fluid.

2. State and prove (i) Pascal's law and (ii) hydrostatic law.

3. Define atmospheric, gauge, vacuum and absolute pressures.

4. What are manometers? Give its classification, advantages and limitations.

5. What do you mean by piezometer? Also give its applications and limitations.

6. What are U-tube manometers? How it measures gauge and vacuum pressures?

7. What do you mean by single column manometers? Discuss its types?

8. Explain how vertical and inclined single column manometers measures pressure?

9. What is the difference between U-tube differential manometer and inverted U-tube differential manometer? Also give their application areas.

10. Explain how micromanometer measures small pressure difference?

11. Derive an expression for the pressure p at a height h from sea level for a static air when the compression of the air is assumed isothermal and the pressure and temperature at sea level are p_o and T_o, respectively.

12. Briefly explain the constructional and working details of a Bourdon pressure gauge with a neat sketch.

13. Derive expressions for pressure, temperature and density for an adiabatic process at a height h from sea level for static air when p_o, T_o and ρ_o are the pressure, temperature and density at ground level, respectively.

14. Define the term temperature lapse rate and also derive an expression for the same.

Problems

1. A hydraulic press has a ram and a plunger of diameters 200 mm and 30 mm, respectively. Find the force required at the plunger to lift a weight of 20 kN.

[**Ans.** 450 N]

2. Find the pressure at a point 4 m below the free surface in a liquid that has a variable density given by the relation $\rho = (200 + Ah)$ kg/m^3, where $A = 5$ kg/m^4 and h metres is the distance measured from free surface.

[**Ans.** 8.24 kN/m^2]

3. Determine the pressure due to a column of 0.5 m (i) of water, (ii) an oil of specific gravity 0.85 and (iii) mercury of specific gravity 13.6.

[**Ans.** 4.905 kN/m^2, 4.169 kN/m^2, 66.708 kN/m^2]

4. The pressure at the base and top of a mountain are 750 mmHg and 620 mmHg, respectively. Calculate the height of mountain if specific weight of air is 12 N/m^3.

[**Ans.** 144.534 m]

5. If the pressure intensity at a point in a fluid is 4 N/cm^2, then determine the corresponding height of fluid when the fluid is (i) water and (ii) oil of specific gravity 0.86.

[**Ans.** 4.08 m of water, 4.74 m of water]

6. Determine the minimum depth of water in the tank when the inlet to pump which draws water through suction pipe is 12 m above the bottom of tank and the pressure at the pump inlet is not to fall below 2×10^4 N/m^2 (abs). The atmospheric pressure is given as 10^5 N/m^2.

[**Ans.** 3.845 m]

7. A hydraulic jack filled with water has a large piston and a small piston of diameters 12 cm and 4 cm, respectively. If a force of 100 N is applied on the small piston, then determine the load lifted by the large piston in the following cases, (i) when the pistons are at the same level and (ii) when small piston is 20 cm above the large piston.

[**Ans.** 900 N, 922.18 N]

8. A cylindrical vessel 4 m high and $5{\times}10^{-4}$ m^2 cross-sectional area is filled with water up to a height of 3 m and remaining with oil (specific gravity = 0.8). If this vessel is open to atmosphere, then determine (i) pressure at the interface and (ii) absolute and gauge pressures on the base of the tank in terms of water head.

[**Ans.** 7.85 kN/m^2, 3.8 m of water, 4.75 m of oil, 14.129 m of water (abs), 17.661 m of oil (abs)]

9. A pressure gauge consists of two cylindrical bulbs P and Q each of 1000 mm^2 cross-sectional area are connected by a U-tube with vertical limbs each of 20 mm^2 cross-sectional area. A coloured liquid (specific gravity = 0.9) is filled into Q and clear water is filled into P, the surface of separation being in the limb attached to Q. Determine the displacement of the surface of separation when the pressure on the surface in Q is greater than that in P by an amount equal to 15 mm head of water.

[**Ans.** 10.87 cm]

10. Determine the pressure intensity at a depth of 20 m below the surface of sea water having a specific weight of 9.95×10^3 N/m^3. Also determine the absolute pressure if the barometer reads 760 mm Hg.

[**Ans.** 199 kN/m^2, 300.39 kN/m^2]

11. A U-tube manometer measures the water pressure in excess of atmospheric pressure in a pipe. The right limb opened to atmosphere contains mercury and the contact between water and mercury is in left limb of the manometer. Find the water pressure in the pipe, if the difference in level of mercury in the limbs of U-tube is 12 cm and the free surface of mercury is in level with centre of pipe. Also find the new difference in the level of mercury when the water pressure in pipe is reduced to 10 kN/m^2.

[**Ans.** 14832.72 N/m^2, 8.24 cm]

12. Determine the height of liquid column in the three piezometer tubes by taking the bottom of tank as datum. The top liquid column (specific gravity = 0.8) of tank is up to 1 m, followed by 2 m high liquid column (specific gravity = 0.9) and the bottom column is of water column of height 3 m. The piezometer tubes are fitted at a distance of 1 m, 3 m and 6 m from the top of the tank.

[**Ans.** 5.6 m, 5.89 m, 6 m]

13. A simple U-tube manometer is installed across an orifice-meter. The manometer is filled with mercury (specific gravity = 13.6) and the liquid above mercury is carbon tetrachloride (specific gravity = 1.6). If the manometer reads 200 mm, then determine the pressure difference over the manometer.

[**Ans.** 23.544 kN/m^2]

14. An inclined single column water manometer with 25° inclination to the horizontal is used to measure the pressure of oil (specific gravity = 0.9) kept in a tank. The area ratio of reservoir to tube is 25. Determine the absolute pressure of oil in the tank if the deflection length of inclined tube is 50 cm and the centre height of the oil tank is at 10 cm from the original water level.

[**Ans.** 1.207 kN/m^2]

15. An inclined glass tube manometer is connected to a cylinder standing upright and the manometric fluid fills the apparatus to a fixed zero mark on the tube when both cylinder and the tube are open to atmosphere. The upper end of the cylinder is then connected to a gas supply at a pressure p and the liquid rises in the tube. Obtain an expression for the pressure in cm of water when the liquid reads l cm in the tube, in terms of the inclination α of the tube and specific gravity S and the ratio R of the diameter of the cylinder to the diameter of the tube. Also determine the value of R so that the error due to disregarding the change in the level in the cylinder will not exceed 0.2% when $\alpha = 20°$.

[**Ans.** $p = \rho g l (R^{-2} + \sin\alpha)$, 38.2]

16. A U-tube differential manometer containing mercury is connected on one side to a pipe X containing a liquid (specific gravity = 1.6) under a pressure of 125 kN/m^2 and on the other side of pipe Y containing another liquid (specific gravity = 0.9) under a pressure of 200 kN/m^2. The pipe X lies 2 m above pipe Y and the mercury level in the limb communicating with pipe X lies 4 m below the pipe X. Determine the difference in the levels of mercury in the two limbs of the manometer.

[**Ans.** 24 cm]

17. Determine the gauge pressure at P in Figure 2.32 which shows a compound manometer. Take all the dimensions in metres.

[**Ans.** 320 kPa]

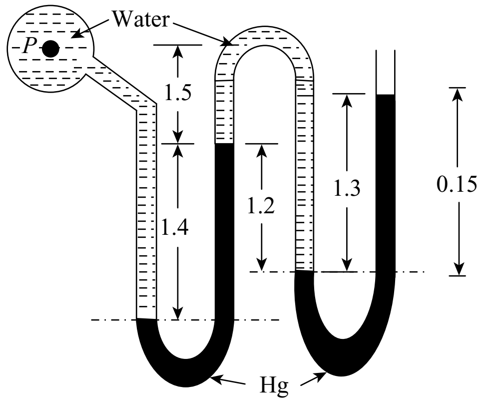

Figure 2.32

18. The pressure head h measured by a single column mercury manometer is within 1% of the true height corresponding to a pressure differential. Find out the diameter D of the reservoir required with respect to tube diameter d. If the ratio of tank area to the tube area is 400, then also find out percentage error involved in the difference in pressure by reading the single column height h.

[**Ans.** $D = 10\,d$, 0.25%]

19. An inverted U-tube manometer (specific gravity = 0.8) is connected to two horizontal water pipes A and B. Pipe A lies 150 mm above than pipe B. The vertical height of water columns in the two limbs of the inverted manometer measured from the respective centres of the pipes are measured to be same equal to 175 mm. Find the difference of pressure between the pipes.

[**Ans.** 1177.2 N/m^2]

20. An inverted U-tube manometer containing a manometric light fluid (specific gravity = 0.7) is connected to the two pipes at the points A and B. Pipe A carries liquid of specific gravity 1.2 and pipe B carries water. The pipes are at the same level. The height of the liquid of specific gravity 1.2 from the centre of the pipe is 15 cm. If all liquids are immiscible and the pressure in pipe A and B is equal, then determine the differential reading of the manometer.

[**Ans.** 10 cm]

21. Derive an expression for the pressure ratio in the troposphere if the absolute temperature is given by $T = T_o - \alpha(h - h_o)$, where T_o is the absolute temperature at ground level and α is the temperature gradient.

$$\textbf{Ans.}\ \frac{p}{p_o} = \left[\frac{T_o - \alpha(h - h_o)}{T_o}\right]^{g/(R\alpha)}$$

22. The barometric pressure at ground level is 76 cmHg while that on a mountain top is 73.5 cmHg. Determine the height of the mountain top if the density of air is 1.23 kg/m^3.

[**Ans.** 276.42 m]

23. Determine the pressure at an elevation of 8000 m above ground level if the atmospheric pressure is 101430 N/m^2, density of air is 1.2 kg/m^3, temperature is 288 K and there is no variation of 'g' with altitude, when (i) air is incompressible, (ii) pressure variation is as per isothermal law and (iii) pressure variation is as per adiabatic law.

[**Ans.** 7.254 kN/m^2, 40.08 kN/m^2, 34.48 kN/m^2]

24. Determine the pressure around an aeroplane flying at an altitude of 6000 m when the pressure and temperature at sea level are 101430 N/m^2 and 290 K, respectively. Assume that there is no variation of 'g' with altitude and the temperature lapse rate in atmosphere is 0.0065 K/m. Take density of air as 1.23 kg/m^3.

[**Ans.** 47.13 kN/m^2]

25. Figure 2.33 shows an inverted differential manometer having an oil (specific gravity = 0.8) connected to two pipes A and B carrying water under pressure. Determine the pressure in the pipe B if the pressure in the pipe A is 2 m of water. In the figure take all the dimensions in metres.

[**Ans.** 19.23 kN/m^2]

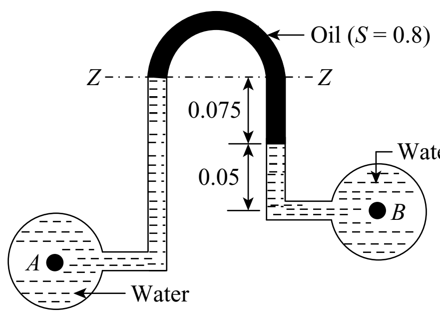

Figure 2.33

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (d) **2.** (a) **3.** (c) **4.** (b) **5.** (d)
6. (c) **7.** (b) **8.** (b) **9.** (a)

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Chapter **3**

Hydrostatic Forces on Submerged Surfaces

3.1 ❒ INTRODUCTION

Hydrostatics deals with the behaviour of fluids at rest. A static fluid element may be subjected to two forces, namely a body force and normal surface forces. The body force means the force of gravity, whereas the normal surface forces are the forces exerted on the fluid element by the surrounding fluid or other sources. Force exerted on immersed surfaces by the static fluid is due to pressure distribution on the surfaces. In static fluids, shear stresses are completely absent. Thus, the surface forces are only due to the action of normal stress, i.e., hydrostatic pressure.

In many engineering applications, it becomes necessary to determine the pressure forces on the entire surface of a hydraulic device and structures, such as submarines, ships, pipes, dams, gates, containers, balloons, tanks, etc. This chapter describes the hydrostatic equations and methods required to determine the magnitude, location and the direction of resultant force acting on a submerged surface under static fluid conditions. The submerged surface may be horizontal plane surface, vertical plane surface, inclined plane surface and curved surface.

3.2 ❒ TOTAL PRESSURE, CENTRE OF PRESSURE AND CENTRE OF GRAVITY

In the design of several hydraulic machines and structures, it is often required to calculate the magnitude of total pressure and to locate its point of application.

3.2.1 Total Pressure

A static mass of fluid when comes in contact with a solid surface (plane or curved) exerts a force on it. This force always acts normal to the surface and it is known as total pressure. The total pressure is denoted by p.

3.2.2 Centre of Pressure

The point of application of total pressure on the surface is known as centre of pressure and it is denoted by C. In this chapter, the total pressure and centre of pressure are computed for (i) horizontal submerged plane surface, (ii) vertical submerged plane surface, (iii) inclined submerged plane surface and (iv) curved submerged surface.

3.2.3 Centre of Gravity

The centre of gravity (or centroid) is the point where the whole weight of the body lies and it is denoted by G. The submerged surface does not experience the same intensity of pressure because pressure intensity at a point varies with the depth of the liquid. The lower portion of the submerged surface is subjected to higher pressure and thus, the centre of pressure lies below the centre of gravity.

3.3 ❒ MOMENTS OF AREA AND GEOMETRICAL PROPERTIES

The determination of first and second moment of areas is necessary in the evaluation of the resultant force and centre of pressure.

3.3.1 First Moment of Area

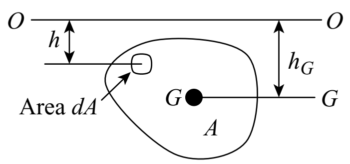

Figure 3.1 *First and second moments of an area*

Consider the area A and the moments of area about the line O–O as shown in Figure 3.1.

Let h_G be the distance of the centre of gravity of area from the line O–O. The moment of area with respect to the line O–O can be obtained by summing up the moments of elementary areas (dA) all over the surface with respect to the given axis. The first moment of area about the line O–O is given below.

$$\Rightarrow \boxed{\int h dA = Ah_G} \tag{3.1}$$

The first moment of area is used to locate the centroid of the area. The moment of area about any line passing through the centroid will be zero.

3.3.2 Second Moment of Area (or Area Moment of Inertia)

The second moment of area is also known as area moment of inertia and it is given below.

$$\boxed{I_O = \int h^2 dA} \tag{3.2}$$

By parallel axis theorem, we get:

$$\boxed{I_O = I_G + Ah_G^{\,2}} \tag{3.3}$$

Here, I_G is the moment of inertia about an axis G–G passing through the centre of gravity G and parallel to the line O–O.

Thus, moment of inertia (M.O.I.) of an area about any axis is equal to the sum of the moment of inertia about a parallel axis through the centroid and the product of the area and the square of the distance between this axis and the axis passing through centroid. The second moment of area is used in the determination of centre of pressure for plane areas submerged in liquids.

The moments of inertia and other geometrical properties of some important plane surfaces are given in Table 3.1 in which CG is the centre of gravity, I_G is the moment of inertia about an axis passing through CG and parallel to base and I_O is the moment of inertia about base.

Table 3.1 *Moments of inertia and geometrical properties*

Plane surface	CG	Area	I_G	I_O
1. Rectangle of width b and depth d.	$d/2$	bd	$bd^3/12$	$bd^3/3$
2. Triangle of side b, height h and base zero of axis.	$h/3$	$bh/2$	$bh^3/36$	$bh^3/12$

(*Continued*)

Table 3.1 (*Continued*)

Plane surface	CG	Area	I_G	I_O
3. Triangle of side b, height h and vortex zero of axis.	$2h/3$	$bh/2$	$bh^3/36$	$bh^3/12$
4. Circle of diameter d.	$d/2$	$\pi d^2/4$	$\pi d^4/64$	–
5. Semicircle with diameter horizontal and zero of axis.	$2d/(3\pi)$	$\pi d^2/8$	$0.11r^4$	$\pi d^4/128$
6. Trapezium of parallel sides a and b with height h and axis along b.	$\left(\frac{2a+b}{a+b}\right)\frac{h}{3}$	$\left(\frac{a+b}{2}\right)h$	$\left[\frac{a^2+4ab+b^2}{36(a+b)}\right]h^3$	–

3.4 ❒ HORIZONTAL SUBMERGED PLANE SURFACE

3.4.1 Total Pressure on a Horizontal Submerged Plane Surface

Consider a horizontal plane surface submerged in a static liquid as shown in Figure 3.2.

Let A be the surface area, C be the centre of pressure, G be the centroid, h_C be the distance of centre of pressure from the free surface of liquid, h_G be the distance of centre of gravity from the free surface of liquid, p be the pressure intensity and F be the total pressure force on the surface. For a submerged horizontal plane surface, the points C and G coincides with each other and thus, $h_C = h_G$.

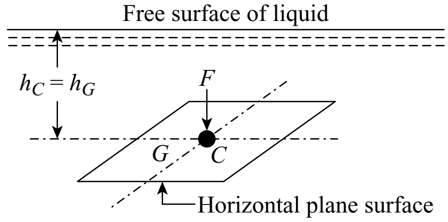

Figure 3.2 *Horizontal submerged plane surface*

Since all the points on the horizontal plane surface are at the same depth from the free surface of liquid, the pressure intensity is constant over the entire surface and it is given below.

$$p = \rho g h_G \tag{3.4}$$

The total pressure force on the surface is given,

$$\boxed{F = p \times A = \rho g h_G \times A} \tag{3.5}$$

3.5 ❒ VERTICALLY SUBMERGED PLANE SURFACE

3.5.1 Total Pressure on a Vertical Submerged Plane Surface

Consider a plane vertical surface with random shape submerged in a static liquid as shown in Figure 3.3.

Let A be the surface area, C be the centre of pressure, G be the centroid, h_C be the distance of centre of pressure from the free surface of liquid, h_G be the distance of centre of gravity from the free surface of liquid, p be the pressure intensity and F be the total force on the surface.

Consider an elementary strip of area dA at a depth h from the free surface of liquid and parallel to it. The pressure force on the strip is expressed below.

$$dF = p \times dA = \rho g h \times dA$$

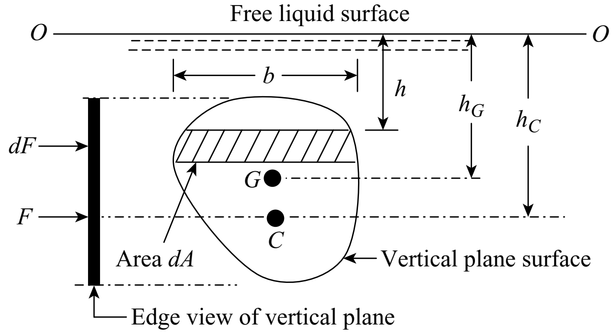

Figure 3.3 *Vertical submerged plane surface*

Total pressure force on the whole surface is given by,

$$F = \int dF = \int \rho gh\, dA = \rho g \int h dA = \rho g \times Ah_G \quad (3.6)$$

3.5.2 Centre of Pressure on a Vertical Submerged Plane Surface

The pressure force on the strip is given by,

$$dF = \rho gh\, dA$$

Moment of this pressure force about the free liquid surface is given by,

$$= \rho gh\, dA \times h = \rho gh^2 dA$$

Sum of moments of all such pressure forces about the free liquid surface becomes,

$$= \int \rho gh^2 dA = \rho g \int h^2 dA = \rho g \times I_O \quad (3.7)$$

Now moment of total force F acting at point C at a distance h_C is given by,

$$= F \times h_C = \rho gAh_G \times h_C \quad (3.8)$$

Principle of moments states that the moment of the resultant force about an axis is equal to the sum of moments of the components about the same axis. Thus, by equating (3.7) and (3.8), we get the following expression.

$$\rho gAh_G \times h_C = \rho g \times I_O$$

$$h_C = \frac{\rho g I_O}{\rho g A h_G} = \frac{I_O}{Ah_G} \quad (3.9)$$

Now substituting the value of I_O from Equation (3.3) in Equation (3.9), we get:

$$\boxed{h_C = \frac{I_G + Ah_G^2}{Ah_G} = \frac{I_G}{Ah_G} + \frac{Ah_G^2}{Ah_G} = \frac{I_G}{Ah_G} + h_G} \quad (3.10)$$

Thus, Equation (3.10) gives the position of the centre of pressure on a plane surface submerged vertically in a static mass of liquid. From Equation (3.10), it is observed that the centre of pressure h_C lies below the centroid of the area and it is independent of the density of the liquid.

Example 3.1 A rectangular plate 0.4 m × 1.6 m is immersed in water. Determine the hydrostatic force and the centre of pressure when the plate is kept (i) vertical with 0.4 m side coinciding with water surface, (ii) vertical with 0.4 m side kept 2 m below and parallel to water surface and (iii) vertical with 1.6 m side kept 2 m below and parallel to water surface.

Solution

Refer Figure 3.4. Let $b = 0.4$ m and $d = 1.6$ m.

(i) Refer Figure 3.4(a).

$$A = bd = 0.4 \times 1.6 = 0.64 \text{ m}^2$$

$$h_G = \frac{d}{2} = \frac{1.6}{2} = 0.8 \text{ m}$$

$$F = \rho_w gAh_G = 1000 \times 9.81 \times 0.64 \times 0.8 = \mathbf{5022.72\ N}$$

$$I_G = \frac{bd^3}{12} = \frac{0.4 \times 1.6^3}{12} = 0.1365 \text{ m}^4$$

$$h_C = \frac{I_G}{Ah_G} + h_G = \frac{0.1365}{0.64 \times 0.8} + 0.8 = \mathbf{1.067\ m}$$

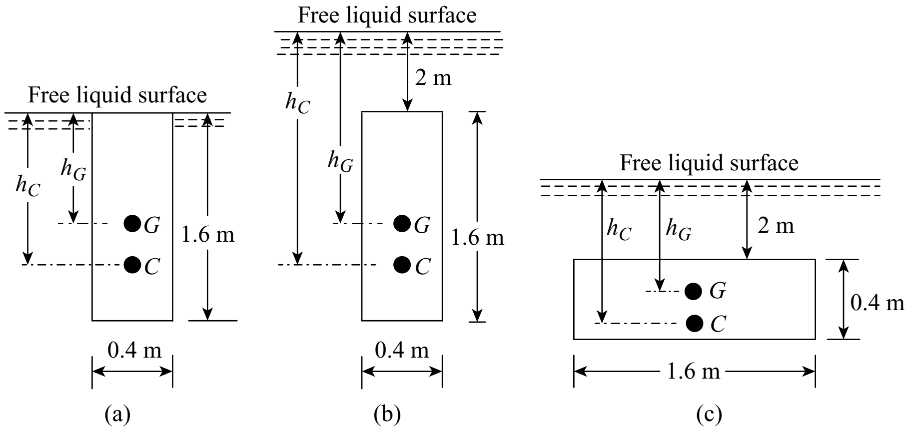

Figure 3.4

(ii) Refer Figure 3.4(b).

$$h_G = 2 + \frac{1.6}{2} = 2.8 \text{ m}$$

$$F = \rho_w g A h_G = 1000 \times 9.81 \times 0.64 \times 2.8 = \mathbf{17579.52\ N}$$

$$h_C = \frac{I_G}{A h_G} + h_G = \frac{0.1365}{0.64 \times 2.8} + 2.8 = \mathbf{2.8762\ m}$$

(iii) Refer Figure 3.4(c).

$$h_G = 2 + \frac{0.4}{2} = 2.2 \text{ m}$$

$$F = \rho_w g A h_G = 1000 \times 9.81 \times 0.64 \times 2.2 = \mathbf{13812.48\ N}$$

$$I_G = \frac{db^3}{12} = \frac{1.6 \times 0.4^3}{12} = 8.533 \times 10^{-3} \text{ m}^4$$

$$h_C = \frac{I_G}{A h_G} + h_G = \frac{8.533 \times 10^{-3}}{0.64 \times 2.2} + 2.2 = \mathbf{2.2061\ m}$$

Example 3.2 A circular thin plate of diameter 600 mm is immersed in water vertically such that its top edge is 2 m below free water surface. Determine the total pressure acting on the plate and the position of its centre of pressure.

Solution

Refer Figure 3.5. Let $d = 600$ mm $= 0.6$ m.

$$A = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 0.6^2 = 0.09\pi \text{ m}^2$$

$$h_G = 2 + \frac{0.6}{2} = 2.3 \text{ m}$$

$$F = \rho_w g A h_G = 1000 \times 9.81 \times 0.09\pi \times 2.3 = \mathbf{6379.538\ N}$$

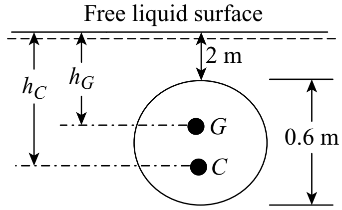

Figure 3.5

$$I_G = \frac{\pi d^4}{64} = \frac{\pi \times 0.6^4}{64} = 6.362 \times 10^{-3}\ \text{m}^4$$

$$h_C = \frac{I_G}{Ah_G} + h_G = \frac{6.362 \times 10^{-3}}{0.09\pi \times 2.3} + 2.3 = \mathbf{2.3098\ m}$$

Example 3.3 A triangular thin plate of base 1 m and height 1.5 m is hinged vertically inside a tank containing a liquid (specific gravity = 1.2) such that the base coincides with the free surface. Determine the total pressure acting on the plate and the depth of its centre of pressure.

Solution

Refer Figure 3.6. Let $b = 1$ m, $h = 1.5$ m and $S = 1.2$.

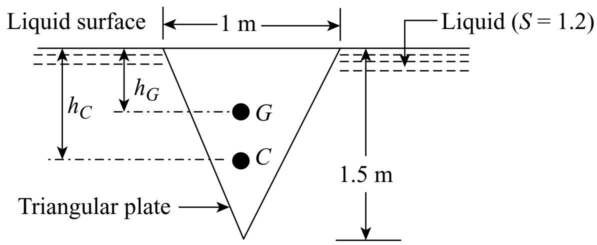

Figure 3.6

$$A = \frac{bh}{2} = \frac{1 \times 1.5}{2} = 0.75\ \text{m}^2$$

$$h_G = \frac{h}{3} = \frac{1.5}{3} = 0.5\ \text{m}$$

$$F = S\rho_w gAh_G = 1.2 \times 1000 \times 9.81 \times 0.75 \times 0.5 = \mathbf{4414.5\ N}$$

$$I_G = \frac{bh^3}{36} = \frac{1 \times 1.5^3}{36} = 0.09375\ \text{m}^4$$

$$h_C = \frac{I_G}{Ah_G} + h_G = \frac{0.09375}{0.75 \times 0.5} + 0.5 = \mathbf{0.75\ m}$$

Example 3.4 A disc of diameter 2 m which can rotate about a horizontal diameter is used to close a circular opening of the same size in the vertical side of a tank. If the head of water above the horizontal diameter of the disc is 3 m, then find (i) force on the disc, (ii) position of centre of pressure and (iii) torque required to maintain the disc in equilibrium in the vertical position.

Solution

Refer Figure 3.7. Let $d = 2$ m, $h_G = 3$ m and T be the torque required.

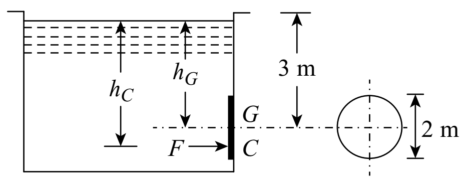

Figure 3.7

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4} \times 2^2 = \pi\ \text{m}^2$$

(i) $F = \rho_w gAh_G = 1000 \times 9.81 \times \pi \times 3 = \mathbf{92457.072\ N}$

(ii) $I_G = \dfrac{\pi d^4}{64} = \dfrac{\pi \times 2^4}{64} = 0.7854\ \text{m}^4$

$$h_C = \frac{I_G}{Ah_G} + h_G = \frac{0.7854}{\pi \times 3} + 3 = \mathbf{3.0833\ m}$$

(iii) $T = F \times (h_C - h_G) = 92457.072 \times (3.0833 - 3) = \mathbf{7701.674\ Nm}$

Example 3.5 A pipeline 4 m in diameter containing an oil (specific gravity = 0.9) has a gate valve. The pressure at the centre of the pipe is 200 kPa. Find (i) the force exerted on the gate and (ii) the position of centre of pressure.

Solution

Refer Figure 3.8. Let $d = 4$ m, $S = 0.9$ and $p = 200$ kPa.

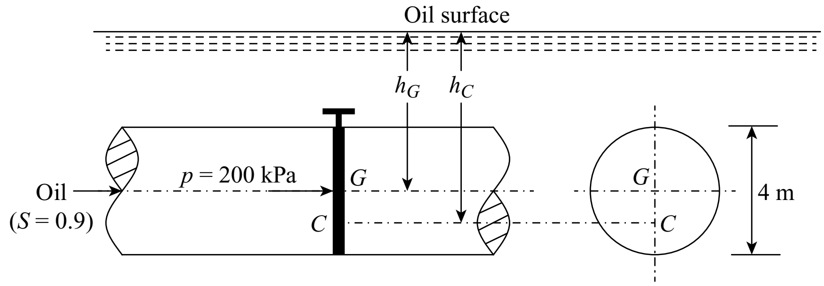

Figure 3.8

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4} \times 4^2 = 4\pi \text{ m}^2$$

$$\rho = S\rho_w = 0.9 \times 1000 = 900 \text{ kg/m}^3$$

Pressure head at the centre is given by,

$$h = h_G = \frac{p}{\rho g} = \frac{200 \times 10^3}{900 \times 9.81} = 22.653 \text{ m}$$

$$F = \rho g A h_G = 900 \times 9.81 \times 4\pi \times 22.653 = \mathbf{2513.316\ kN}$$

$$I_G = \frac{\pi d^4}{64} = \frac{\pi \times 4^4}{64} = 4\pi \text{ m}^4$$

$$h_C = \frac{I_G}{A h_G} + h_G = \frac{4\pi}{4\pi \times 22.653} + 22.653 = \mathbf{22.697\ m}$$

Thus, the centre of pressure lies (22.697 – 22.653) = 0.044 m below the centre of pipe.

Example 3.6 A square aperture in the vertical side of a tank has one diagonal and is completely covered by a plane plate hinged along one of the upper sides of the aperture. The diagonals of the aperture are 2 m long and the tank contains glycerine (specific gravity = 1.26). The centre of aperture is 1.4 m below the free surface. Determine (i) thrust exerted on the plate by the glycerine and (ii) position of its centre of pressure.

Solution

Refer Figure 3.9. Diagonals $QS = PR = 2$ m, $S = 1.26$ and $h_G = 1.4$ m.

Let A be the area of aperture.

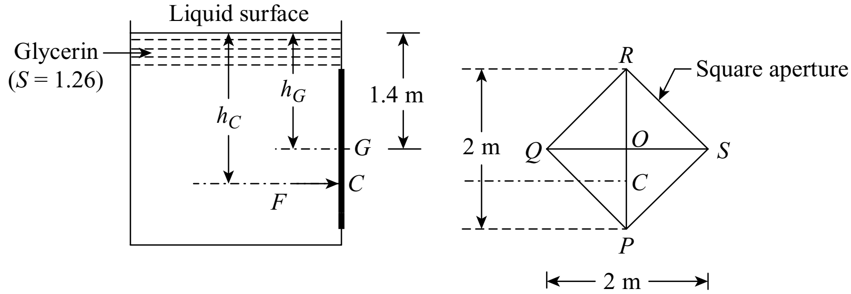

Figure 3.9

$$A = \frac{1}{2} \times QS \times OR + \frac{1}{2} \times QS \times OP = \frac{1}{2} \times 2 \times 1 + \frac{1}{2} \times 2 \times 1 = 2 \text{ m}^2$$

$$F = S\rho_w gAh_G = 1.26 \times 1000 \times 9.81 \times 2 \times 1.4 = \mathbf{34609.68\ N}$$

Since I_G = M.O.I. of ΔQRS about QS + M.O.I. of ΔQPS about QS

$$I_G = \frac{2 \times 1^3}{12} + \frac{2 \times 1^3}{12} = 0.333 \text{ m}^2 \quad [\because \text{M.O.I.} = bh^3/12]$$

$$h_C = \frac{I_G}{Ah_G} + h_G = \frac{0.333}{2 \times 1.4} + 1.4 = \mathbf{1.519\ m}$$

Example 3.7 A dry dock is closed by a gate of trapezoidal shape having top and bottom lengths 18 m and 12 m, respectively and a height of 7.5 m. Determine the total water pressure and the depth of centre of pressure on the gate if the sea water (specific gravity = 1.02) level is up to the top of the gate on one side and the other side is empty.

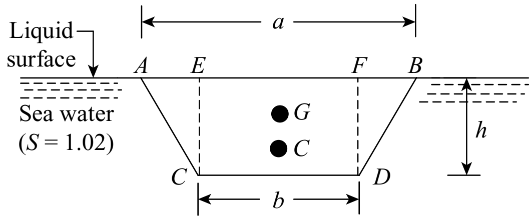

Figure 3.10

Solution

Refer Figure 3.10. Let $a = 18$ m, $b = 12$ m, $h = 7.5$ m and $S = 1.02$.

Let A be the area of the gate. The distance of CG of the trapezoidal gate from the top surface AB is given below.

$$h_G = \left(\frac{2a+b}{a+b}\right)\frac{h}{3} = \left(\frac{2 \times 18 + 12}{18 + 12}\right) \times \frac{7.5}{3} = 4 \text{ m}$$

$$A = \frac{(a+b)h}{2} = \frac{(18+12) \times 7.5}{2} = 112.5 \text{ m}^2$$

$$F = S\rho_w gAh_G = 1.02 \times 1000 \times 9.81 \times 112.5 \times 4 = \mathbf{4502.79\ kN}$$

$$I_G = \left[\frac{a^2 + 4ab + b^2}{36(a+b)}\right]h^3 = \left[\frac{18^2 + 4 \times 18 \times 12 + 12^2}{36(18+12)}\right] \times 7.5^3 = 520.31 \text{ m}^4$$

$$h_C = \frac{I_G}{Ah_G} + h_G = \frac{520.31}{112.5 \times 4} + 4 = \mathbf{5.156\ m}$$

Example 3.8 A sluice gate is placed across a trapezoidal channel that is 20 m wide at the top and 8 m at a depth of 5 m. Calculate (i) total pressure on the gate and (ii) position of the centre of pressure when the depth of water on the gate is 3 m.

Solution

Refer Figure 3.11. Let $AB = 20$ m, $CD = b = 8$ m, $h_1 = 5$ m, $h = 3$ m and $GH = a$ m. Let A be the area of the gate part submerged into water.

$$a = GH = IJ + 2(GI) = 8 + 2(GI)$$

$$AE = \frac{20 - 8}{2} = 6 \text{ m}$$

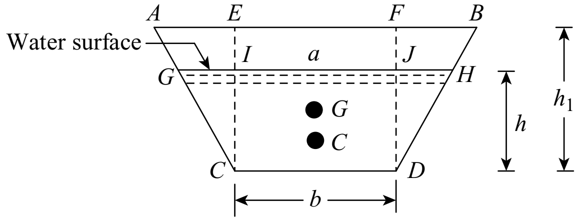

Figure 3.11

From similar triangles *AEC* and *GIC*, we get:

$$\frac{GI}{AE} = \frac{CI}{CE} \Rightarrow GI = \frac{CI}{CE} \times AE = \frac{3}{5} \times 6 = 3.6 \text{ m}$$

$$a = GH = 8 + 2(GI) = 8 + 2 \times 3.6 = 15.2 \text{ m}$$

$$A = \frac{(GH + CD)h}{2} = \frac{(15.2 + 8) \times 3}{2} = 34.8 \text{ m}^2$$

$$h_G = \left(\frac{2a + b}{a + b}\right)\frac{h}{3} = \left(\frac{2 \times 15.2 + 8}{15.2 + 8}\right) \times \frac{3}{3} = 1.6552 \text{ m}$$

$$F = \rho_w g A h_G = 1000 \times 9.81 \times 34.8 \times 1.6552 = \mathbf{565.065 \text{ kN}}$$

$$I_G = \left[\frac{a^2 + 4ab + b^2}{36(a + b)}\right] h^3 = \left[\frac{15.2^2 + 4 \times 15.2 \times 8 + 8^2}{36 \times (15.2 + 8)}\right] \times 3^3 = 25.262 \text{ m}^4$$

$$h_C = \frac{I_G}{A h_G} + h_G = \frac{25.262}{34.8 \times 1.6552} + 1.6552 = \mathbf{2.094 \text{ m}}$$

Example 3.9 Calculate the total pressure and the position of the centre of pressure in an isosceles triangular plate of base 3 m and 6 m height immersed vertically in oil (specific gravity = 0.85), (i) when the axis of symmetry of the plate passing through the apex being horizontal and 9 m below the water surface and (ii) when the base of the plate is 9 m below the water surface and apex is above the base.

Solution

Let $b = 3$ m, $h = 6$ m and $S = 0.85$.

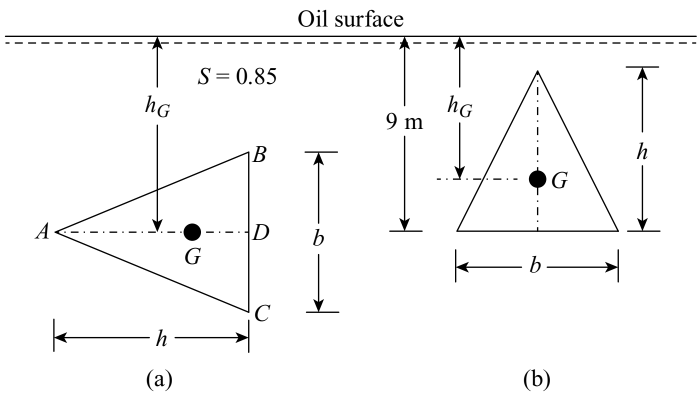

Figure 3.12

(i) Refer Figure 3.12(a). Given that: $h_G = 9$ m

$$A = \frac{bh}{2} = \frac{3 \times 6}{2} = 9 \text{ m}^2$$

$$F = S\rho_w g A h_G = 0.85 \times 1000 \times 9.81 \times 9 \times 9 = \mathbf{675.42 \text{ kN}}$$

Since $\quad I_G = \text{M.O.I. of } \Delta ABD \text{ about } AD + \text{M.O.I. of } \Delta ACD \text{ about } AD$

$$I_G = \frac{6 \times 1.5^3}{12} + \frac{6 \times 1.5^3}{12} = 3.375 \text{ m}^4 \quad [\because I_G = bh^3/12]$$

$$h_C = \frac{I_G}{Ah_G} + h_G = \frac{3.375}{9 \times 9} + 9 = \mathbf{9.042\ m}$$

(ii) Refer Figure 3.12(b).

$$h_G = 9 - \frac{6}{3} = 7 \text{ m}$$

$$F = S\rho_w gAh_G = 0.85 \times 1000 \times 9.81 \times 9 \times 7 = \mathbf{525.325\ kN}$$

$$I_G = \frac{bh^3}{36} = \frac{3 \times 6^3}{36} = 18 \text{ m}^4$$

$$h_C = \frac{I_G}{Ah_G} + h_G = \frac{18}{9 \times 7} + 7 = \mathbf{7.286\ m}$$

Example 3.10 A trapezoidal channel 4 m wide at the bottom and 2 m deep has sides inclined at 45° to the horizontal (side slopes 1 : 1). Calculate (i) total pressure force and (ii) centre of pressure on the vertical gate closing the channel when it is full of water.

Solution

Refer Figure 3.13. Let $CD = b = 4$ m, $h = 2$ m, $\alpha = 45°$. Let $A_1 = A_3$ be the area of the triangles *AEC* and *BFD* and A_2 be the area of rectangle *EFDC*.

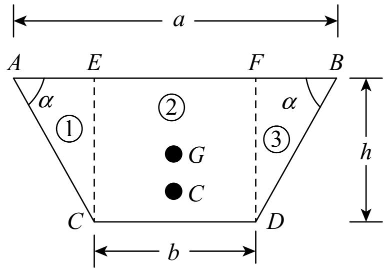

Figure 3.13

$$AE = BF = \frac{h}{\tan\alpha} = \frac{2}{\tan 45°} = 2 \text{ m}$$

$$A_1 = A_3 = \frac{2 \times 2}{2} = 2 \text{ m}^2$$

$$h_{G1} = h_{G3} = \frac{h}{3} = \frac{2}{3} \text{ m}$$

$$F_1 = F_3 = \rho_w gA_1h_{G1} = 1000 \times 9.81 \times 2 \times \frac{2}{3} = 13080 \text{ N}$$

This force acts at a depth of $h_{C1} = h_{C3}$.

$$h_{C1} = h_{C3} = \frac{I_{G1}}{A_1h_{G1}} + h_{G1} = \frac{(2 \times 2^3)/36}{2 \times (2/3)} + \frac{2}{3} = 1 \text{ m}$$

$$A_2 = bh = 4 \times 2 = 8 \text{ m}^2$$

$$h_{G2} = \frac{h}{2} = \frac{2}{2} = 1 \text{ m}$$

$$F_2 = \rho_w gA_2h_{G2} = 1000 \times 9.81 \times 8 \times 1 = 78480 \text{ N}$$

This force acts at a depth of h_{C2} which is derived as given below.

$$h_{C2} = \frac{I_{G2}}{A_2h_{G2}} + h_{G2} = \frac{(4 \times 2^3)/12}{8 \times 1} + 1 = \frac{4}{3} \text{ m}$$

(i) Thus, total pressure force is given by,

$$F = F_1 + F_2 + F_3 = 13080 + 78480 + 13080 = \mathbf{104640\ N}$$

(ii) Taking moments about the top, we get:

$$F \times h_C = F_1 \times h_{C1} + F_2 \times h_{C2} + F_3 \times h_{C3}$$

$$h_C = \frac{F_1 h_{C1} + F_2 h_{C2} + F_3 h_{C3}}{F}$$

$$\therefore h_C = \frac{13080 \times 1 + 78480 \times (4/3) + 13080 \times 1}{104640} = \mathbf{1.25\ m}$$

Example 3.11 A tank 1.2 m high contains water up to a height of 0.4 m above the base and an immiscible oil (specific gravity = 0.9) on the top of water for the remaining height. Determine the total pressure and the position of centre of pressure on one side of the tank which has a width of 2 m.

Solution

Refer Figure 3.14. Let $h = 1.2$ m, $h_1 = 0.4$ m, $h_2 = 1.2 - 0.4 = 0.8$ m, $S = 0.9$ and $b = 2$ m.

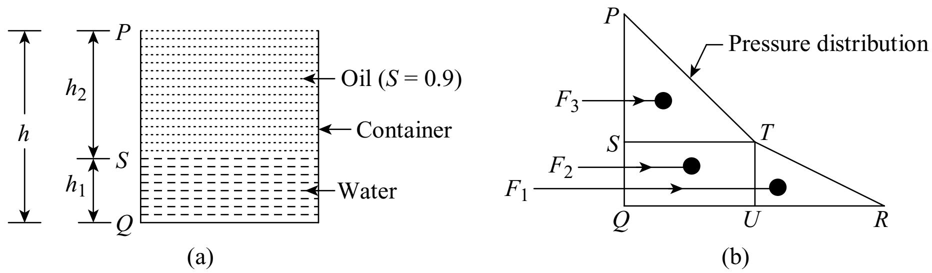

Figure 3.14

To determine the given objectives, the pressure diagram is drawn as shown in Figure 3.14(b). Let F be the total pressure force and h_C be the position of centre of pressure from the free oil surface.

Intensity of pressure at top is given by,

$$p_P = 0$$

Intensity of pressure on S is given by,

$$p_S = ST = QU = S\rho_w g h_2 = 0.9 \times 1000 \times 9.81 \times 0.8 = 7063.2\ \text{N/m}^2$$

Intensity of pressure on Q is given by,

$$p_Q = QU + UR = p_S + \rho_w g h_1 = 7063.2 + 1000 \times 9.81 \times 0.4$$

$$p_Q = QU + UR = 7063.2 + 3924 = 10987.2\ \text{N/m}^2$$

$$F_3 = \frac{1}{2} \times h_2 \times ST \times b = \frac{1}{2} \times 0.8 \times 7063.2 \times 2 = 5650.56\ \text{N}$$

$$F_2 = h_1 \times ST \times b = 0.4 \times 7063.2 \times 2 = 5650.56\ \text{N}$$

$$F_1 = \frac{1}{2} \times h_1 \times UR \times b = \frac{1}{2} \times 0.4 \times 3924 \times 2 = 1569.6\ \text{N}$$

$$\therefore F = F_1 + F_2 + F_3 = 1569.6 + 5650.56 + 5650.56 = \mathbf{12870.72\ N}$$

$$h_{C1} = 0.8 + \frac{2}{3} \times 0.4 = \frac{3.2}{3}\ \text{m}$$

$$h_{C2} = 0.8 + \frac{0.4}{2} = 1\ \text{m}$$

$$h_{C3} = \frac{2}{3} \times 0.8 = \frac{1.6}{3}\ \text{m}$$

Taking moments about the top, we get:

$$F \times h_C = F_1 \times h_{C1} + F_2 \times h_{C2} + F_3 \times h_{C3}$$

$$h_C = \frac{F_1 h_{C1} + F_2 h_{C2} + F_3 h_{C3}}{F}$$

$$\therefore h_C = \frac{1569.6 \times (3.2/3) + 5650.56 \times 1 + 5650.56 \times (1.6/3)}{12870.72} = \mathbf{0.8032\ m}$$

Example 3.12 A vertical rectangular gate 3.5 m wide and 5 m high contains water on one side to a depth of 2.4 m and an oil (specific gravity = 0.9) to a depth of 1.5 m on the other side. Determine the resultant hydrostatic pressure force on the gate and its point of application with respect to the bottom.

Solution

Refer Figure 3.15. Let $b = 3.5$ m, $h = 5$ m, $h_1 = 2.4$ m, $S = 0.9$ and $h_2 = 1.5$ m. Pressure force on the left side of the gate is given by,

$$F_1 = \rho_w g A_1 h_{G1} = 1000 \times 9.81 \times (3.5 \times 2.4) \times \frac{2.4}{2} = 98884.8\ \text{N}$$

This force acts at a distance of $h_{C1} = \frac{2.4}{3} = 0.8$ m from the bottom.

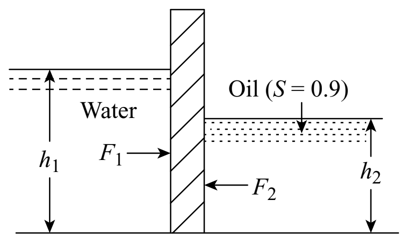

Figure 3.15

Pressure force on the right side of the gate is given by,

$$F_2 = S\rho_w g A_2 h_{G2} = 0.9 \times 1000 \times 9.81 \times (3.5 \times 1.5) \times \frac{1.5}{2} = 34764.19\ \text{N}$$

This force acts at a distance of $h_{C2} = \frac{1.5}{3} = 0.5$ m from the bottom.

Resultant pressure force is given by,

$$F = F_1 - F_2 = 98884.8 - 34764.19 = \mathbf{64120.61\ N}$$

Let the resultant pressure force act at a distance of h_C from the bottom. Taking moments about the bottom, we get the following expression.

$$F \times h_C = F_1 \times h_{C1} - F_2 \times h_{C2}$$

$$\therefore h_C = \frac{F_1 h_{C1} - F_2 h_{C2}}{F} = \frac{98884.8 \times 0.8 - 34764.19 \times 0.5}{64120.61} = \mathbf{0.9626\ m}$$

Example 3.13 A 4 m × 2 m wide rectangular gate is vertical and is hinged at point 0.2 m below the centre of gravity of the gate. The total depth of water is 6 m. Find out the horizontal force required at the bottom of the gate to keep it in closed position.

Solution

Refer Figure 3.16. Let $h = 4$ m, $b = 2$ m, $x = 0.2$ m and $h_1 = 6$ m.

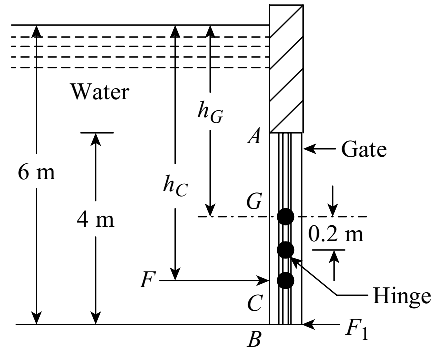

Figure 3.16

$$h_G = 6 - \frac{4}{2} = 4 \text{ m}$$

Pressure force acting on the plane surface of the gate is given by,

$$F = \rho_w gAh_G = 1000 \times 9.81 \times (4 \times 2) \times 4 = 313920 \text{ N}$$

$$h_C = \frac{I_G}{Ah_G} + h_G = \frac{(bh^3)/12}{Ah_G} + h_G = \frac{(2 \times 4^3)/12}{(4 \times 2) \times 4} + 4 = 4.333 \text{ m}$$

Let F_1 be the force required to be applied at the bottom of the gate to keep it closed. Taking moments of all forces about the hinge, we get the following expression.

$$F_1 \times (2 - 0.2) = 313920 \times (0.333 - 0.2)$$

$$\therefore F_1 = \frac{313920 \times (0.333 - 0.2)}{(2 - 0.2)} = \mathbf{23195.2\ N}$$

Example 3.14 A sliding gate of height 1.4 m and width 2.8 m lies in vertical plane that weighs 25 kN. Determine the vertical force required to lift the gate when its upper edge is 6 m below the free water surface and the coefficient of friction between the gate and guides is 0.15. Determine the position of centre of pressure acting on the gate.

Solution

Refer Figure 3.17. Let $h = 1.4$ m, $b = 2.8$ m, $W = 25$ kN, $h_1 = 6$ m and $\mu = 0.15$. Let F_v be the vertical force required to lift the gate.

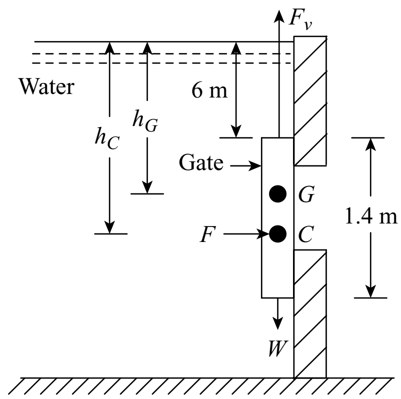

Figure 3.17

$$h_G = 6 + \frac{1.4}{2} = 6.7 \text{ m}$$

$$F = \rho_w gAh_G = 1000 \times 9.81 \times (1.4 \times 2.8) \times 6.7 = 257.65 \text{ kN}$$

$$I_G = \frac{bh^3}{12} = \frac{2.8 \times 1.4^3}{12} = 0.6403 \text{ m}^4$$

$$h_C = \frac{I_G}{Ah_G} + h_G = \frac{0.6403}{(2.8 \times 1.4) \times 6.7} + 6.7 = \mathbf{6.724\ m}$$

$$F_v = \mu F + W = 0.15 \times 257.65 + 25 = \mathbf{63.6475\ kN}$$

Example 3.15 A circular plate of diameter 1 m with a hole of diameter 0.25 m is immersed vertically in a liquid (specific gravity = 0.9) with its upper edge 0.5 m below the free surface of the liquid. The centre of hole is 0.25 m vertically below the centre of the plate. Determine the pressure force acting on the plate and the centre of pressure.

Solution

Refer Figure 3.18. Let $d_1 = 1$ m, $d_2 = 0.25$ m, $S = 0.9$, $h_1 = 0.5$ m and $h_2 = 0.25$ m. Let O_1 and O_2 be the centres of the plate and the hole, respectively.

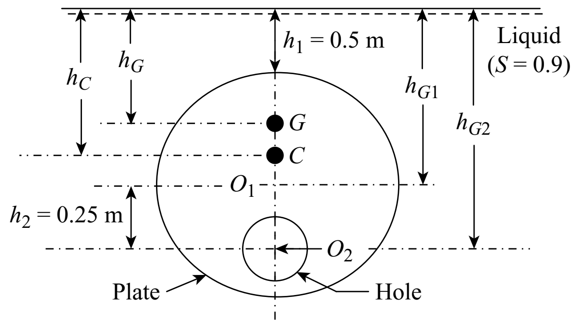

Figure 3.18

$$A_1 = \frac{\pi}{4}d_1^2 = \frac{\pi}{4}\times 1^2 = 0.7854 \text{ m}^2$$

$$A_2 = \frac{\pi}{4}d_2^2 = \frac{\pi}{4}\times 0.25^2 = 0.0491 \text{ m}^2$$

$$h_{G1} = 0.5 + \frac{1}{2} = 1 \text{ m}$$

$$h_{G2} = 0.5 + 0.5 + 0.25 = 1.25 \text{ m}$$

$$h_G = \frac{A_1 h_{G1} - A_2 h_{G2}}{A_1 - A_2} = \frac{0.7854\times 1 - 0.0491\times 1.25}{0.7854 - 0.0491} = 0.9833 \text{ m}$$

Since $$F = S\rho_w g(A_1 - A_2)h_G$$

$$\therefore F = 0.9\times 1000\times 9.81\times(0.7854 - 0.0491)\times 0.9833 = \mathbf{6392.23\ N}$$

Since $$I_G = \left[\frac{\pi}{64}d_1^2 + A_1(h_{G1} - h_G)^2\right] - \left[\frac{\pi}{64}d_2^2 + A_2(h_{G2} - h_G)^2\right]$$

$$I_G = \left[\frac{\pi}{64}\times 1^2 + 0.7854\times(1 - 0.9833)^2\right] - \left[\frac{\pi}{64}\times 0.25^2 + 0.0491\times(1.25 - 0.9833)^2\right]$$

$$\therefore I_G = 0.0493 - 0.0066 = 0.0427 \text{ m}^4$$

$$h_C = \frac{I_G}{Ah_G} + h_G = \frac{0.0427}{(0.7854 - 0.0491)\times 0.9833} + 0.9833 = \mathbf{1.0423\ m}$$

3.6 ❐ INCLINED SUBMERGED PLANE SURFACE

3.6.1 Total Pressure on an Inclined Plane Submerged Surface

Consider a plane inclined vertical surface with random shape submerged in a static liquid as shown in Figure 3.19. Let A be the surface area, h_C be the distance of centre of pressure from the free surface of liquid, h_G be the distance of centre of gravity from the free surface of liquid, p be the pressure intensity, F be the total force on the surface, C be the centre of pressure, G be the centroid and α be the angle made by the plane of the surface with free liquid surface. When the plane of the surface is produced from point 'B', then it meets the free liquid surface at point 'A' and it will be perpendicular to the plane of the surface. Let l_G and l_C be the distances of G and C, respectively, from the axis AB.

Consider an elementary strip of area dA at a depth h from the free surface of liquid and at a distance l from the axis AB shown in Figure 3.19.

Pressure force on the strip is given by,

$$dF = p\times dA = \rho gh\,dA \quad [\because p = \rho gh]$$

Total pressure force on the whole surface is given by,

$$F = \int dF = \int \rho gh\,dA = \rho g\int h dA \tag{3.11}$$

Since $h = l\sin\alpha$, $\int l dA = Al_G$ and $h_G = l_G \sin\alpha$

$$F = \rho g\int l\sin\alpha dA = \rho g\sin\alpha\int l dA = \rho g\sin\alpha\times Al_G = \rho g A h_G \tag{3.12}$$

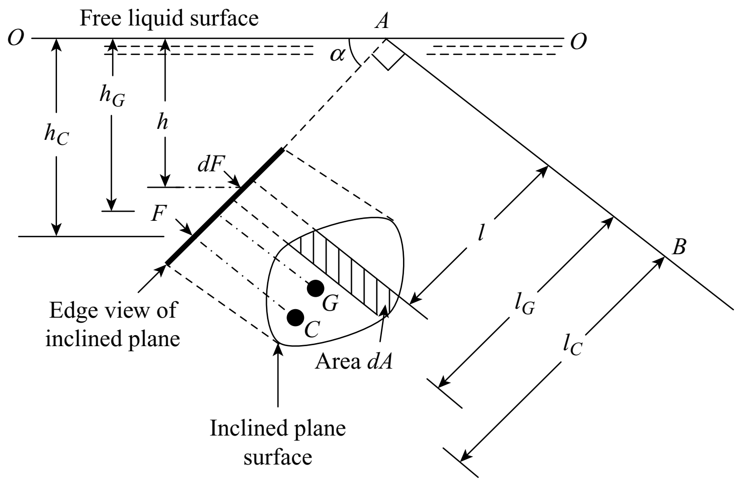

Figure 3.19 *Inclined submerged plane surface*

3.6.2 Centre of Pressure on an Inclined Plane Submerged Surface

The pressure force on the strip is given by,

$$dF = \rho gh \times dA = \rho g \times l \sin\alpha \times dA \quad [\because h = l\sin\alpha]$$

Moment of this pressure force about the axis AB is given by,

$$= \rho gl \sin\alpha dA \times l = \rho g \sin\alpha\, l^2 dA$$

Sum of moments of all such pressure forces about the axis AB is given by,

$$= \int \rho g \sin\alpha\, l^2 dA = \rho g \sin\alpha \int l^2 dA = \rho g \sin\alpha \times I_O = \rho g \sin\alpha \times [I_G + Al_G{}^2] \tag{3.13}$$

Where $\int l^2 dA$ = M.O.I. of the surface about $AB = I_O$ and $I_O = I_G + Al_G{}^2$

Moment of total force F acting at point C at a distance l_C is given by,

$$= F \times l_C = \rho g A h_G \times l_C = \rho g A h_G \times \frac{h_C}{\sin\alpha} \tag{3.14}$$

By solving equations (3.13) and (3.14), we get:

$$\rho g A h_G \times \frac{h_C}{\sin\alpha} = \rho g \sin\alpha \times [I_G + Al_G{}^2]$$

$$\therefore h_C = \frac{\sin^2\alpha}{Ah_G}[I_G + Al_G{}^2] = \frac{\sin^2\alpha}{Ah_G}\left[I_G + A\frac{h_G{}^2}{\sin^2\alpha}\right] = \frac{I_G \sin^2\alpha}{Ah_G} + h_G \tag{3.15}$$

Here, Equation (3.15) gives the vertical depth of centre of pressure for the inclined surface submerged below the free surface of static liquid. If $\alpha = 90°$, then Equation (3.15) becomes the same as Equation (3.10) which is applicable for vertically submerged plane surfaces.

Example 3.16 A rectangular plane surface that is 1.5 m wide and 4 m deep is immersed in a liquid (specific gravity = 0.9) in such a way that its plane makes an angle of 30° with the free surface of liquid. Determine the total pressure and position of centre of pressure when the upper edge is 1 m below the free liquid surface.

Solution

Refer Figure 3.20. Let $b = 1.5$ m, $d = 4$ m, $S = 0.9$, $\alpha = 30°$ and $h_1 = 1$ m.

$$h_G = SU + UT = h_1 + TR\sin\alpha = h_1 + \frac{d}{2}\sin\alpha = 1 + \frac{4}{2}\sin 30° = 2 \text{ m}$$

$$F = S\rho_w gAh_G = 0.9 \times 1000 \times 9.81 \times (1.5 \times 4) \times 2 = \mathbf{105948\ N}$$

$$I_G = \frac{bd^3}{12} = \frac{1.5 \times 4^3}{12} = 8 \text{ m}^4$$

$$h_C = \frac{I_G \sin^2\alpha}{Ah_G} + h_G = \frac{8 \times \sin^2 30°}{(1.5 \times 4) \times 2} + 2 = \mathbf{2.167\ m}$$

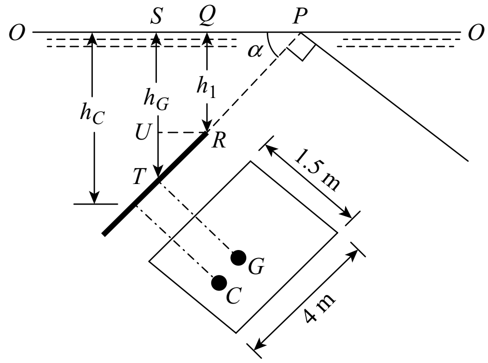

Figure 3.20

Example 3.17 Determine the total pressure and position of centre of pressure of a circular plate that has diameter of 2 m submerged in water whose greatest and least depths below the surface are 1.5 m and 0.5 m, respectively.

Solution

Refer Figure 3.21. Let $d = 2$ m, $h_2 = 1.5$ m and $h_1 = 0.5$ m.

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4} \times 2^2 = \pi \text{ m}^2$$

$$\sin\alpha = \frac{VX}{VR} = \frac{VU - UX}{VR} = \frac{h_2 - h_1}{d} = \frac{1.5 - 0.5}{2} = 0.5$$

$$h_G = h_1 + WT = h_1 + TR\sin\alpha = 0.5 + \frac{2}{2} \times 0.5 = 1 \text{ m}$$

$$F = \rho_w gAh_G = 1000 \times 9.81 \times \pi \times 1 = \mathbf{30819.024\ N}$$

$$I_G = \frac{\pi d^4}{64} = \frac{\pi \times 2^4}{64} = \frac{\pi}{4} \text{ m}^4$$

$$h_C = \frac{I_G \sin^2\alpha}{Ah_G} + h_G = \frac{(\pi/4) \times 0.5^2}{\pi \times 1} + 1 = \mathbf{1.0625\ m}$$

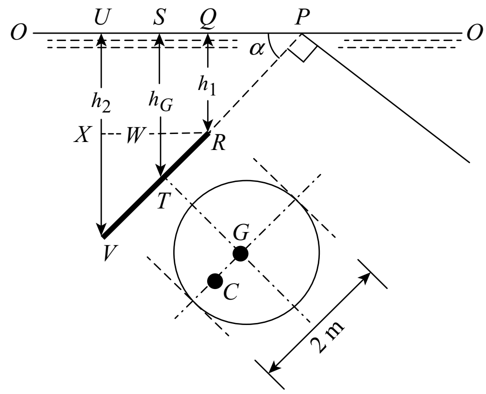

Figure 3.21

Example 3.18 An annular plate having external and internal diameters of 2 m and 1 m, respectively is submerged in an oil (specific gravity = 0.92) in such a way that its greatest and least depths below the oil surface are 3 m and 2 m, respectively. Determine the total pressure and the position of centre of pressure on one face of the plate.

Solution

Refer Figure 3.22. Let $d_1 = 2$ m, $d_2 = 1$ m, $S = 0.92$, $h_2 = 3$ m and $h_1 = 2$ m.

$$A = \frac{\pi}{4}(d_1^2 - d_2^2) = \frac{\pi}{4} \times (2^2 - 1^2) = 2.3562 \text{ m}^2$$

$$\sin\alpha = \frac{VX}{VR} = \frac{VU - UX}{VR} = \frac{h_2 - h_1}{d_1} = \frac{3 - 2}{2} = 0.5$$

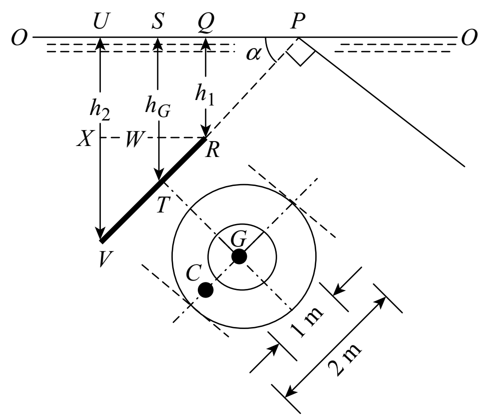

Figure 3.22

$$h_G = h_1 + WT = h_1 + TR\sin\alpha = 2 + \frac{2}{2}\times 0.5 = 2.5 \text{ m}$$

$$F = S\rho_w gAh_G = 0.92\times1000\times9.81\times2.3562\times2.5 = \mathbf{53162.941\ N}$$

$$I_G = \frac{\pi(d_1^4 - d_2^4)}{64} = \frac{\pi\times(2^4 - 1^4)}{64} = 0.7363 \text{ m}^4$$

$$h_C = \frac{I_G\sin^2\alpha}{Ah_G} + h_G = \frac{0.7363\times0.5^2}{2.3562\times2.5} + 2.5 = \mathbf{2.5312\ m}$$

Example 3.19 A triangular plate of base 1.5 m and height 2 m is submerged in oil (specific gravity = 0.92). The plane of the plate is inclined at 30° with free oil surface and the base is parallel and it is at a depth of 1 m from the oil surface. Determine the total pressure and position of centre of pressure on one face of the plate.

Solution

Refer Figure 3.23. Let $b = 1.5$ m, $h = 2$ m, $S = 0.92$, $\alpha = 30°$ and $h_1 = 1$ m.

$$A = \frac{bh}{2} = \frac{1.5\times2}{2} = 1.5 \text{ m}^2$$

$$h_G = h_1 + WT = h_1 + TR\sin\alpha = 1 + \frac{2}{3}\sin30° = 1.333 \text{ m}$$

$$F = S\rho_w gAh_G = 0.92\times1000\times9.81\times1.5\times1.333 = \mathbf{18045.89\ N}$$

$$I_G = \frac{bh^3}{36} = \frac{1.5\times2^3}{36} = 0.333 \text{ m}^4$$

$$h_C = \frac{I_G\sin^2\alpha}{Ah_G} + h_G = \frac{0.333\times\sin^2 30°}{1.5\times1.333} + 1.333 = \mathbf{1.3746\ m}$$

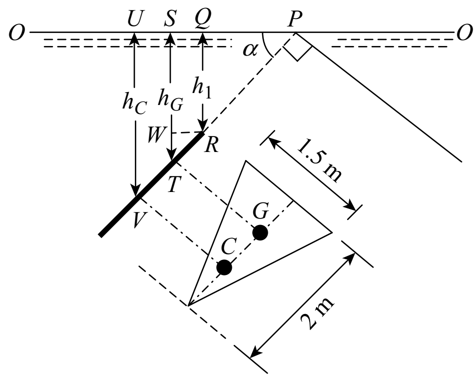

Figure 3.23

Example 3.20 A trapezoidal plate of height 2.2 m and sides of 2.4 m and 3.6 m is immersed in water at an inclination of 30° to the free surface of the water. The depth of top edge of the plate is at 2 m from the free surface. Determine the hydrostatic force on the given plate and the centre of pressure.

Solution

Refer Figure 3.24. Let $h = 2.2$ m, $a = 2.4$ m, $b = 3.6$ m, $\alpha = 30°$ and $h_1 = 2$ m.

$$TV = \left(\frac{2a+b}{a+b}\right)\frac{h}{3} = \left(\frac{2\times2.4+3.6}{2.4+3.6}\right)\times\frac{2.2}{3} = 1.03 \text{ m}$$

$$\therefore TR = h - TV = 2.2 - 1.03 = 1.17 \text{ m}$$

$$h_G = h_1 + UT = h_1 + TR\sin\alpha = 2 + 1.17\sin30° = 2.585 \text{ m}$$

$$A = \frac{(a+b)h}{2} = \frac{(2.4+3.6)\times2.2}{2} = 6.6 \text{ m}^2$$

$$F = \rho_w gAh_G = 1000\times9.81\times6.6\times2.585 = \mathbf{167368.41\ N}$$

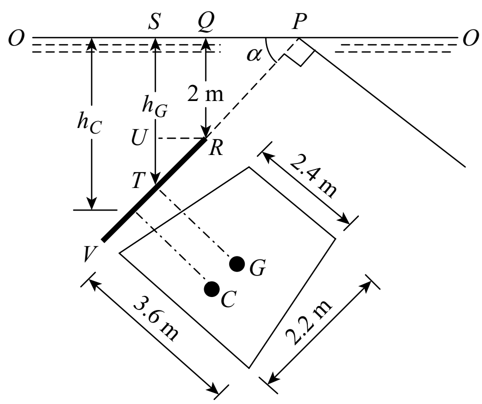

Figure 3.24

$$I_G = \frac{a^2+4ab+b^2}{36(a+b)}h^3 = \left[\frac{2.4^2+4\times 2.4\times 3.6+3.6^2}{36\times(2.4+3.6)}\right]\times 2.2^3 = 2.6265 \text{ m}^4$$

$$h_C = \frac{I_G \sin^2\alpha}{Ah_G} + h_G = \frac{2.6265\sin^2 30°}{6.6\times 2.585} + 2.585 = \mathbf{2.6235\ m}$$

Example 3.21 A circular plate of diameter 8 m has a circular hole of diameter 2 m with its centre above the centre of plate as shown in Figure 3.25. The plate is immersed in a liquid (specific gravity = 0.9) at an angle of 30° to the horizontal and with its top edge 4 m below the surface of the liquid. Determine the hydrostatic force on the given plate and the position of centre of pressure.

Solution

Refer Figure 3.25. Let $d_1 = 8$ m, $d_2 = 2$ m, $S = 0.9$, $\alpha = 30°$ and $h_1 = 4$ m.

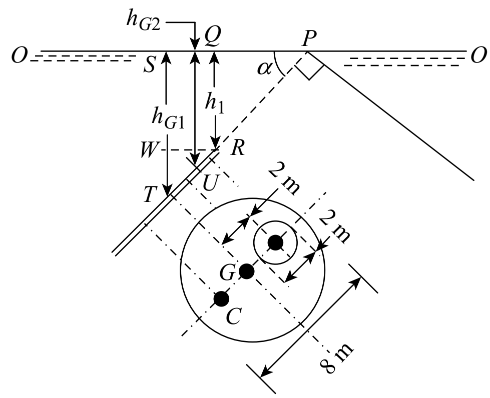

Figure 3.25

$$h_{G1} = h_1 + WT = h_1 + TR\sin\alpha = 4 + 4\sin 30° = 6 \text{ m}$$

$$h_{G2} = h_1 + UR\sin\alpha = 4 + 2\sin 30° = 5 \text{ m}$$

Pressure force on the plate is given by,

$$F_1 = S\rho_w g A_1 h_{G1} = 0.9\times 1000\times 9.81\times\frac{\pi}{4}\times 8^2\times 6 = 2662.764 \text{ kN}$$

Since $h_{C1} = \dfrac{I_{G1}\sin^2\alpha}{A_1 h_{G1}} + h_{G1} = \dfrac{(\pi/64)d_1^4\sin^2\alpha}{(\pi/4)d_1^2 h_{G1}} + h_{G1} = \dfrac{d_1^2\sin^2\alpha}{16h_{G1}} + h_{G1}$

$$\therefore h_{C1} = \frac{8^2\sin^2 30°}{16\times 6} + 6 = 6.167 \text{ m}$$

Pressure force on the hole is given by,

$$F_2 = S\rho_w g A_2 h_{G2} = 0.9\times 1000\times 9.81\times\frac{\pi}{4}\times 2^2\times 5 = 138.686 \text{ kN}$$

Since
$$h_{C2} = \frac{I_{G2}\sin^2\alpha}{A_2 h_{G2}} + h_{G2} = \frac{(\pi/64)d_2^4\sin^2\alpha}{(\pi/4)d_2^2 h_{G2}} + h_{G2} = \frac{d_2^2\sin^2\alpha}{16h_{G2}} + h_{G2}$$

$$\therefore h_{C2} = \frac{2^2\sin^2 30°}{16\times 5} + 5 = 5.0125 \text{ m}$$

Thus, force on the plate is given by,

$$F = F_1 - F_2 = 2662.764 - 138.686 = \mathbf{2524.078\ kN}$$

Taking moments about the liquid surface, we get:

$$F\times h_C = F_1\times h_{C1} - F_2\times h_{C2}$$

$$\therefore h_C = \frac{F_1 h_{C1} - F_2 h_{C2}}{F} = \frac{2662.764\times 6.167 - 138.686\times 5.0125}{2524.078} = \mathbf{6.23\ m}$$

Example 3.22 A 6 m × 2 m rectangular gate is hinged at the base and it is inclined at an angle of 60° with the horizontal. The upper end of the gate is kept in position by a weight of 55000 N acting perpendicularly to the gate through a pulley system. If the weight of the gate and the friction at the hinge and pulley is neglected, then find the level of water when the gate begins to fall.

Solution

Refer Figure 3.26. Let $PQ = d = 6$ m, $b = 2$ m, $\alpha = 60°$, $W = 55000$ N, h be the level of water surface when the gate begins to fall, PR be the length of gate submerged in water and A be the wetted area of the gate.

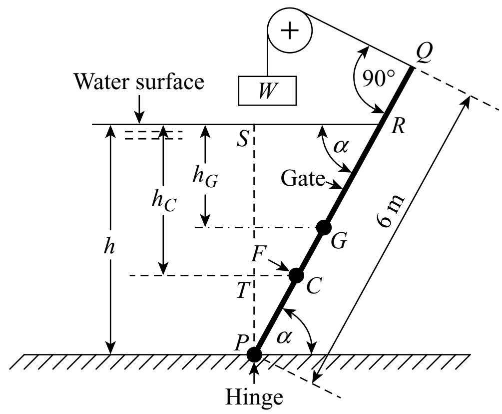

Figure 3.26

$$PR = \frac{h}{\sin\alpha} = \frac{h}{\sin 60°} = 1.1547h$$

$$A = PR \times b = 1.1547h \times 2 = 2.309h \text{ m}^2$$

$$h_G = \frac{h}{2} = 0.5h$$

$$F = \rho_w g A h_G = 1000 \times 9.81 \times 2.309h \times 0.5h = 11325.645h^2 \text{ N}$$

$$I_G = \frac{b(PR)^3}{12} = \frac{2 \times (1.1547h)^3}{12} = 0.2566h^3 \text{ m}^4$$

$$h_C = \frac{I_G \sin^2\alpha}{Ah_G} + h_G = \frac{0.2566h^3 \sin^2 60°}{2.309h \times 0.5h} + 0.5h = 0.667h$$

$$PC = \frac{PT}{\sin\alpha} = \frac{h - h_C}{\sin 60°} = \frac{h - 0.667h}{\sin 60°} = 0.3845h$$

Taking moments about the hinge, we get:

$$F \times PC = W \times 6$$

$$11325.645h^2 \times 0.3845h = 55000 \times 6$$

$$\therefore h = \left(\frac{55000 \times 6}{11325.645 \times 0.3845}\right)^{1/3} = \mathbf{4.232 \text{ m}}$$

Example 3.23 A 4 m × 2.5 m rectangular sluice gate PQ hinged at point P (Figure 3.27) and inclined at an angle of 45° with the horizontal is kept closed by a weight fixed to the gate. The total weight of the gate and weight fixed to the gate is 450 kN. The centre of gravity of the weight and gate is at G. Determine the height of the water h which will cause the gate to open.

Solution

Refer Figure 3.27. Let $PQ = d = 4$ m, $b = 2.5$ m, $\alpha = 45°$, $W = 450$ kN, $TG = 0.75$ m and h be the height of water surface when the gate begins to open.

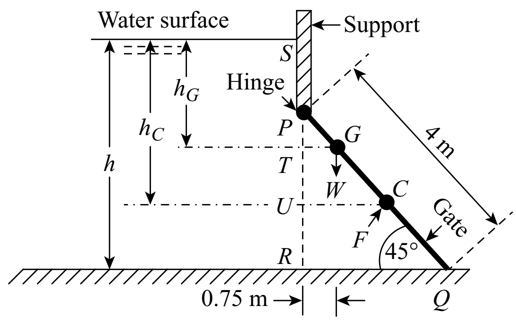

Figure 3.27

$$h_G = h - TR = h - (PR - PT) = h - (PQ \sin 45° - TG \tan 45°)$$

$$\therefore h_G = h - (4\sin 45° - 0.75\tan 45°) = (h - 2.08) \text{ m}$$

$$A = PQ \times b = 4 \times 2.5 = 10 \text{ m}^2$$

$$F = \rho_w g A h_G = 1000 \times 9.81 \times 10 \times (h-2.08) = 98.1(h-2.08) \text{ kN}$$

$$I_G = \frac{bd^3}{12} = \frac{2.5 \times 4^3}{12} = 13.333 \text{ m}^4$$

Since
$$h_C = \frac{I_G \sin^2 \alpha}{A h_G} + h_G$$

$$h_C = \frac{13.333 \times \sin^2 45^\circ}{10 \times (h-2.08)} + (h-2.08) = \frac{0.667}{(h-2.08)} + (h-2.08)$$

Taking moments about the hinge P, we get:

$$F \times PC = W \times TG$$

$$98.1(h-2.08) \times \frac{PU}{\sin 45^\circ} = 450 \times 0.75$$

$$PU = \frac{450 \times 0.75 \sin 45^\circ}{98.1(h-2.08)} = \frac{2.433}{(h-2.08)} \qquad \text{(i)}$$

$$PU = h_C - SP = h_C - (SR - PR) = h_C - (h - PQ \sin 45^\circ)$$

Substituting the values of h_C and PQ in the above expression, we get:

$$PU = \frac{0.667}{(h-2.08)} + (h-2.08) - (h - 4\sin 45^o) = \frac{0.667}{(h-2.08)} + 0.748 \qquad \text{(ii)}$$

From expressions (i) and (ii), we get:

$$\frac{0.667}{(h-2.08)} + 0.748 = \frac{2.433}{(h-2.08)}$$

$$0.667 + 0.748(h-2.08) = 2.433$$

$$0.748h = 2.433 - 0.667 + 1.556$$

$$\therefore h = \frac{3.322}{0.748} = \mathbf{4.44 \text{ m}}$$

Example 3.24 Figure 3.28 illustrates a gate hinged at point C. Determine the height of water so that the gate tips about the hinged point. Consider the width of gate as unity.

Solution

Refer Figure 3.28. Let $PQ = d$ m, $b = 1$ m, $\alpha = 60^\circ$, and $h_C = (h - 2.5)$ m.

The tipping of the gate is possible when the centre of pressure h_C of the resultant pressure force F acts at point C.

$$h_G = \frac{d}{2}\sin\alpha = \frac{d}{2}\sin 60^\circ$$

Since
$$h_C = \frac{I_G \sin^2 \alpha}{A h_G} + h_G$$

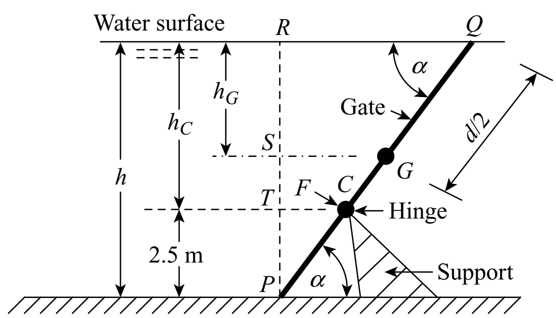

Figure 3.28

$$h - 2.5 = \frac{[(1\times d^3)/12]\sin^2 60°}{(1\times d)\times(d/2)\sin 60°} + \frac{d}{2}\sin 60°$$

$$d\sin 60° - 2.5 = \frac{d}{6}\sin 60° + \frac{d}{2}\sin 60°$$

$$\frac{d}{3}\sin 60° = 2.5$$

$$\therefore d = \frac{2.5\times 3}{\sin 60°} = 8.66 \text{ m}$$

$$\therefore h = d\sin 60° = 8.66\sin 60° \approx \mathbf{7.5\ m}$$

Example 3.25 Figure 3.29 illustrates an inclined rectangular sluice gate PQ of size 1 m × 3 m hinged at point P which controls the discharge of water. Determine the force normal to the gate applied at Q to open it.

Solution

Refer Figure 3.29. Let $PQ = d = 1$ m, $b = 3$ m, $\alpha = 45°$, $h = 4$ m and F_1 be the force required to open the gate.

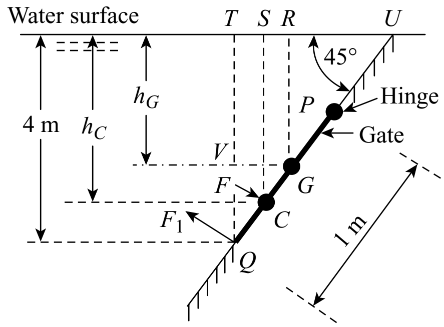

Figure 3.29

$$h_G = h - VQ = h - QG\sin\alpha = 4 - \frac{1}{2}\sin 45° = 3.646 \text{ m}$$

$$F = \rho_w gAh_G = 1000\times 9.81\times(1\times 3)\times 3.646 = 107301.78 \text{ N}$$

$$I_G = \frac{bd^3}{12} = \frac{3\times 1^3}{12} = 0.25 \text{ m}^4$$

$$h_C = \frac{I_G\sin^2\alpha}{Ah_G} + h_G = \frac{0.25\sin^2 45°}{(1\times 3)\times 3.646} + 3.646 = 3.657 \text{ m}$$

$$PC = PQ - QC = PQ - (QU - CU)$$

$$PC = 1 - \left[\frac{QT}{\sin 45°} - \frac{CS}{\sin 45°}\right] = 1 - \left[\frac{h}{\sin 45°} - \frac{h_c}{\sin 45°}\right]$$

$$\therefore PC = 1 - \left[\frac{4}{\sin 45°} - \frac{3.657}{\sin 45°}\right] = 0.5149 \text{ m}$$

Taking moments about the hinge P, we get:

$$F_1 \times PQ = F \times PC$$

$$F_1 \times 1 = 107301.78 \times 0.5149$$

$$\therefore F_1 = \mathbf{55249.69\ N}$$

Example 3.26 Figure 3.30(a) illustrates a container filled with an oil (specific gravity = 0.9) under a pressure of 17658 Pa. The opening of the container is covered by an inclined square plate 1 m × 1 m hinged at point P by a force F_1. Determine the force F_1 and the reaction R_p at the hinge point P.

Solution

Refer Figure 3.30. Let $S = 0.9$, $p = 17658$ Pa, $PQ = d = 1$ m, $b = 1$ m, $\alpha = 30°$. F_1 be the force required to close the plate and R_p be the reaction at the hinge point P.

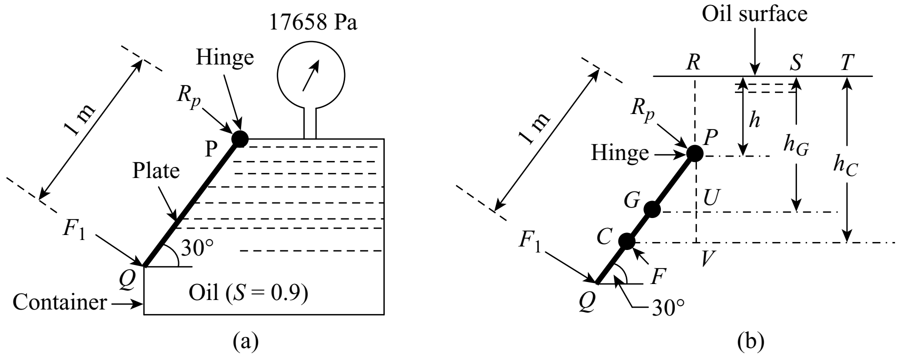

Figure 3.30

The water head equivalent to the given gauge pressure of $p = 17658$ Pa is given by,

$$h = \frac{p}{S\rho_w g} = \frac{17658}{0.9 \times 1000 \times 9.81} = 2 \text{ m}$$

Thus, the oil surface may be considered 2 m above the hinge as shown in Figure 3.30(b).

$$h_G = h + PU = h + PG \sin\alpha = 2 + \frac{1}{2}\sin 30° = 2.25 \text{ m}$$

$$F = S\rho_w g A h_G = 0.9 \times 1000 \times 9.81 \times (1 \times 1) \times 2.25 = 19865.25 \text{ N}$$

$$I_G = \frac{bd^3}{12} = \frac{1 \times 1^3}{12} = 0.0833 \text{ m}^4$$

$$h_C = \frac{I_G \sin^2\alpha}{A h_G} + h_G = \frac{0.0833 \sin^2 30°}{(1 \times 1) \times 2.25} + 2.25 = 2.259 \text{ m}$$

$$PC = \frac{PV}{\sin 30°} = \frac{RV - RP}{\sin 30°} = \frac{h_C - 2}{\sin 30°} = \frac{2.259 - 2}{\sin 30°} = 0.518 \text{ m}$$

Taking moments about the hinge P, we get:

$$F_1 \times PQ = F \times PC$$

$$F_1 \times 1 = 19865.25 \times 0.518$$

$$\therefore F_1 = \mathbf{10290.2\ N}$$

$$R_p = F - F_1 = 19865.25 - 10290.2 = \mathbf{9575.05\ N}$$

Example 3.27 A 3 m square gate provided in an oil tank containing petrol (specific gravity = 0.7) up to a height of 1.5 m above the top edge of the gate is hinged at its top edge as shown in Figure 3.31(a). The space above the oil is subjected to a negative pressure of 6.867 kPa. Determine the necessary vertical pull to be applied at the lower edge to open the gate.

Solution

Refer Figure 3.31. Let $PQ = d = 3$ m, $b = 3$ m, $S = 0.7$, $h_t = 1.5$ m, $p = 6.867$ kPa, $\alpha = 45°$ and F_1 be the vertical force required to open the gate.

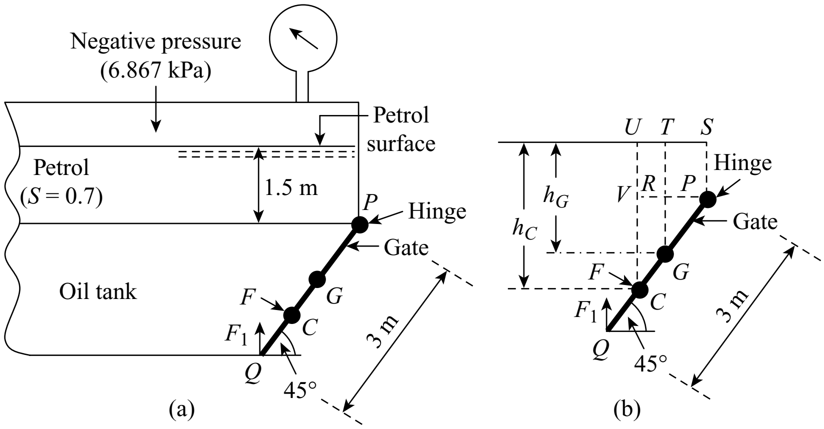

Figure 3.31

The oil head equivalent to the given negative pressure above the petrol surface is as follows.

$$h = \frac{p}{S\rho_w g} = \frac{6.867 \times 1000}{0.7 \times 1000 \times 9.81} = 1 \text{ m}$$

Thus, negative pressure reduces the head above the top edge of the gate from $h_t = 1.5$ m to $(1.5 - 1) = 0.5$ of oil. Therefore, the calculations for the magnitude and location of the pressure force are to be made corresponding to a head of $PS = 0.5$ m of oil as shown in Figure 3.31(b).

$$h_G = TR + RG = PS + PG \sin\alpha = 0.5 + \frac{3}{2}\sin 45° = 1.561 \text{ m}$$

$$F = S\rho_w gAh_G = 0.7 \times 1000 \times 9.81 \times (3 \times 3) \times 1.561 = 96474.483 \text{ N}$$

$$I_G = \frac{bd^3}{12} = \frac{3 \times 3^3}{12} = 6.75 \text{ m}^4$$

$$h_C = \frac{I_G \sin^2\alpha}{Ah_G} + h_G = \frac{6.75 \sin^2 45°}{(3 \times 3) \times 1.561} + 1.561 = 1.801 \text{ m}$$

$$VC = UC - UV = h_C - PS = 1.801 - 0.5 = 1.301 \text{ m}$$

$$PC = \frac{VC}{\sin 45°} = \frac{1.301}{\sin 45°}$$

Taking moments about the hinge P, we get:

$$F_1 \sin 45° \times PQ = F \times PC$$

$$F_1 \sin 45° \times 3 = 96474.483 \times \frac{1.301}{\sin 45°}$$

$$\therefore F_1 = \frac{96474.483 \times 1.301}{3\sin^2 45°} = \mathbf{83675.535\ N}$$

3.7 ❐ CURVED SUBMERGED PLANE SURFACE

Figure 3.32 illustrates a curved surface PQ submerged in a static fluid. Let dA be the elemental area at a vertical depth of h below free liquid surface. The differential force dF acting on the elemental area is given below.

$$dF = \text{Pressure} \times \text{Area} = \rho gh \times dA \tag{3.16}$$

In case of curved surface submerged in liquid, the direction of the pressure force on the elementary areas varies from point to point. Therefore, we do not use a direct method of integration to find the force due to hydrostatic pressure. Thus, the force dF can be resolved in two components dF_H and dF_V in the horizontal and vertical directions, respectively.

$$dF_H = dF \sin\alpha = \rho ghdA \sin\alpha \tag{3.17}$$

$$dF_V = dF \cos\alpha = \rho ghdA \cos\alpha \tag{3.18}$$

In the above equations, $dA \sin\alpha$ and $dA \cos\alpha$ are the vertical and horizontal projections respectively of the elemental area dA shown in Figure 3.32(b).

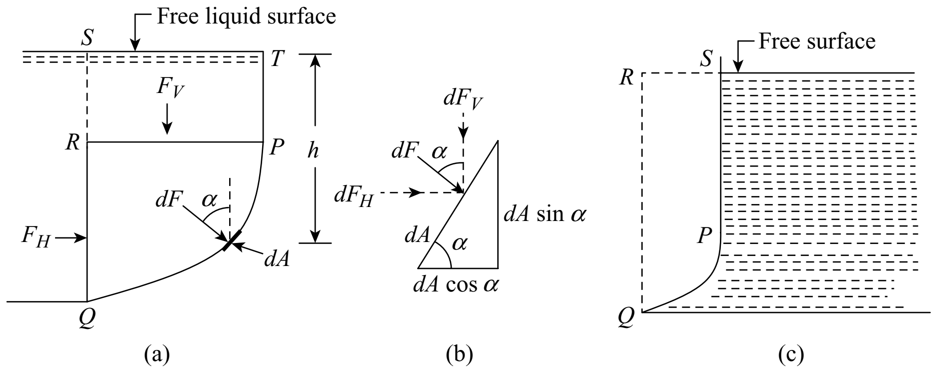

Figure 3.32

The total forces in the horizontal and vertical directions, namely F_H and F_V respectively can be obtained by integrating Equations (3.17) and (3.18). The force F_H is the total pressure force which acts on the imaginary vertical projection RQ of the curved surface on vertical plane through the centre of pressure of the plane surface SRQ. The force F_V is the total pressure force which acts on the imaginary horizontal projection RP of the curved surface on horizontal plane. The magnitude of F_V is the weight of the liquid supported by the curved surface up to the free liquid surface and it passes through the centre of gravity of the volume $PQRSTP$.

The total resultant force on the curved surface is given,

$$\boxed{F = \sqrt{F_H{}^2 + F_V{}^2}} \tag{3.19}$$

The resultant force F passes through the intersection of its two components and its inclination with horizontal is given below.

$$\boxed{\beta = \tan^{-1}\left(\frac{F_V}{F_H}\right)} \tag{3.20}$$

When the underside of a curved surface is subjected to hydrostatic pressure as shown in Figure 3.32 (c), the force F_V will be equal to the weight of the imaginary fluid supported by PQ upto the free surface of liquid and its direction will be taken in upward direction.

Example 3.28 Determine the resultant pressure force per unit length acting on the curved corner PQ of the container having gasoline (specific gravity = 0.7) upto a depth of 4 m as shown in Figure 3.33.

Solution

Refer Figure 3.33. Let $OP = OQ = 2$ m, $b = 1$ m, $S = 0.7$, F be the resultant force and β be its inclination with the horizontal.

Let A_p be the projected area of the curved surface on vertical plane.

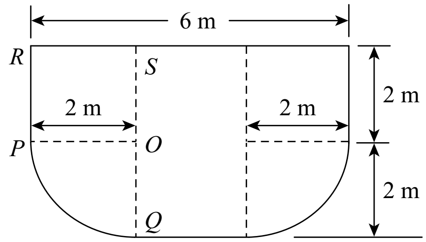

Figure 3.33

$$F_H = S\rho_w g A_p h_G = 0.7\times1000\times9.81\times(2\times1)\times\left(2+\frac{2}{2}\right) = 41.202 \text{ kN}$$

Since F_V = Weight of gasoline above PQ upto RS

or F_V = Weight of gasoline × [Volume $OPRS$ + Volume OPQ]

($OPRS$ is a cuboid and OPQ is a quadrant of a circular cylinder)

Thus
$$F_V = 0.7\times1000\times9.81\times\left[2\times2\times1+\frac{1}{4}\pi\times2^2\times1\right] = 49.041 \text{ kN}$$

$$F = \sqrt{F_H{}^2 + F_V{}^2} = \sqrt{41.202^2 + 49.041^2} = \mathbf{64.052\ kN}$$

$$\beta = \tan^{-1}\left(\frac{F_V}{F_H}\right) = \tan^{-1}\left(\frac{49.041}{41.202}\right) = \mathbf{49.96°}$$

Example 3.29 A door in the form of a quadrant of a cylinder of 3 m radius and 4 m width is fitted in a water tank shown in Figure 3.34. Determine the magnitude and direction of the resultant force on the door.

Solution

Refer Figure 3.34. Let $OP = OQ = R = 3$ m, $b = 4$ m, F be the resultant force and β be its inclination with the horizontal.

Let A_p be the projected area of the curved surface on vertical plane.

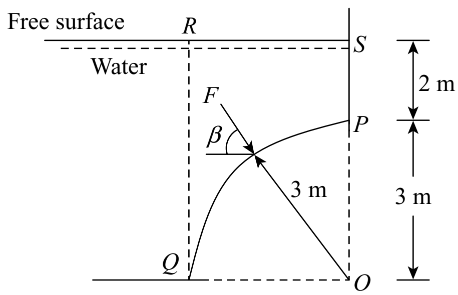

Figure 3.34

$$F_H = \rho_w g A_p h_G = 1000\times9.81\times(3\times4)\times\left(2+\frac{3}{2}\right) = 412.02 \text{ kN}$$

Since F_V = Weight of water block $PQRS = \rho_w \times (\text{Volume } PQRS)\times g$

$$\therefore F_V = 1000\times\left[5\times3\times4-\frac{1}{4}\pi\times3^2\times4\right]\times9.81 = 311.23 \text{ kN}$$

$$F = \sqrt{F_H{}^2 + F_V{}^2} = \sqrt{412.02^2 + 311.23^2} = \mathbf{516.357\ kN}$$

$$\beta = \tan^{-1}\left(\frac{F_V}{F_H}\right) = \tan^{-1}\left(\frac{311.23}{412.02}\right) = \mathbf{37.07°}$$

Example 3.30 Figure 3.35 illustrates a cylindrical roller gate of diameter 2 m and 5 m in length which is placed on the dam in such a way that water is going to spill over it. Determine the magnitude and direction of the resultant force acting on the gate due to water.

Solution

Refer Figure 3.35. Let $PR = 2$ m, $b = 5$ m, F be the resultant force and β be its inclination with the horizontal. Let A_p be the projected area of the curved surface on vertical plane.

$$F_H = \rho_w g A_p h_G = 1000 \times 9.81 \times (2 \times 5) \times \frac{2}{2} = 98.1 \text{ kN}$$

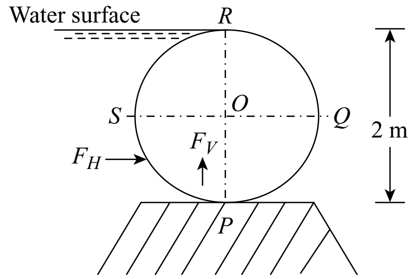

Figure 3.35

Since F_V = Weight of water block $PSR = \rho_w \times$(Volume of curved surface PSR)$\times g$

Thus $$F_V = \rho_w \times \frac{\pi}{2} r^2 b \times g = 1000 \times \frac{\pi}{2} \times 1^2 \times 5 \times 9.81 = 77.05 \text{ kN}$$

$$F = \sqrt{F_H^2 + F_V^2} = \sqrt{98.1^2 + 77.05^2} = \mathbf{124.74\ kN}$$

$$\therefore \beta = \tan^{-1}\left(\frac{F_V}{F_H}\right) = \tan^{-1}\left(\frac{77.05}{98.1}\right) = \mathbf{38.15°}$$

Example 3.31 Figure 3.36 illustrates a cylinder of diameter 4 m and 5 m in length which retains water on one side. If the cylinder weighs 350.19 kN and the frictional effects are neglected, then determine the vertical reaction at P and the horizontal reaction at Q.

Solution

Refer Figure 3.36. Let $PR = 4$ m, $b = 5$ m, $W = 350.19$ kN, R_P be the vertical reaction at P and R_Q be the horizontal reaction at Q.

Let A_p be the projected area of the curved surface on vertical plane.

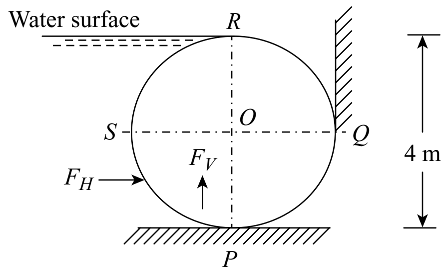

Figure 3.36

$$F_H = \rho_w g A_p h_G = 1000 \times 9.81 \times (4 \times 5) \times \frac{4}{2} = 392.4 \text{ kN}$$

Since $$F_V = \text{Weight of water block } PSR = \rho_w \times \frac{\pi}{2} r^2 b \times g$$

$$\therefore F_V = 1000 \times \frac{\pi}{2} \times 2^2 \times 5 \times 9.81 = 308.19 \text{ kN}$$

The vertical component F_V is acting in the upward direction and for equilibrium, we get the following result.

$$R_P = W - F_V = 350.19 - 308.19 = \mathbf{42\ kN}$$

$$R_Q = F_H = \mathbf{392.4\ kN}$$

Example 3.32 Figure 3.37 illustrates a 45° sector gate of 3 m radius mounted on a support. One of the radial arms of the gate and its hinge are at same horizontal level as the liquid surface (specific gravity = 0.9). If the length of the gate is 2.6 m, then find the magnitude and direction of the resultant pressure force on the gate.

Solution

Refer Figure 3.37. Let $\alpha = 45° = (1/8)$th of a circle, $OP = OQ = r = 3$ m, $S = 0.9$ and $b = 2.6$ m. Let A_p be the projected area of the curved surface on vertical plane RQ and $RQ = h$.

$$RQ = h = OQ \sin 45° = 3 \sin 45° = 2.12 \text{ m}$$

$$A_p = RQ \times b = 2.12 \times 2.6 = 5.512 \text{ m}^2$$

$$F_H = S\rho_w g A_p h_G = 0.9 \times 1000 \times 9.81 \times 5.512 \times \frac{2.12}{2} = 51585.37 \text{ N}$$

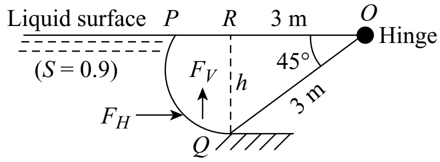

Figure 3.37

$$\text{Area } PRQ = \text{Area } POQ - \text{Area } ROQ = \frac{\pi}{8}r^2 - \frac{1}{2}\times OR \times RQ$$

$$\text{Area } PRQ = \frac{\pi}{8}\times 3^2 - \frac{1}{2}\times 3\cos 45° \times 2.12 = 1.286 \text{ m}^2$$

Since $F_V = \text{Weight of liquid block } PRQ = S\rho_w \times \text{Area } PRQ \times b \times g$

$$\therefore F_V = 0.9\times 1000\times 1.286\times 2.6\times 9.81 = 29520.64 \text{ N}$$

$$F = \sqrt{{F_H}^2 + {F_V}^2} = \sqrt{51585.37^2 + 29520.64^2} = \mathbf{59434.99\ N}$$

The angle of inclination of resultant with the horizontal is given by,

$$\beta = \tan^{-1}\left(\frac{F_V}{F_H}\right) = \tan^{-1}\left(\frac{29520.64}{51585.37}\right) = \mathbf{29.78°}$$

Example 3.33 A gate having a shape of a quadrant of circle 1 m radius has to resist liquid (specific gravity = 0.92) force as shown in Figure 3.38. If the width of the gate is unity, then determine the magnitude and direction of the resultant pressure force on the gate.

Solution

Refer Figure 3.38. Let $OP = OQ = r = 1$ m, $S = 0.92$ and $b = 1$ m.

Let A_p be the projected area of the curved surface on vertical plane OQ.

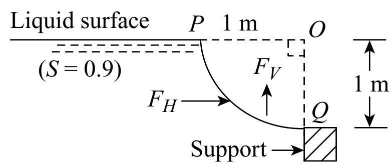

Figure 3.38

$$F_H = S\rho_w g A_p h_G = 0.92\times 1000\times 9.81\times (1\times 1)\times \frac{1}{2} = 4512.6 \text{ N}$$

Since $F_V = \text{Weight of water block } POQ = S\rho_w \times \frac{\pi}{4}r^2 b\times g$

$$\therefore F_V = 0.92\times 1000\times \frac{\pi}{4}\times 1^2\times 1\times 9.81 = 7088.37 \text{ N}$$

$$F = \sqrt{{F_H}^2 + {F_V}^2} = \sqrt{4512.6^2 + 7088.37^2} = \mathbf{8402.89\ N}$$

The angle of inclination of resultant with the horizontal is given by,

$$\beta = \tan^{-1}\left(\frac{F_V}{F_H}\right) = \tan^{-1}\left(\frac{7088.37}{4512.6}\right) = \mathbf{57.52°}$$

Example 3.34 Figure 3.39 illustrates a gate of radius 4 m and of 1 m width. Find the magnitude and direction of the resultant pressure force acting on the gate.

Solution

Refer Figure 3.39. Let $\alpha = 30° = (1/12)$th of a circle, $OP = OQ = r = 4$ m and $b = 1$ m.

Let A_p be the projected area of the curved surface on vertical plane RQ and $RQ = h$.

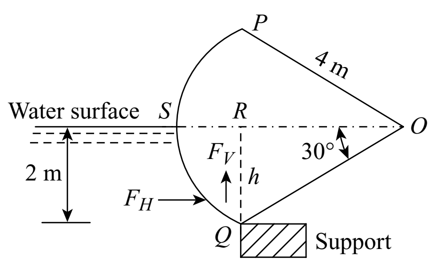

Figure 3.39

$$RQ = h = OQ\times \sin 30° = 4\sin 30° = 2 \text{ m}$$

$$A_p = RQ\times b = 2\times 1 = 2 \text{ m}^2$$

$$F_H = \rho_w g A_p h_G = 1000 \times 9.81 \times 2 \times \frac{2}{2} = 19620 \text{ N}$$

$$\text{Area } RSQ = \text{Area } OSQ - \text{Area } ORQ = \frac{\pi}{12} r^2 - \frac{1}{2} \times OR \times RQ$$

$$\text{Area } RSQ = \frac{\pi}{12} \times 4^2 - \frac{1}{2} \times 4 \cos 30° \times 2 = 0.7247 \text{ m}^2$$

Since F_V = Weight of liquid block $RQS = \rho_w \times$ Area $RQS \times b \times g$

$$\therefore F_V = 1000 \times 0.7247 \times 1 \times 9.81 = 7109.31 \text{ N}$$

$$F = \sqrt{F_H^2 + F_V^2} = \sqrt{19620^2 + 7109.31^2} = \mathbf{20868.32\ N}$$

The angle of inclination of resultant with the horizontal is given by,

$$\beta = \tan^{-1}\left(\frac{F_V}{F_H}\right) = \tan^{-1}\left(\frac{7109.31}{19620}\right) = \mathbf{19.92°}$$

Example 3.35 The pressure gauge fitted on a water tank (Figure 3.40) shows a reading of 19620 N/m^2. The curved surface PQ of the top is quarter of a circular cylinder of radius 1.4 m. Find the magnitude and direction of the resultant pressure force acting on the curved surface if the width of the tank is unity.

Solution

Refer Figure 3.40. Let $p = 19620$ N/m^2, $OP = OQ = r = 1.4$ m and $b = 1$ m.

Let A_p be the projected area of the curved surface on vertical plane OQ.

The water head equivalent to the given pressure is given by,

$$h = \frac{p}{\rho_w g} = \frac{19620}{1000 \times 9.81} = 2 \text{ m}$$

Thus, the free water surface can be imagined to be 2 m above the top of the tank.

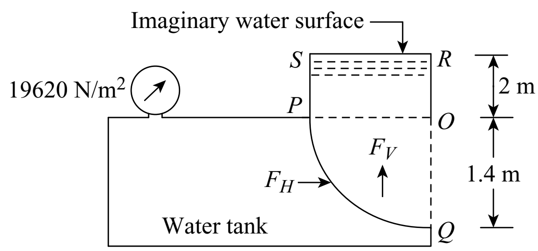

Figure 3.40

$$A_p = OQ \times b = 1.4 \times 1 = 1.4 \text{ m}^2$$

$$F_H = \rho_w g A_p h_G = 1000 \times 9.81 \times 1.4 \times \left(2 + \frac{1.4}{2}\right) = 37081.8 \text{ N}$$

Since F_V = Weight of liquid block above PQ

or $F_V = \rho_w \times (\text{Area } OPSR + \text{Area } OPQ) \times b \times g$

$$\therefore F_V = 1000 \times \left[(2 \times 1.4) + \frac{\pi}{4} \times 1.4^2\right] \times 1 \times 9.81 = 42569.32 \text{ N}$$

$$F = \sqrt{F_H^2 + F_V^2} = \sqrt{37081.8^2 + 42569.32^2} = \mathbf{56455.35\ N}$$

The angle of inclination of resultant with the horizontal is given by,

$$\beta = \tan^{-1}\left(\frac{F_V}{F_H}\right) = \tan^{-1}\left(\frac{42569.32}{37081.8}\right) = \mathbf{48.94°}$$

Example 3.36 Figure 3.41 illustrates the water level on the two sides of a cylindrical gate 2 m in diameter and 1 m in length. If the weight of the cylinder is 15000 N, then determine the magnitude and the location of the horizontal and vertical components of the force that keeps the cylinder just touching the floor. Also determine the magnitude and direction of the resultant force.

Solution

Refer Figure 3.41. Let $POQ = h_1 = 2$ m, $OQ = h_2 = 1$ m, $b = 1$ m and $W = 15000$ N.

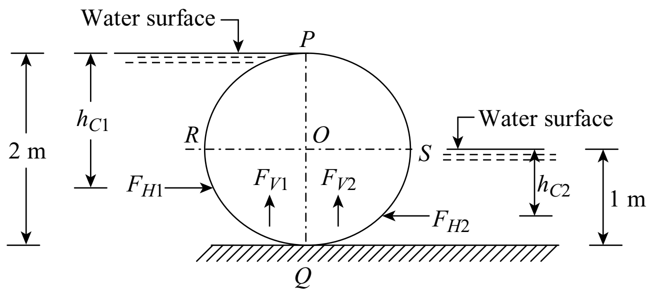

Figure 3.41

$$F_{H1} = \rho_w g A_{p1} h_{G1} = 1000 \times 9.81 \times (2 \times 1) \times \frac{2}{2} = 19620 \text{ N}$$

$$F_{H2} = \rho_w g A_{p2} h_{G2} = 1000 \times 9.81 \times (1 \times 1) \times \frac{1}{2} = 4905 \text{ N}$$

Thus, the net horizontal force is given by,

$$F_H = F_{H1} - F_{H2} = 19620 - 4905 = \mathbf{14715\ N}$$

Therefore, 14715 N force acting towards left is required to keep the gate in position.

$$h_{C1} = \frac{I_{G1}}{A_1 h_{G1}} + h_{G1} = \frac{(bh_1^3)/12}{A_1 h_{G1}} + h_{G1} = \frac{(1 \times 2^3)/12}{2 \times 1 \times (2/2)} + \frac{2}{2} = 1.333 \text{ m}$$

$$h_{C2} = \frac{I_{G2}}{A_2 h_{G2}} + h_{G2} = \frac{(bh_2^3)/12}{A_2 h_{G2}} + h_{G2} = \frac{(1 \times 1^3)/12}{1 \times 1 \times (1/2)} + \frac{1}{2} = 0.667 \text{ m}$$

The line of action of net horizontal force from Q (bottom) can be obtained by taking moments about Q.

$$F_H \times h = F_{H1} \times (2 - h_{C1}) - F_{H2} \times (1 - h_{C2})$$

$$14715 \times h = 19620 \times (2 - 1.333) - 4905 \times (1 - 0.667)$$

$$\therefore h = \frac{11453.175}{14715} = \mathbf{0.7783\ m} \quad \textbf{(From bottom)}$$

Since $F_{V1} = \text{Weight of water block } PRQ = \rho_w \times \text{Area } PQR \times b \times g$

$$\therefore F_{V1} = 1000 \times \frac{\pi}{2} \times 1^2 \times 1 \times 9.81 = 15409.512 \text{ N}$$

$$F_{V2} = \rho_w \times Area\ SOQ \times b \times g = 1000 \times \frac{\pi}{4} \times 1^2 \times 1 \times 9.81 = 7704.756 \text{ N}$$

Net upward force is given by,

$$F_V = F_{V1} + F_{V2} = 15409.512 + 7704.756 = \mathbf{23114.268\ N}$$

Thus, the additional force (F_a) required to keep the cylinder in position is given by,

$$F_a = F_V - W = 23114.268 - 15000 = \mathbf{8114.268\ N}$$

From vertical diameter, F_{V1} acts at a distance as given below.

$$\frac{4r}{3\pi} = \frac{4\times 1}{3\pi} = 0.4244 \text{ m} \quad \text{(From } PQ \text{ towards left)}$$

From vertical diameter, F_{V2} acts at a distance as given below,

$$\frac{4r}{3\pi} = \frac{4\times 1}{3\pi} = 0.4244 \text{ m} \quad \text{(From } PQ \text{ towards right)}$$

The line of action of net vertical force from POQ can be obtained by taking moments about POQ.

$$F_V \times x = F_{V1} \times 0.424 - F_{V2} \times 0.424$$

$$23114.268 \times x = 15409.512 \times 0.4244 - 7704.756 \times 0.4244$$

$$\therefore x = \frac{3269.898}{23114.268} = \mathbf{0.1415\ m} \quad \textbf{(From } \boldsymbol{POQ}\textbf{)}$$

$$F = \sqrt{F_H^{\,2} + F_V^{\,2}} = \sqrt{14715^2 + 23114.268^2} = \mathbf{27400.74\ N}$$

The angle of inclination of resultant with the horizontal is given by,

$$\beta = \tan^{-1}\left(\frac{F_V}{F_H}\right) = \tan^{-1}\left(\frac{23114.268}{14715}\right) = \mathbf{57.52^\circ}$$

Example 3.37 Figure 3.42 illustrates the curved surface PQ which is a quadrant of a circular cylinder. The radius of the curved surface is 3 m and length of the cylinder is 4 m. If water is 2 m above the curved surface PQ, then find the magnitude and location of the horizontal and vertical components of the force exerted by the water from Q. Also determine the magnitude and direction of the resultant force acting on the curved surface.

Solution

Refer Figure 3.42. Let $OP = OQ = r = 3$ m, $b = 4$ m and $OR = 2$ m.

Let A_p be the projected area of the curved surface on vertical plane OQ.

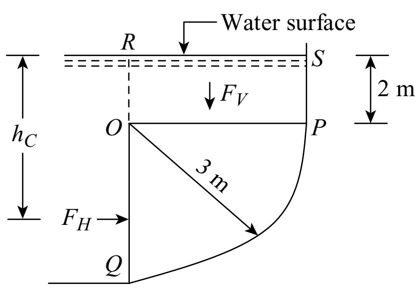

Figure 3.42

$$A_p = 3 \times 4 = 12 \text{ m}^2$$

$$h_G = 2 + \frac{3}{2} = 3.5 \text{ m}$$

$$F_H = \rho_w g A_p h_G = 1000 \times 9.81 \times 12 \times 3.5 = \mathbf{412.02\ kN}$$

$$I_G = \frac{bd^3}{12} = \frac{4\times 3^3}{12} = 9 \text{ m}^4$$

$$h_C = \frac{I_G}{Ah_G} + h_G = \frac{9}{12\times 3.5} + 3.5 = 3.7143 \text{ m}$$

Thus, the horizontal component of the resultant force acts (5 – 3.7143) = 1.2857 m vertically above from *Q*.

Since F_V = Weight of water block $PQORSP = F_{V1} + F_{V2}$

or $F_V = \rho_w \times$ Volume of part $PORS \times g + \rho_w \times$ Volume of part $PQO \times g$

$$F_V = 1000 \times 2 \times 3 \times 4 \times 9.81 + 1000 \times \frac{\pi}{4} \times 3^2 \times 4 \times 9.81$$

$$\therefore F_V = 235.44 + 277.37 = \mathbf{512.81\ kN}$$

The line of action of F_V can be obtained by taking moments of its two components say $F_{V1} = 235.44$ kN and $F_{V2} = 277.37$ kN about the line *ROQ* and we get the following expression.

$$F_V \times x = F_{V1} \times \frac{3}{2} + F_{V2} \times \frac{4r}{3\pi}$$

$$512.81x = 235.44 \times 1.5 + 277.37 \times \frac{4 \times 3}{3\pi}$$

$$\therefore x = \frac{706.3184}{512.81} = \mathbf{1.3773\ m} \quad \textbf{(From } \boldsymbol{ROQ} \textbf{ towards right)}$$

$$F = \sqrt{F_H^{\,2} + F_V^{\,2}} = \sqrt{412.02^2 + 512.81^2} = \mathbf{657.826\ kN}$$

The angle of inclination of resultant with the horizontal is given by,

$$\beta = \tan^{-1}\left(\frac{F_V}{F_H}\right) = \tan^{-1}\left(\frac{512.81}{412.02}\right) = \mathbf{51.22°}$$

Example 3.38 Determine the magnitude and direction of the resultant water pressure force acting on the curved face of the dam (Figure 3.43) which is shaped according to the relation $y = (x^2/2)$. The height of the water retained by the dam is 10 m and the width of the dam is 2 m.

Solution

Refer Figure 3.43. Let $y = x^2/2$ or $x = \sqrt{2}\, y^{1/2}$, $h = 10$ m and $b = 2$ m.

Let A_p be the projected area of the curved surface on vertical plane *QR*.

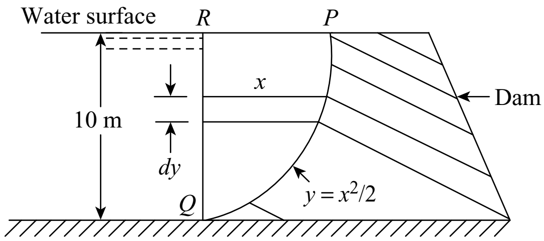

Figure 3.43

$$F_H = \rho_w g A_p h_G = 1000 \times 9.81 \times (10 \times 2) \times \frac{10}{2} = 981 \text{ kN}$$

Since F_V = Weight of water block $PQR = \rho_w \times$ Area $PQR \times b \times g$

$$F_V = \rho_w g \int_0^{10} x\,dy\,b = \rho_w g \int_0^{10} \sqrt{2} y^{1/2} dy \times 2 = 1000 \times 9.81 \times 2\sqrt{2} \times \left[\frac{y^{3/2}}{3/2}\right]_0^{10}$$

$$\therefore F_V = 1000 \times 9.81 \times 2\sqrt{2} \times \left[\frac{10^{3/2}}{3/2}\right] = 584.955 \text{ kN}$$

$$\therefore F = \sqrt{F_H^{\,2} + F_V^{\,2}} = \sqrt{981^2 + 584.955^2} = \mathbf{1142.162\ kN}$$

The angle of inclination of resultant with the horizontal is given by,

$$\beta = \tan^{-1}\left(\frac{F_V}{F_H}\right) = \tan^{-1}\left(\frac{584.955}{981}\right) = \mathbf{30.81°}$$

3.8 ❐ ANALYSIS OF FORCES ON DAMS

A dam is a structure built across a river or a stream to store water whose cross section may be triangular, rectangular or trapezoidal. The upstream face of the dam experiences force due to water pressure and it resists due to its dead weight. Consider a dam as shown in Figure 3.44 of trapezoidal section. The height of the dam is H and the widths of the top and bottom section is a and b, respectively. The main forces that act on a gravity dam are hydrostatic force and its weight.

1. **Hydrostatic force:** Let h be the depth of water stored on the upstream side for unit length. The horizontal thrust of water on the dam is given by the following expression.

$$F_H = \rho_w gAh_G = \rho_w g(h\times 1)\frac{h}{2} = \rho_w g\frac{h^2}{2} \tag{3.21}$$

This force will act at a depth of h_C below the free surface which is given by,

$$h_C = \frac{I_G}{Ah_G} + h_G = \frac{(1\times h^3)/12}{(1\times h)(h/2)} + \frac{h}{2} = \frac{2h}{3} \tag{3.22}$$

2. **Weight of the dam:** Let ρ_d be the density of the material used in the dam. Thus, the weight of the dam is given by the following expression.

$$W = \rho_d gH\frac{(a+b)}{2} \tag{3.23}$$

Let the centre of gravity of the section lies at a distance x from the vertical face PS. Divide the trapezium $PQRS$ into a rectangle $PQUS$ and a triangle QUR. Now take moments about the vertical face, we get the following expression.

$$a\times H\times\frac{a}{2} + \frac{1}{2}(b-a)\times H\left[a+\left(\frac{b-a}{3}\right)\right] = \left[a\times H + \frac{(b-a)}{2}\times H\right]\times x$$

Thus, by rearranging the above expression, the distance of line of action of the force due to weight of the dam from the vertical face PS can be calculated.

$$\boxed{x = \frac{(a^2+b^2+ab)}{3(a+b)}} \tag{3.24}$$

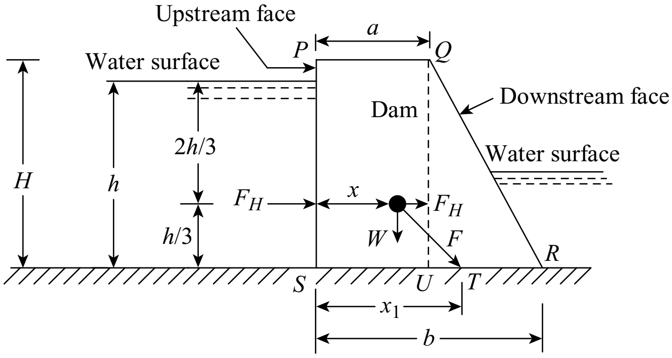

Figure 3.44

For the equilibrium and stability of the dam, the resultant of F_H and W must be balanced by the reaction force R_F at the base. Let the resultant force F pass through a point T at a distance of x_1 from the vertical face PS. The eccentricity of this force is expressed below.

$$e = x_1 - \frac{b}{2} \tag{3.25}$$

The horizontal component F_H ends to slide the dam. The sliding is resisted by the frictional force μW acting between the bottom of the dam and the soil on which it is resting. In order to avoid the sliding of the dam it is taken that $\mu W > F$, where μ is the coefficient of friction.

Example 3.39 A dam with vertical upstream face 17 m high retains water to a depth of 15 m. Determine the total pressure force per metre length due to water on the upstream face of the dam and the location of centre of pressure.

Solution

Refer Figure 3.45. Let $H = 17$ m, $h = 15$ m and $b = 1$ m.

Total water pressure force is given by,

$$F_H = \rho_w gAh_G = 1000 \times 9.81 \times (15 \times 1) \times \frac{15}{2} = \mathbf{1103.625\ kN}$$

$$I_G = \frac{bd^3}{12} = \frac{1 \times 15^3}{12} = 281.25\ \text{m}^4$$

$$h_C = \frac{I_G}{Ah_G} + h_G = \frac{281.25}{15 \times 1 \times (15/2)} + \frac{15}{2} = \mathbf{10\ m}$$

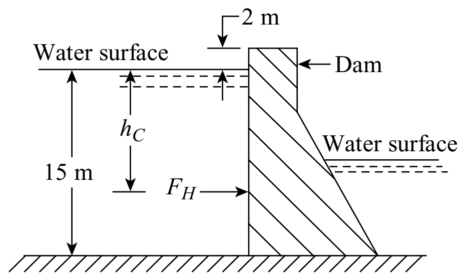

Figure 3.45

Example 3.40 A dam retains water to a depth of 10 m. The face of the dam in contact with water is vertical to 4 m from the top of the dam and thereafter, it is inclined at 60° to the horizontal to increase the thickness of the dam at the base. Determine the total pressure per metre length due to water on the upstream face of the dam.

Solution

Refer Figure 3.46. Let $h = 10$ m, $h_1 = 4$ m, $\alpha = 60°$ and $b = 1$ m.

Horizontal component of the water pressure force is given by,

$$F_H = \rho_w gAh_G = 1000 \times 9.81 \times (10 \times 1) \times \frac{10}{2} = 490.5\ \text{kN}$$

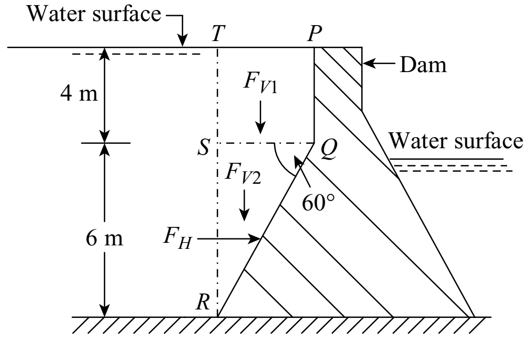

Figure 3.46

Vertical components of the water pressure force is given by,

$$F_{V1} = \text{Weight of water block } PQST = \rho_w \times \text{Volume of } PQST \times g$$

$$F_{V1} = \rho_w \times (bh_1 \times QS) \times g = \rho_w g(bh_1 \times RS \cot 60°)$$

$$\therefore F_{V1} = 1000 \times 9.81 \times (1 \times 4 \times 6 \cot 60°) = 135.93\ \text{kN}$$

$$F_{V2} = \text{Weight of water block } QSR = \rho_w \times \text{ Volume of } QSR \times g$$

$$F_{V2} = \rho_w g \times \frac{1}{2} \times RS \times QS \times b = \rho_w g \times \frac{1}{2} \times RS \times RS \cot 60° \times b$$

$$\therefore F_{V2} = 1000 \times 9.81 \times \frac{1}{2} \times 6 \times 6 \cot 60° \times 1 = 101.95\ \text{kN}$$

Thus, total vertical component of water pressure force is given by,

$$F_V = F_{V1} + F_{V2} = 135.93 + 101.95 = 237.88 \text{ kN}$$

The resultant pressure force is given by,

$$F = \sqrt{F_H^2 + F_V^2} = \sqrt{490.5^2 + 237.88^2} = \mathbf{545.14\ kN}$$

The angle of inclination of resultant with the horizontal is given by,

$$\beta = \tan^{-1}\left(\frac{F_V}{F_H}\right) = \tan^{-1}\left(\frac{237.88}{490.5}\right) = \mathbf{25.87°}$$

3.9 ❐ LOCK GATES

Lock gates are used to change the water level in a canal or a river for the purpose of navigation. Let AB and BC be the two set of gates, one set on either side of the chamber (Figure 3.47). Each gate is supported on two hinges fixed on their top and bottom. Figure 3.47 illustrates the front view and top view of the closed position of the lock gates with its two wings butting each other at B. Let α be the inclination of the lock gate with the normal to the side of the lock and h be the distance between two hinges.

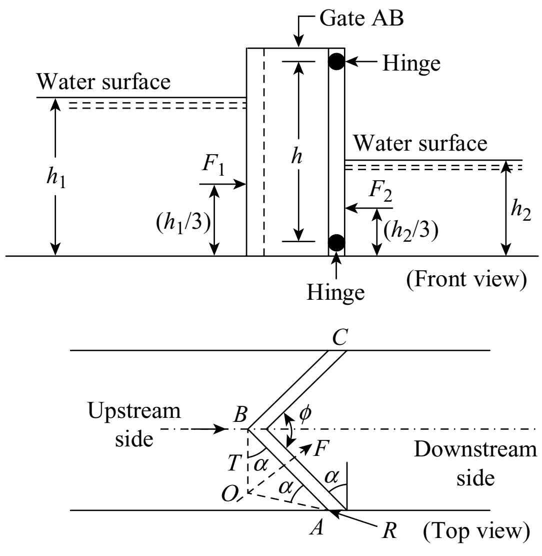

Figure 3.47

Consider the gate AB. Let h_1 and h_2 be the heights of water on the upstream and downstream sides, respectively, F_1 and F_2 be the water pressure forces on the gates and b be the width of the gate.

$$F_1 = \rho_w gA_1h_{G1} = \rho_w g(h_1b)\frac{h_1}{2} = \frac{\rho_w gbh_1^2}{2}$$

The force F_1 acts at $\frac{2h_1}{3}$ below the water surface or at $\frac{h_1}{3}$ from bottom.

$$F_2 = \rho_w gA_2h_{G2} = \rho_w g(h_2b)\frac{h_2}{2} = \frac{\rho_w gbh_2^2}{2}$$

The force F_2 acts at $\frac{2h_2}{3}$ below the water surface or at $\frac{h_2}{3}$ from bottom.

The resultant force F acting at the right angle to the gate is given by,

$$F = F_1 - F_2 = \frac{\rho_w gbh_1^2}{2} - \frac{\rho_w gbh_2^2}{2} \tag{3.26}$$

The other forces acting on each gate are (i) thrust T exerted by the other gate, which is normal to the point of contact B of the gates and (ii) reaction R at the lower and upper hinges, which is given by $R = R_T + R_B$, where R_T is the reaction at the top hinge and R_B is the reaction at the bottom hinge, respectively.

Let the forces T and F intersect at O. Due to equilibrium condition of the gate, the reaction R must also pass through O. Resolving the forces along the gate AB, we get the following expression.

$$T\cos\alpha = R\cos\alpha$$

$$T = R \tag{3.27}$$

Resolving the forces normal to the gate AB, we get:

$$F = T\sin\alpha + R\sin\alpha = 2R\sin\alpha \quad [\because T = R]$$

$$\therefore \boxed{R = T = \frac{F}{2\sin\alpha}} \tag{3.28}$$

After finding the total reaction R at the hinges, the reactions R_T and R_B at the top and bottom hinges can be determined by taking moments about the lower hinge and it is expressed as follows.

$$R_T \times \sin\alpha \times h = \frac{F_1}{2}\times\frac{h_1}{3} - \frac{F_2}{2}\times\frac{h_2}{3} \tag{3.29}$$

Generally, the angle between the two lock gates (ϕ) is given, so angle α can be calculated using $\phi = (180° - 2\alpha)$.

Example 3.41 The end gates of a lock are 5 m high and when closed, it includes an angle of 120°. Each gate is carried on two hinges placed at the top and the bottom of the gate. If the water levels are 4 m and 2 m on the upstream and downstream sides, respectively and the width of the lock is 6.6 m, then find the magnitudes of the forces on the hinges due to the water pressure.

Solution

Refer Figure 3.48. Let $h = 5$ m, $\phi = (180° - 2\alpha) = 120° \Rightarrow \alpha = 30°$, $h_1 = 4$ m, $h_2 = 2$ m and width of lock = 6.6 m. Let b be the width of each gate.

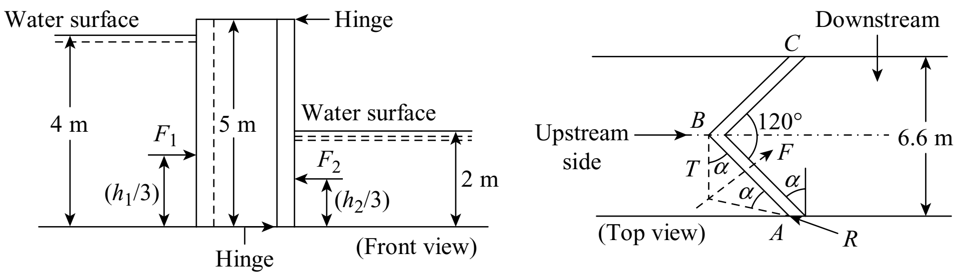

Figure 3.48

$$b = \frac{(6.6/2)}{\cos 30°} = 3.81 \text{ m}$$

$$F_1 = \rho_w g A_1 h_{G1} = 1000 \times 9.81 \times (4 \times 3.81) \times \frac{4}{2} = 299.01 \text{ kN}$$

$$h_{C1} = \frac{4}{3} = 1.333 \text{ m} \quad \text{(From bottom)}$$

$$F_2 = \rho_w g A_2 h_{G2} = 1000 \times 9.81 \times (2 \times 3.81) \times \frac{2}{2} = 74.752 \text{ kN}$$

$$h_{C2} = \frac{2}{3} = 0.667 \text{ m} \quad \text{(From bottom)}$$

$$F = F_1 - F_2 = 299.01 - 74.752 = 224.258 \text{ kN}$$

Let x be the distance of F from the bottom and taking moment of the forces, we get the following expression.

$$F \times x = F_1 \times h_{C1} - F_2 \times h_{C2}$$

$$224.258 \times x = 299.01 \times 1.333 - 74.752 \times 0.667$$

$$\therefore x = \frac{347.6008}{224.26} = 1.555 \text{ m}$$

$$R = T = \frac{F}{2\sin\alpha} = \frac{F}{2\sin 30°} = F$$

$$T = R = F = 224.258 \text{ kN}$$

For determining the reactions R_T and R_B at the top and bottom hinges taking moments about the bottom hinge, we get the following expression.

$$R_T \times 5 = R \times 1.555$$

$$\therefore R_T = \frac{224.258 \times 1.555}{5} = \mathbf{69.744\ kN}$$

$$R_B = R - R_T = 224.258 - 69.744 = \mathbf{154.514\ kN}$$

Summary

1. Hydrostatics deals with the behaviour of fluids at rest.
2. A static mass of fluid when comes in contact with a solid surface (plane or curved), it exerts a normal force on it and it is known as total pressure (p).
3. **Centre of pressure (C):** The point of application of total pressure on the surface.
4. **Centre of gravity or centroid (G):** The point where the whole weight of the surface lies.
5. **Total pressure force on the vertical, horizontal or an inclined plane immersed surface:** $F = \rho g h_G A$, here h_G is the distance of the centre of gravity of the immersed surface from the free surface of the liquid.
6. **The position of the centre of pressure on a plane surface submerged vertically in a static mass of liquid:** $h_C = I_G/(Ah_G) + h_G$, here I_G is the moment of inertia about an axis passing through centre of gravity.
7. For a submerged horizontal plane surface, the points C and G coincides with each other and thus, $h_C = h_G$.
8. The vertical depth of centre of pressure for the inclined surface submerged below the free surface of static liquid: $h_C = I_G \sin^2\alpha/(Ah_G + h_G)$, here α is the inclination of the inclined surface with free liquid surface.
9. **Resultant force on curved surface:** $F = \sqrt{F_H^2 + F_V^2}$, here F_H and F_V are the horizontal and vertical forces on curved surface, respectively. F_H = total pressure force which acts on the imaginary vertical projection of the curved surface on vertical plane and F_V = weight of the liquid supported by the curved surface up to the free liquid surface. The inclination of F with horizontal: $\beta = \tan^{-1}(F_V/F_H)$.
10. A dam is a structure built across a river or a stream to store water whose cross section may be triangular, rectangular or trapezoidal.
11. **Horizontal thrust of water on the dam:** $F_H = (1/2)\rho_w g h^2$, here h is the depth of water stored on the upstream side. F_H acts at a depth of $h_C = 2h/3$ below the free surface.
12. Lock gates are used to change the water level in a canal or a river for the purpose of navigation. The resultant force F acting at the right angle to the gate: $F = F_1 - F_2$, here F_1 and F_2 are the water pressure forces on the gates on the upstream and downstream sides, respectively.
13. **Reaction between two gates:** $R = T = F/(2\sin\alpha)$, here T is the thrust exerted by the other gate and α is the inclination of the lock gate with normal to the side of the lock.

Multiple-choice Questions

1. The point of application of the total pressure on the surface is called
(a) Centre of pressure.
(b) Centroid.
(c) Both (a) and (b).
(d) None of the above.

2. Depth of the centre of pressure from the free surface of a liquid on a vertical wall in terms of vertical height (h) is given by
(a) h.
(b) $h/2$.
(c) $(2h)/3$.
(d) All the above.

3. The hydrostatic pressure force on a plane surface in terms of density of liquid (ρ), surface area (A), acceleration due to gravity (g), distance of centre of gravity from the free surface of liquid (h_G) is given by
(a) $F = (1/2)\rho g A h_G$.
(b) $F = \rho g A h_G$.
(c) $F = (2/3)\rho g A h_G$.
(d) None of the above.

4. Centre of pressure of a plane surface immersed in water lies
(a) At the centre of gravity of the plane surface.
(b) Above the centre of gravity of the plane surface.
(c) Below the centre of gravity of the plane surface.
(d) None of the above.

5. When a vertical wall is subjected to pressure due to water on both sides, the resultant pressure is due to
(a) Arithmetic mean of pressures.
(b) Sum of pressures.
(c) Logarithmic mean of pressures.
(d) Difference of pressures.

6. The water pressure per metre of length on a vertical wall of a gravity dam for the depth of water stored (h) is given by
(a) $F_H = (1/2)\rho_w g h^2$.
(b) $F_H = (1/3)\rho_w g h^2$.
(c) $F_H = (2/3)\rho_w g h^2$.
(d) None of the above.

7. For a submerged curved surface, the vertical component of the hydrostatic force is equal to
(a) Force on the projected area of the curved surface on vertical plane.
(b) Mass of the liquid supported by the curved surface.
(c) Weight of the liquid supported by the curved surface.
(d) None of the above.

8. For a submerged curved surface, the horizontal component of the hydrostatic force is equal to
(a) Force on the projected area of the curved surface on vertical plane.
(b) Mass of the liquid supported by the curved surface.
(c) Weight of the liquid supported by the curved surface.
(d) None of the above.

9. Lock gate is used to
(a) Store water.
(b) To divert water.
(c) Alter water level for navigation.
(d) All the above.

10. The possibility of dam failure may be due to
(a) Sliding.
(b) Tension or compression.
(c) Overturning.
(d) All the above.

Review Questions

1. Define the terms (i) hydrostatics, (ii) total pressure and (iii) centre of pressure.

2. Define first moment and second moment of areas.

3. Derive expressions for total pressure and centre of pressure for a vertical submerged surface.

4. Derive expressions for total pressure and centre of pressure for an inclined plane submerged surface.

5. Determine from the first principle, the horizontal and vertical components of the total pressure force on a vessel which is a quadrant of a circle of radius r.

6. Derive an expression for resultant hydrostatic force on a curved surface.

7. Derive an expression for the reaction between the gates as $R = T = F/(2\sin\alpha)$, where T is the thrust exerted by the other gate and α is the inclination of the lock gate with the normal to the side of the lock.

Problems

1. A rectangular plate 2 m × 4 m is submerged vertically in water such that the 2 m side is parallel to the water surface. Determine the hydrostatic force and the centre of pressure if the top edge of the surface is (a) flush with the water surface and (b) 1 m below the water surface.
[**Ans.** (a) 156.96 kN, 2.67 m (b) 235.44 kN, 3.44 m]

2. A rectangular plate 2 m × 1 m is held in water at a depth of 1 m below the free water surface (i) if 2 m height is vertical, then find the total pressure force on the plate and depth of centre of pressure, (ii) if 1 m side lies in vertical plane at the same depth, then find the change in total pressure force acting and depth of centre of pressure, and (iii) another circular plate having the same area of rectangle is also kept at 1 m below the free water surface. Also determine the total pressure force acting on the circular plate and centre of pressure.
[**Ans.** (i) 39.24 kN, 2.17 m, (ii) 29.43 kN, 1.55 m, 9.81 kN, 0.62 m and (iii) 35.32 kN, 1.889 m]

3. A tank has a square opening on one of its vertical sides. The opening is such that one of its diagonal is vertical. It is closed by a plate. Find the thrust exerted by the oil (specific gravity = 0.9) stored in the tank and the position of its centre of pressure, if the diagonal of the square opening is 0.6 m and the centre of the opening is 0.4 m below the oil surface.
[**Ans.** 635.69 N, 0.4375 m]

4. An isosceles triangular lamina of 2 m base and 4 m height is immersed vertically in water. Determine the total pressure on the plate and the centre of pressure when (i) base of the lamina coincides with the free surface of water, (ii) base of the lamina is 8 m below the water surface (apex above the base), and (iii) axis of symmetry passing through the apex being horizontal and 8 m below the water surface.
[**Ans.** (i) 52.32 kN, 2 m, (ii) 261.6 kN, 6.8 m, and (iii) 313.92 kN, 8.021 m]

5. A circular plate of diameter 400 mm is placed vertically in water in such a way that the centre of plate is 2 m below the free surface of water. Determine the total pressure and the position of centre of pressure.
[**Ans.** 2.466 kN, 2.005 m]

6. An isosceles triangular lamina of base 3 m and altitude 3 m is immersed vertically in water. If base of the plate coincides with the free surface of water determine the total pressure and centre of pressure.
[**Ans.** 44.145 kN, 1.5 m]

7. A square aperture in the vertical side of a tank has one diagonal vertical and is completely covered by a plane plate hinged along one of the upper sides of the aperture. The diagonals of the aperture are 2.4 m long and the tank contains a liquid of specific gravity 1.2. The centre of the aperture is 1.8 m below the free surface. Determine the thrust exerted on the plate by the liquid and the position of its centre of pressure.
[**Ans.** 61.03 kN, 1.93 m]

8. Figure 3.14(a) illustrates a tank containing water and oil (specific gravity = 0.9) up to a height of 0.25 m and 0.5 m, respectively. Determine the total pressure on the side of the tank and the position of centre of pressure from one side of the tank which is 1.5 m wide.
[**Ans.** 3.77 kN, 0.502 m from the top]

9. An opening 2 m wide and 1.2 m high in a dam is covered by the use of a vertical sluice gate. On the upstream of the gate a liquid (specific gravity = 1.45) is available up to a height of 1.5 m above the top of the gate, whereas on the downstream side the water is available up to a height touching the top of the gate. If the gate is hinged at the bottom, then determine (i) the resultant force acting on the gate and the position of centre of pressure and (ii) the force acting horizontally at the top of the gate which is capable of opening it.
[**Ans.** (i) 57.56 kN, 0.578 m above the hinge and (ii) 27.72 kN]

10. A rectangular plate 1.2 m deep and 0.6 m wide is immersed in water. The minimum and maximum depths of the plate are 0.75 m and 1.6 m from the free surface. Determine the hydrostatic force on one face of the plate and the depth of centre of pressure.
[**Ans.** 7.95 kN, 1.167 m]

11. Figure 3.49 shows a circular opening in the sloping wall of a reservoir which is closed by 1 m diameter disc valve. The side is hinged at point P and a balance weight W is just sufficient to hold the valve close when the reservoir is empty. Determine (i) the total force on the valve and (ii) the additional weight required to be placed in order that the valve remains closed until the water level is 0.8 m above the centre of the valve.
[**Ans.** 6.164 kN, 2495.89 N]

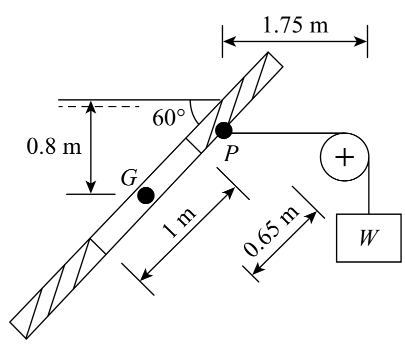

Figure 3.49

12. Figure 3.50 shows an opening in a dam which is closed by a 1 m square plate hinged at the upper horizontal edge. The weight of the plate is 1000 N. The plate is inclined at an angle of 60° to the horizontal and its top edge is 2 m below the water surface in the reservoir. If this plate is pulled by means of a chain attached to the centre of the lower edge, then determine

the necessary pull in the chain. The line of action of the chain makes an angle of 45° with the plate.

[**Ans.** 18.243 kN]

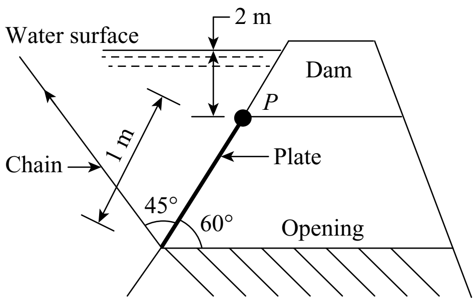

Figure 3.50

13. A rectangular door covering an opening 3 m wide and 2 m high in a vertical wall is hinged about its vertical edge by two pivots placed symmetrically 250 mm from either end. The door is locked by a clamp placed at the centre of the vertical edge. The height of water is 1.5 m above the top edge of the opening. Find (i) the total pressure on the door, (ii) the position of centre of pressure and (iii) the reactions at the two hinges and the clamp.

[**Ans.** (i) 147.15 kN, (ii) 2.633 m and (iii) 49.835 kN, 23.74 kN]

14. A circular plate 2.5 m in diameter is submerged in a liquid (specific gravity = 1.42), its least and greatest depths below the free surface are 1 m and 3 m, respectively. Determine (i) the total pressure on one face of the plate and (ii) the position of centre of pressure.

[**Ans.** 136.77 kN, 2.125 m]

15. A rectangular opening which is 3 m long and 1.2 m high in the vertical side of a reservoir is closed by a plate with the help of four bolts fixed at the corners of the opening. The water is stored up to a height of 1.6 m above the top edge of the horizontal opening. Determine (i) the total pressure on the plate, (ii) the position of centre of pressure and (iii) tension in the bolts.

[**Ans.** (i) 77.69 kN, 2.2545 m, 21.187 kN, 17.658 kN]

16. An annular plate of external diameter 3 m and internal diameter 1.5 m is submerged in an oil (specific gravity = 0.9) with the least and greatest depths below oil surface as 1.2 m and 3.6 m, respectively. Determine the total pressure and the position of the centre of pressure on one face of the plate.

[**Ans.** 112.305 kN, 2.59 m]

17. Find the horizontal and vertical components of water pressure acting on the face of a tainter gate *PQ* of 90° sector of radius 5 m and width 1 m (Figure 3.51).

[**Ans.** 245.176 kN, 69.994 kN]

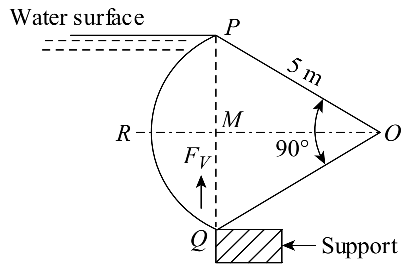

Figure 3.51

18. Figure 3.52 shows a gate *PQ* of quadrant shape with 2 m radius supporting water. If the gate is 3 m long and the height of the water above the lowest point of the gate is 5 m, then find the total resultant pressure force acting on the gate and its relative direction with the horizontal surface.

[**Ans.** 357.51 kN, 48.81°]

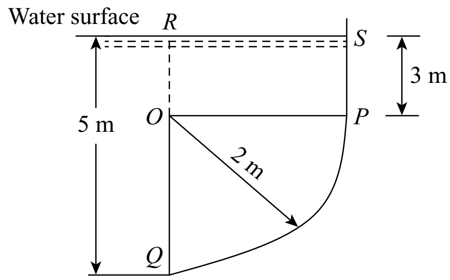

Figure 3.52

19. The curved shape of a dam is given by the relation $y = (x^2/16)$ If the width of dam is unity and the height of water is stored up to 20 m, then calculate the magnitude and direction of the resultant water pressure acting over the curved surface.

[**Ans.** 3053.55 kN, 50.02°]

20. Each gate of a lock is 5 m high and 4 m wide which is supported on one side by two hinges, each 0.5 m from the top and from bottom. The angle between the gates in closed position is 120°. The water levels are 4 m and 1 m on the upstream and downstream sides, respectively and the reaction between the gates to be in the same horizontal plane as that of the resultant water pressure. Determine (i) the magnitude and position of the resultant water pressure on each gate and (ii) the magnitudes of the reactions at the hinges.

[**Ans.** (i) 294.3 kN, 1.4 m (ii) 52.97 kN, 241.33 kN]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (a)	**2.** (c)	**3.** (b)	**4.** (c)	**5.** (d)
6. (a)	**7.** (c)	**8.** (a)	**9.** (c)	**10.** (d)

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Chapter 4

Liquids in Relative Equilibrium

4.1 ❒ INTRODUCTION

For a liquid to be in absolute rest or equilibrium there must be no relative motion between the particles of a liquid or between the liquid and the vessel containing it. If there is any relative motion between the liquid and the vessel but there is no relative motion among the liquid particles and the entire liquid moves as a single unit, then that liquid is said to be in a state of relative rest or relative equilibrium. In such cases, there is no relative motion between the liquid particles and thus, there is no shear stress. Therefore, the liquid pressure is normal everywhere to the surface on which it acts. Hence, hydrostatic law can be used to evaluate the liquid pressure by taking into account the effect of acceleration.

A liquid contained in a container may be subjected to horizontal acceleration, vertical acceleration and radial acceleration. D'Alembert's principle states that a moving liquid mass may be brought to a static equilibrium position by applying an imaginary inertia force of the same magnitude as that of the accelerating force but in opposite directions. In this chapter, liquid in a container subjected to uniform acceleration in the horizontal and vertical directions and liquid in a container subjected to constant rotation under relative equilibrium conditions are explained briefly.

4.2 ❒ LIQUID CONTAINERS SUBJECTED TO CONSTANT HORIZONTAL ACCELERATION

Consider an open tank partly filled with liquid at absolute rest, the free surface of which is horizontal. Let the tank be subjected to a horizontal acceleration a towards right as shown in Figure 4.1(a). Due to acceleration, the liquid mass redistributes itself to the upwards slope in the direction opposite to that of horizontal acceleration, i.e., the liquid falls at the front end and rise at the back end of the container as shown in Figure 4.1(b). The fuel tank in an aeroplane during take-off is a good example of a liquid in a container subjected to a constant horizontal acceleration.

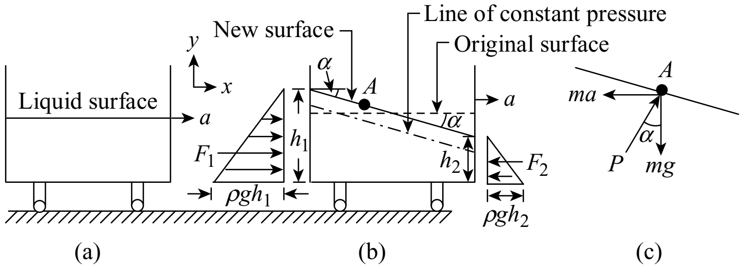


Figure 4.1 *Liquid under constant horizontal acceleration*

An equation for the free liquid surface can be written by considering the equilibrium of a fluid particle A on the free surface. As there is no relative motion between particles, there exists no shear stress. Thus, the fluid particle A is subjected to the following forces (Figure 4.1(c)), (i) pressure force P exerted by the surrounding fluid particles which acts normal to the free surface, (ii) accelerating force which is equal to ma acting horizontally in a direction opposite to the direction of acceleration and (iii) gravitational force which is equal to the weight of the fluid element (mg) acting vertically downward. Resolving the forces horizontally and vertically, we get the following expressions.

$$P \sin \alpha = ma \tag{i}$$

$$P \cos \alpha = mg \tag{ii}$$

Dividing expression (i) by expression (ii), we get:

$$\boxed{\tan \alpha = \frac{a}{g}} \tag{4.1}$$

The term (a/g) is constant at all points on the free surface and thus, $\tan \alpha$ is constant. Therefore, the free surface is a straight plane inclined down at α along the direction of acceleration.

To find the pressure intensity at any point, say at depth h from the free liquid surface consider a fluid element of cross sectional area dA. The forces acting on the fluid element are (i) atmospheric pressure force ($p_{atm}dA$) acting downwards, (ii) pressure force (pdA) acting upwards and (iii) weight of the element ($\rho ghdA$) acting downwards. From the equilibrium, we get:

$$pdA - p_{atm}dA - \rho ghdA = 0$$

$$p = p_{atm} + \rho gh \tag{4.2}$$

If pressure is written in gauge units, (i.e., $p_{atm} = 0$), then we have the following expression.

$$p = \rho gh \tag{4.2a}$$

Therefore, the pressure distribution is same as hydrostatic pressure distribution. Thus, the lines of constant pressure are parallel to the inclined liquid surface as shown in Figure 4.1(b). Let h_1 and h_2 be the depths of liquid at the rear and front ends of the container, respectively, h_G be the depth of centre of gravity from free surface, b be the width of the container and m be the total mass of liquid. The total pressure force exerted on the rear and front ends are respectively given by the following expressions.

$$F_1 = \rho gAh_G = \rho g \times (bh_1) \times \frac{h_1}{2} = \frac{\rho gbh_1^2}{2} \tag{4.3}$$

$$F_2 = \rho gAh_G = \rho g \times (bh_2) \times \frac{h_2}{2} = \frac{\rho gbh_2^2}{2} \tag{4.4}$$

Therefore, net force is given by,

$$F = F_1 - F_2 \tag{4.5}$$

Thus, according to Newton's second law, we get the following expression.

$$ma = F_1 - F_2 \tag{4.5a}$$

For a closed container completely filled with liquid and subjected to horizontal acceleration, there will be no adjustment in the liquid surface. The pressure building up at the rear end will be greater than that at the front end. The slope of the constant pressure lines will be governed by Equation (4.1). A portion of liquid will spill out from an open container filled with liquid subjected to horizontal acceleration and a new free surface will form.

Example 4.1 A rectangular water tank moves horizontally in the direction of its length with a constant acceleration of 2.2 m/s^2. The tank is 7 m long, 2 m wide, 2 m deep and it contains water to a depth of 1 m. Determine (i) slope of the free surface, (ii) maximum and minimum pressure intensities at bottom, (iii) total force acting at the front and back ends of the tank, (iv) net force due to water acting on each end of the tank and (v) volume of water getting spilt, if the tank is completely filled.

Solution

Refer Figure 4.2(a). Let $a = 2.2\ \text{m/s}^2,\ l = 7\ \text{m},\ b = 2\ \text{m},\ d = 2\ \text{m}$ and $h = 1\ \text{m}$.

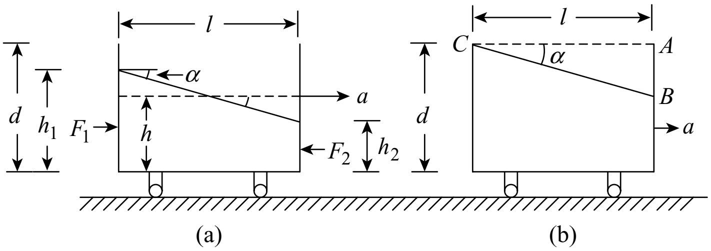


Figure 4.2

(i) $\alpha = \tan^{-1}\left(\frac{a}{g}\right) = \tan^{-1}\left(\frac{2.2}{9.81}\right) = \mathbf{12.64°}$

(ii) $h_1 = h + \frac{l}{2}\tan\alpha = h + \frac{l}{2}\times\frac{a}{g} = 1 + \frac{7}{2}\times\frac{2.2}{9.81} = 1.785\ \text{m}$

$h_2 = h - \frac{l}{2}\tan\alpha = h - \frac{l}{2}\times\frac{a}{g} = 1 - \frac{7}{2}\times\frac{2.2}{9.81} = 0.215\ \text{m}$

Maximum pressure intensity occurs at the bottom of the back end as given by,

$$p_{max} = \rho_w g h_1 = 1000 \times 9.81 \times 1.785 = \mathbf{17510.85\ N/m^2}$$

Minimum pressure intensity occurs at the bottom of the front end as given by,

$$p_{min} = \rho_w g h_2 = 1000 \times 9.81 \times 0.215 = \mathbf{2109.15\ N/m^2}$$

(iii) Force at the back end is given by,

$$F_1 = \frac{\rho_w g b h_1^2}{2} = \frac{1000 \times 9.81 \times 2 \times 1.785^2}{2} = \mathbf{31256.87\ N}$$

Force at the front end is given by,

$$F_2 = \frac{\rho_w g b h_2^2}{2} = \frac{1000 \times 9.81 \times 2 \times 0.215^2}{2} = \mathbf{453.47\ N}$$

(iv) Net force is given by,

$$F = F_1 - F_2 = 31256.87 - 453.47 = \mathbf{30803.4\ N}$$

(v) When the tank is filled completely, then the water surface after spilling would be as shown in Figure 4.2(b). Drop in water level at the front end will be equal to distance *AB* and is given by the following expression.

$$AB = l\tan\alpha = l \times \frac{a}{g} = 7 \times \frac{2.2}{9.81} = 1.57\ \text{m}$$

Volume of water spilt is given by,

$$v_s = b \times \text{area}(\Delta CAB) = 2 \times \frac{1}{2} \times 7 \times 1.57 = \mathbf{10.99\ m^3}$$

Example 4.2 A water tank is 3 m square and contains 1 m of water. How high must its sides be if no water is to be spilt when the uniform horizontal acceleration is 5 m/s^2.

Solution

Refer Figure 4.3. Let $l = b = 3$ m, $h = 1$ m and $a = 5$ m/s^2.

Rise or fall in water surface, i.e., $AB = CD$ is given by,

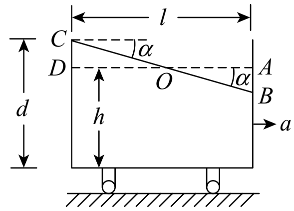


Figure 4.3

$$AB = CD = \frac{l}{2}\tan\alpha = \frac{l}{2} \times \frac{a}{g} = \frac{3}{2} \times \frac{5}{9.81} = 0.7645 \text{ m}$$

Thus, the height of the tank must be at least,

$$d = h + CD = 1 + 0.7645 = \mathbf{1.7645\ m}$$

Example 4.3 An open water tank which is 8 m long, 4 m wide and 2 m deep containing water up to a depth of 1.4 m of water is uniformly accelerated from rest to 10 m/s. Determine the shortest time in which the tank may be accelerated without the water spilling over the edge.

Solution

Refer Figure 4.3. Let $l = 8$ m, $b = 4$ m, $d = 2$ m, $h = 1.4$ m and $v = 10$ m/s. Let t be the shortest time.

$$AB = CD = d - h = 2 - 1.4 = 0.6 \text{ m}$$

$$a = g\tan\alpha = g \times \frac{CD}{OD} = 9.81 \times \frac{0.6}{4} = 1.4715$$

$$t = \frac{v-u}{a} = \frac{10-0}{1.4715} = \mathbf{6.796\ s} \qquad [\because v = u + at]$$

Example 4.4 A spherical water tank of radius 2 m is half filled and it is given a horizontal acceleration of 6 m/s^2. Determine the inclination of the water surface to the horizontal and maximum pressure on the tank.

Solution

Let $r = 2$ m and $a = 6$ m/s^2.

$$\alpha = \tan^{-1}\left(\frac{a}{g}\right) = \tan^{-1}\left(\frac{6}{9.81}\right) = \mathbf{31.45°}$$

The maximum pressure acting on the boundary where depth is maximum, i.e., at radius is given by the following expression.

$$\therefore p_{\max} = \rho_w gh = 1000 \times 9.81 \times 2 = \mathbf{19.62\ kN}$$

Example 4.5 A rectangular water tank which is 5 m long, 3 m wide and 2.5 m deep contains water to a depth of 2 m. Determine the horizontal acceleration which should be imparted to the tank in the direction of its length so that (i) spilling of water from the tank is just on the verge of taking place, (ii) the front bottom corner of the tank is just exposed and (iii) the bottom of the tank is exposed up to its midpoint. Also determine the total forces on each end of the tank in each case.

Solution

Refer Figure 4.4. Let $l = 5$ m, $b = 3$ m, $d = 2.5$ m and $h = 2$ m. Let a be the required horizontal acceleration to be imparted to the tank, F_1 be the force acting on the back end, F_2 be the force acting on the front end and F be the net force.

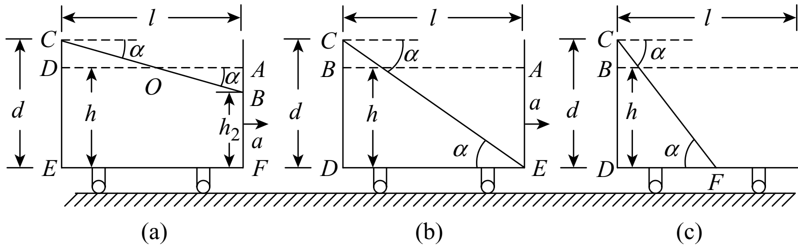


Figure 4.4

(i) Refer Figure 4.4(a).

$$h_2 = h - (d - h) = 2 - (2.5 - 2) = 1.5 \text{ m}$$

$$\tan\alpha = \frac{CD}{OD} = \frac{d - h}{(l/2)} = \frac{2.5 - 2}{(5/2)} = 0.2$$

$$a = g\tan\alpha = 9.81 \times 0.2 = \mathbf{1.962\ m/s^2}$$

$$F_1 = \frac{\rho_w g b d^2}{2} = \frac{1000 \times 9.81 \times 3 \times 2.5^2}{2} = \mathbf{91.97\ kN}$$

$$F_2 = \frac{\rho_w g b h_2^2}{2} = \frac{1000 \times 9.81 \times 3 \times 1.5^2}{2} = \mathbf{33.11\ kN}$$

$$F = F_1 - F_2 = 91.97 - 33.11 = \mathbf{58.86\ kN}$$

(ii) Refer Figure 4.4(b).

$$\tan\alpha = \frac{CD}{DE} = \frac{d}{l} = \frac{2.5}{5} = 0.5$$

$$a = g\tan\alpha = 9.81 \times 0.5 = \mathbf{4.905\ m/s^2}$$

$$F_1 = \frac{\rho_w g b d^2}{2} = \frac{1000 \times 9.81 \times 3 \times 2.5^2}{2} = \mathbf{91.97\ kN}$$

Since there is no water against face AE, we derive the following expression.

$$F_2 = \mathbf{0}$$

$$F = F_1 - F_2 = 91.97 - 0 = \mathbf{91.97\ kN}$$

(iii) Refer Figure 4.4(c).

$$\tan\alpha = \frac{CD}{DF} = \frac{d}{(l/2)} = \frac{2.5}{2.5} = 1$$

$$a = g\tan\alpha = 9.81 \times 1 = \mathbf{9.81\ m/s^2}$$

$$F_1 = \frac{\rho_w g b d^2}{2} = \frac{1000 \times 9.81 \times 3 \times 2.5^2}{2} = \mathbf{91.97\ kN}$$

Since no water against face *AE*, we get the following expression.

$$F_2 = \mathbf{0}$$

$$F = F_1 - F_2 = 91.97 - 0 = \mathbf{91.97\ kN}$$

Example 4.6 A closed rectangular water tank which is 6 m long, 2 m wide and 1.6 m deep contains water to a depth of 1 m and its top has an opening in the front part to have air space at atmospheric pressure. Determine the total pressure force on the top of the tank when it is given a constant horizontal acceleration of 2.5 m/s^2 along its length.

Solution

Refer Figure 4.5. Let $l = 6$ m, $b = 2$ m, $d = 1.6$ m, $h = 1$ m and $a = 2.5$ m/s^2.

Let *F* be the force acting on the top of the tank.

$$\tan\alpha = \frac{a}{g} = \frac{2.5}{9.81} = 0.255$$

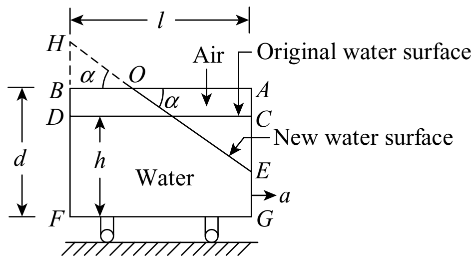


Figure 4.5

Volume of air in triangle *OAE* = Volume of air in rectangle *ABDC*

$$\frac{1}{2} OA \times OA \tan\alpha \times b = l \times b \times (d - h)$$

$$[\because AE = OA \tan\alpha]$$

$$\frac{1}{2}(OA)^2 \times 0.255 \times 2 = 6 \times 2 \times (1.6 - 1)$$

$$\therefore OA = \sqrt{\frac{7.2}{0.255}} = 5.314 \text{ m}$$

$$BO = l - OA = 6 - 5.314 = 0.686 \text{ m}$$

$$HB = BO \tan\alpha = 0.686 \times 0.255 = 0.175 \text{ m}$$

The pressure on the top of the tank is shown by the imaginary water weight in triangle *HBO* extending over the width which is given by the following expression.

$$F = \rho_w g \times \frac{1}{2} BO \times HB \times b$$

$$\therefore F = 1000 \times 9.81 \times \frac{1}{2} \times 0.686 \times 0.175 \times 2 = \mathbf{1177.69\ N}$$

4.3 ❒ LIQUID CONTAINERS SUBJECTED TO CONSTANT VERTICAL ACCELERATION

Consider an open tank containing liquid and moving vertically upward with a constant acceleration '*a*'. The free liquid surface in the tank will remain horizontal but the pressure intensity at any point in the liquid will be different from that when the tank is stationary. An equation for pressure distribution can be obtained by considering the equilibrium of forces acting on an imaginary elementary prism of height *h* and cross-sectional area *dA* as shown in Figure 4.6(a).

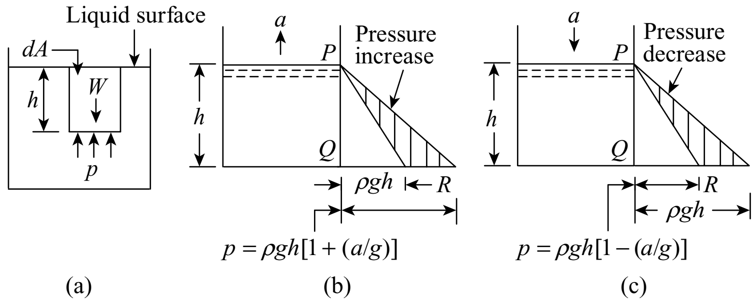


Figure 4.6

The forces acting on the elementary prism are pressure force equal to pdA acting upwards and weight W of the prism element equal to $\rho ghdA$ acting downwards. By applying Newton's second law of motion, according to which the net force in the vertical upward direction will be equal to the product of mass and acceleration. Thus, the expression is derived as given below.

$$\text{Pressure force on element} - \text{Weight of element} = \text{Mass} \times \text{Acceleration}$$

$$pdA - \rho ghdA = \rho hdA \times a$$

$$pdA = \rho ghdA + \rho hdA \times a$$

$$\boxed{p = \rho gh\left[1 + \frac{a}{g}\right]} \tag{4.6}$$

Equation (4.6) shows that the pressure variation is linear and it is greater than static pressure ρgh by an amount $[(\rho gha)/g]$. It is also shown in Figure 4.6(b).

If the tank accelerates in downward direction, then the pressure variation is given by the following expression.

$$\boxed{p = \rho gh\left[1 - \frac{a}{g}\right]} \tag{4.7}$$

Equation (4.7) indicates that the pressure intensity is lower than static pressure ρgh by an amount $[(\rho gha)/g]$ as shown in Figure 4.6(c).

When the tank is lowered vertically at the gravitational acceleration, then a becomes equal to g and Equation (4.7) becomes $p = 0$. It means that pressure throughout the liquid mass is same and it is equal to that of the surrounding atmosphere. It is evident that there is no force on the walls or at the base of the tank.

Example 4.7 A cubical water tank of side 2 m contains water to a depth of 1.5 m. Determine the force acting on the side of the tank when (i) it is accelerated vertically upward at 5.5 m/s^2 and (ii) it is accelerated vertically downward at 5.5 m/s^2.

Solution

Let $l = b = d = 2$ m, $h = 1.5$ m and $a = 5.5$ m/s^2.

(i) Refer Figure 4.6(b).

When the water tank accelerates upwards, the pressure intensity is derived as follows.

$$p = \rho_w gh\left[1 + \frac{a}{g}\right] = 1000 \times 9.81 \times 1.5 \times \left(1 + \frac{5.5}{9.81}\right) = 22965 \text{ N/m}^2$$

Force on the side PQ is given by,

$$F = p_{av} \times A = \frac{(QR + 0)}{2} \times PQ \times b \qquad [\because \text{pressure at point P} = 0]$$

$$\therefore F = \frac{22965}{2} \times 1.5 \times 2 = \mathbf{34447.5\ N}$$

(ii) Refer Figure 4.6(c).
When the water tank accelerates downwards, the pressure intensity is derived as follows.

$$p = \rho_w gh\left[1-\frac{a}{g}\right] = 1000\times 9.81\times 1.5\times\left(1-\frac{5.5}{9.81}\right) = 6465 \text{ N/m}^2$$

Force on the side PQ is given by,

$$F = p_{av}\times A = \frac{QR}{2}\times PQ\times b = \frac{6465}{2}\times 1.5\times 2 = \mathbf{9697.5\ N}$$

Example 4.8 An open rectangular tank which is 4 m long and 3 m wide contains a liquid (specific gravity = 0.9) up to a depth of 1.5 m. Determine the total force acting on the base of the tank when (i) it is moving vertically upward with an acceleration of $(g/2)$ m/s^2 and (ii) it is moving vertically downwards with an acceleration of $(g/2)$ m/s^2.

Solution

Let $l = 4$ m, $b = 3$ m, $S = 0.9$, $h = 1.5$ m and $a = (g/2)$ m/s^2.

(i) When the water tank is accelerated upwards then pressure intensity is given by,

$$p = S\rho_w gh\left[1+\frac{a}{g}\right] = 0.9\times 1000\times 9.81\times 1.5\times\left[1+\frac{(g/2)}{g}\right] = 19865.25 \text{ N/m}^2$$

Force on the base of the tank is given by,

$$F = p\times \text{base area} = 19865.25\times 4\times 3 = \mathbf{238.383\ kN}$$

(ii) When the water tank is accelerated downwards then pressure intensity is given by,

$$p = S\rho_w gh\left[1-\frac{a}{g}\right] = 0.9\times 1000\times 9.81\times 1.5\times\left[1-\frac{(g/2)}{g}\right] = 6621.75 \text{ N/m}^2$$

Force on the base of the tank is given by,

$$F = p\times \text{base area} = 6621.75\times 4\times 3 = \mathbf{79.461\ kN}$$

4.4 ❒ LIQUID CONTAINERS SUBJECTED TO CONSTANT ACCELERATION ALONG INCLINED PLANE

Consider an open tank containing liquid and moving upwards along the inclined plane with a constant acceleration a as illustrated in Figure 4.7.

Let β be the inclination of the direction of acceleration with the horizontal surface, then the horizontal and vertical components of acceleration are derived as given below.

$$a_x = a\cos\beta \tag{4.8}$$

$$a_y = a\sin\beta \tag{4.9}$$

Let α be the angle of the liquid surface with the horizontal after the redistribution of the liquid. The liquid surface falls on the front side and rises up on the back end of the tank.

An equation for the free liquid surface can be written by considering the equilibrium of a fluid particle M on the free surface. As there is no relative motion between the particles, there exists no shear stress. Thus, the fluid particle M is subjected to the following forces (Figure 4.7(b)), such as (i) pressure force p exerted by the surrounding fluid particles which acts normal to the fluid element, (ii) accelerating forces, i.e., ma_x acting horizontally and ma_y acting vertically

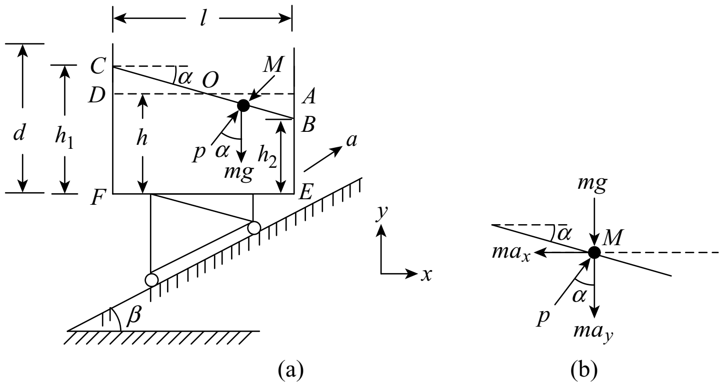


Figure 4.7

downwards and (iii) gravitational force, that is equal to the weight of the fluid element (mg) acting vertically downward. Resolving the forces horizontally and vertically, we have the following expressions.

$$p\sin\alpha = ma_x \tag{i}$$

$$p\cos\alpha = ma_y + mg \tag{ii}$$

Dividing expression (i) by expression (ii), we get:

$$\boxed{\tan\alpha = \frac{a_x}{g+a_y}} \tag{4.10}$$

Similarly, when the vessel is moving in the downward direction then Equation (4.10) is derived as given below.

$$\boxed{\tan\alpha = \frac{a_x}{g-a_y}} \tag{4.11}$$

Example 4.9 An open rectangular tank which is 6 m long and 3 m wide contains water up to a depth of 2.5 m. Determine the angle made by the free surface of the liquid with the horizontal when the tank moves with an acceleration of 4.5 m/s^2 (i) up towards 30° inclined plane and (ii) down towards 30° inclined plane.

Solution

Refer Figure 4.7. Let $l = 6$ m, $b = 3$ m, $h = 2.5$ m, $a = 4.5$ m/s^2 and $\beta = 30°$.

Let α be the slope of the free liquid surface, a_x and a_y be the horizontal and vertical components of acceleration, respectively.

$$a_x = a\cos\beta = 4.5\cos 30° = 3.897 \text{ m/s}^2$$

$$a_y = a\sin\beta = 4.5\sin 30° = 2.25 \text{ m/s}^2$$

(i) For the tank moving up the inclined plane with constant acceleration, we have the following expression.

$$\alpha = \tan^{-1}\left(\frac{a_x}{g+a_y}\right) = \tan^{-1}\left(\frac{3.897}{9.81+2.25}\right) = \mathbf{17.91°}$$

(ii) For the tank moving down the inclined plane with constant acceleration, we have the following expression.

$$\alpha = \tan^{-1}\left(\frac{a_x}{g-a_y}\right) = \tan^{-1}\left(\frac{3.897}{9.81-2.25}\right) = \mathbf{27.27°}$$

Example 4.10 An open rectangular tank which is 4 m long and 2 m wide contains a liquid (specific gravity = 0.9) up to a depth of 1.5 m. The tank is moving up towards a 30° inclined plane with a constant acceleration of 5 m/s^2. Determine (i) the angle made by the free surface of the liquid with the horizontal and (ii) pressure at the bottom of the tank at the front and back ends.

Solution

Refer Figure 4.7. Let $l = 4$ m, $b = 2$ m, $S = 0.9$, $h = 1.5$ m, $\beta = 30°$ and $a = 5$ m/s^2.

Let h_2 be the height of liquid at front end, h_1 be the height of liquid at back end, p_2 be the pressure at the bottom of the tank at front end and p_1 be the pressure at the bottom of the tank at back end.

$$a_x = a\cos\beta = 5\cos 30° = 4.33 \text{ m/s}^2$$

$$a_y = a\sin\beta = 5\sin 30° = 2.5 \text{ m/s}^2$$

(i) When the water tank is accelerated upwards, we derive as follows.

$$\tan\alpha = \frac{a_x}{g + a_y} = \frac{4.33}{9.81 + 2.5} = 0.35175$$

$$\therefore \alpha = \tan^{-1}(0.35175) = \mathbf{19.38°}$$

(ii) $CD = OD\tan\alpha = \frac{4}{2} \times 0.35137 = 0.7035$ m

$$\therefore h_1 = h + CD = 1.5 + 0.7035 = 2.2035 \text{ m}$$

$$h_2 = h - CD = 1.5 - 0.7035 = 0.7965 \text{ m}$$

Since $p_2 = S\rho_w g h_2\left(1 + \frac{a_y}{g}\right)$

$$\therefore p_2 = 0.9 \times 1000 \times 9.81 \times 0.7965 \times \left(1 + \frac{2.5}{9.81}\right) = \mathbf{8.8244\ kN/m^2}$$

Since $p_1 = S\rho_w g h_1\left(1 + \frac{a_y}{g}\right)$

$$\therefore p_1 = 0.9 \times 1000 \times 9.81 \times 2.2035 \times \left(1 + \frac{2.5}{9.81}\right) = \mathbf{24.4126\ kN/m^2}$$

4.5 ❒ LIQUID CONTAINERS SUBJECTED TO CONSTANT ROTATION

Consider an open cylindrical container partly filled with liquid at absolute rest whose free surface is horizontal. Let the container along with the liquid be subjected to rotation about its vertical axis with a constant angular velocity ω. The shape of the free surface of liquid becomes concave. This phenomenon is due to the liquid rising above the original surface near the walls and it falls down below the original free surface at the centre of the container. When a steady state of rotation is reached, the liquid attains equilibrium condition and it rotates as a solid mass with the container at the same angular velocity.

In Figure 4.8(a), the path *AB* shows the free liquid surface before rotation and $A'OB'$ shows the new free surface after attaining steady state.

Consider a small fluid element *M* at a distance *x* from the axis of rotation. The liquid element will be in equilibrium under the action of the forces, such as (i) weight of fluid element (*W*) acting downwards, (ii) inertia force given by

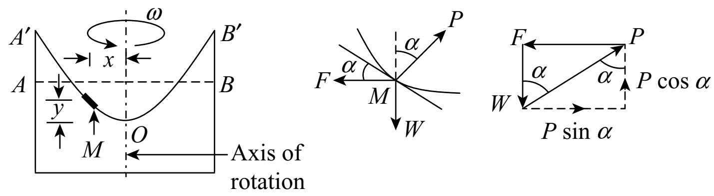


Figure 4.8 *Liquid container subjected to constant rotation*

$F = (W\omega^2 x)/g$ which acts radially from the axis of rotation at a distance x and (iii) a force P acting normal to the surface of the element due to action of surrounding liquid particles as shown in Figure 4.8(b).

Resolving the forces horizontally and vertically, we get the following expressions.

$$P \sin \alpha = \frac{W}{g} \omega^2 x \qquad \text{(i)}$$

$$P \cos \alpha = W \qquad \text{(ii)}$$

Dividing expression (i) by expression (ii), we get:

$$\tan \alpha = \frac{\omega^2 x}{g} \qquad (4.12)$$

The angle α is the slope of the tangent drawn to the free surface of the fluid element M.

$$\frac{dy}{dx} = \frac{\omega^2 x}{g} \qquad (4.13)$$

Integrating the above expression, we get:

$$y = \int_0^x \frac{\omega^2 x}{g} dx = \frac{\omega^2 x^2}{2g} + C \qquad \text{(iii)}$$

Here, C is the constant of integration. Let the vortex O of the curve be the origin, then the boundary conditions become $y = 0$ at $x = 0$. Substituting this condition in expression (iii), we get the following expression.

$$0 = \frac{\omega^2 (0)^2}{2g} + C, \text{ i.e., } C = 0$$

Substituting the value of C in expression (iii), we get:

$$\boxed{y = \frac{\omega^2 x^2}{2g}} \qquad (4.14)$$

From Equation (4.14), it is observed that the shape of the surface of liquid developed is a paraboloid of revolution with its section being a parabola. It can also be observed from this equation that the concavity of the surface depends only on the angular velocity ω and the distance x from the axis of rotation. In case of a closed cylinder filled with liquid subjected to constant rotation, the lines of constant pressure will also be governed by Equation (4.14).

Example 4.11 An open cylindrical tank which is 2 m in diameter and 4 m high contains water up to 3 m depth. If the cylinder rotates about its vertical axis, then what maximum angular velocity can be attained without any spillage?

Solution

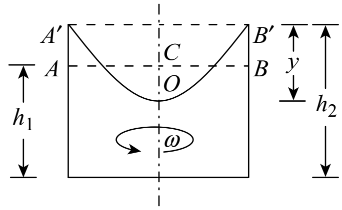


Figure 4.9

Refer Figure 4.9. Let $d = 2$ m, $h_2 = 4$ m and $h_1 = 3$ m.

Let ω be the angular speed and N be the corresponding speed in rpm. At maximum speed, the water surface will just touch the top of the rim of the cylinder. Let AB be the free liquid surface before rotation and $A'OB'$ shows the new free surface after attaining steady state. Let y be the water surface elevation at the outer edge above vortex O and the expression is $OC = y/2$.

$$y = 2(h_2 - h_1) = 2(4-3) = 2 \text{ m}$$

But
$$y = \frac{\omega^2 x^2}{2g}$$

Thus
$$2 = \frac{\omega^2 \times 1^2}{2 \times 9.81} \qquad [\because x = r = d/2]$$

$$\therefore \omega = \sqrt{2 \times 2 \times 9.81} = 6.2642 \text{ rad/s}$$

$$\frac{2\pi N}{60} = 6.2642$$

$$\therefore N = \frac{6.2642 \times 60}{2\pi} = \mathbf{59.82\ rpm}$$

Example 4.12 An open cylindrical tank 0.5 m in diameter and 3 m high contains water up to a depth of 1.6 m. If the cylinder rotates about its vertical axis at a speed of 250 rpm, then determine the height of the paraboloid formed at the free surface. Also determine the speed of rotation required for water to just start spilling, the total pressure force on the bottom of the tank and side walls.

Solution

Refer Figure 4.9. Let $d = 0.5$ m, $h_2 = 3$ m, $h_1 = 1.6$ m and $N = 250$ rpm. Let ω be the angular speed and $x = r = 0.5/2 =$ 0.25 m.

$$\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 250}{60} = 26.18 \text{ rad/s}$$

The height of the paraboloid formed at the free surface is given by,

$$y = \frac{\omega^2 x^2}{2g} = \frac{26.18^2 \times 0.25^2}{2 \times 9.81} = \mathbf{2.183\ m}$$

For the water to just spill, the water will just touch the top of the rim of the tank. Thus, rise of water above the original become equal to $(h_2 - h_1)$, i.e., $(3 - 1.6) = 1.4$ m.

$$\therefore y = 2(h_2 - h_1) = 2 \times 1.4 = 2.8 \text{ m}$$

But
$$y = \frac{\omega^2 x^2}{2g}$$

Thus
$$2.8 = \frac{\omega^2 \times 0.25^2}{2 \times 9.81}$$

$$\therefore \omega = \sqrt{\frac{2.8 \times 2 \times 9.81}{0.25^2}} = 29.65 \text{ rad/s}$$

$$\frac{2\pi N}{60} = 29.65$$

$$\therefore N = \frac{29.65 \times 60}{2\pi} = \mathbf{283.14\ rpm}$$

Total pressure force on the bottom (F_{bottom}) will be equal to the total weight of the water in the tank and is given by the following expression.

$$F_{\text{bottom}} = \rho_w g \pi r^2 h_1 = 1000 \times 9.81 \times \pi \times 0.25^2 \times 1.6 = \mathbf{3081.9\ N}$$

Total pressure force on the side walls (F_{side}) will be given by,

$$F_{side} = \rho_w g \times 2\pi r h_2 h_G = 1000 \times 9.81 \times 2\pi \times 0.25 \times 3 \times \frac{3}{2} = \mathbf{69342.8\ N}$$

Example 4.13 An open cylindrical tank 0.6 m in diameter and 1 m high is completely filled with water. It spins about its vertical axis at 125 rpm. Determine (i) the water left in the tank when it reaches to its full speed and (ii) the slope of the water surface at the point where it just touches the top of the rim of the tank.

Solution

Refer Figure 4.9. Let $d = 0.6$ m, $h_1 = h_2 = 1$ m and $N = 125$ rpm.

Let ω be the angular speed, y be the height of the paraboloid formed at the free surface and $x = r = 0.6/2 = 0.3$ m.

(i) $$\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 125}{60} = 13.09 \text{ rad/s}$$

$$y = \frac{\omega^2 x^2}{2g} = \frac{13.09^2 \times 0.3^2}{2 \times 9.81} = 0.786 \text{ m}$$

Initial volume of water (v_{initial}) in the tank is given by,

$$v_{\text{initial}} = \pi r^2 h_2 = \pi \times 0.3^2 \times 1 = 0.2827 \text{ m}^3$$

Volume of water spilled (v_{spilled}) is equal to the volume of paraboloid formed and it is derived as follows.

$$v_{\text{spilled}} = \frac{1}{2}\pi r^2 y = \frac{1}{2} \times \pi \times 0.3^2 \times 0.786 = 0.1111 \text{ m}^3$$

Thus, the volume of water left (v) is given by,

$$v = v_{\text{initial}} - v_{\text{spilled}} = 0.2827 - 0.1111 = \mathbf{0.1716\ m^3}$$

(ii) The slope of the water surface at the point where it meets the rim of the tank is given by equations 4.12 and 4.13 and the value is derived as follows.

$$\frac{dy}{dx} = \tan\alpha = \frac{\omega^2 x}{g}$$

$$\therefore \alpha = \tan^{-1}\left(\frac{\omega^2 x}{g}\right) = \tan^{-1}\left(\frac{13.09^2 \times 0.3}{9.81}\right) = \mathbf{79.2°}$$

Example 4.14 An open cylindrical tank 0.3 m in diameter and 1.2 m high contains water up to a height of 0.8 m. Determine the speed at which the cylinder may be rotated about its vertical axis so that the axial depth becomes zero.

Solution

Refer Figure 4.10. Let $d = 0.3$ m, $h_2 = 1.2$ m and $h_1 = 0.8$ m.

Let ω be the angular speed, N be the corresponding speed in rpm and $x = r = 0.3/2 = 0.15$ m.

When axial depth become zero, the depth of paraboloid is calculated as follows.

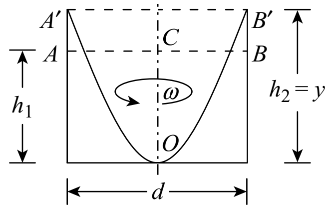


Figure 4.10

$$y = h_2 = 1.2 \text{ m}$$

But
$$y = \frac{\omega^2 x^2}{2g}$$

Thus
$$\frac{\omega^2 \times 0.15^2}{2 \times 9.81} = 1.2$$

$$\therefore \omega = \sqrt{\frac{1.2 \times 2 \times 9.81}{0.15^2}} = 32.35 \text{ rad/s}$$

$$\frac{2\pi N}{60} = 32.35$$

$$\therefore N = \frac{32.35 \times 60}{2\pi} = \mathbf{308.92\ rpm}$$

Summary

1. D'Alembert's principle states that a moving liquid mass may be brought to a static equilibrium position by applying an imaginary inertia force of the same magnitude as that of the accelerating force but in opposite directions.
2. When a tank containing a liquid is accelerated horizontally, then the liquid falls at the front end and rises at the back end of the container. The slope of the free surface of the liquid along the direction of acceleration is given by $\tan\alpha = a/g$, here α is the inclination of free surface of the liquid and a is the horizontal acceleration.
3. Pressure variation for an open tank containing a liquid moves vertically with a constant acceleration a is $p = \rho gh[1+(a/g)]$ (for upward) and $p = \rho gh[1-(a/g)]$ (for downward).
4. For an open tank containing liquid and moving upwards along the inclined plane (inclination angle β) with a constant acceleration a, the horizontal and vertical components of acceleration are $a_x = a\cos\beta$ and $a_y = a\sin\beta$.
5. For an open tank containing a liquid subjected to a constant acceleration a along the inclined plane, the slope of the free liquid surface is $\tan\alpha = a_x/(g+a_y)$ (for upward) and $\tan\alpha = a_x/(g-a_y)$ (for downward).
6. The shape of the surface of the liquid developed in a container subjected to constant rotation is a paraboloid of revolution with its section being a parabola and it is governed by $y = (\omega^2 x^2)/2g$, here ω is the angular velocity and x is the distance from the axis of rotation.

Multiple-choice Questions

1. When an open tank containing a liquid is accelerated in the upward direction, then it causes
   (a) No change in hydrostatic pressure.
   (b) A decrease in hydrostatic pressure.
   (c) An increase in hydrostatic pressure.
   (d) None of the above.
2. When an open tank containing a liquid is accelerated with a constant linear acceleration, then the free surface of the liquid
   (a) Is inclined with smaller depth at the rear.
   (b) Is inclined with larger depth at the rear.
   (c) Remains horizontal.
   (d) None of the above.

3. When an open tank containing a liquid is vertically accelerated downward, then the free surface of the liquid
   (a) Is inclined with smaller depth at the rear.
   (b) Is inclined with larger depth at the rear.
   (c) Remains horizontal.
   (d) None of the above.
4. When the tank is lowered vertically at the gravitational acceleration (g), then pressure intensity (p) at any point in the tank is given by which of the following expression?
   (a) $p = \rho g h[1+(a/g)]$
   (b) $p = \rho g h[1-(a/g)]$
   (c) $p = 0$
   (d) None of the above.
5. An open tank containing liquid is sliding down an inclined plane with uniform velocity, then the free surface of the liquid
   (a) Will be parallel to the plane of inclined plane.
   (b) Will be horizontal.
   (c) Will be inclined to the horizontal whose angle of inclination depends upon the slope of inclined plane.
   (d) None of the above.
6. When an open cubical tank containing liquid is accelerated on a horizontal plane along one of its side, then one fourth of the volume of liquid spills out. The acceleration is
   (a) $(2g)/3$
   (b) $g/3$
   (c) $g/2$
   (d) None of the above.
7. When an open cubical tank containing liquid is accelerated on a horizontal plane along one of its side, then one third of the volume of liquid spills out. The acceleration is equal to
   (a) $(2g)/3$
   (b) $g/3$
   (c) $g/2$
   (d) g

Review Questions

1. Explain the effect of constant horizontal acceleration in a tank containing liquid.
2. What happens when a liquid is subjected to constant acceleration while moving up along an inclined plane? Also give the angle of slope of the free surface of the liquid.
3. Describe the effect of constant vertical acceleration on the pressure distribution for a liquid container.
4. An open cylindrical container partly filled with a liquid rotates about a vertical axis at constant angular velocity. Derive an expression for the free surface of the liquid when the liquid has attained the angular velocity of the container.

Problems

1. An open tank which is 8 m long and 2 m deep is filled with 1.5 m of oil (specific gravity = 0.9). The tank is subjected to horizontal acceleration to the velocity of 10 m/s. Determine the smallest time to attain this velocity without any spillage.
   [**Ans.** 10.194 s]
2. An open tank 2 m high contains 1.8 m water. How high must its side be if no water is to be spilled out when subjected to horizontal acceleration of 5 m/s^2?
   [**Ans.** 2.31 m]
3. An open rectangular tank which is 6 m long, 2.5 m wide and 2 m deep is moving horizontally in the direction of its length with a constant acceleration of 2.2 m/s^2. If the depth of water in tank is 1 m, then calculate (i) angle of the water surface to the horizontal, (ii) maximum and minimum pressure intensities at the bottom and (iii) total force and net force due to water acting on each end of the tank.
   [**Ans.** (i) 12.64^o (ii) 16.41 kN/m^2, 3.21 kN/m^2 and (iii) 34.32 kN, 1.312 kN, 33.01 kN]
4. An open rectangular tank which is 7 m long, 3.6 m wide and 4 m deep contains water to a depth of 3 m. Determine the horizontal acceleration which may be given to the tank along its longer side so that (i) there is no spillage of water from the tank, (ii) the front bottom of the tank is just exposed and (iii) the bottom of the tank is exposed up to its midpoint. Also determine the total forces on each end of the tank.
   [**Ans.** (i) 2.803 m/s^2, 282.53 kN, 70.632 kN, 211.9 kN, (ii) 5.6 m/s^2, 282.53 kN, 0 kN, 282.528 kN and (iii) 11.213 m/s^2, 282.53 kN, 0 kN, 282.53 kN]
5. A cubical tank of side 2 m is filled with 1.5 m of a liquid (specific gravity = 1.6). Find the force acting on the side of the tank when (i) it is accelerated vertically upward at 5 m/s^2 and (ii) it is accelerated downward at 5 m/s^2.
   [**Ans.** 53.32 kN, 17.32 kN]
6. An open cubical tank with each side 2 m contains oil (specific gravity = 0.765) up to a depth of 2 m. Find the force acting on the side of the tank when it is moved with an acceleration of $(g/2)$ m/s^2 in vertically upward and downward directions. Also determine the pressure at the bottom of the tank when the acceleration rate is g m/s^2 vertically downwards.
   [**Ans.** 45.028 kN, 15.01 kN, 0]

7. A tank of length 4 m and width 2 m is filled with oil (specific gravity = 0.9) up to a height of 1 m. Determine the total force on the sides of the tank when (i) it moves vertically upwards with an acceleration of 3 m/s^2 and (ii) it moves vertically downwards with an acceleration of 3 m/s^2.

[**Ans.** 69.17 kN, 36.77 kN]

8. An open cylinder of diameter 0.4 m, height 2 m contains a liquid up to a height of 1.2 m. Determine the depth of parabola formed at the free surface of the liquid when the tank is rotated about its vertical axis at 200 rpm.

[**Ans.** 0.894 m]

9. An open cylinder of diameter 0.2 m, height 1.25 m contains a liquid up to a height of 0.85 m. Determine the maximum speed of the cylinder at which it should be rotated about its vertical axis so that no water spills.

[**Ans.** 378.34 rpm]

10. An open cylindrical tank of diameter 0.9 m, height 1.2 m contains a liquid up to 2/3rd of its height when at rest. If it is spun about its vertical axis with an angular velocity, then determine (i) the speed of rotation when the liquid just starts spilling over the sides of the tank and (ii) what would be the speed of rotation and the percentage of liquid left in the tank when the point at the centre of the base is just exposed?

[**Ans.** (i) 84.03 rpm, (ii) 102.97 rpm, 24.99%]

11. An open cylindrical container of diameter 0.15 m and height 1 m contains a liquid up to a height of 0.76 m from its bottom when at rest. If it is spun about its vertical axis with a speed of 240 rpm, then determine (i) the height of the paraboloid formed at the free surface of liquid, (ii) maximum speed at which the container is rotated so that no water spills out and (iii) speed at which the axial depth of liquid becomes zero.

[**Ans.** (i) 0.181 m, (ii) 390.76 rpm, (iii) 563.98 rpm]

12. An open cylindrical container of diameter 0.15 m and height 1 m contains a liquid up to a height of 0.65 m from its bottom when at rest. If it is spun about its vertical axis, then determine the speed so that the axial depth becomes zero.

[**Ans.** 563.98 rpm]

13. An open cylindrical container of diameter 0.5 m and height 0.85 m contains a liquid up to its full height. If it is spun about its vertical axis at 100 rpm, then determine the liquid left in the container when it reaches to its full speed. Also find the slope of liquid surface at the point where it meets the rim of the container.

[**Ans.** 0.1327 m^3, 2.794]

14. An open cylindrical tank of diameter 0.2 m and length 1 m contains a liquid up to a height of 0.8 m. If it is spun about its vertical axis, then determine the maximum speed so that there is no spillage of the liquid.

[**Ans.** 267.52 rpm]

15. An open cylindrical tank of diameter 0.4 m and length 0.8 m contains a liquid (specific gravity = 1) up to a height of 0.6 m. If it is spun about its vertical axis at a speed of 240 rpm, then determine (i) the height of the paraboloid formed at the free surface of the liquid, (ii) how fast should the tank be rotated so that the liquid is just on the point of spilling and (iii) pressure on the side walls and bottom of the container.

[**Ans.** (i) 1.287 m, (ii) 133.76 rpm, (iii) 739.66 N, 3944.84 N]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (b)	**2.** (b)	**3.** (c)	**4.** (c)	**5.** (b)
6. (c)	**7.** (a)			

Chapter 5

Buoyancy and Floatation

5.1 ❐ INTRODUCTION

When a body is immersed in a fluid it is subjected to two forces, namely gravitational force (or body force) and buoyant force (or an upthrust or surface forces). The gravitational force is due to the weight of the body and it acts vertically downwards. The buoyant force is exerted by the liquid (fluid) on the body and it acts vertically upwards. A body placed on the free surface of a liquid will either sink or float. In this chapter, the equilibrium of floating as well as submerged bodies has been described. The concept of floating and submerged bodies is used in various practical applications, such as boats, ships, submarines and toys. The necessary conditions required for the body to float and for its stability are also discussed in this chapter.

5.2 ❐ BUOYANCY, BUOYANT FORCE AND CENTRE OF BUOYANCY

5.2.1 Buoyancy

A body feels lighter and it weighs less in water (liquid) than it does in air which suggests that water exerts an upward force. Therefore, when a body is immersed in fluid (water) either wholly or partially, an upward force is exerted by the fluid on the body which tends to lift up. This tendency for an immersed body to be lifted up in the fluid is known as buoyancy.

5.2.2 Buoyant Force

The force tending to lift up an immersed body against the gravitational force is called buoyant force (or force of buoyancy) and it is denoted by F_B. It is also known as upthrust which is equal to the weight of the fluid displaced by the body.

5.2.3 Centre of Buoyancy

The point of application of the buoyant force on the body is known as centre of buoyancy and it is denoted by B. The centre of buoyancy will be the centre of gravity of the fluid displaced. The buoyant force exerted by the fluid on the body can be calculated by Archimedes' principle.

5.3 ❐ ARCHIMEDES' PRINCIPLE

The Archimedes' principle states that when a body is immersed in a fluid either wholly or partially, it is lifted up by a force which is equal to the weight of the fluid displaced by the body. It is due to this upward force acting on an immersed body in a fluid, where the body experiences an apparent weight loss. Thus, a body has less weight in a liquid than outside. Since

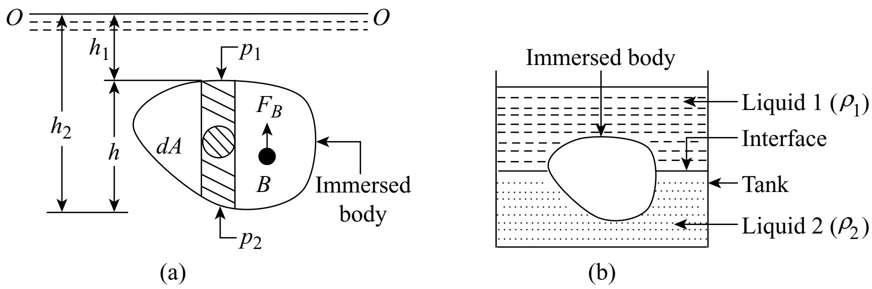


Figure 5.1 *Archimedes' principle*

liquids are heavier than gases (air), we are conscious about their buoyant forces. However, air also exerts buoyancy on any body immersed in it. Therefore, in the design of balloons and blimps, the buoyant force of air (instead of being negligible) is the controlling factor.

5.3.1 Proof

Consider a body completely immersed in a liquid of density ρ as shown in Figure 5.1(a). Let dA be the cross sectional area of a small vertical element, dv be the volume of the small element, p_1 and p_2 be the intensity of pressures at depths h_1 and h_2, respectively.

The force acting on the top face of the element is equal to $p_1 dA = \rho g h_1 \times dA$ which acts vertically downwards. The force on the bottom face of the element is equal to $p_2 dA = \rho g h_2 \times dA$ which acts vertically upwards. The net force on the element is equal to the buoyant force dF_B which acts upwards ($\because h_2 > h_1$) and it is given below.

$$dF_B = \rho g h_2 dA - \rho g h_1 dA = \rho g \times (h_2 - h_1) dA = \rho g \times dv$$

The total buoyant force is given by,

$$F_B = \int dF_B = \int \rho g dv = \rho g v = W_d \tag{5.1}$$

Here, v is the volume of the submerged body which is equal to the volume of fluid displaced by the body and W_d is the weight of the fluid displaced by the body.

Thus, buoyant force is equal to the weight of the fluid displaced by the body. It acts through centre of buoyancy which coincides with the centroid of the displaced volume. For a fully submerged body, the centre of buoyancy (B) coincides with the centre of gravity (G) of the body. Further, the lines of action of both the buoyant force and the weight of the body must be along the same vertical line, so that their moment about any axis is zero.

When a body floats on the surface of separation between two immiscible fluids of different density (assume ρ_1 and ρ_2) (Figure 5.1(b)), then the total buoyancy force is given below.

$$F_B = \rho_1 g v_1 + \rho_2 g v_2 \tag{5.2}$$

Here, v_1 is the volume of the body submerged in the liquid of density ρ_1 and v_2 is the volume of body submerged in the liquid of density ρ_2.

It is observed from the discussion that a body will sink if its weight (W) is greater than the buoyant force (F_B). Thus, if $W > F_B$, then the body would sink, but if $F_B \geq W$ then it would float.

Example 5.1 A cuboidal wooden block (specific gravity = 0.65) that is 3.5 m long, 1.3 m wide and 2 m deep floats horizontally in sea water (specific gravity = 1.025). Determine (i) the volume of liquid displaced and (ii) the position of centre of buoyancy.

Solution

Refer Figure 5.2. Let $S_{wood} = 0.65$, $l = 3.5$ m, $b = 1.3$ m, $d = 2$ m and $S_s = 1.025$. Let W be the weight of the wooden block, v be the volume of water displaced by the block, B be the centre of buoyancy and $\rho_w = 1000$ kg/m^3 be the density of water.

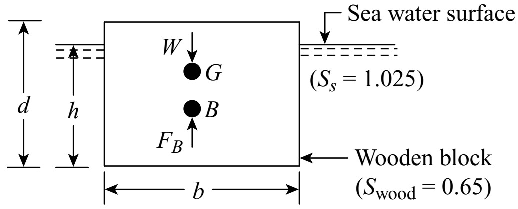


Figure 5.2

(i) Let $W = S_{wood}\rho_w g \times lbd$

$$= 0.65 \times 1000 \times 9.81 \times 3.5 \times 1.3 \times 2 = 58026.15 \text{ N}$$

Weight of the liquid displaced by the body = Weight of the wooden block

Thus $$S_s \rho_w g v = W$$

$$\therefore v = \frac{W}{S_s \rho_w g} = \frac{58026.15}{1.025 \times 1000 \times 9.81} = \mathbf{5.771\ m^3}$$

(ii) Let h be depth of wooden block under water and we get the following result.

$$3.5 \times 1.3 \times h = 5.771 \quad [\because v = lbh]$$

$$\therefore h = \frac{5.771}{3.5 \times 1.3} = 1.268 \text{ m}$$

$$B = \frac{h}{2} = \frac{1.268}{2} = \mathbf{0.634\ m\ from\ base}$$

Example 5.2 A metallic body weighs 500 kN in air and 250 kN in water. Determine the volume of body and its specific gravity.

Solution

Let $W = 500$ kN and $W_1 = 250$ kN. Let v be the volume of body which is equal to the volume of water displaced by it and S be its specific gravity. The reduction in weight of the metallic body when immersed in water is due to the buoyancy force (F_B).

$$F_B = W - W_1 = 500 - 250 = 250 \text{ kN}$$

But $$F_B = \rho_w g v$$

$$\therefore v = \frac{F_B}{\rho_w g} = \frac{250 \times 10^3}{1000 \times 9.81} = \mathbf{25.4842\ m^3}$$

The specific weight of metallic body is given by,

$$w = \frac{W}{v} = \frac{500 \times 10^3}{25.4842} = 19620 \text{ N/m}^3$$

$$S = \frac{w}{\rho_w g} = \frac{19620}{1000 \times 9.81} = \mathbf{2}$$

Example 5.3 A metallic body floats at the interface of mercury (Hg) and water (H_2O) in a tank such that 35% of its volume is submerged in mercury and 65% in water. Find the density of the metallic body. Take density of mercury as 13600 kg/m^3 and density for water as 1000 kg/m^3.

Solution

Refer Figure 5.3. Let v be the volume and ρ be the density of the metallic body.

Let $v_1 = 0.35v$ be the volume of the body submerged into mercury, $v_2 = 0.65v$ be the volume submerged into water, $\rho_{Hg} = 13600$ kg/m^3 and $\rho_w = 1000$ kg/m^3.

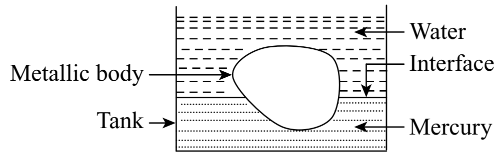


Figure 5.3

Weight of metal piece = Weight of Hg displaced + Weight of H_2O displaced

or
$$\rho g v = \rho_{Hg} g v_1 + \rho_w g v_2$$

$$\rho \times 9.81 \times v = 13600 \times 9.81 \times 0.35v + 1000 \times 9.81 \times 0.65v$$

$$\therefore \rho = 13600 \times 0.35 + 1000 \times 0.65 = \mathbf{5410\ kg/m^3}$$

Example 5.4 If a piece of ice (specific gravity = 0.93) floats in sea water (specific gravity = 1.04), then determine the percentage volume of ice outside the water.

Solution

Let $S_i = 0.93$ and $S_s = 1.04$. Let v be the total volume of the ice piece and x be a fraction of it outside water.

Weight of ice piece = Weight of water displaced by ice piece

$$S_i \rho_w g \times v = S_s \rho_w g \times (1-x)v$$

$$0.93 \times 1000 \times 9.81 \times v = 1.04 \times 1000 \times 9.81 \times (1-x)v$$

$$1 - x = \frac{0.93}{1.04} = 0.8942$$

$$\therefore x = 1 - 0.8942 = \mathbf{0.1058\ or\ 10.58\%}$$

Example 5.5 A wooden body of height 73 mm floats in a water tank of height 25 mm projecting above the water surface. The same wooden body when placed in glycerine tank is projected 37.5 mm above the surface of glycerine. Find (i) the relative density of the wooden body and (ii) the relative density of glycerine.

Solution

Refer Figure 5.4. Let $d = 73$ mm, $y_1 = 25$ mm and $y_2 = 37.5$ mm. Let S_{wood} be the relative density of wooden body and S_g be the relative density of the glycerine.

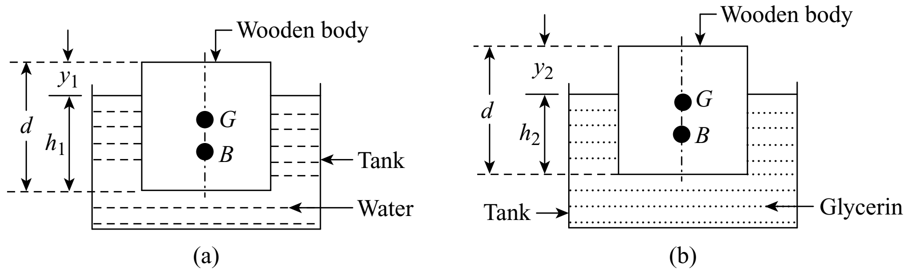


Figure 5.4

(i) Refer Figure 5.4(a). Let $h_1 = d - y_1 = 73 - 25 = 48$ mm

$$\text{Weight of wooden body} = \text{Weight of water displaced}$$

or
$$\rho_{\text{wood}} gAd = \rho_w gAh_1$$

$$\frac{\rho_{\text{wood}}}{\rho_w} = \frac{h_1}{d}$$

$$\therefore S_{\text{wood}} = \frac{h_1}{d} = \frac{48}{73} = \mathbf{0.6575}$$

(ii) Refer Figure 5.4(b). Let $h_2 = d - y_2 = 73 - 37.5 = 35.5$ mm

$$\text{Weight of wooden body} = \text{Weight of glycerine displaced}$$

or
$$\rho_{\text{wood}} gAd = \rho_g gAh_2$$

$$(S_{\text{wood}} \rho_w) gAd = \rho_g gAh_2$$

$$\frac{\rho_g}{\rho_w} = \frac{S_{\text{wood}} d}{h_2}$$

$$\therefore S_g = \frac{S_{\text{wood}} d}{h_2} = \frac{0.6575 \times 73}{35.5} = \mathbf{1.352}$$

Example 5.6 If a wooden log that is 0.8 m in diameter, 7 m long and of specific gravity 0.65 floats in water, then find its depth in water.

Solution

Refer Figure 5.5. Let $d = 0.8$ m, $l = 7$ m and $S_{\text{wood}} = 0.65$.

Let h be the depth of the wooden log in water and v be the volume of water displaced.

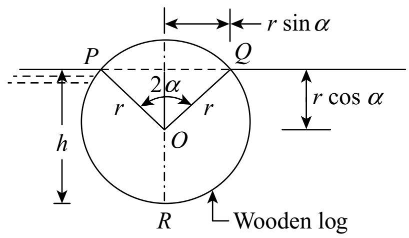


Figure 5.5

$$r = \frac{d}{2} = \frac{0.8}{2} = 0.4 \text{ m}$$

$$\text{Weight of wooden log} = \text{Weight of water displaced}$$

or
$$S_{\text{wood}} \rho_w gAl = \rho_w gv$$

$$v = S_{\text{wood}} Al = S_{\text{wood}} \times \frac{\pi}{4} d^2 \times l$$

$$\therefore v = 0.65 \times \frac{\pi}{4} \times 0.8^2 \times 7 = 2.2871 \text{ m}^3$$

$$\text{Area } (PQRP) \times l = 2.2871 \text{ m}^3$$

$$\text{Area } (PQRP) = \frac{2.2871}{l} = \frac{2.2871}{7} = 0.3267 \text{ m}^2$$

$$\text{Area } (POQRP) + \text{Area } (\Delta POQ) = 0.3267 \text{ m}^2$$

$$\pi r^2\left[\frac{360-2\alpha}{360}\right]+2\left[\frac{1}{2}r\sin\alpha\times r\cos\alpha\right]=0.3267$$

$$\pi r^2\left[1-\frac{\alpha}{180}\right]+r^2\sin\alpha\cos\alpha=0.3267$$

$$\pi\times0.4^2\times\left[1-\frac{\alpha}{180}\right]+0.4^2\sin\alpha\cos\alpha=0.3267$$

$$\pi\times0.4^2-\frac{\pi\times0.4^2\times\alpha}{180}+0.4^2\sin\alpha\cos\alpha=0.3267$$

$$0.503-2.792\times10^{-3}\alpha+0.16\sin\alpha\cos\alpha=0.3267$$

Let $\alpha=76.3°$ [Hit and trial]

$$0.503-2.792\times10^{-3}\times76.3+0.16\sin76.3°\cos76.3°=0.3267$$

$$0.3267\approx0.3267$$

Since L.H.S. ≈ R.H.S.

Thus $\alpha=76.3°$

$$h=r+r\cos\alpha=0.4+0.4\cos76.3°=\mathbf{0.4947\ m}$$

Example 5.7 A football of diameter 40 cm fell into a water tank, 20% of its volume is found under water. Determine the density of the football.

Solution

Let $d=40\text{ cm}=0.4\text{ m}$. Let ρ_f be the density of football and v be its volume.

$$\text{Volume of water displaced}=20\%\text{ of }v=0.2v$$

$$\text{Weight of football}=\text{Weight of water displaced}$$

or $\rho_f gv=\rho_w g\times0.2v$

$$\therefore\rho_f=\rho_w\times0.2=1000\times0.2=\mathbf{200\ kg/m^3}$$

Example 5.8 An iceberg of relative density 0.92 floats in sea water (specific gravity = 1.03). Find the weight of the iceberg if the volume of ice above the water surface is 10 m^3.

Solution

Let $S_i=0.92$, $S_{\text{sea}}=1.03$ and $(v-v_b)=10\text{ m}^3$, where v is the total volume of the iceberg and v_b is the volume of iceberg below the water surface.

$$\text{Weight of iceberg}=\text{Weight of water displaced by iceberg}$$

or $S_i\rho_w gv=S_{\text{sea}}\rho_w gv_b$

Thus $0.92\times1000\times9.81\times v=1.03\times1000\times9.81\times v_b$

$$\therefore v_b=\frac{0.92}{1.03}v=0.8932v$$

Since
$$v - v_b = 10 \text{ m}^3$$
$$v - 0.8932v = 10$$
$$0.1068v = 10$$
$$\therefore v = \frac{10}{0.1068} = 93.633 \text{ m}^3$$

The weight of iceberg is given by,

$$W = S_i \rho_w g v = 0.92 \times 1000 \times 9.81 \times 93.633 = \mathbf{845.06\ kN}$$

Example 5.9 A wooden block (specific gravity = 0.65) that is 2.5 m long, 1 m wide and 0.5 m high floats in a water tank. Determine the volume of concrete of specific weight 24.5 kN/m^3, that may be kept on the block and immerse the (i) block completely in water and (ii) block and the concrete completely in water. Take weight density of water as 9.81 kN/m^3.

Solution

Let $S_{\text{wood}} = 0.65$, $l = 2.5$ m, $b = 1$ m, $d = 0.5$ m, $w_c = 24.5$ kN/m^3 and $w_{\text{water}} = 9.81$ kN/m^3.

Let v_c be the volume of the concrete and v_{wood} be the volume of wooden block which is equal to the volume of water displaced.

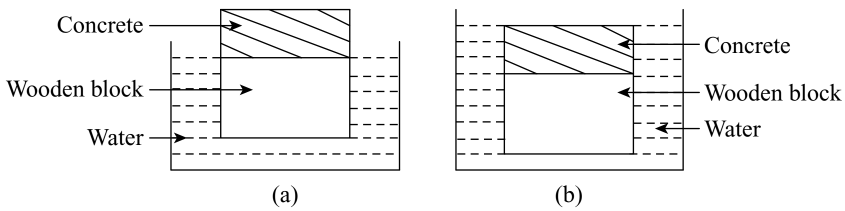


Figure 5.6

(i) Refer Figure 5.6(a). We get the following relation when the block is completely immersed.

Weight of the concrete + Weight of wooden block = Weight of water displaced

or
$$w_c v_c + S_{\text{wood}} w_{\text{water}} v_{\text{wood}} = w_{\text{water}} v_{\text{wood}}$$
$$24.5 \times v_c + 0.65 \times 9.81 \times (2.5 \times 1 \times 0.5) = 9.81 \times (2.5 \times 1 \times 0.5)$$
$$\therefore v_c = \frac{9.81 \times (2.5 \times 1 \times 0.5) - 0.65 \times 9.81 \times (2.5 \times 1 \times 0.5)}{24.5} = \mathbf{0.1752\ m^3}$$

(ii) Refer Figure 5.6(b). We get the following relation when the block and concrete is completely immersed.

Weight of the concrete and wooden block = Weight of water displaced

$$w_c v_c + S_{\text{wood}} w_{\text{water}} v_{\text{wood}} = w_{\text{water}} (v_{\text{wood}} + v_c)$$
$$24.5 \times v_c + 0.65 \times 9.81 \times (2.5 \times 1 \times 0.5) = 9.81 \times [2.5 \times 1 \times 0.5 + v_c]$$
$$24.5v_c - 9.81v_c = 9.81 \times (2.5 \times 1 \times 0.5) - 0.65 \times 9.81 \times (2.5 \times 1 \times 0.5)$$
$$\therefore v_c = \frac{9.81 \times (2.5 \times 1 \times 0.5) - 0.65 \times 9.81 \times (2.5 \times 1 \times 0.5)}{24.5 - 9.81} = \mathbf{0.2922\ m^3}$$

Example 5.10 A wooden block (specific gravity = 0.64) that is 0.12 m square in cross-section and 2.6 m long floats in a water tank. Determine how much lead (specific gravity = 12.5) is to be attached at the lower end of the block so that it floats vertically in water with 0.6 m length out of the water.

Solution

Refer Figure 5.7. Let $S_{\text{wood}} = 0.64$, $b = 0.12$ m, $l = 2.6$ m, $S_l = 12.5$ and $y = 0.6$ m.

Let v_{wood} be the volume of the wooden block in water and v_l be the volume of lead, W_l be the weight of the lead and W_{wood} be the weight of the wooden block.

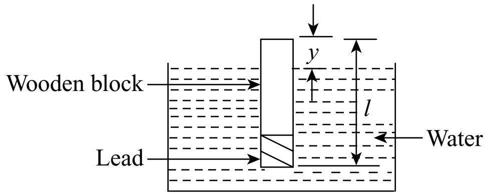


Figure 5.7

$$W_{\text{wood}} = S_{\text{wood}}\rho_w g \times bbl = 0.64 \times 1000 \times 9.81 \times 0.12 \times 0.12 \times 2.6 = 235.063 \text{ N}$$

The volume of wooden block in water which is equal to the volume of water displaced by it is given below.

$$v_{\text{wood}} = b \times b \times (l - y) = 0.12 \times 0.12 \times (2.6 - 0.6) = 0.0288 \text{ m}^3$$

The volume of lead in water which is equal to the volume of water displaced by it is given below.

$$v_l = \frac{\text{Weight}}{S_l \rho_w g} = \frac{W_l}{12.5 \times 1000 \times 9.81} = \frac{W_l}{122625} \text{ m}^3$$

Thus, $(W_{\text{wood}} + W_l)$ is equal to the sum of weight of water displaced by block and weight of water displaced by the lead.

$$W_{\text{wood}} + W_l = \rho_w g v_{\text{wood}} + \rho_w g v_l$$

Thus

$$235.063 + W_l = 1000 \times 9.81 \times 0.0288 + 1000 \times 9.81 \times \frac{W_l}{122625}$$

$$235.063 + W_l = 282.528 + 0.08 W_l$$

$$W_l - 0.08 W_l = 282.528 - 235.063$$

$$0.92 W_l = 47.465$$

$$\therefore W_l = \frac{47.465}{0.92} = \mathbf{51.5924 \text{ N}}$$

Example 5.11 A metallic cube has side 0.25 m and it weighs 250 N when lowered into a tank containing a two-fluid layer of water and mercury. Determine the position of block at mercury-water interface when it has reached equilibrium.

Solution

Refer Figure 5.8. Let $b = 0.25$ m and $W = 250$ N.

The metallic cube sinks beneath the water surface and comes to rest at the water-mercury interface. Let h_1 be the depth of cube in water and h_2 be the depth of cube in mercury in metres. The weight of cubical block is equal to the sum of weight of water displaced and the weight of mercury displaced.

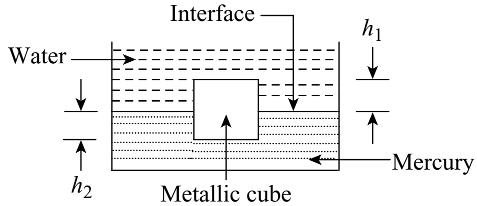


Figure 5.8

$$W = \rho_w g v_{\text{water}} + S_{\text{Hg}} \rho_w g v_{\text{Hg}}$$

Thus

$$250 = 1000 \times 9.81 \times (h_1 \times 0.25 \times 0.25) + 13.6 \times 1000 \times 9.81 \times (h_2 \times 0.25 \times 0.25)$$

$$250 = 1000 \times 9.81 \times 0.25 \times 0.25 \times (h_1 + 13.6 h_2)$$

$$h_1 + 13.6h_2 = \frac{250}{1000 \times 9.81 \times 0.25 \times 0.25}$$

$$h_1 + 13.6h_2 = 0.40775 \quad \text{(i)}$$

Since $$h_1 + h_2 = 0.25 \quad \text{(ii)}$$

Subtracting expression (ii) from (i), we get:

$$12.6h_2 = 0.40775 - 0.25 = 0.15775$$

$$\therefore h_2 = \frac{0.15775}{12.6} = \mathbf{0.01252\ m\ or\ 12.52\ mm}$$

$$\therefore h_1 = 0.25 - 0.01252 = \mathbf{0.23748\ m\ or\ 237.48\ mm}$$

5.4 ❐ METACENTRE

Metacentre (M) is defined as the point about which a floating body starts oscillating when it is given a small angular displacement. A floating body in static equilibrium is acted upon by two forces, namely the weight of the body W acting at G and the buoyant force F_B acting at B as shown in Figure 5.9(a). These two forces are equal and opposite and the points G and B lie along the same vertical line which is the normal axis.

When this body is given a small angular displacement (or angle of heel), such as α in clockwise direction, the centre of buoyancy moves to a new position B_1 and thus, the buoyant force acts in a vertical upward direction at this new point. Now if a vertical line is drawn through the new centre of buoyancy B_1, then it intersects the normal axis of the body through BG at point M, which is called the metacentre.

Thus, metacentre may also be defined as the point of intersection between the normal axis of the floating body which passes through the points B and G and a vertical line passing through the new centre of buoyancy B_1. The position of the metacentre practically remains constant for small values of angular displacement.

5.5 ❐ METACENTRIC HEIGHT AND METHODS OF ITS DETERMINATION

Metacentric height is the distance between the centre of gravity G and the metacentre M of a floating body. In Figure 5.9(b), GM is the metacentric height. The normal ranges of metacentric heights for different ships are (i) sailing ships: 0.45 to 1.25 m, (ii) battle ships: 1 to 1.5 m, (iii) merchants ships: 0.3 to 1 m and (iv) river crafts: up to 3.5 m.

The metacentric height of a floating body can be determined by either of the methods, such as (i) analytical method and (ii) experimental method.

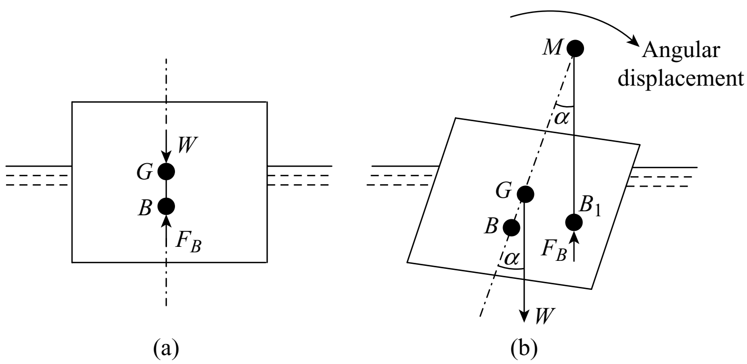


Figure 5.9 *Metacentre and metacentric height*

5.5.1 Analytical Method

Figure 5.10(a and c) illustrates the front view and top view of a floating body (boat or a ship) in equilibrium at the water surface in which points G and B lie on the normal vertical axis. Let l be the length, b be the width or breadth, d be the depth or height and h be the depth of immersion of the floating body.

Figure 5.10(b) illustrates the position of the floating body after it has been given a small angular displacement α in the clockwise direction in which B_1 is the new centre of buoyancy. The vertical line through B_1 cuts the normal axis (axis of symmetry) at M, i.e., the metacentre and GM is the metacentric height. In the tilted position, the wedge shaped portion POP' comes out of the water on the left of the axis, whereas the wedge shaped portion QOQ' goes inside water on the right of the axis. The wedge shaped portions represent gain or loss in buoyant force on either side of the axis of symmetry. The buoyancy force dF_B acts at the centre of gravity of the displaced portions POP' in the downward direction, thus there is a loss in buoyancy force on the left side. The buoyancy force (dF_B) acts at the centre of gravity of the displaced portions QOQ' in the upward direction. Thus, there is a gain in buoyancy force on the right side. The angle of heel α is such that $\alpha \approx \tan\alpha \approx \sin\alpha$.

The weight of the body W acting at G and the buoyant force F_B acting at B (Figure 5.10(a)) are equal and opposite. If v is the volume of water displaced by the body, we get the following expression.

$$F_B = W = \rho_w g v$$

The moment due to displacement of centre of buoyancy from B to B_1 is given by,

$$= F_B \times BB_1 = F_B \times BM \sin\alpha = F_B BM\alpha = \rho_w g v \times BM\alpha \quad (5.3)$$

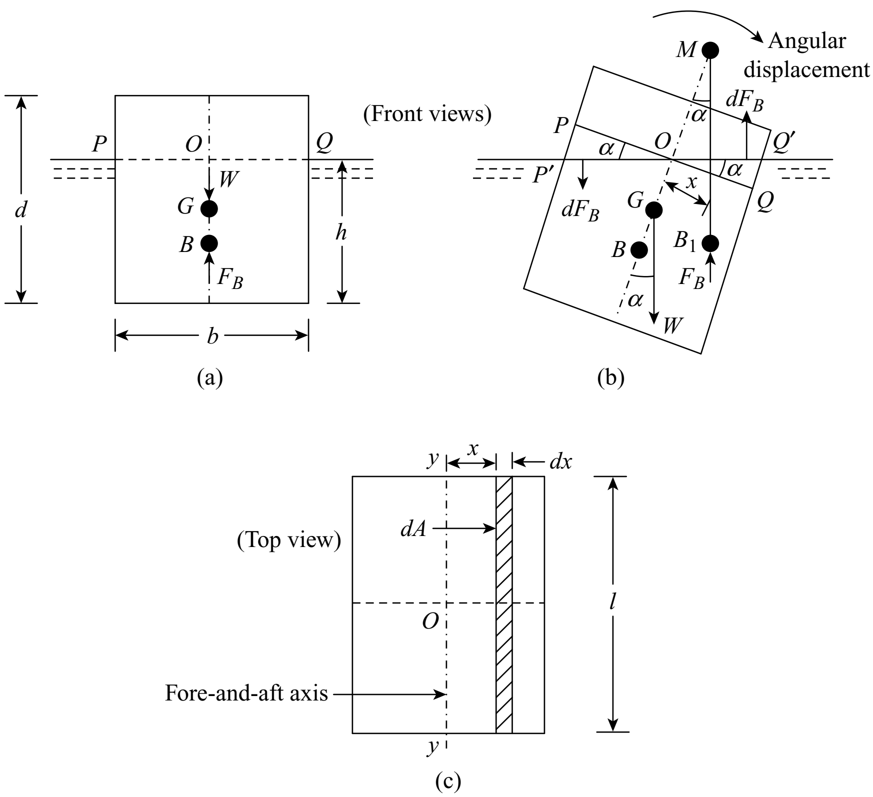


Figure 5.10 *Metacentric height of a floating body*

Now consider an elemental strip of thickness dx and area $dA = l \times dx$ in the top view at a distance x from the axis y–y as shown in Figure 5.10(c), where l is the length of the floating body. The height of the elemental strip (dh) in the front view at a distance x (Figure 5.10(b)) is given by the following expression.

$$dh = x \times \angle QOQ' = x \times \alpha$$

The volume of the elemental strip (dv) is given by,

$$dv = dA \times dh = dA \times x\alpha$$

The weight of the elemental strip (dW) is given by,

$$dW = \rho_w \times dv \times g = \rho_w \times dAx\alpha \times g$$

Thus, the gain in buoyant force is given by,

$$dF_B = dW = \rho_w g\alpha x dA \tag{5.4}$$

Similarly, if a small elemental area dA is considered at a distance x from the axis $y-y$ towards the left of the axis, then the weight of the strip will be the same. The loss in buoyant force acting on this element is the weight dW of the element below the waterline and it is also given by Equation (5.4). These two buoyant forces acting in opposite directions constitute a couple which is given by the following expression.

$$\Rightarrow dF_B \times (x + x) = 2dF_B x = 2 \times \rho_w g\alpha x dA \times x = 2\rho_w g\alpha x^2 dA$$

Therefore, the moment of the couple for the whole wedge is as follows.

$$\Rightarrow \int 2\rho_w g\alpha x^2 dA \tag{5.5}$$

In Equation (5.5), $2\int x^2 dA$ is the second moment of area (or moment of inertia) of horizontal sectional area of the body at the water surface about its longitudinal axis $y-y$ and it is denoted by I. It is pertinent to mention here that the body of ships or boats is more stable about $x-x$ axis (transverse axis) than $y-y$ axis (longitudinal axis). Therefore, the rotational stability about $y-y$ axis is considered in practice.

From Equation (5.5), we get:

$$\Rightarrow \rho_w g\alpha \times I \tag{5.6}$$

The moment of the couple due to buoyant forces dF_B must be equal to the moment caused by the displacement of the centre of buoyancy from B to B_1. From Equations (5.3) and (5.6), we get:

$$\rho_w gvBM\alpha = \rho_w g\alpha I$$

$$\boxed{BM = \frac{I}{v}} \tag{5.7}$$

Here, sometimes BM is also known as metacentric radius.

The metacentric height is given by,

$$\boxed{GM = BM - BG = \frac{I}{v} - BG} \tag{5.8}$$

If the centre of gravity G lies below the centre of buoyancy B, then the metacentric height is given below.

$$\boxed{GM = \frac{I}{v} + BG} \tag{5.9}$$

5.5.2 Experimental Method

The Figure 5.11(a) illustrates a floating body (a vessel or a ship or a boat) in equilibrium at the water surface in which points G and B lie on the normal vertical axis and the top surface of the body is horizontal. Let w_1 be a movable weight placed centrally on the floating body and W be the total weight of the body including the movable weight w_1.

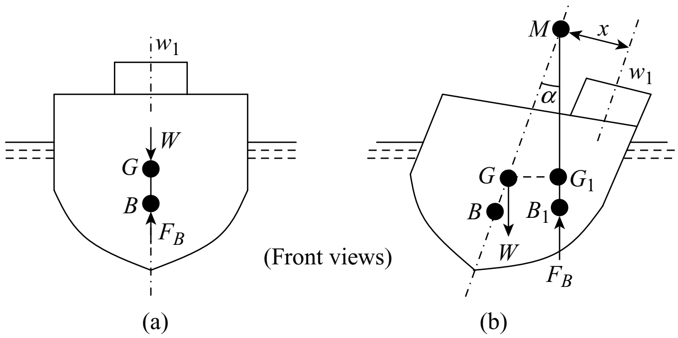


Figure 5.11 *Experimental method for metacentric height of a floating body*

Now the weight w_1 is moved transversely through a distance x so that the body tilts through a small angle α and attains a new equilibrium position. The angle α can be measured with the help of a plumb line and a protractor provided on the floating body. The movement of w_1 through distance x to the right of the axis changes the centre of gravity of the body from G to G_1 and the centre of buoyancy from B to B_1 as shown in Figure 5.11(b). Under equilibrium, the moment due to change in position of w_1 and the moment due to change of G to G_1 will be equal.

Thus
$$w_1 \times x = W \times GG_1 = W \times GM \tan\alpha$$

$$\boxed{\therefore GM = \frac{w_1 x}{W \tan\alpha}} \tag{5.10}$$

Example 5.12 If a wooden block (specific gravity = 0.65) of size 4 m × 2 m × 1.6 m floats in water, then determine (i) the weight of the wooden block and (ii) its metacentric height.

Solution

Refer Figure 5.12. Let $S_{\text{wood}} = 0.65$, $l = 4$ m, $b = 2$ m and $d = 1.6$ m.

Let W be the weight of the wooden block which is equal to the weight of water displaced, GM be the metacentric height and h be the depth of immersion.

(i) Weight of the wooden block = Weight of water displaced

$$S_{\text{wood}}\,\rho_w g \times \text{Volume of block} = \rho_w g \times \text{Volume of block in water}$$

$$S_{\text{wood}}\rho_w g \times (lbd) = \rho_w g \times (lbh)$$

Thus $0.65 \times 1000 \times 9.81 \times (4 \times 2 \times 1.6) = 1000 \times 9.81 \times (4 \times 2 \times h)$

$$\therefore h = 0.65 \times 1.6 = 1.04 \text{ m}$$

$$W = S_{\text{wood}}\,\rho_w g(lbh) = 0.65 \times 1000 \times 9.81 \times (4 \times 2 \times 1.04) = \mathbf{53052.48\ N}$$

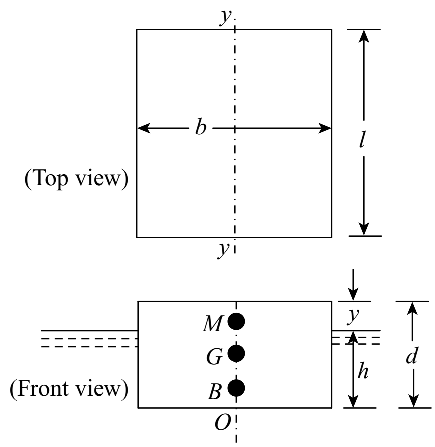


Figure 5.12

(ii) Moment of inertia of the top view at water surface about $y-y$ is given by,

$$I = \frac{lb^3}{12} = \frac{4 \times 2^3}{12} = 2.667 \text{ m}^4$$

$$v = lbh = 4 \times 2 \times 1.04 = 8.32 \text{ m}^3$$

$$BM = \frac{I}{v} = \frac{2.667}{8.32} = 0.3205 \text{ m}$$

$$OM = OB + BM = \frac{h}{2} + BM = \frac{1.04}{2} + 0.3205 = 0.8405 \text{ m}$$

$$OG = \frac{d}{2} = \frac{1.6}{2} = 0.8 \text{ m}$$

$$\therefore GM = OM - OG = 0.8405 - 0.8 = \mathbf{0.0405\ m}$$

Example 5.13 A rectangular barge of dimensions 10 m × 3 m weighs 75 tons and its centre of gravity lies 1.3 m above the bottom. Determine the metacentric height when it floats in fresh water.

Solution

Refer Figure 5.12. Let $l = 10$ m, $b = 3$ m, $m = 75 \text{ tons} = 75 \times 10^3$ kg and $OG = 1.3$ m.

Let GM be the metacentric height and h be the depth of immersion.

$$W = mg = (75 \times 10^3 \times 9.81)\ N$$

Volume of water displaced by the barge is given by,

$$v = \frac{W}{\rho_w g} = \frac{75 \times 10^3 \times 9.81}{1000 \times 9.81} = 75 \text{ m}^3$$

$$10 \times 3 \times h = 75 \quad [\because v = lbh]$$

$$\therefore h = \frac{75}{10 \times 3} = 2.5 \text{ m}$$

Thus $$OB = \frac{h}{2} = \frac{2.5}{2} = 1.25 \text{ m}$$

Moment of inertia of the top view at water surface about $y - y$ is given by,

$$I = \frac{lb^3}{12} = \frac{10 \times 3^3}{12} = 22.5 \text{ m}^4$$

$$BM = \frac{I}{v} = \frac{22.5}{75} = 0.3 \text{ m}$$

$$OM = OB + BM = 1.25 + 0.3 = 1.55 \text{ m}$$

$$GM = OM - OG = 1.55 - 1.3 = \mathbf{0.25\ m}$$

Example 5.14 A rectangular pontoon of length 20 m and weight 2750 kN floats in fresh water of specific weight 10 kN/m^3. Its centre of gravity lies 25 cm above the centre of cross section and for 10° angle of heel its metacentric height is 1 m. If 0.6 m height portion of the pontoon is lying outside water, then determine its breadth and height.

Solution

Refer Figure 5.12. Let $l = 20$ m, $W = 2750$ kN, $w = 10 \text{ kN/m}^3, OG = (d/2) + 0.25$, $\alpha = 10°$, $GM = 1$ m and $y = 0.6$ m, where d is the depth or height of the pontoon and b is its width or breadth and h is the depth of immersion.

Total volume of water displaced is given by,

$$v = \frac{W}{w} = \frac{2750}{10} = 275 \text{ m}^3$$

$$OB = \frac{h}{2} = \frac{d-y}{2} = \frac{d-0.6}{2}$$

$$BG = OG - OB = \left(\frac{d}{2} + 0.25\right) - \left(\frac{d-0.6}{2}\right) = 0.55$$

$$BM = \frac{I}{v} = \frac{(lb^3)/12}{v} = \frac{(20b^3)/12}{275} = \frac{b^3}{165}$$

Since $GM = BM - BG$

$$1 = \frac{b^3}{165} - 0.55$$

$$\therefore b = \sqrt[3]{1.55 \times 165} = \mathbf{6.35\ m}$$

Since $v = l \times b \times h = l \times b \times (d-y) = l \times b \times (d-0.6)$

$$275 = 20 \times 6.35 \times (d-0.6)$$

$$\therefore d = \frac{275}{20 \times 6.35} + 0.6 = \mathbf{2.765\ m}$$

Example 5.15 A body has cylindrical upper portion of diameter 2.5 m and it is 1.5 m deep. The lower portion is a curved one which displaces a volume of 500 litres of water and its centre of buoyancy is at a distance of 1.6 m below the top of the cylinder. The centre of gravity of the whole body is 1 m below the top of the cylinder and the total displacement of water is 32.5 kN. Determine the metacentric height of the body if the specific weight of sea water is 10 kN/m^3.

Solution

Refer Figure 5.13. Let $D = 2.5$ m, $d_1 = 1.5$ m, $v_1 = 500$ litres $= 0.5$ m^3, $OB_1 = 1.6$ m, $OG = 1$ m, $W = 32.5$ kN and $w = 10$ kN/m^3.

Let GM be the metacentric height, B be the centre of buoyancy of the whole body and y be the distance between the water surface and top of the body.

The total volume of water displaced is given by,

$$v = \frac{W}{w} = \frac{32.5}{10} = 3.25\ \text{m}^3$$

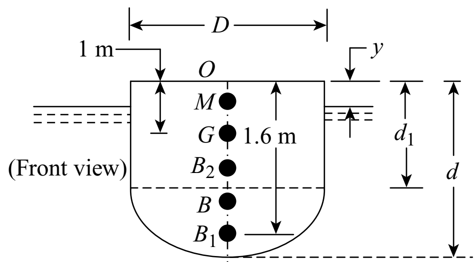


Figure 5.13

The volume of water displaced by cylindrical portion is given by,

$$v_2 = v - v_1 = 3.25 - 0.5 = 2.75\ \text{m}^3$$

$$v_2 = \frac{\pi}{4} D^2 \times (d_1 - y)$$

Thus $2.75 = \frac{\pi}{4} \times 2.5^2 \times (1.5 - y)$

$$\therefore y = 1.5 - \frac{2.75 \times 4}{\pi \times 2.5^2} = 0.94\ \text{m}$$

The distance of the centre of buoyancy of the cylindrical portion from top of the body is given below.

$$OB_2 = y + \frac{d_1 - y}{2} = 0.94 + \frac{1.5 - 0.94}{2} = 1.22 \text{ m}$$

Now $$v \times OB = v_1 \times OB_1 + v_2 \times OB_2$$

or $$(v_1 + v_2) \times OB = v_1 \times OB_1 + v_2 \times OB_2$$

Thus $$OB = \frac{v_1 \times OB_1 + v_2 \times OB_2}{v_1 + v_2} = \frac{0.5 \times 1.6 + 2.75 \times 1.22}{0.5 + 2.75} = 1.2785 \text{ m}$$

$$BG = OB - OG = 1.2785 - 1 = 0.2785 \text{ m}$$

$$I = \frac{\pi D^4}{64} = \frac{\pi \times 2.5^4}{64} = 1.9175 \text{ m}^4$$

$$BM = \frac{I}{v} = \frac{1.9175}{3.25} = 0.59 \text{ m}$$

$$GM = BM - BG = 0.59 - 0.2785 = \mathbf{0.3115 \text{ m}}$$

Example 5.16 A vessel that is 49 m long and 7 m broad has a displacement of 12800 kN in sea water (specific gravity = 1.02). The vessel tilts through 6.1° when a weight of 160 kN moves through a distance of 5 m. If the second moment of area of the waterline section about its fore-and-aft axis is 75% than that of the circumscribing rectangle and centre of buoyancy is 1.5 m below the waterline, then determine the metacentric height and the position of centre of gravity of the vessel.

Solution

Refer Figure 5.14. Let $l = 49$ m, $b = 7$ m, $W = 12800$ kN, $S_{sea} = 1.02$, $\alpha = 6.1°$, $w_1 = 160$ kN, $x = 5$ m, $I = 0.75 \times 2^{nd}$ M.I. of water line section and $OB = 1.5$ m.

Let v be the volume of water displaced by the vessel and GM be its metacentric height.

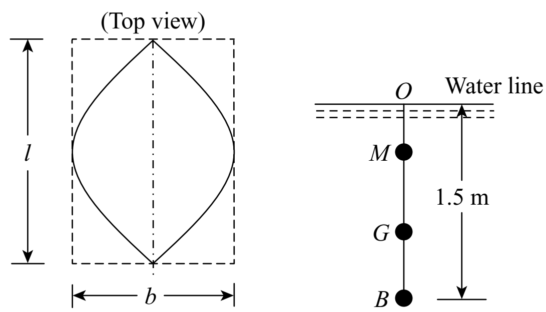


Figure 5.14

$$GM = \frac{w_1 x}{W \tan\alpha} = \frac{160 \times 5}{12800 \tan 6.1°} = \mathbf{0.585 \text{ m}}$$

$$I = 0.75 \times \frac{lb^3}{12} = 0.75 \times \frac{49 \times 7^3}{12} = 1050.44 \text{ m}^4$$

$$v = \frac{W}{S_{sea} \rho_w g} = \frac{12800 \times 10^3}{1.02 \times 1000 \times 9.81} = 1279.21 \text{ m}^3$$

$$BM = \frac{I}{v} = \frac{1050.44}{1279.21} = 0.8212 \text{ m}$$

Since $OG = OM + GM = (OB - BM) + GM$

$\therefore OG = (1.5 - 0.8212) + 0.585 =$ **1.2638 m below the waterline**

Example 5.17 A cube of side 2 m floats in a liquid with half of its volume immersed and the bottom face being horizontal. The weight 360 N is moved on to the middle point of one of the top edges of the cube. Find the angle through which the cube tilts under the action of weight, if the centre of gravity of the cube is 0.65 m below the geometric centre in a vertical line through it.

Solution

Refer Figure 5.15. Let $l = b = d = 2$ m, $v = (l^3/2)$ m^3, $w_1 = 360$ N, $x = 2/2 = 1$ m and $h_1 = 0.65$ m.

Let v be the volume of water displaced by the cube, W be the weight of water displaced, GM be its metacentric height and h be the depth of immersion.

Since half of the volume of cube is immersed, we get the below expression.

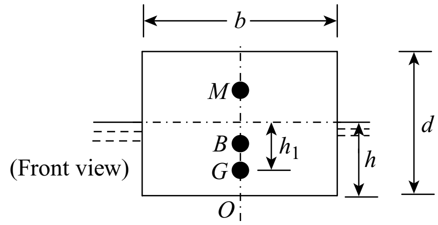


Figure 5.15

$$h = \frac{d}{2} = \frac{2}{2} = 1 \text{ m}$$

$$OB = \frac{h}{2} = \frac{1}{2} = 0.5 \text{ m}$$

$$OG = \frac{d}{2} - h_1 = 1 - 0.65 = 0.35 \text{ m}$$

$$GB = OB - OG = 0.5 - 0.35 = 0.15 \text{ m}$$

$$v = \frac{l^3}{2} = \frac{2^3}{2} = 4 \text{ m}^3 \quad \text{(Since half of the volume of cube is immersed)}$$

$$W = \rho_w g v = 1000 \times 9.81 \times 4 = 39240 \text{ N}$$

$$I = \frac{lb^3}{12} = \frac{2 \times 2^3}{12} = 1.333 \text{ m}^4$$

$$BM = \frac{I}{v} = \frac{1.333}{4} = 0.333 \text{ m}$$

$$GM = BM + BG = 0.333 + 0.15 = 0.483 \text{ m}$$

Since

$$GM = \frac{w_1 x}{W \tan\alpha}$$

$$0.483 = \frac{360 \times 1}{39240 \tan\alpha}$$

$$\tan\alpha = \frac{360 \times 1}{39240 \times 0.483} = 0.018994$$

$$\therefore \alpha = \tan^{-1}(0.018994) = \mathbf{1.09°}$$

5.6 ❒ STABILITY OF SUBMERGED AND FLOATING BODIES

The stability of a submerged or a floating body means the tendency of the body to return to its original position after a slight displacement caused by any external force. A submerged or a floating body may have any of the following three equilibrium conditions.

1. **Stable equilibrium:** The body will have stable equilibrium when a small angular displacement of the body sets up a restoring couple tending to bring back the body to its original equilibrium position.
2. **Unstable equilibrium:** The body will have unstable equilibrium when a small angular displacement of the body sets up a couple that tends to displace the body further and thereby, not allowing the body to its original equilibrium position.

3. **Neutral equilibrium:** The body will have neutral equilibrium when a small angular displacement of the body does not set up couple of any kind and the body takes new position without either returning to its original position or increasing the displacement.

5.6.1 Stability of a Submerged Body

The stability of a submerged body such as a balloon submerged in air or a submarine submerged in water is determined by the relative position of the centre of gravity G and centre of buoyancy B of the body. The centre of gravity and the centre of buoyancy of a wholly submerged body remain fixed. The conditions for stability of a submerged body are listed below.

1. The body remains in stable equilibrium when G lies below B.
2. The body remains in unstable equilibrium when G lies above B.
3. The body remains in neutral equilibrium when G coincides with B.

The Figure 5.16(a) shows a balloon fully submerged in air whose lower portion contains a heavier material, so that its centre of gravity is lower than its centre of buoyancy. The balloon is in equilibrium due to its weight W acting at G is equal to the buoyant force F_B acting at B. When a clockwise angular displacement is given to the balloon, W and F_B makes an anticlockwise couple (or restoring couple) that brings the balloon to its original position as shown in Figure 5.16(a). Thus, the balloon has a stable equilibrium.

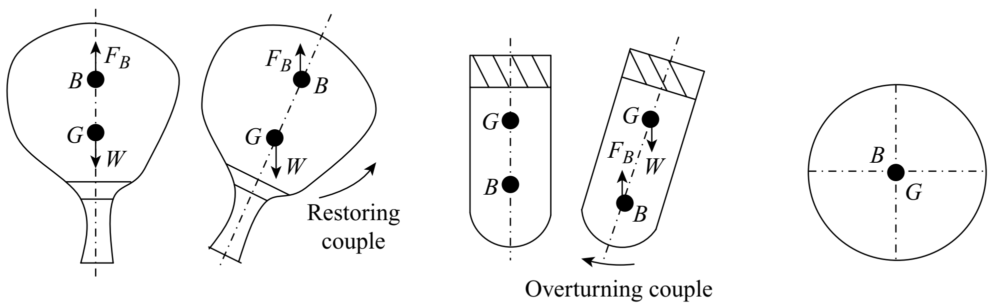
The Figure 5.16(b) shows a test tube fitted with a heavy stopper and immersed in a liquid. In this case, the centre of gravity lies above the centre of buoyancy. When a small displacement is given to the body, an overturning couple is formed which tends to tilt the tube further. Thus, the body has unstable equilibrium.

The Figure 5.16(c) shows a homogeneous spherical body submerged in a liquid whose centre of gravity and centre of buoyancy coincide. When a small displacement is given to the body, it assumes a new position. Thus, the body has neutral equilibrium.

5.6.2 Stability of a Floating Body

The stability of a floating body is determined by the relative position of the centre of gravity G and the metacentre M of the body. Thus, the stability of a floating body differs from the stability of a submerged body, where the body may be in stable equilibrium even when its centre of gravity lies above the centre of buoyancy. The conditions for stability of a floating body are listed below.

1. The body remains in stable equilibrium when G lies below M.
2. The body remains in unstable equilibrium when G lies above M.
3. The body remains in neutral equilibrium when G coincides with M.



Stable equilibrium (a) Unstable equilibrium (b) Neutral equilibrium (c)

Figure 5.16

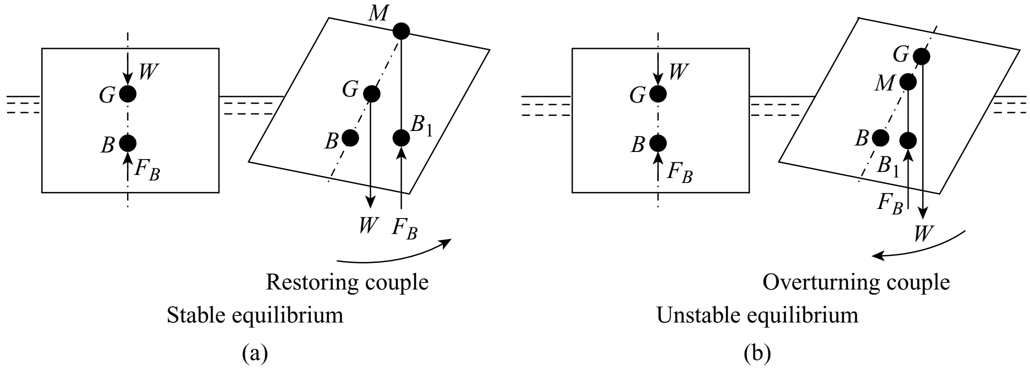


Figure 5.17

Figure 5.17(a) shows a floating body which has undergone a small angular displacement in the clockwise direction so that its new centre of buoyancy B_1 is such that the metacentre M lies above the centre of gravity G of the body. The body weight W and the buoyant force F_B make an anticlockwise couple (or restoring couple) that brings the body to its original position as shown in Figure 5.17(a). Thus, the body has a stable equilibrium.

Figure 5.17(b) shows a floating body in which the centre of gravity lies above the metacentre of the body. When a small displacement is given to the body, an overturning couple is formed which tends to tilt the body further. Thus, the body has unstable equilibrium.

For a floating body when the centre of gravity coincides with metacentre of the body, there will be neither a restoring couple nor an overturning couple formed when a small angular displacement is given to the body. Thus, the body will adapt to new position and it will have neutral equilibrium.

Example 5.18 A wooden solid cylinder (specific gravity = 0.65) of diameter 2 m and height 1.5 m floats in water with its axis vertical. Determine its metacentric height and also comment on its equilibrium.

Solution

Refer Figure 5.18. Let $S_{\text{wood}} = 0.65$, $D = 2$ m and $d = 1.5$ m. Let v be the volume of water displaced by the cylinder, GM be its metacentric height and h be the depth of immersion.

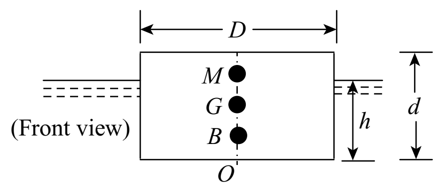


Figure 5.18

Weight of the wooden cylinder = Weight of water displaced

or $\quad S_{\text{wood}}\,\rho_w g \times \text{Volume of block} = \rho_w g \times \text{Volume of water displaced}$

$$S_{\text{wood}}\,\rho_w g \times \frac{\pi}{4} D^2 d = \rho_w g \times \frac{\pi}{4} D^2 h$$

Thus $\quad h = S_{\text{wood}}\, d = 0.65 \times 1.5 = 0.975$ m

Moment of inertia of the top view at water surface about $y-y$ is given by,

$$I = \frac{\pi D^4}{64} = \frac{\pi \times 2^4}{64} = 0.7854 \text{ m}^4$$

$$v = \frac{\pi}{4} D^2 \times h = \frac{\pi}{4} \times 2^2 \times 0.975 = 3.063 \text{ m}^3$$

$$BM = \frac{I}{v} = \frac{0.7854}{3.063} = 0.2564 \text{ m}$$

$$BG = OG - OB = \frac{d}{2} - \frac{h}{2} = \frac{1.5}{2} - \frac{0.975}{2} = 0.2625 \text{ m}$$

$$GM = BM - BG = 0.2564 - 0.2625 = \mathbf{-0.0061 \text{ m}}$$

The negative sign shows that the metacentre lies below the centre of gravity and thus, the cylinder is in unstable equilibrium.

Example 5.19 A rectangular pontoon that is 10 m long, 7.5 m wide and 2.5 m deep weighs 750 kN and floats in water having a specific weight of 10 kN/m^3. The pontoon carries on its upper deck a boiler of diameter 5 m and it weighs 600 kN. The centre of gravity of each unit coincides with the geometrical centre of the arrangement and it lies in the same vertical line. Determine the metacentric height and comment on the stability of this arrangement.

Solution

Refer Figure 5.19. Let $l = 10$ m, $b = 7.5$ m, $d = 2.5$ m, $W_1 = 750$ kN, $w = 10$ kN/m^3, $D = 5$ m and $W_2 = 600$ kN.

Let v be the volume of water displaced by the arrangement and W be its weight, GM be its metacentric height and h be the depth of immersion.

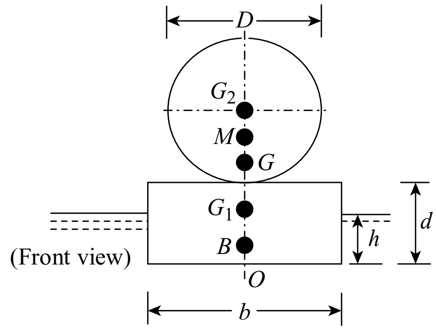


Figure 5.19

$$W = W_1 + W_2 = 750 + 600 = 1350 \text{ kN}$$

$$v = \frac{W}{w} = \frac{1350}{10} = 135 \text{ m}^3$$

Weight of the arrangement = Weight of water displaced by the arrangement

or $\quad W = w \times$ Volume of arrangement in water $= w \times lbh$

Thus $\quad 1350 = 10 \times (10 \times 7.5 \times h)$

$$\therefore h = \frac{1350}{10 \times 10 \times 7.5} = 1.8 \text{ m}$$

$$I = \frac{lb^3}{12} = \frac{10 \times 7.5^3}{12} = 351.5625 \text{ m}^4$$

$$BM = \frac{I}{v} = \frac{351.5625}{135} = 2.6042 \text{ m}$$

$$OB = \frac{h}{2} = \frac{1.8}{2} = 0.9 \text{ m}$$

$$\therefore OM = OB + BM = 0.9 + 2.6042 = 3.5042 \text{ m}$$

$$OG_1 = \frac{d}{2} = \frac{2.5}{2} = 1.25 \text{ m}$$

$$OG_2 = d + \frac{D}{2} = 2.5 + \frac{5}{2} = 5 \text{ m}$$

The position of the combined centre of gravity G above the base point O can be determined by taking the moments about point O and we get the below expression.

$$W \times OG = W_1 \times OG_1 + W_2 \times OG_2$$

$$\therefore OG = \frac{W_1 \times OG_1 + W_2 \times OG_2}{W} = \frac{750 \times 1.25 + 600 \times 5}{1350} = 2.9167 \text{ m}$$

Since $OM > OG$, i.e., M lies above G and thus, the arrangement is stable.

$$GM = OM - OG = 3.5042 - 2.9167 = \mathbf{0.5875\ m}$$

Example 5.20 A solid cylinder of diameter 3 m and length 3 m consists of two parts made of different materials. Its base part is 0.1 m long and has specific gravity 6. The remaining part of the cylinder is of specific gravity 0.5. State whether it can float vertically in water.

Solution

Refer Figure 5.20. Let $D = 3$ m, $d = 3$ m, $d_1 = 0.1$ m, $S_1 = 6$, $d_2 = 2.9$ m and $S_2 = 0.5$.

Let G_1 and G_2 be the centre of gravity of the two parts, W_1 and W_2 be the weight of the two parts, G be the combined centre of gravity, W be the weight of the body, v be the volume of water displaced, GM be its metacentric height and h be the depth of immersion.

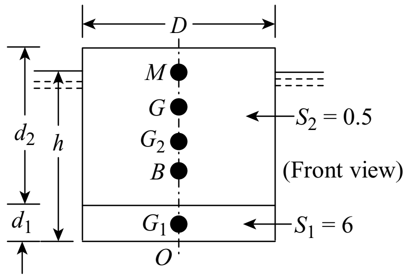


Figure 5.20

$$W = W_1 + W_2 = S_1 \rho_w g \times \frac{\pi}{4} D^2 d_1 + S_2 \rho_w g \times \frac{\pi}{4} D^2 d_2$$

$$W = 6 \times 1000 \times 9.81 \times \frac{\pi}{4} \times 3^2 \times 0.1 + 0.5 \times 1000 \times 9.81 \times \frac{\pi}{4} \times 3^2 \times 2.9$$

$$\therefore W = 41.606 + 100.547 = 142.153 \text{ kN}$$

$$OG_1 = \frac{d_1}{2} = \frac{0.1}{2} = 0.05 \text{ m}$$

$$OG_2 = d_1 + \frac{d_2}{2} = 0.1 + \frac{2.9}{2} = 1.55 \text{ m}$$

Taking moments about point O, we get:

$$W \times OG = W_1 \times OG_1 + W_2 \times OG_2$$

$$\therefore OG = \frac{W_1 \times OG_1 + W_2 \times OG_2}{W} = \frac{41.606 \times 0.05 + 100.547 \times 1.55}{142.153} = 1.111 \text{ m}$$

Weight of the body = Weight of water displaced

or $$W = \rho_w g \times \text{Volume of water displaced} = \rho_w g \times \frac{\pi}{4} D^2 h$$

$$142.153 \times 10^3 = 1000 \times 9.81 \times \frac{\pi}{4} \times 3^2 \times h$$

$$\therefore h = \frac{142.153 \times 10^3 \times 4}{1000 \times 9.81 \times \pi \times 3^2} = 2.05 \text{ m}$$

$$v = \frac{\pi}{4} D^2 \times h = \frac{\pi}{4} \times 3^2 \times 2.05 = 14.49 \text{ m}^3$$

$$OB = \frac{h}{2} = \frac{2.05}{2} = 1.025 \text{ m}$$

$$BG = OG - OB = 1.111 - 1.025 = 0.086 \text{ m}$$

$$I = \frac{\pi D^4}{64} = \frac{\pi \times 3^4}{64} = 3.976 \text{ m}^4$$

$$BM = \frac{I}{v} = \frac{3.976}{14.49} = 0.2744 \text{ m}$$

$$GM = BM - BG = 0.2744 - 0.086 = \mathbf{0.1884 \text{ m}}$$

As GM is positive which indicates that the metacentre lies above the centre of gravity and thus, the cylinder is in stable equilibrium and it can float vertically in water.

Example 5.21 A wooden cylinder (specific gravity = 0.6) of diameter d and length l is required to float in oil (specific gravity = 0.8). Show that l cannot exceed about $0.8165d$ for the cylinder to float when its longitudinal axis being vertical.

Solution

Refer Figure 5.21. Let $S_{\text{wood}} = 0.6$, diameter $= d$, length $= l$, and $S_{\text{oil}} = 0.8$.

Let v be the volume of oil displaced, GM be the metacentric height of cylinder and h be the depth of immersion.

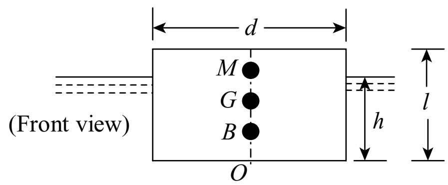


Figure 5.21

Weight of the cylinder = Weight of oil displaced

or $S_{\text{wood}} \rho_w g \times$ Volume of cylinder $= S_{\text{oil}} \rho_w g \times$ Volume of oil displaced

$$0.6 \times 1000 \times 9.81 \times \frac{\pi}{4} d^2 l = 0.8 \times 1000 \times 9.81 \times \frac{\pi}{4} d^2 h$$

$$\therefore h = \frac{0.6 \times l}{0.8} = \frac{3}{4} l$$

$$OB = \frac{h}{2} = \frac{1}{2} \times \frac{3}{4} l = \frac{3}{8} l$$

$$OG = \frac{l}{2}$$

$$BG = OG - OB = \frac{l}{2} - \frac{3}{8} l = \frac{l}{8}$$

$$BM = \frac{I}{v} = \frac{(\pi/64) d^4}{(\pi/4) d^2 \times (3/4) l} = \frac{d^2}{12l}$$

$$GM = BM - BG = \frac{d^2}{12l} - \frac{l}{8}$$

For stable equilibrium, $GM > 0$ and we get:

$$\frac{d^2}{12l} - \frac{l}{8} > 0$$

$$l^2 < \frac{8}{12} d^2$$

$$l < \sqrt{\frac{8}{12} d^2}$$

$$\therefore \boldsymbol{l < 0.8165d}$$

Example 5.22 A wooden cylinder (specific gravity = 0.6) of diameter *d* and length *l* twice of its diameter is required to float in water with its longitudinal axis vertical. Comment on its stability and also determine the metacentric height and locate the metacentre with reference to water surface.

Solution

Refer Figure 5.22. Let $S_{\text{wood}} = 0.6$, diameter $= d$ and length $= l = 2d$.

Let v be the volume of water displaced, GM be the metacentric height of cylinder and h be the depth of immersion.

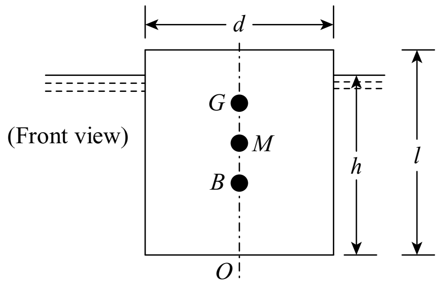


Figure 5.22

$$\text{Weight of the cylinder} = \text{Weight of water displaced}$$

or $\quad S_{\text{wood}}\,\rho_w g \times \text{Volume of cylinder} = \rho_w g \times \text{Volume of water displaced}$

$$0.6 \times 1000 \times 9.81 \times \frac{\pi}{4} d^2 \times 2d = 1000 \times 9.81 \times \frac{\pi}{4} d^2 h$$

$$\therefore h = 0.6 \times 2d = 1.2d$$

$$OB = \frac{h}{2} = \frac{1.2d}{2} = 0.6d$$

$$OG = \frac{l}{2} = \frac{2d}{2} = d$$

$$BG = OG - OB = d - 0.6d = 0.4d$$

$$BM = \frac{I}{v} = \frac{(\pi/64)d^4}{(\pi/4)d^2 \times 1.2d} = 0.0521d$$

Since *BM* < *BG*, i.e., the metacentre lies below the centre of gravity and thus, the cylinder is in unstable equilibrium.

$$GM = BM - BG = 0.0521d - 0.4d = \mathbf{-0.3479}\boldsymbol{d}$$

The depth of the metacentre below the water surface is given by,

$$= 0.6d - 0.0521d = \mathbf{0.5479}\boldsymbol{d}$$

Example 5.23 If a hollow cylinder (specific gravity = 7.7) closed at both ends of outside diameter 1.25 m and length 3.5 m floats with its axis vertical just in equilibrium in sea water (specific gravity = 1.02), then find its minimum permissible thickness.

Solution

Refer Figure 5.23. Let $S_c = 7.7$, $D = 1.25$ m, $d = 3.5$ m and $S_{\text{sea}} = 1.02$.

Let t be the thickness of cylinder, v be the volume of water displaced, W be the weight of water displaced and h be the depth of immersion.

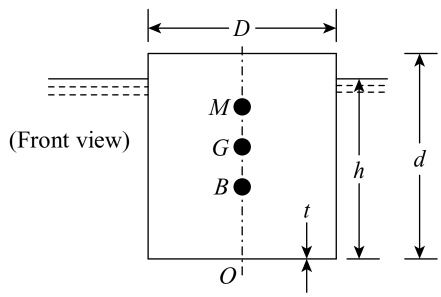


Figure 5.23

$$W = S_{\text{sea}}\rho_w g \times v = S_{\text{sea}}\rho_w g \times \frac{\pi}{4} D^2 h$$

$$W = 1.02 \times 1000 \times 9.81 \times \frac{\pi}{4} \times 1.25^2 \times h = 12279.455h \qquad \text{(i)}$$

The weight of the cylinder is given by,

$$W_c = S_c \rho_w g \left[2 \times \frac{\pi}{4} D^2 t + \frac{\pi}{4}\{D^2 - (D - 2t)^2\} d \right]$$

$$W_c = S_c \rho_w g \left[2 \times \frac{\pi}{4} D^2 t + \pi D t d \right] \quad \text{[Neglecting } t^2 \text{]}$$

$$\therefore W_c = 7.7 \times 1000 \times 9.81 \times \left[2 \times \frac{\pi}{4} \times 1.25^2 t + \pi \times 1.25 t \times 3.5 \right] = 1223611.56t \quad \text{(ii)}$$

For equilibrium, equating the expressions (i) and (ii), we get:

$$12279.455h = 1223611.56t$$

$$h = \frac{1223611.56t}{12279.455} = 99.647t$$

$$OB = \frac{h}{2} = \frac{99.647t}{2} = 49.8235t$$

$$v = \frac{W}{S_{sea} \rho_w g} = \frac{1223611.56t}{1.02 \times 1000 \times 9.81} = 122.285t \text{ m}^3$$

$$BM = \frac{I}{v} = \frac{(\pi/64)D^4}{v} = \frac{(\pi/64) \times 1.25^4}{122.285t} = \frac{9.8 \times 10^{-4}}{t}$$

$$BG = OG - OB = 1.75 - 49.8235t \quad [\because OG = d/2]$$

For the cylinder to float just in stable equilibrium: $BG = BM$

$$1.75 - 49.8235t = \frac{9.8 \times 10^{-4}}{t}$$

$$49.8235t^2 - 1.75t + 9.8 \times 10^{-4} = 0$$

$$t = \frac{1.75 \pm \sqrt{(-1.75)^2 - 4 \times 49.8235 \times 9.8 \times 10^{-4}}}{2 \times 49.8235}$$

$$\therefore t = \frac{1.75 \pm 1.6933}{99.647} = 0.03455 \text{ m or } 0.000569 \text{ m}$$

Thus, the minimum permissible thickness of the cylinder is given below.

$$t_{\min} = \mathbf{0.000569 \text{ m or } 0.569 \text{ mm}}$$

Example 5.24 A hollow wooden cylinder (specific gravity = 0.65) has outer and inner diameters as 0.6 m and 0.3 m, respectively. If it is required to float in an oil (specific gravity = 0.8), then determine the maximum height of the cylinder so that it would be stable when it floats with its axis vertical. Also determine the depth to which it will sink.

Solution

Refer Figure 5.24. Let $S_{\text{wood}} = 0.65$, $D_o = 0.6$ m, $D_i = 0.3$ m and $S_{\text{oil}} = 0.8$.

Let d be the height or depth of cylinder, v be the volume of oil displaced and h be the depth of immersion.

$$\text{Weight of the cylinder} = \text{Weight of oil displaced}$$

$$S_{\text{wood}} \rho_w g \times \text{Volume of cylinder} = S_{\text{oil}} \rho_w g \times \text{Volume of oil displaced}$$

$$0.65 \times 1000 \times 9.81 \times \frac{\pi}{4}(D_o^2 - D_i^2)d = 0.8 \times 1000 \times 9.81 \times \frac{\pi}{4}(D_o^2 - D_i^2)h$$

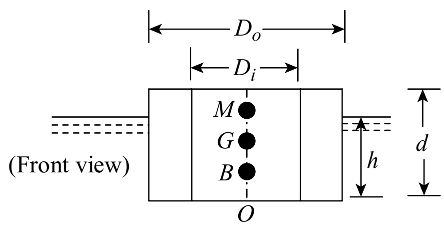


Figure 5.24

$$\therefore h = \frac{0.65d}{0.8} = 0.8125d$$

$$OB = \frac{h}{2} = \frac{0.8125d}{2} = 0.40625d$$

$$OG = \frac{d}{2} = 0.5d$$

$$BM = \frac{I}{v} = \frac{(\pi/64)(D_o^4 - D_i^4)}{(\pi/4)(D_o^2 - D_i^2)h}$$

$$= \frac{(\pi/64)(0.6^4 - 0.3^4)}{(\pi/4)(0.6^2 - 0.3^2)\times 0.8125d} = \frac{0.0346}{d}$$

$$OM = OB + BM = 0.40625d + \frac{0.0346}{d}$$

For stable equilibrium, M should be at a level higher than G, i.e., $OM > OG$ and we get the below expression.

$$0.40625d + \frac{0.0346}{d} > 0.5d$$

$$\frac{0.0346}{d} > 0.09375d \Rightarrow 0.0346 > 0.09375d^2$$

$$d < \sqrt{\frac{0.0346}{0.09375}}$$

$$d < 0.6075 \text{ m}$$

The maximum height of the cylinder is given by,

$$d_{max} = \mathbf{0.6075\ m}$$

$$h = 0.8125d = 0.8125 \times 0.6075 = \mathbf{0.4936\ m}$$

Example 5.25 A cylinder buoy of diameter 1.4 m, length 1 m and weight 4400 N floats in sea water (specific weight = 10 kN/m^3) with its axis being vertical. If a 440 N load is placed centrally at the top of the buoy, then find the maximum permissible height of the centre of gravity of the load above the top of the buoy so that it remains in stable equilibrium.

Solution

Refer Figure 5.25. Let $D = 1.4$ m, $d = 1$ m, $W_1 = 4400$ N, $w = 10$ kN/m^3 and $W_2 = 440$ N.

Let v be the volume of water displaced, h_G be the height of centre of gravity G_2 above the base and h be the depth of immersion.

$$W = W_1 + W_2 = 4400 + 440 = 4840\ N$$

$$v = \frac{W}{w} = \frac{4840}{10\times 10^3} = 0.484\ m^3$$

Weight of the arrangement = Weight of water displaced by the buoy in water

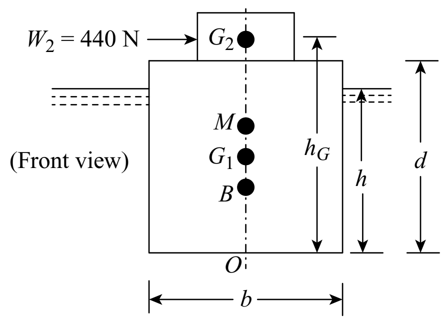


Figure 5.25

or $$W = w \times \text{Volume of water displaced} = w \times \frac{\pi}{4} D^2 h$$

Thus $$4840 = 10 \times 10^3 \times \frac{\pi}{4} \times 1.4^2 \times h$$

$$\therefore h = \frac{4840}{10 \times 10^3 \times (\pi/4) \times 1.4^2} = 0.3144 \text{ m}$$

$$OB = \frac{h}{2} = \frac{0.3144}{2} = 0.1572 \text{ m}$$

$$BM = \frac{I}{v} = \frac{(\pi/64)D^4}{v} = \frac{(\pi/64) \times 1.4^4}{0.484} = 0.3896 \text{ m}$$

$$OM = OB + BM = 0.1572 + 0.3896 = 0.5468 \text{ m}$$

$$OG_1 = \frac{d}{2} = \frac{1}{2} = 0.5 \text{ m}$$

$$OG_2 = h_G$$

The position of the combined centre of gravity G above the base point O can be determined by taking the moments about point O and we get the expression as follows.

$$W \times OG = W_1 \times OG_1 + W_2 \times OG_2$$

$$\therefore OG = \frac{W_1 \times OG_1 + W_2 \times OG_2}{W} = \frac{4400 \times 0.5 + 440 h_G}{4840} = 0.4545 + 0.091 h_G$$

For stable equilibrium, M should lie above G, i.e., $OM > OG$ and we get:

$$0.5468 > 0.4545 + 0.091 h_G$$

$$0.091 h_G < 0.0923$$

$$h_G < \frac{0.0923}{0.091}$$

$$h_G < 1.0143$$

Thus, the height of centre of gravity of the load above the buoy should not be more than

$$= 1.0143 - 1 = \mathbf{0.0143 \text{ m}}$$

Example 5.26 A cylindrical buoy of diameter 1.8 m, length 2.4 m and weight 20 kN is in sea water (specific weight = 10 kN/m^3). (i) Show that the buoy does not float with its axis being vertical. (ii) What minimum pull should be applied to a chain attached to the centre of the base to keep the buoy vertical?

Solution

(i) Refer Figure 5.26(a). Let $D = 1.8$ m, $d = 2.4$ m, $W = 20$ kN and $w = 10$ kN/m^3.

Let v be the volume of water displaced and h be the depth of immersion.

$$v = \frac{W}{w} = \frac{20}{10} = 2 \text{ m}^3$$

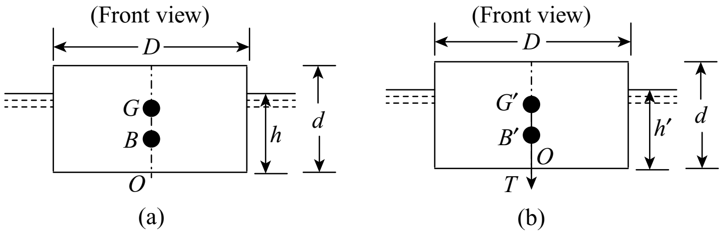


Figure 5.26

Since $$W = w \times \text{Volume of buoy in water} = w \times \frac{\pi}{4} D^2 h$$

$$20 \times 10^3 = 10 \times 10^3 \times \frac{\pi}{4} \times 1.8^2 \times h$$

$$\therefore h = \frac{20 \times 10^3}{10 \times 10^3 \times (\pi/4) \times 1.8^2} = 0.786 \text{ m}$$

$$OB = \frac{h}{2} = \frac{0.786}{2} = 0.393 \text{ m}$$

$$BM = \frac{I}{v} = \frac{(\pi/64)D^4}{v} = \frac{(\pi/64) \times 1.8^4}{2} = 0.2576 \text{ m}$$

$$OM = OB + BM = 0.393 + 0.2576 = 0.6506 \text{ m}$$

$$OG = \frac{d}{2} = \frac{2.4}{2} = 1.2 \text{ m}$$

Since $OM < OG$, i.e., M lies below G and thus, the buoy is unstable and it does not float with vertical axis.

(ii) Refer Figure 5.26(b). Let T be the minimum pull in chain to keep the buoy vertical, F be the total downward force, v' be the volume of water displaced and h' be the new depth of immersion.

$$F = W + T = (20 + T)$$

$$v' = \frac{F}{w} = \frac{F}{10} \text{ m}^3$$

$$F = w \times \text{Volume of buoy in water} = 10 \times \frac{\pi}{4} \times 1.8^2 \times h'$$

$$\therefore h' = \frac{F}{10 \times (\pi/4) \times 1.8^2} = \frac{F}{25.447} \text{ m}$$

$$OB' = \frac{h'}{2} = \frac{F}{2 \times 25.447} = \frac{F}{50.894} \text{ m}$$

$$B'M' = \frac{I}{v'} = \frac{(\pi/64) \times 1.8^4}{(F/10)} = \frac{5.153}{F} \text{ m}$$

Taking moments about point O to obtain the new centre of gravity G', we get the following expression.

$$F \times OG' = W \times OG$$

$$F \times OG' = 20 \times 1.2 \qquad [\because OG = d/2]$$

Thus $$OG' = \frac{24}{F}$$

$$B'G' = OG' - OB' = \frac{24}{F} - \frac{F}{50.894}$$

For stable equilibrium, M' should lie above G' and thus, $B'M' > B'G'$, we get:

$$\frac{5.153}{F} > \frac{24}{F} - \frac{F}{50.894}$$

or $$\frac{F}{50.894} > \frac{24}{F} - \frac{5.153}{F}$$

$$F^2 > 959.2$$

or $$F > \sqrt{959.2}$$

$$\therefore F > 30.971$$

$$T = F - W = 30.971 - 20 = \mathbf{10.971\ kN}$$

Example 5.27 A float valve regulates the flow of oil (specific gravity = 0.85) into a cistern (Figure 5.27). The spherical float is 0.15 m in diameter. *AOB* is weightless link carrying the float at one end and a valve at the other end which closes the pipe through which oil flows into the cistern. The link is mounted in a frictionless hinge at O and the angle AOB is 135°. The length of *OA* is 0.2 m and the distance between the centre of the float and the hinge is 0.5 m. When the flow is stopped *AO* will be vertical. The valve is to be pressed on to the seat with a force of 10 N to completely stop the flow of oil into the cistern. It was observed that the flow of oil is stopped when the free surface of oil in the cistern is 0.35 m below the hinge. Determine the weight of the float.

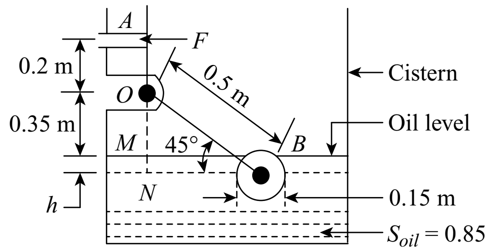


Figure 5.27

Solution

Refer Figure 5.27. Let $S_{\text{oil}} = 0.85$, $D = 0.15$ m, $\angle AOB = 135°$, $OA = 0.2$ m, $OB = 0.5$ m, $F = 10$ N and $OM = 0.35$ m.

Let W be the weight of the float, F_B be the buoyant force which passes through B, h be the depth of its centre below the oil level and v be the volume of oil displaced.

$$\angle NOB = \angle OBN = 180° - 135° = 45°$$

From ΔONB, we get:

$$\frac{0.35 + h}{0.5} = \sin 45°$$

$$h = 0.5 \sin 45° - 0.35 = 0.00355 \text{ m}$$

$$BN = OB \cos 45° = 0.5 \times \frac{1}{\sqrt{2}} = 0.3535 \text{ m}$$

Since $$v = \frac{2}{3}\pi R^3 + \pi R^2 h$$

$$\therefore v = \frac{2}{3}\pi \times \left(\frac{0.15}{2}\right)^3 + \pi \times \left(\frac{0.15}{2}\right)^2 \times 0.00355 = 0.000946\ \text{m}^3$$

$$F_B = S_{\text{oil}}\,\rho_w g v = 0.85 \times 1000 \times 9.81 \times 0.000946 = 7.8882\ \text{N}$$

Thus, the net vertical force on the float is given by,

$$R = F_B - W = (7.8882 - W)\ \text{N}$$

Taking moments about the hinge O, we get:

$$F \times AO = R \times BN$$

$$10 \times 0.2 = (7.8882 - W) \times 0.3535$$

$$\therefore W = 7.8882 - \frac{10 \times 0.2}{0.3535} = \mathbf{2.2305\ N}$$

Example 5.28 A cone of base radius R and height l floats in water with its vertex downwards. Show that for stable equilibrium of the cone: (i) $\sec^2\alpha = l/h$ and (ii) $l < [R^2 S^{1/3}/(1 - S^{1/3})]^{1/2}$, where h is the depth of immersion, α is the semi-vertex angle of the cone and S be the specific gravity of the cone material.

Solution

Refer Figure 5.28.

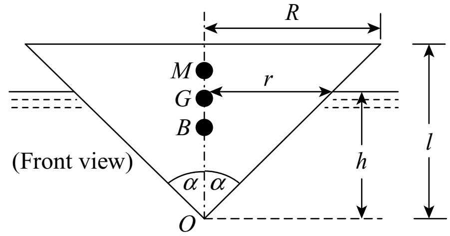


Figure 5.28

(i) $OG = \frac{3}{4}l$ and $OB = \frac{3}{4}h$

$$BM = \frac{I}{v} = \frac{(\pi/4)r^4}{(1/3)\pi r^2 h} = \frac{3}{4}\frac{r^2}{h}$$

$$BM = \frac{3}{4}\frac{(h\tan\alpha)^2}{h} = \frac{3}{4}h\tan^2\alpha \qquad [\because r = h\tan\alpha]$$

$$OM = OB + BM = \frac{3}{4}h + \frac{3}{4}h\tan^2\alpha$$

$$OM = \frac{3}{4}h \times (1 + \tan^2\alpha) = \frac{3}{4}h \times \sec^2\alpha$$

For equilibrium, M should be above G, i.e., $OM > OG$ and we get the following expression.

$$\frac{3}{4}h\sec^2\alpha > \frac{3}{4}l$$

$$\therefore \sec^2\alpha > \frac{l}{h}\ \textbf{(Proved)}$$

(ii) Let w_{cone} be the specific weight of the cone and w_{water} be the specific weight of water.

Weight of the cone = Weight of water displaced

$$w_{\text{cone}} \times \frac{1}{3}\pi R^2 l = w_{\text{water}} \times \frac{1}{3}\pi r^2 h$$

$$w_{\text{cone}} \times \frac{1}{3}\pi \times (l\tan\alpha)^2 \times l = w_{\text{water}} \times \frac{1}{3}\pi \times (h\tan\alpha)^2 \times h$$

$$h = l\left(\frac{w_{\text{cone}}}{w_{\text{water}}}\right)^{1/3} = l\,S^{1/3}$$

$$\frac{l}{h} = \frac{1}{S^{1/3}}$$

$$\sec^2\alpha > \frac{1}{S^{1/3}} \qquad [\therefore \sec^2\alpha > (l/h)]$$

$$(1+\tan^2\alpha) > \frac{1}{S^{1/3}} \Rightarrow \tan^2\alpha > \frac{1}{S^{1/3}} - 1$$

$$\left(\frac{R}{l}\right)^2 > \frac{1-S^{1/3}}{S^{1/3}} \qquad [\therefore \tan\alpha = R/l]$$

$$\left(\frac{l}{R}\right)^2 < \frac{S^{1/3}}{1-S^{1/3}} \Rightarrow l^2 < \frac{R^2 S^{1/3}}{1-S^{1/3}}$$

$$\therefore \boldsymbol{l < \left(\frac{R^2 S^{1/3}}{1-S^{1/3}}\right)^{1/2}} \quad \textbf{(Proved)}$$

Example 5.29 A solid cone of base radius R and height l floats in a liquid (specific gravity = 0.8) with its vertex downwards. If specific weight of the cone material is 0.6, then determine the least apex angle of cone for stable equilibrium.

Solution

Refer Figure 5.28. Let $S_{\text{liquid}} = 0.8$ and $S_{\text{cone}} = 0.6$.

Let 2α be the least apex angle of cone for stable equilibrium.

Weight of the cone = Weight of liquid displaced

$$w_{\text{cone}} \times \frac{1}{3}\pi R^2 l = w_{\text{liquid}} \times \frac{1}{3}\pi r^2 h$$

$$w_{\text{cone}} \times \frac{1}{3}\pi \times (l\tan\alpha)^2 \times l = w_{\text{liquid}} \times \frac{1}{3}\pi \times (h\tan\alpha)^2 \times h$$

$$\frac{l}{h} = \left(\frac{w_{\text{liquid}}}{w_{\text{cone}}}\right)^{1/3}$$

$$\sec^2\alpha > \left(\frac{w_{\text{liquid}}}{w_{\text{cone}}}\right)^{1/3} \qquad [\therefore \sec^2\alpha > (l/h)]$$

$$\sec^2\alpha > \left(\frac{w_{\text{liquid}}}{w_{\text{water}}} \times \frac{w_{\text{water}}}{w_{\text{cone}}}\right)^{1/3}$$

$$\sec^2\alpha > \left(\frac{S_{\text{liquid}}}{S_{\text{cone}}}\right)^{1/3}$$

$$\sec^2\alpha > \left(\frac{0.8}{0.6}\right)^{1/3}$$

$$\sec\alpha > 1.0491 \Rightarrow \cos\alpha > 0.9532$$

$$\alpha > \cos^{-1} 0.9532 \Rightarrow \alpha > 17.6°$$

$$\therefore \mathbf{2\alpha > 35.2°}$$

Example 5.30 A conical buoy that is 0.6 m long and base diameter 0.5 m floats in water with its apex downwards. Determine the minimum weight of the buoy for its stable equilibrium.

Solution

Refer Figure 5.28. Let $l = 0.6$ m and $D = 0.5$ m.

Let v be the volume of water displaced, h be the depth of immersion and W be the weight of the buoy.

$$v = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times (h\tan\alpha)^2 \times h$$

$$= \frac{1}{3}\pi \times (\tan\alpha)^2 \times h^3 = \frac{1}{3}\pi \times \left(\frac{R}{l}\right)^2 \times h^3$$

$$v = \frac{1}{3}\pi \times \left(\frac{0.25}{0.6}\right)^2 \times h^3 = 0.1818h^3$$

$$I = \frac{\pi}{64}d^4 = \frac{\pi}{64}\times(2h\tan\alpha)^4 \quad [\therefore d = 2r = 2h\tan\alpha]$$

$$I = \frac{\pi}{64}\times\left(2h\times\frac{0.25}{0.6}\right)^4 = 0.0237h^4 \quad [\therefore \tan\alpha = R/l]$$

$$BM = \frac{I}{v} = \frac{0.0237h^4}{0.1818h^3} = 0.1304h$$

$$OG = \frac{3}{4}l = \frac{3}{4}\times 0.6 = 0.45 \text{ m}$$

$$OB = \frac{3}{4}h = 0.75h$$

For stable equilibrium, M should lie above G, i.e., $OM \geq OG$ and we get the following expression.

$$BM \geq BG$$

$$BM \geq OG - OB$$

$$0.1304h \geq 0.45 - 0.75h$$

$$0.8804h \geq 0.45 \Rightarrow h \geq \frac{0.45}{0.8804}$$

$$\therefore h \geq 0.511$$

$$v = 0.1818h^3 = 0.1818 \times 0.511^3 = 0.02426 \text{ m}^3$$

$$W = \rho_w g v = 1000 \times 9.81 \times 0.02426 = \mathbf{237.9906\ N}$$

5.7 ❐ OSCILLATION OF A FLOATING BODY

A floating body (Figure 5.29(a)) may be set in a state of oscillation when an overturning couple by which it is tilted through a small angle is suddenly removed. The body starts oscillating about its metacentre just like a pendulum oscillating about its point of suspension as shown in Figure 5.29(b).

Let W be the weight of the floating body, α be the small angular displacement in radians, $-(d^2\alpha/dt^2)$ be the angular acceleration in rad/s^2 in which the negative sign shows that it tends to decrease the angle α, T be the time of one complete oscillation in seconds, k be the radius of gyration about a longitudinal axis passing through G, GM be the metacentric height and $I = (Wk^2)/g$ be the moment of inertia of the body about its axis passing through G.

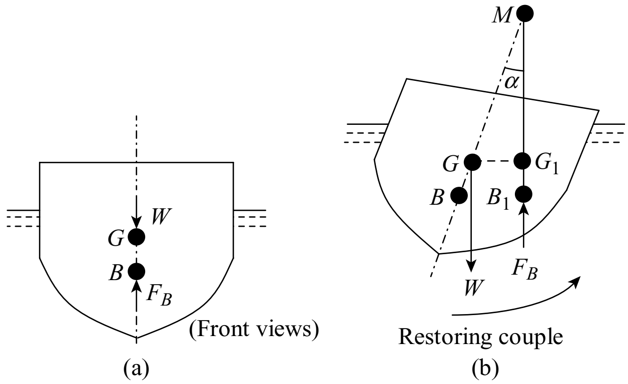


Figure 5.29 *Oscillation of a floating body*

$$\text{Inertia torque} = \frac{W}{g}k^2\left(-\frac{d^2\alpha}{dt^2}\right) \quad \text{(i)}$$

$$\text{Restoring couple} = W \times GG_1 = W \times GM \sin\alpha \quad \text{(ii)}$$

Equating the expressions (i) and (ii), we get:

$$\frac{W}{g}k^2\left(-\frac{d^2\alpha}{dt^2}\right) = WGM\sin\alpha$$

$$\frac{k^2}{g}\frac{d^2\alpha}{dt^2} + GM\alpha = 0 \quad [\because \sin\alpha \approx \alpha]$$

$$\frac{d^2\alpha}{dt^2} + \frac{gGM}{k^2}\alpha = 0$$

This is a second-degree differential equation, whose solution is given by,

$$\alpha = A\sin\omega t + B\cos\omega t$$

$$\alpha = A\sin\sqrt{\frac{gGM}{k^2}} \times t + B\cos\sqrt{\frac{gGM}{k^2}} \times t$$

Here, A and B are the constants of integration which can be determined by applying the following conditions.

(i) At $t = 0$, $\alpha = 0°$ and (ii) At $t = (T/2)$, $\alpha = 0°$

Substituting the first boundary condition, we get:

$$0 = A \times 0 + B \times 1$$

$$B = 0$$

Now substituting second boundary conditions, we get:

$$0 = A\sin\sqrt{\frac{gGM}{k^2}} \times \frac{T}{2}$$

$$\because A \neq 0$$

$$\therefore \sin\sqrt{\frac{gGM}{k^2}} \times \frac{T}{2} = 0 = \sin\pi$$

$$\sqrt{\frac{gGM}{k^2}} \times \frac{T}{2} = \pi$$

$$\boxed{\therefore T = 2\pi\sqrt{\frac{k^2}{gGM}}} \tag{5.11}$$

The oscillatory motion of a ship (or a boat) about its longitudinal axis is called rolling, whereas the oscillatory motion of a ship about its transverse axis is called pitching. The Equation (5.11) specifies the time period of oscillation for rolling motion of ships. However, if a ship has a safe metacentric height for rolling motion, then it will also be safe in pitching motion. The increase in metacentric height gives better stability to a floating body but reduces the time period of the rolling body. A smaller value of the time period of a rolling ship is not desirable because it is not comfortable to its passengers and the ship is also subjected to undue strains which may damage its structures.

Example 5.31 A ship of weight 30000 kN floats in sea water (specific weight = 10 kN/m^3) whose centre of buoyancy is 1.8 m below its centre of gravity. The radius of gyration of the ship is 3 m and its moment of inertia at the water level is 8500 m^4. Determine the rolling period of ship.

Solution

Let $W = 30000$ kN, $w = 10$ kN/m^3, $BG = 1.8$ m, $k = 3$ m and $I = 8500$ m^4. Let T be the rolling period of the ship and v be the volume of water displaced.

$$v = \frac{W}{w} = \frac{30000}{10} = 3000 \text{ m}^3$$

$$BM = \frac{I}{v} = \frac{8500}{3000} = 2.833 \text{ m}$$

$$GM = BM - BG = 2.833 - 1.8 = 1.033 \text{ m}$$

$$T = 2\pi\sqrt{\frac{k^2}{gGM}} = 2\pi\sqrt{\frac{3^2}{9.81 \times 1.033}} = \mathbf{5.92\ s}$$

Example 5.32 A log of wood (specific gravity = 0.85) of square section 0.4 m × 0.4 m floats in water. Determine the period of rolling when its one edge is depressed and released.

Solution

Refer Figure 5.30. Let $S_{\text{wood}} = 0.85$ and $b = d = 0.4$ m. Let l be the length of the log, h be the depth of immersion in water and v be the volume of water displaced.

Weight of the wood log = Weight of water displaced

or $$S_{\text{wood}}\,\rho_w g \times (lbd) = \rho_w g \times (lbh)$$

$$h = S_{\text{wood}}\, d = 0.85 \times 0.4 = 0.34 \text{ m}$$

$$OB = \frac{h}{2} = \frac{0.34}{2} = 0.17 \text{ m}$$

$$OG = \frac{d}{2} = \frac{0.4}{2} = 0.2 \text{ m}$$

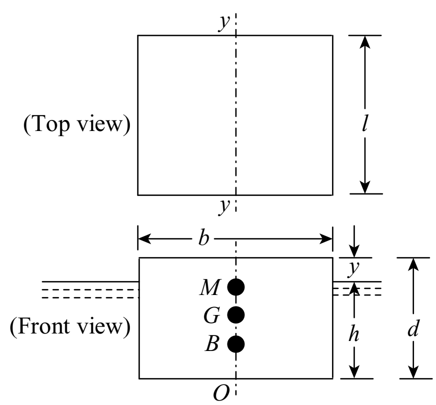


Figure 5.30

$$BG = OG - OB = 0.2 - 0.17 = 0.03 \text{ m}$$

$$I = \frac{lb^3}{12} = \frac{l \times 0.4^3}{12} = 5.333 \times 10^{-3} l \text{ m}^4$$

$$v = lbh = l \times 0.4 \times 0.34 = 0.136l \text{ m}^3$$

$$GM = \frac{I}{v} - BG = \frac{5.333 \times 10^{-3} l}{0.136l} - 0.03 = 0.00921 \text{ m}$$

$$k^2 = \frac{b^2}{12} = \frac{0.4^2}{12} = 0.0133 \text{ m}^2$$

$$T = 2\pi\sqrt{\frac{k^2}{gGM}} = 2\pi\sqrt{\frac{0.0133}{9.81 \times 0.00921}} = \mathbf{2.41 \text{ s}}$$

Example 5.33 A ship of weight 28000 kN floats in sea water (specific weight = 10.2 kN/m^3) whose centre of buoyancy is 1.4 m below its centre of gravity. If the rolling period of ship is 4.31 seconds and the moment of inertia of the ship at the waterline about fore-and-aft axis is 9200 m^4, then determine its radius of gyration.

Solution

Let $W = 28000$ kN, $w = 10.2$ kN/m^3, $BG = 1.4$ m, $T = 4.31$ s and $I = 9200$ m^4. Let k be the radius of gyration of the ship and v be the volume of water displaced.

$$v = \frac{W}{w} = \frac{28000}{10.2} = 2745.1 \text{ m}^3$$

$$GM = \frac{I}{v} - BG = \frac{9200}{2745.1} - 1.4 = 1.95 \text{ m}$$

$$k = \sqrt{\left(\frac{T}{2\pi}\right)^2 \times gGM} \quad \text{[From Equation (5.11)]}$$

$$\therefore k = \sqrt{\left(\frac{4.31}{2\pi}\right)^2 \times 9.81 \times 1.95} = \mathbf{3 \text{ m}}$$

Example 5.34 A 5 m wide ship coming into port has a draught of 1.2 m and after unloading its cargo it has a draught of 1 m. If its centre of gravity remains at the waterline, then determine the ratio of the periodic times before and after leaving the cargo.

Solution

Let the subscripts 1 and 2 denote the parameters before and after leaving the cargo respectively, $b = 5$ m, $(OG)_1 = h_1 = 1.2$ m and $(OG)_2 = h_2 = 1$ m.

Let T_1 and T_2 be the periodic times for the ship before and after leaving the cargo, respectively, l be its length and v be the volume of water displaced.

$$(BG)_1 = \frac{(OG)_1}{2} = \frac{1.2}{2} = 0.6 \text{ m}$$

$$(GM)_1 = \frac{I}{v} - (BG)_1 = \frac{(l \times 5^3)/12}{l \times 5 \times 1.2} - 0.6 = 1.136 \text{ m}$$

$$T_1 = 2\pi\sqrt{\frac{k^2}{g(GM)_1}} = 2\pi\sqrt{\frac{k^2}{9.81\times 1.136}} = 1.882\,k$$

$$(BG)_2 = \frac{(OG)_2}{2} = \frac{1}{2} = 0.5 \text{ m}$$

$$(GM)_2 = \frac{I}{v} - (BG)_2 = \frac{(l\times 5^3)/12}{l\times 5\times 1} - 0.5 = 1.583 \text{ m}$$

$$T_2 = 2\pi\sqrt{\frac{k^2}{g(GM)_2}} = 2\pi\sqrt{\frac{k^2}{9.81\times 1.583}} = 1.5944\,k$$

$$\therefore \frac{T_1}{T_2} = \frac{1.882k}{1.5944k} = \mathbf{1.18}$$

Summary

1. The tendency for an immersed body to be lifted up in the fluid due to an upward force exerted by the fluid is known as buoyancy.
2. The force tending to lift up an immersed body against the gravitational force is called buoyant force (or force of buoyancy).
3. The point of application of the buoyant force on the body is known as the centre of buoyancy which is denoted by B.
4. The Archimedes' principle states that when a body is immersed in a fluid either wholly or partially, it is lifted up by a force which is equal to the weight of the fluid displaced by the body.
5. Metacentre is the point about which a floating body starts oscillating when it is given a small angular displacement and it is denoted by M.
6. The distance between the centre of gravity G and the metacentre M of a floating body is called metacentric height. It is given by the expression $GM = (I/v) - BG$, here I is the M.I. (in top view) at the water surface about vertical axis (axis of symmetry), v is the volume of body immersed in water and BG is the distance between the centre of gravity and centre of buoyancy.
7. The experimental value of metacentric height is $GM = (w_1 x)/(W \tan \alpha)$, here w_1 is a movable weight which moves through a distance x, W is the total weight of the body including the movable weight w_1 and α is the small angle through which the body tilts.
8. The stability of a submerged body such as a balloon submerged in air or a submarine submerged in water is determined by the relative position of the centre of gravity G and the centre of buoyancy B of the body.
9. The conditions for stability of a submerged body are listed below.
   - (i) The body remains in stable equilibrium when G lies below B.
   - (ii) The body remains in unstable equilibrium when G lies above B.
   - (iii) The body remains in neutral equilibrium when G coincides with B.
10. The stability of a floating body is determined by the relative position of the centre of gravity G and the metacentre M of the body.
11. The conditions for stability of a floating body are:
    - (i) The body remains in stable equilibrium when G lies below M.
    - (ii) The body remains in unstable equilibrium when G lies above M.
    - (iii) The body remains in neutral equilibrium when G coincides with M.
12. The oscillatory motion of a ship about its longitudinal axis is called rolling.
13. The oscillatory motion of a ship about its transverse axis is called pitching.
14. The time period of oscillation (T) of a floating body is $T = 2\pi\sqrt{k^2/(gGM)}$, here k is the radius of gyration and GM is the metacentric height.

Multiple-choice Questions

1. When a block of ice floats on the surface of water contained in a vessel the water level will
(a) Rise.
(b) Fall.
(c) Remains same.
(d) None of the above.

2. When a ship enters sea from a river then its depth of submergence will
(a) Decrease.
(b) Increase.
(c) Remains same.
(d) None of the above.

3. A floating body is in stable equilibrium when its
(a) Metacentric height is zero.
(b) Centre of gravity is below the centre of buoyancy.
(c) Metacentric height is negative.
(d) Metacentric height is positive.

4. The metacentric height is the distance between the
(a) Metacentre and the centre of buoyancy.
(b) Centre of gravity of the floating body and the centre of buoyancy.
(c) Centre of gravity of the floating body and the metacentre.
(d) None of the above.

5. The relation for stable equilibrium of a floating body in terms of metacentric height *GM*, distance between centre of gravity (*G*) and centre of buoyancy (*B*) *BG*, the moment of inertia *I* and the volume of water displaced by the body *v* is given by
(a) $BG + GM = I/v$.
(b) $BG - GM = I/v$.
(c) $BG = (I/v)/GM$.
(d) None of the above.

6. For merchants ships, the metacentric height lies in the range of
(a) 0 to 0.25 m.
(b) 0.3 to 1 m.
(c) 1 to 2.5 m.
(d) None of the above.

7. The metacentric heights of two floating bodies *P* and *Q* are 0.5 m and 1 m, respectively, then which one of the following statements is correct?
(a) Body *P* is more stable.
(b) Body *Q* is more stable.
(c) Both *P* and *Q* are equally stable.
(d) None of the above.

8. For a submerged body, if the metacentre is below the centre of gravity, then equilibrium is
(a) Stable.
(b) Unstable.
(c) Neutral.
(d) None of the above.

9. For a submerged body, if the centre of buoyancy coincides with the centre of gravity, then the equilibrium is
(a) Stable.
(b) Unstable.
(c) Neutral.
(d) None of the above.

10. For a floating body, if the metacentre is below the centre of gravity, then the equilibrium is
(a) Stable.
(b) Unstable.
(c) Neutral.
(d) None of the above.

Review Questions

1. Define buoyancy, centre of buoyancy, metacentre and metacentric height.

2. Prove the Archimedes' principle of buoyancy.

3. Explain how will you determine the metacentric height analytically?

4. Explain how will you determine the metacentric height experimentally?

5. Define oscillation of a floating body. Also derive an expression for the time period of the oscillations (T) of a floating body in terms of metacentric height (GM) and radius of gyration (k) of the floating body.

6. Explain the conditions of equilibrium for floating and submerged bodies.

Problems

1. A wooden block (specific gravity = 0.64) of cuboidal shape 4 m long, 1.25 m wide and 2 m deep floats horizontally in water. Determine the volume of liquid displaced and position of centre of buoyancy.

[**Ans.** 6.4 m^3, 0.64 m above base]

2. A stone weighs 400 N in air and 200 N in water. Determine the volume of body and its specific gravity.

[**Ans.** 0.0204 m^3, 2]

3. A metallic body floats at the interface of mercury and water in a tank such that 40% of its volume is submerged in mercury and 60% in water. Find the specific weight of the metallic body.

[**Ans.** 59252.4 N/m^3]

4. A wooden body floats in water tank with 50 mm height projecting above the water surface. The same wooden body when placed in glycerine tank it projects 75 mm above the surface of glycerine. If the specific gravity of glycerine is 1.35, then find (i) the height of the wooden body and (ii) relative density of wooden body.

[**Ans.** 146 mm, 0.657]

5. When a football of 30 cm diameter fell into a water tank, 10% of its volume found under water. Determine the density of the football.

[**Ans.** 100 kg/m^3]

6. A wooden block (specific gravity = 0.7) that is 2 m long, 0.4 m wide and 0.2 m high floats in a water tank. Determine the volume of concrete of specific weight 20 kN/m^3, that may be kept on the block and thus, immerse the (i) block completely in water and (ii) block and the concrete completely in water.

[**Ans.** 0.0235 m^3, 0.0462 m^3]

7. A spherical body of diameter 1 m is completely immersed in a water tank and chained to the bottom of the tank. If the chain has a tension of 2 kN, then determine the weight of the body when it is taken out of the tank into the air.

[**Ans.** 3.136 kN]

8. Find the volume and density of a body that weighs 4 N in water and 5 N in oil of specific gravity 0.85.

[**Ans.** 6.796×10^{-4} m^3, 1600.45 kg/m^3]

9. A cube 1 m side is inserted in a two-layer fluid with specific gravities 1.2 and 0.9. The upper and lower halves of the cube are composed of materials with specific gravity 0.6 and 1.4, respectively. What is the distance of the top of the cube above interface?

[**Ans.** 0.667 m]

10. An iceberg (specific gravity = 0.9) floats in sea water (specific gravity = 1.023). If the volume of ice above the water surface is 30 m^3, then determine the weight of the iceberg.

[**Ans.** 2203.54 kN]

11. A 6.5 m long, 2.5 m wide and 1.5 m high rectangular pontoon floats in seawater (specific gravity = 1.023) with a depth of immersion of 1 m. Find the metacentric height, if the centre of gravity is 0.7 m above the bottom of the pontoon.

[**Ans.** 0.321 m]

12. If a wooden block (specific gravity = 0.72) of size 1 m × 0.5 m × 0.4 m floats in water, then determine its metacentric height.

[**Ans.** 0.016 m]

13. A uniform body of the size 4 m × 2 m × 1 m floats in water. Determine its metacentric height and the weight if the depth of immersion is 0.6 m.

[**Ans.** 0.3555 m, 47.09 kN]

14. A vessel that is 70 m long and 10 m broad has a displacement of 19620 kN in sea water (specific gravity = 1.03). The vessel tilts through 6^o when a weight of 343 kN moves through a distance of 6 m. If the second moment of area of the waterline section about its fore-and-aft axis is 70% than that of the circumscribing rectangle and centre of buoyancy is 2.2 m below the water line, then determine the metacentric height and the position of centre of gravity of the vessel.

[**Ans.** 0.998 m, 1.095 m]

15. A rectangular pontoon of length 25 m and weight 2450 kN floats in fresh water of specific weight 10 kN/m^3. Its centre of gravity lies 0.3 m above the centre of cross section and for 10° angle of heel its metacentric height is 1.3 m. If 0.65 m height portion of the pontoon is lying outside water, then determine its breadth and height.

[**Ans.** 6.095 m, 2.26 m]

16. A rectangular pontoon that is 8 m long, 6 m wide and 2 m deep weighs 500 kN and it floats in water having a specific weight of 10 kN/m^3. The pontoon carries on its upper deck a boiler of diameter 5 m and weight 200 kN. The centre of gravity of each unit coincides with the geometrical centre of the arrangement and it lies in the same vertical line. Determine the metacentric height.

[**Ans.** 0.786 m]

17. A solid wooden cube (specific gravity = 0.6) of sides 0.6 m floats in a liquid (specific gravity = 0.9) with two of its faces being horizontal. Find its metacentric height and state about its stability.

[**Ans.** –0.025 m, unstable]

18. A wooden cylinder (specific gravity = 0.6) of diameter d and length l is required to float in oil (specific gravity = 0.9). Show that l cannot exceed about $0.75d$ for the cylinder to float when its longitudinal axis being vertical.

19. A log of wood (specific gravity = 0.86) of square section 0.35 m × 0.35 m floats in water. One edge is depressed and released causing the log to roll. Find the period of a roll.

[**Ans.** 2.09 s]

20. A wooden cone (specific gravity = 0.6) weighing 86 N floats with its apex downwards in a liquid (specific gravity = 0.82). Determine the weight of a steel piece (specific gravity = 7.86) suspended from the apex of the cone by a rope which will just suffice to submerge the cone. Also determine the tension in the rope.

[**Ans.** 35.2 N, 31.528 N]

21. A ship has a displacement of 50000 kN and its radius of gyration is 2 m when it floats in sea water (specific weight = 10 kN/m^3). If the centre of buoyancy is 2.5 m below the centre of gravity and the moment of inertia about its fore-and-aft axis is 15000 m^4, then determine the metacentric height and the time period of oscillation.

[**Ans.** 0.5 m, 5.67 s]

22. A ship weighing 29.43 MN in sea water (specific weight = 10.1 kN/m^3) has a rolling time period of 5 seconds. The moment of inertia of the ship about its fore-and-aft axis is 10^4 m^4 and its centre of buoyancy is 1.5 m below the centre of gravity. Calculate the radius of gyration of the ship.

[**Ans.** 3.46 m]

23. A rectangular pontoon is 5 m long, 3 m wide and 1.2 m high. The depth of immersion of the pontoon is 0.8 m in sea water (specific gravity = 1.023). Determine the metacentric height if the centre of gravity is 0.6 m above the bottom of the pontoon.

[**Ans.** 0.7375 m]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (c)	**2.** (a)	**3.** (d)	**4.** (c)	**5.** (a)
6. (b)	**7.** (b)	**8.** (d)	**9.** (c)	**10.** (b)

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Chapter 6

Fluid Kinematics

6.1 ❐ INTRODUCTION

Fluid kinematics deals with the geometry of fluid motion in terms of displacement, velocity and acceleration without considering the forces causing the motion. After knowing the velocity it becomes possible to calculate the pressure distribution and consequently, the force acting on the fluid can also be estimated. The methods by which the motion of a fluid may be described are Lagrangian method and the Eulerian method. The Eulerian method is commonly used in fluid mechanics. In this chapter, the basic concepts related to fluid kinematics and the methods for determining velocity and acceleration are described.

6.2 ❐ VELOCITY OF FLUID PARTICLES

The velocity of fluid flow is a function of space and time. Let ds be the distance travelled by a fluid particle in time dt in the space occupied by a fluid in motion. The velocity (V) of the fluid particle at any point is given by the following expression.

$$V = \lim_{dt \to 0} \frac{ds}{dt} \tag{6.1}$$

The velocity is a vector quantity and thus, it has both magnitude as well as direction. The velocity V at any point in the fluid can be resolved into three components, such as u, v and w in the three mutually perpendicular directions x, y and z respectively. If dx, dy and dz be the components of displacement ds in x, y and z directions, respectively, then the velocity components can be expressed as follows.

$$u = \lim_{dt \to 0} \frac{dx}{dt} = f_1(x, y, z, t) \tag{6.2a}$$

$$v = \lim_{dt \to 0} \frac{dy}{dt} = f_2(x, y, z, t) \tag{6.2b}$$

$$w = \lim_{dt \to 0} \frac{dz}{dt} = f_3(x, y, z, t) \tag{6.2c}$$

Thus, at a particular instant of time, the velocity components u, v and w vary at different points and each individual particle has its own velocity that varies both with respect to time and position.

In vector form, the velocity of the fluid particle at any point can be represented as follows.

$$\boxed{\vec{V} = ui + vj + wk} \tag{6.3}$$

Here, i, j and k are the unit vectors in the direction of coordinate axes.

The magnitude of velocity (or the resultant velocity) is given by,

$$\boxed{V = \sqrt{u^2 + v^2 + w^2}} \tag{6.4}$$

Example 6.1 Find the velocity vector and its magnitude for the velocity components $u = (3xy - t)$, $v = (2yz + t + 1)$ and $w = (1 + 3ty)$ at point A (3, 2, 1) m at $t = 2$ s.

Solution

Let $u = (3xy - t)$, $v = (2yz + t + 1)$, $w = (1 + 3ty)$, $x = 3$ m, $y = 2$ m, $z = 1$ m and $t = 2$ s. Let $\vec{V}$ be the velocity vector and V be its magnitude.

The velocity components at point A (3, 2, 1) are calculated in the following expressions.

$$u = 3xy - t = 3 \times 3 \times 2 - 2 = 16 \text{ m/s}$$

$$v = 2yz + t + 1 = 2 \times 2 \times 1 + 2 + 1 = 7 \text{ m/s}$$

$$w = 1 + 3ty = 1 + 3 \times 2 \times 2 = 13 \text{ m/s}$$

$$\vec{V} = ui + vj + wk = \mathbf{16}\boldsymbol{i} + \mathbf{7}\boldsymbol{j} + \mathbf{13}\boldsymbol{k}$$

$$V = \sqrt{u^2 + v^2 + w^2} = \sqrt{16^2 + 7^2 + 13^2} = \mathbf{21.77 \ m/s}$$

6.3 ❐ TYPES OF FLUID FLOW

The fluid flow may be of the following types classified as (i) steady flow and unsteady flow, (ii) uniform flow and non-uniform flow, (iii) laminar flow and turbulent flow, (iv) compressible flow and incompressible flow, (v) rotational flow and irrotational flow and (vi) one dimensional flow, two dimensional flow and three dimensional flow.

6.3.1 Steady and Unsteady Flows

Steady flow The flow in which the fluid characteristics like velocity (V), pressure (p), density (ρ), etc., do not change with time is called steady flow. However, these characteristics may be different at different points in the flowing fluid. Most of the practical engineering problems involve steady flow and thus, it can be analysed easily. Mathematically, the steady flow at any point in the flowing fluid may be given by the following expressions.

$$\frac{\partial V}{\partial t} = 0, \ \frac{\partial p}{\partial t} = 0, \ \frac{\partial \rho}{\partial t} = 0, \text{ etc.}$$

Unsteady flow The flow in which the fluid characteristics like velocity, pressure, density, etc., change with time is called unsteady flow (or transient flow). It is very difficult to analyse the unsteady flow except under special conditions. Mathematically, the unsteady flow at any point in the flowing fluid may be expressed as given below.

$$\frac{\partial V}{\partial t} \neq 0, \ \frac{\partial p}{\partial t} \neq 0, \ \frac{\partial \rho}{\partial t} \neq 0, \text{ etc.}$$

6.3.2 Uniform and Non-uniform Flows

Uniform flow The flow in which the fluid velocity does not change with location over a specified region at any given time is called uniform flow. Mathematically, the uniform flow may be expressed as given below.

$$\frac{\partial V}{\partial s} = 0$$

Here, s denotes any space coordinates. Therefore, for a uniform flow, the space rate of change of the flow parameters at any given time is equal to zero.

Non-uniform flow The flow in which the fluid velocity at any given time changes with location over a specified region is called non-uniform flow. Mathematically, the non-uniform flow may be expressed as given below.

$$\frac{\partial V}{\partial s} \neq 0$$

Therefore, for a non-uniform flow, the space rate of change of the flow parameters at any given time is not equal to zero.

6.3.3 Laminar and Turbulent Flows

Laminar flow A laminar flow is characterized by a smooth flow of one layer (or lamina) of fluid over the adjacent layer. The fluid particles move in well-defined paths (or streamlines) and keep the same path at successive cross sections of the flow passage. The laminar flow is also known as viscous flow or streamline flow. Generally, laminar flow occurs in highly viscous liquids flow and in smooth pipes when the flow velocity is low.

Turbulent flow A fluid motion is said to be turbulent when the fluid particles move in a zigzag manner, i.e., entirely in a disorderly manner. This causes rapid and continuous mixing of the fluid leading to momentum transfer when the flow occurs. Eddies or vortices are formed in turbulent flow and it causes energy losses. More often turbulent flow occurs than laminar flow, for examples, flow in water supply pipes, flow in natural streams, sewers, etc.

Reynolds number (Re) is defined as the ratio of inertia force to the viscous force. Laminar and turbulent flows are characterized on the basis of Reynolds number. Flow through a pipe is laminar when $\text{Re} < 2000$, turbulent when $\text{Re} > 4000$ and transitional when Re lies between 2000 and 4000.

6.3.4 Compressible and Incompressible Flows

Compressible flow The flow in which the density of the fluid does not remain constant is called compressible flow. Thus, for compressible flow, $\rho \neq \text{Constant}$.

Incompressible flow The flow in which the density remains constant is called incompressible flow. Thus, for incompressible flow, $\rho = \text{Constant}$. The densities of liquids are constant and thus, the flow of liquids for practical purposes can be considered as incompressible.

Mach number (M) is defined as the square root of the ratio of the inertia force to the elastic force or the ratio of local flow velocity to the sonic velocity in the fluid. The compressibility effects are generally ignored for $M < 0.3$. Based on the Mach number, the flow may be subsonic flow ($M < 1$), sonic flow ($M = 1$), supersonic flow ($M > 1$) and hypersonic flow ($M >> 1$).

6.3.5 One-dimensional, Two-dimensional and Three-dimensional Flows

One-dimensional flow The flow in which the parameter such as velocity is a function of time and has only one space coordinate is called one-dimensional flow. Thus, the flow parameters vary only in one direction and mathematically, it is expressed as given below.

$$V = f(x, t) \text{ or } u = f_1(x, t),\ v = 0 \text{ and } w = 0$$

For a steady one-dimensional flow, the velocity is a function of one space coordinate only and the variation of velocities in other two directions is zero. Mathematically, it is expressed as given below.

$$V = f(x) \text{ or } u = f_1(x),\ v = 0 \text{ and } w = 0$$

Two-dimensional flow The flow in which the parameter such as velocity is a function of time and contains two space coordinates is called two-dimensional flow. The two-dimensional flow is mathematically expressed as given below.

$$V = f(x, y, t) \text{ or } u = f_1(x, y, t),\ v = f_2(x, y, t) \text{ and } w = 0$$

For a steady two-dimensional flow, the velocity is a function of two space coordinates only and the variation of velocity in third direction is zero. Mathematically, it is expressed as given below.

$$V = f(x, y) \text{ or } u = f_1(x, y),\ v = f_2(x, y) \text{ and } w = 0$$

Three-dimensional flow The flow in which the parameter such as velocity is a function of time and contains three space coordinates is called three-dimensional flow. The three-dimensional flow is mathematically expressed as given below.

$$V = f(x, y, z, t) \text{ or } u = f_1(x, y, z, t),\ v = f_2(x, y, z, t) \text{ and } w = f_3(x, y, z, t)$$

For a steady three-dimensional flow, the velocity is a function of three space coordinates and it is mathematically expressed as given below.

$$V = f(x, y, z) \text{ or } u = f_1(x, y, z),\ v = f_2(x, y, z) \text{ and } w = f_3(x, y, z)$$

6.3.6 Rotational and Irrotational Flows

Rotational flow The flow in which the fluid particles rotate about their own axis is called rotational flow.

Irrotational flow The flow in which the fluid particles do not rotate about their own axis is called irrotational flow.

6.4 ❐ DESCRIPTION OF FLUID FLOW PATTERN (FLOW VISUALIZATION)

The fluid flow pattern may be described by means of streamlines, stream-tubes, pathlines, streaklines and timelines which are described below.

1. **Streamline:** A streamline may be defined as an imaginary line drawn through a flowing fluid in such a way that the tangent to it at any point gives the direction of the velocity of flow at that point. Thus, streamlines indicate the direction of motion of particles at each point. Since the streamlines are tangent to the velocity vector at every point in the flow field, there can be no flow across a streamline. Figure 6.1(a) illustrates streamlines in a two-dimensional flow field in which one of the streamlines passing through a point $A(x, y)$ is tangential to the velocity vector V at point A.

   Let ds be the distance travelled by a fluid particle along the streamline during the time interval dt. Here, dx and dy be the components of the displacement along x and y directions, respectively and u and v be the components of the velocity V along x and y directions, respectively.

   $$\frac{v}{u} = \tan \alpha = \frac{dy}{dx}$$

   Thus, the differential equation for streamlines in two-dimensional flow field is as follows.

   $$\boxed{\frac{dx}{u} = \frac{dy}{v}} \qquad (6.5)$$

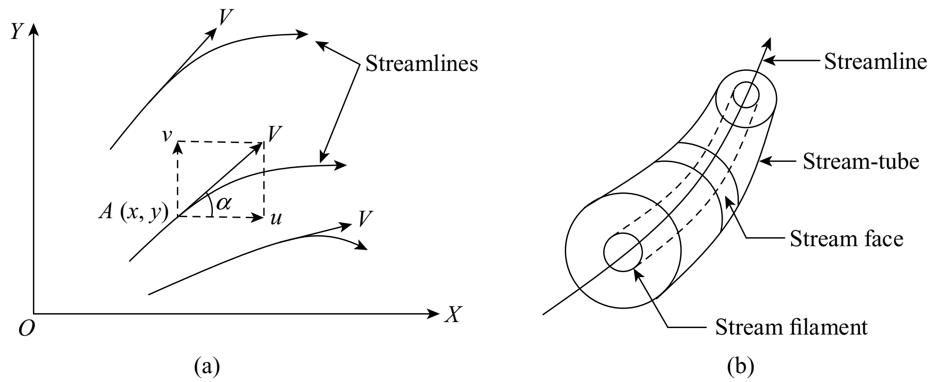

Figure 6.1 *Streamlines and stream-tube*

Similarly, the general differential equation for three-dimensional flow for streamlines as given below.

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \tag{6.6}$$

The streamlines for steady flow do not vary with time and it is constant for a given set of conditions. In unsteady flow, the streamline pattern would change from time to time.

2. **Stream-tube:** A stream-tube is a cylindrical passage or tube which may be imagined to form by a bundle of neighbouring streamlines through which the fluid flows. The surface of the stream-tube is known as the stream surface. Since the stream-tube is bounded on all sides by streamlines, there can be no flow across the surface. Therefore, the flow can be only through the ends of a stream-tube. A schematic diagram of stream-tube is shown in Figure 6.1(b). A stream-tube with a small enough cross-sectional area is known as stream filament. A clear picture of the actual flow pattern can be known from stream-tubes. The shape of a stream-tube changes from one instant to another due to change in the position of streamlines. However, in a steady flow it is fixed in space.
3. **Pathline:** A pathline is the trace of the path of a single particle over a period of time. It shows the direction of the velocity of a fluid particle at successive instants of time. In a steady flow, the pathlines and streamlines are identical. However, in case of unsteady flow, the pathlines and streamlines are different as shown in Figure 6.2. A pathline can intersect itself at different times. The streamline shows the velocity vectors for particles P and Q at time t_1. The particle P takes different positions at different times (t_2, t_3 and t_4) and the line connecting these positions of P occupied at different instant of times signifies its pathline.

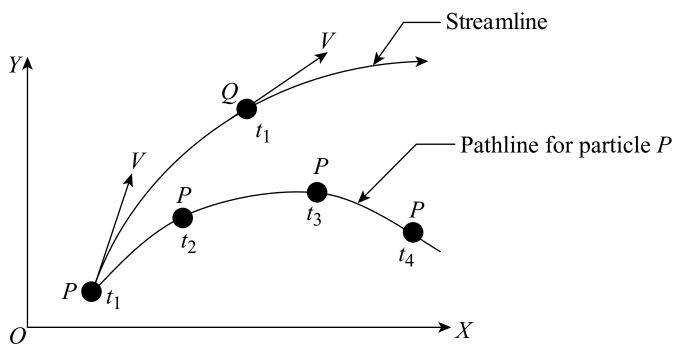

Figure 6.2 *Pathline and streamline*

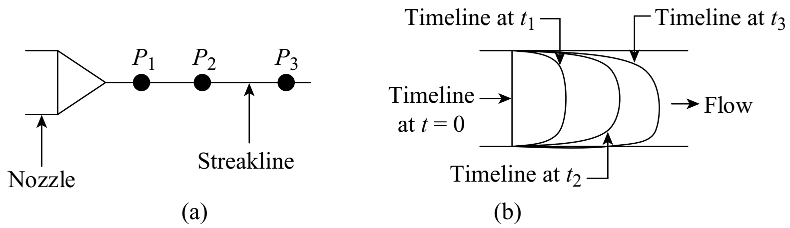

Figure 6.3 *Streakline and timelines*

4. **Streakline:** A line traced by a fluid particle passing through a fixed point in a flow field is known as streakline. If a dye or a colour is injected into a flowing field, then the resulting trails of colour are known as streaklines. A line formed by the smoke particles (assume P_1, P_2 and P_3) emanating from a fixed nozzle also forms a streaklines as shown in Figure 6.3(a).

   In a steady flow, there is no change in the flow pattern and thus, a streakline also represents a streamline and a pathline. Thus, it can be concluded that in a steady flow, a streakline, a streamline and a pathline are identical. However, in unsteady flow, the fluid particles may not remain on the same streamline.

5. **Timeline:** A timeline is the line formed by a number of adjacent fluid particles in a flow field marked at a given instant. The timelines are used for examining the uniformity of a flow and it can also be used to study the deformation of a fluid under shear force. Here, the timelines generated in a water channel through the use of hydrogen bubble wire at different times are shown in Figure 6.3(b).

Example 6.2 The velocity vector in two different flow fields are given by the equations (i) $\vec{V} = 2xi - 2yj$ and (ii) $\vec{V} = 2x^3 i - 6x^2 yj$. Determine the equations of streamline in each case when it passes through a point $A(3, 2)$.

Solution

(i) $\vec{V} = 2xi - 2yj$

$$u = 2x \text{ and } v = -2y$$

Since $$\frac{dx}{u} = \frac{dy}{v} \quad \text{[Streamline equation]}$$

Thus $$\frac{dx}{2x} = \frac{dy}{-2y}$$

$$\int \frac{dx}{x} = -\int \frac{dy}{y}$$

$$\ln x = -\ln y + \ln C$$

$$\ln x + \ln y = \ln C$$

$$\ln(xy) = \ln C$$

$$xy = C \qquad \text{(i)}$$

The streamline passes through point $A(3, 2)$ and thus, it must satisfy the expression (i).

$$3 \times 2 = C \text{ or } C = 6$$

Therefore, the required streamline is calculated as given below.

$$xy = \mathbf{6}$$

(ii) $\vec{V} = 2x^3 i - 6x^2 yj$

Thus $$u = 2x^3 \text{ and } v = -6x^2 y$$

Since $$\frac{dy}{dx} = \frac{v}{u} \quad \text{[Streamline equation]}$$

$$\frac{dy}{dx} = \frac{-6x^2 y}{2x^3}$$

$$\frac{dy}{y} = -3\frac{dx}{x}$$

$$\int \frac{dy}{y} = -3\int \frac{dx}{x}$$

$$\ln y = -3\ln x + \ln C$$

$$\ln y + 3\ln x = \ln C$$

$$\ln(yx^3) = \ln C$$

Thus $$yx^3 = C \qquad \text{(ii)}$$

The streamline passes through the point $A(3, 2)$ and thus, it must satisfy the expression (ii).

$$2 \times 3^3 = C \text{ or } C = 54$$

Therefore, the required streamline is calculated as given below.

$$yx^3 = \mathbf{54}$$

6.5 ❐ ACCELERATION OF A FLUID PARTICLE

The rate of change of velocity with respect to time is called acceleration. At any instant of time, each fluid particle has its own velocity and acceleration that varies with respect to time and position. Thus, the motion of fluid particles at various points and at successive instants of time is to be observed for complete description of a fluid flow. The fluid motion is described by two methods, namely Lagrangian method and Eulerian method. Generally, the Eulerian method is used in fluid mechanics.

6.5.1 Lagrangian Method

In this method, a single particle is followed over the flow field during its course of motion by a moving rectangular coordinate system and its behaviour is observed. Let the initial coordinate of a fluid particle be a, b and c which change to x, y and z after time interval t. The position of the fluid particle can be expressed as given below.

$$x = f_1(a, b, c, t),\ y = f_2(a, b, c, t),\ z = f_3(a, b, c, t) \qquad (6.7)$$

From the above equations, the velocity and acceleration components of the fluid particles can be obtained by taking derivatives with respect to time.

The velocity components can be obtained by the following expression.

$$u = \frac{dx}{dt}, \; v = \frac{dy}{dt}, \; w = \frac{dz}{dt} \tag{6.8}$$

The acceleration components can be obtained by the following expression.

$$a_x = \frac{du}{dt} = \frac{d^2x}{dt^2}, \; a_y = \frac{dv}{dt} = \frac{d^2y}{dt^2}, \; a_z = \frac{dw}{dt} = \frac{d^2z}{dt^2} \tag{6.9}$$

The magnitude of velocity (or the resultant velocity) is given by the following expression.

$$V = \sqrt{u^2 + v^2 + w^2} \tag{6.10}$$

The magnitude of acceleration (or the resultant acceleration) is given by the following expression.

$$a = \sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2} \tag{6.11}$$

Similarly, other quantities, such as pressure, density, etc., can be determined. However, the motion of single fluid particle is not sufficient to describe the entire flow field. Moreover, the solution of equations of motion is difficult due to its non-linear nature and therefore, this method is rarely used.

6.5.2 Eulerian Method

In this method, a finite volume called control volume (or flow domain) is defined through which the fluid flows in and out. The motion of fluid is specified by velocity components expressed as functions of space and time in the control volume. Thus, Eulerian method does not follow an individual particle. Let V be the resultant velocity at any point in a fluid flow with u, v and w being its components in x, y and z directions, respectively. Thus, the velocity components can be mathematically expressed as given below.

$$u = f_1(x, y, z, t), \; v = f_2(x, y, z, t), \; w = f_3(x, y, z, t) \tag{6.12}$$

Let a be the resultant acceleration at any point in a fluid flow with a_x, a_y and a_z being its components in x, y and z directions, respectively. The components of acceleration of the fluid particles can be worked out by partial differentiation. Therefore, for x component of acceleration, we have the following expressions.

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz + \frac{\partial u}{\partial t}dt$$

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} + \frac{\partial u}{\partial t}\frac{dt}{dt}$$

But
$$\frac{dx}{dt} = u, \; \frac{dy}{dt} = v, \; \frac{dz}{dt} = w$$

$$\therefore \boxed{a_x = \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}} \tag{6.13a}$$

However, in some textbooks, total acceleration (du/dt) is denoted by (Du/Dt) and the expression is as follows.

$$a_x = \frac{du}{dt} = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

Similarly, the y and z components of acceleration are given in the below expressions.

$$\boxed{a_y = \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}} \tag{6.13b}$$

$$\boxed{a_z = \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}} \tag{6.13c}$$

Acceleration vector is given by,

$$\boxed{\vec{a} = a_x i + a_y j + a_z k} \tag{6.14}$$

Here, i, j and k are the unit vectors in the directions of coordinate axes.

In vector notation, acceleration may be represented as given below.

$$\vec{a} = \frac{dV}{dt} = \frac{\partial V}{\partial t} + V \cdot \nabla V \tag{6.15}$$

The magnitude of acceleration (or the resultant acceleration) is given by the following expression.

$$\boxed{a = \sqrt{a_x^{\,2} + a_y^{\,2} + a_z^{\,2}}} \tag{6.16}$$

Local acceleration It is defined as the rate of change of velocity of the fluid particles with respect to time at a given point in a fluid flow. In Equations 6.13 (a), 6.13(b) and 6.13(c), the following given expressions are called local acceleration or temporal acceleration.

$$\frac{\partial u}{\partial t};\ \frac{\partial v}{\partial t};\ \frac{\partial w}{\partial t} \tag{6.17}$$

Convective acceleration It is defined as the rate of change of velocity due to the change in position of the fluid particles in a flow field. In Equations 6.13(a), 6.13(b) and 6.13(c), the following given expressions are called covective acceleration or advective acceleration.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z},\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z},\ u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \tag{6.18}$$

The total acceleration, i.e., the sum of local acceleration and convective acceleration of the fluid particle is called material or substantial acceleration. In a steady flow, the local acceleration is zero, since the velocity at any point is invariant with time. However, in uniform flow, the convective acceleration is zero, since the velocity components are not the functions of space coordinates. In steady and uniform flow, both the local and convective acceleration are zero and hence, there exists no total acceleration.

Example 6.3 Find the velocity and acceleration components at point $A(1, 2, 3)$ m and at $t = 2$ s for the fluid flow described by the velocity vector $\vec{V} = 2x^3 i - 5x^2 yj + 3tk$.

Solution

Let $x = 1$ m, $y = 2$ m, $z = 3$ m, $t = 2$ s, $u = 2x^3$, $v = -5x^2 y$ and $w = 3t$. Let $\vec{V}$ be the velocity vector, V be the resultant velocity, $\vec{a}$ be the acceleration vector and a be the resultant acceleration.

The velocity components at point $A(1, 2, 3)$ m and at time $t = 2$ s are calculated as follows.

$$u = 2x^3 = 2 \times 1^3 = 2 \text{ m/s}$$

$$v = -5x^2 y = -5 \times 1^2 \times 2 = -10 \text{ m/s}$$

$$w = 3t = 3 \times 2 = 6 \text{ m/s}$$

$$\vec{V} = ui + vj + wk = \mathbf{2i - 10j + 6k}$$

$$V = \sqrt{u^2 + v^2 + w^2} = \sqrt{2^2 + (-10)^2 + 6^2} = \mathbf{11.83\ m/s}$$

Now $\frac{\partial u}{\partial x}=\frac{\partial(2x^3)}{\partial x}=6x^2;\ \frac{\partial u}{\partial y}=\frac{\partial(2x^3)}{\partial y}=0;\ \frac{\partial u}{\partial z}=\frac{\partial(2x^3)}{\partial z}=0;\ \frac{\partial u}{\partial t}=\frac{\partial(2x^3)}{\partial t}=0$

Since $a_x=u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}$

Thus $a_x=2x^3(6x^2)+(-5x^2y)(0)+(3t)(0)+0=12x^5$

The acceleration in x-direction at point $A(1, 2, 3)$ m and at $t = 2$ s is calculated as follows.

$$a_x=12x^5=12(1)^5=12\text{ m/s}^2$$

Now $\frac{\partial v}{\partial x}=\frac{\partial(-5x^2y)}{\partial x}=-10xy;\ \frac{\partial v}{\partial y}=\frac{\partial(-5x^2y)}{\partial y}=-5x^2;\ \frac{\partial v}{\partial z}=\frac{\partial(-5x^2y)}{\partial z}=0;\ \frac{\partial v}{\partial t}=\frac{\partial(-5x^2y)}{\partial t}=0$

Since $a_y=u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{\partial v}{\partial t}$

Thus $a_y=2x^3(-10xy)+(-5x^2y)(-5x^2)+(3t)(0)+0=5x^4y$

The acceleration in y-direction at point $A(1, 2, 3)$ m and at $t = 2$ s is calculated as follows.

$$a_y=5x^4y=5\times1^4\times2=10\text{ m/s}^2$$

Now $\frac{\partial w}{\partial x}=\frac{\partial(3t)}{\partial x}=0,\ \frac{\partial w}{\partial y}=\frac{\partial(3t)}{\partial y}=0,\ \frac{\partial w}{\partial z}=\frac{\partial(3t)}{\partial z}=0,\ \frac{\partial w}{\partial t}=\frac{\partial(3t)}{\partial t}=3$

Since $a_z=u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{\partial w}{\partial t}$

Thus $a_z=2x^3(0)+(-5x^2y)(0)+(3t)(0)+3=3$

The acceleration in z-direction at point $A(1, 2, 3)$ m and at $t = 2$ s is calculated as follows.

$$a_z=3\text{ m/s}^2$$

$$\vec{a}=a_xi+a_yj+a_zk=\mathbf{12i+10j+3k}$$

$$a=\sqrt{a_x{}^2+a_y{}^2+a_z{}^2}=\sqrt{12^2+10^2+3^2}=\mathbf{15.9\ m/s^2}$$

Example 6.4 Find the velocity and acceleration components of a fluid particle at position $\vec{r}(x, y, z)=2i+j+3k$, when $t=1.5$ for the fluid flow described by the velocity vector $\vec{V}(x, y, z, t)=5xyi+3x^2j+2(t^2x+z)k$.

Solution

Let $x = 2$, $y = 1$, $z = 3$, $t = 1.5$, $u = 5xy$, $v = 3x^2$ and $w = 2(t^2x + z)$. Let $\vec{V}$ be the velocity vector, V be the resultant velocity, $\vec{a}$ be the acceleration vector and a be the resultant acceleration.

The velocity components at point (2, 1, 3) and at time $t = 1.5$ are calculated in the below expressions.

$$u = 5xy = 5\times 2\times 1 = 10 \text{ units}$$

$$v = 3x^2 = 3\times 2^2 = 12 \text{ units}$$

$$w = 2(t^2 x + z) = 2(1.5^2 \times 2 + 3) = 15 \text{ units}$$

$$\vec{V} = ui + vj + wk = \mathbf{10}\boldsymbol{i} + \mathbf{12}\boldsymbol{j} + \mathbf{15}\boldsymbol{k}$$

$$V = \sqrt{u^2 + v^2 + w^2} = \sqrt{10^2 + 12^2 + 15^2} = \mathbf{21.66 \text{ units}}$$

Now $$\frac{\partial u}{\partial x} = \frac{\partial(5xy)}{\partial x} = 5y,\ \frac{\partial u}{\partial y} = \frac{\partial(5xy)}{\partial y} = 5x,\ \frac{\partial u}{\partial z} = \frac{\partial(5xy)}{\partial z} = 0,\ \frac{\partial u}{\partial t} = \frac{\partial(5xy)}{\partial t} = 0$$

Since $$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

Thus $$a_x = 5xy(5y) + (3x^2)(5x) + 2(t^2 x + z)(0) + 0 = 25xy^2 + 15x^3$$

The acceleration in x-direction at point (2, 1, 3) and at $t = 1.5$ is derived as given below.

$$a_x = 25xy^2 + 15x^3 = 25\times 2\times 1^2 + 15\times 2^3 = 170 \text{ units}$$

Now $$\frac{\partial v}{\partial x} = \frac{\partial(3x^2)}{\partial x} = 6x,\ \frac{\partial v}{\partial y} = \frac{\partial(3x^2)}{\partial y} = 0,\ \frac{\partial v}{\partial z} = \frac{\partial(3x^2)}{\partial z} = 0,\ \frac{\partial v}{\partial t} = \frac{\partial(3x^2)}{\partial t} = 0$$

Since $$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

Thus $$a_y = 5xy(6x) + (3x^2)(0) + 2(t^2 x + z)(0) + 0 = 30x^2 y$$

The acceleration in y-direction at point (2, 1, 3) and at $t = 1.5$ is derived as given below.

$$a_y = 30x^2 y = 30\times 2^2 \times 1 = 120 \text{ units}$$

Now $$\frac{\partial w}{\partial x} = \frac{\partial[2(t^2 x + z)]}{\partial x} = 2t^2;\ \frac{\partial w}{\partial y} = \frac{\partial(3t)}{\partial y} = 0;\ \frac{\partial w}{\partial z} = \frac{\partial[2(t^2 x + z)]}{\partial z} = 2;\ \frac{\partial w}{\partial t} = \frac{\partial[2(t^2 x + z)]}{\partial t} = 4tx$$

Since $$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Thus $$a_z = 5xy(2t^2) + (3x^2)(0) + 2(t^2 x + z)(2) + 4tx$$

$$\therefore a_z = 10xyt^2 + 4(t^2 x + z) + 4tx$$

The acceleration in z-direction at point (2, 1, 3) and at $t = 1.5$ is derived as given below.

$$a_z = 10 \times 2 \times 1 \times 1.5^2 + 4(1.5^2 \times 2 + 3) + 4 \times 1.5 \times 2 = 87 \text{ units}$$

$$\vec{a} = a_x i + a_y j + a_z k = \mathbf{170}\boldsymbol{i} + \mathbf{120}\boldsymbol{j} + \mathbf{87}\boldsymbol{k}$$

$$a = \sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2} = \sqrt{170^2 + 120^2 + 87^2} = \mathbf{225.54 \text{ units}}$$

6.6 ❐ TANGENTIAL AND NORMAL ACCELERATIONS

Let s and n be the respective tangential and normal directions at any point on the streamline (Figure 6.4 (a)) and r be its radius of curvature (Figure 6.4 (b)).

Let at any point on a streamline, V_s and V_n be the velocity components along the tangential (s) and normal (n) directions, respectively.

$$V_s = f_1(s, n, t) \text{ and } V_n = f_2(s, n, t)$$

The acceleration components a_s and a_n in the tangential and normal directions, respectively (Figure 6.4(a)) are given in the below expressions.

$$a_s = \frac{dV_s}{dt} = \frac{\partial V_s}{\partial s} \cdot \frac{ds}{dt} + \frac{\partial V_s}{\partial n} \cdot \frac{dn}{dt} + \frac{\partial V_s}{\partial t}$$

$$a_n = \frac{dV_n}{dt} = \frac{\partial V_n}{\partial s} \cdot \frac{ds}{dt} + \frac{\partial V_n}{\partial n} \cdot \frac{dn}{dt} + \frac{\partial V_n}{\partial t}$$

But

$$\frac{ds}{dt} = V_s \text{ and } \frac{dn}{dt} = V_n$$

Thus

$$a_s = V_s \frac{\partial V_s}{\partial s} + V_n \frac{\partial V_s}{\partial n} + \frac{\partial V_s}{\partial t} \text{ and}$$

$$a_n = V_s \frac{\partial V_n}{\partial s} + V_n \frac{\partial V_n}{\partial n} + \frac{\partial V_n}{\partial t}$$

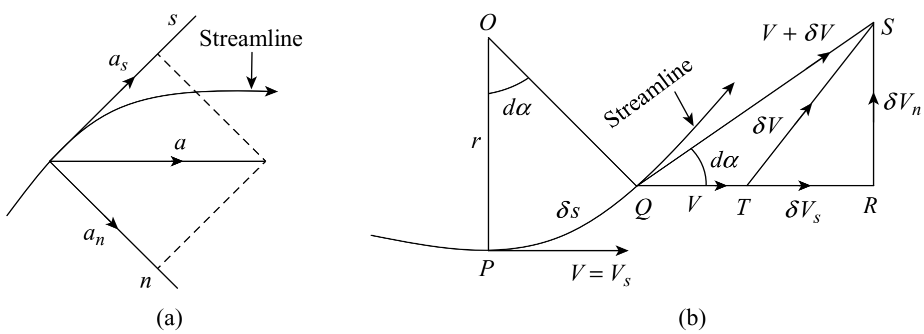

Figure 6.4 *Tangential and normal accelerations*

For any streamline, there is no flow across it and thus, $V_n = 0$, but $(\partial V_n / \partial s)$ need not be zero. Although $V_n = 0$ at any point on the streamline, but at any other point on the streamline, the component of the velocity in the direction parallel to that of V_n need not always be zero. Therefore, we derive the following expressions.

$$a_s = V_s \frac{\partial V_s}{\partial s} + \frac{\partial V_s}{\partial t} \tag{6.19}$$

$$a_n = V_s \frac{\partial V_n}{\partial s} + \frac{\partial V_n}{\partial t} \tag{6.20}$$

Let at any point P on the streamline, the fluid particle has a velocity V tangential to the streamline (i.e., $V = V_s$). In an infinitesimal time dt the fluid particle moves through a distance δs and it takes new position at point Q by attaining the velocity $(V + \delta V)$. The total change in velocity δV for any fluid particle which moves through a distance δs can be resolved into two components, such as δV_s and δV_n along the tangential and normal directions, respectively as shown in Figure 6.4(b).

From ΔPOQ:

$$\delta s = r d\alpha \Rightarrow d\alpha = \frac{\delta s}{r} \tag{i}$$

From ΔQRS:

$$\delta V_n = (V + \delta V)\sin d\alpha = V \sin d\alpha + \delta V \sin d\alpha$$

But for small angular travel $d\alpha$, $\sin d\alpha = d\alpha$, where $\delta V \sin d\alpha$ is negligible.

Thus

$$\delta V_n \approx V d\alpha \Rightarrow d\alpha = \frac{\delta V_n}{V} \tag{ii}$$

Therefore, from expressions (i) and (ii), we derive as follows.

$$\frac{\delta s}{r} = \frac{\delta V_n}{V} \quad \text{or} \quad \frac{\delta V_n}{\delta s} = \frac{V}{r} = \frac{V_s}{r}$$

From Equation (6.20), we get:

$$a_n = \frac{V_s^2}{r} + \frac{\partial V_n}{\partial t} \tag{6.21}$$

From Equations (6.19) to (6.21), the term $(\partial V_s / \partial t)$ is called local tangential acceleration and $(\partial V_n / \partial t)$ is called local normal acceleration. Similarly, in these equations, the term $V_s (\partial V_s / \partial s)$ is called the convective tangential acceleration and (V_s^2 / r) is called convective normal acceleration.

For a steady flow, there is no local acceleration and it means there is only convective acceleration. Thus, for a steady flow, the respective Equations (6.19) and (6.21) are expressed as given below.

$$\boxed{a_s = V_s \frac{\partial V_s}{\partial s}} \tag{6.22}$$

$$\boxed{a_n = \frac{V_s^2}{r}} \tag{6.23}$$

When the streamlines are straight and parallel to each other, then both the normal and tangential convective acceleration are zero. If the streamlines are straight and converging, then there will be tangential convective acceleration only. When the streamlines are curved and equidistant, then there will be normal convective acceleration only. However, when streamlines are curved and converging, then there will be both normal and tangential convective accelerations.

6.7 ❐ RATE OF FLOW (DISCHARGE)

The rate of flow (or discharge) is defined as the quantity of a fluid flowing per second through a section of a pipe or a channel. Generally, the value of rate of flow is denoted by Q.

Let A be the area of cross section of the pipe and V be the average velocity of the liquid flowing through the pipe. The discharge value is given by the following expression.

$$\boxed{Q = AV} \tag{6.24}$$

The value of discharge is measured in m^3/s (or cumecs) and it may also be measured in litres per second (l/s) and $1\ m^3/s = 1000\ l/s$.

6.8 ❐ CONTINUITY EQUATION

The equation based on the principle of conservation of mass is called the continuity equation. According to continuity equation, the mass of a fluid passing through different sections of a pipe is the same if no fluid is added or removed from it. Consider section 1–1 and section 2–2 of a pipe as shown in Figure 6.5.

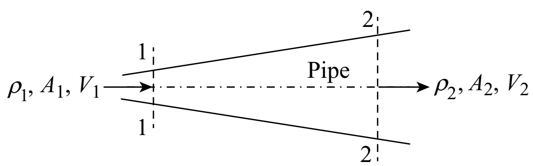

Figure 6.5 *Fluid flow through a pipe*

Let ρ_1 be the density of the fluid at section 1-1, A_1 be the area of the pipe at section 1–1, V_1 be the velocity of the fluid at section 1–1 and ρ_2, A_2, V_2 be the corresponding values at section 2–2.

The mass flow rate at section 1–1 is given by,

$$m_1 = \text{Density} \times \text{Discharge} = \rho_1 \times A_1V_1$$

The mass flow rate at section 2–2 is given by,

$$m_2 = \rho_2 \times A_2V_2$$

According to the law of conservation of mass $m_1 = m_2$, we derive the following expression.

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \tag{6.25}$$

Equation (6.25) is called continuity equation and it is applicable to compressible as well as incompressible fluids. For incompressible fluids, $\rho_1 = \rho_2$ and thus, the continuity equation is expressed as given below.

$$\boxed{A_1V_1 = A_2V_2} \tag{6.26}$$

The Equation (6.26) is applicable to a steady one-dimensional flow of incompressible fluid.

Example 6.5 If 1200 litres of water flows per minute through a 0.2 m diameter pipe which reduces to 0.1 m diameter, then determine the velocities of water flow in the two pipes.

Solution

Let $Q = 1200\ l/\text{min} = 1.2\ m^3/\text{min}$, $d_1 = 0.2$ m and $d_2 = 0.1$ m.

$$Q = \frac{1.2}{60} = 0.02\ m^3/s$$

Since $$Q = A_1V_1 = A_2V_2 \qquad \text{[Continuity equation]}$$

$$\therefore V_1 = \frac{Q}{A_1} = \frac{Q}{(\pi/4)d_1^2} = \frac{0.02}{(\pi/4)\times 0.2^2} = \mathbf{0.637\ m/s}$$

$$\therefore V_2 = \frac{Q}{A_2} = \frac{Q}{(\pi/4)d_2^2} = \frac{0.02}{(\pi/4)\times 0.1^2} = \mathbf{2.55\ m/s}$$

Example 6.6 A water pipe of enlarging cross section has diameters 0.4 m and 1.2 m at sections 1–1 and 2–2, respectively. If the average flow velocity is 2 m/s at section 1–1, then find the velocity at section 2–2. Also determine the discharge and the mass flow rate of water.

Solution

Let $d_1 = 0.4$ m, $d_2 = 1.2$ m and $V_1 = 2$ m/s.

Let Q be the discharge and m be the mass flow rate.

$$A_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} \times 0.4^2 = 0.1257 \text{ m}^2$$

$$A_2 = \frac{\pi}{4} d_2^2 = \frac{\pi}{4} \times 1.2^2 = 1.131 \text{ m}^2$$

Since $$A_1 V_1 = A_2 V_2 \qquad \text{[Continuity equation]}$$

$$\therefore V_2 = \frac{A_1 V_1}{A_2} = \frac{0.1257 \times 2}{1.131} = \mathbf{0.222 \text{ m/s}}$$

$$Q = A_1 V_1 = 0.1257 \times 2 = \mathbf{0.2514 \text{ m}^3\text{/s}}$$

$$m = \rho_w Q = 1000 \times 0.2514 = \mathbf{251.4 \text{ kg/s}}$$

Example 6.7 A pipe 1 m in diameter carrying water at a velocity of 4 m/s is branched into two pipes. The first branch is 0.6 m in diameter and it carries one-third of the water flow. If water flows in the second branch with a velocity of 3 m/s, then determine the flow velocity in the first branch pipe and the diameter of the second branch.

Solution

Let $d = 1$ m, $V = 4$ m/s, $d_1 = 0.6$ m, $Q_1 = (Q/3)$ m^3/s and $V_2 = 3$ m/s.

Let Q be the total discharge, V_1 be the water velocity in the first branched pipe and d_2 be the diameter of the second branched pipe.

$$Q = AV = \frac{\pi}{4} d^2 \times V = \frac{\pi}{4} \times 1^2 \times 4 = 3.1416 \text{ m}^3\text{/s}$$

$$Q_1 = \frac{Q}{3} = \frac{3.1416}{3} = 1.0472 \text{ m}^3\text{/s}$$

$$V_1 = \frac{Q_1}{A_1} = \frac{Q_1}{(\pi/4) d_1^2} = \frac{1.0472}{(\pi/4) \times 0.6^2} = \mathbf{3.704 \text{ m/s}}$$

$$Q_2 = Q - Q_1 = 3.1416 - 1.0472 = 2.0944 \text{ m}^3\text{/s}$$

$$A_2 = \frac{Q_2}{V_2} = \frac{2.0944}{3} = 0.6981 \text{ m}^2$$

Thus $$\frac{\pi}{4} d_2^2 = 0.6981$$

$$\therefore d_2 = \sqrt{\frac{4 \times 0.6981}{\pi}} = \mathbf{0.9428 \text{ m}}$$

Example 6.8 A main pipe of diameter 0.4 m carrying water at an average velocity of 3 m/s is branched into two pipes of diameters 0.25 m and 0.15 m. If water flows in the 0.25 m diameter pipe with a velocity of 2.5 m/s, then determine the water flow velocity in the other branched pipe. Also determine the discharge through the main pipe.

Solution

Let $d = 0.4$ m, $V = 3$ m/s, $d_1 = 0.25$ m, $d_2 = 0.15$ m and $V_1 = 2.5$ m/s.

Let Q be the total discharge, Q_1 be the discharge through d_1 diameter pipe, Q_2 be the discharge through d_2 diameter pipe and V_2 be the water velocity in the pipe of diameter d_2.

$$Q = AV = \frac{\pi}{4}d^2 \times V = \frac{\pi}{4}\times 0.4^2 \times 3 = \mathbf{0.377\ m^3/s}$$

$$Q_1 = \frac{\pi}{4}d_1^2 \times V_1 = \frac{\pi}{4}\times 0.25^2 \times 2.5 = 0.123\ \text{m}^3/\text{s}$$

$$Q_2 = Q - Q_1 = 0.377 - 0.123 = 0.254\ \text{m}^3/\text{s}$$

$$V_2 = \frac{Q_2}{A_2} = \frac{Q_2}{(\pi/4)d_2^2} = \frac{0.254}{(\pi/4)\times 0.15^2} = \mathbf{14.373\ m/s}$$

Example 6.9 A nozzle fitted at the end of a pipe is required to supply 0.3 kg/s of air at pressure 2 bar and temperature 20°C, respectively. If the air flow velocity is limited to 5 m/s, then determine the minimum diameter of the nozzle. Take $R = 287$ J/kgK .

Solution

Let $m = 0.3$ kg/s, $p = 2$ bar $= 2\times 10^5$ N/m^2, $T = 20°\text{C} = 293.15$ K, $V = 5$ m/s and $R = 287$ J/kgK. Let d be the minimum diameter of the nozzle.

$$\rho = \frac{p}{RT} = \frac{2\times 10^5}{287\times 293.15} = 2.377\ \text{kg/m}^3$$

Since $$m = \rho A V$$

Thus $$0.3 = 2.377\times A\times 5$$

$$\therefore A = \frac{0.3}{2.377\times 5} = 0.0252\ \text{m}^2$$

$$\frac{\pi}{4}d^2 = 0.0252\ \text{m}^2$$

$$\therefore d = \sqrt{\frac{0.0252\times 4}{\pi}} = \mathbf{0.1791\ m}$$

Example 6.10 A 10 mm water jet leaves the tip of the nozzle fitted at the end of a pipe with 10 m/s velocity in the vertically upward direction. If there is no energy loss and the jet remains circular, then determine its diameter at a point 3 m above the nozzle tip.

Solution

Refer Figure 6.6. Let the subscripts 1 and 2 denote the conditions at the nozzle tip and at 3 m height from it. Let $d_1 = 10$ mm $= 0.01$ m, $V_1 = 10$ m/s and $h = 3$ m. Let d_2 be the diameter of the jet at 3 m above the nozzle.

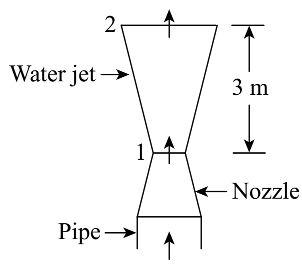

Figure 6.6

Since $$V_2^2 - V_1^2 = 2gh \quad [\because v^2 - u^2 = 2gh]$$

Thus $$V_2^2 - 10^2 = 2\times(-9.81)\times 3$$

$$\therefore V_2 = \sqrt{100 - 58.86} = 6.414\ \text{m/s}$$

From continuity equation, we get:

$$A_2 = \frac{A_1 V_1}{V_2} = \frac{\pi d_1^2 \times V_1}{4 \times V_2} = \frac{\pi \times 0.01^2 \times 10}{4 \times 6.414} = 1.2245 \times 10^{-4} \text{ m}^2$$

Thus $$\frac{\pi}{4} d_2^2 = 1.2245 \times 10^{-4}$$

$$\therefore d_2 = \sqrt{\frac{1.2245 \times 10^{-4} \times 4}{\pi}} = \mathbf{0.0125 \text{ m}}$$

Example 6.11 A cylindrical container of radius 3 m and height 9 m is to be filled completely with water by a number of pipes in 50 minutes. Determine the required water inflow of the container in m^3/s and also determine the number of pipes required if the container is to be filled by 5 cm diameter water pipes in which water flows with a velocity of 5 m/s.

Solution

Let $r = 3$ m, $h = 9$ m, $t = 50$ min $= 3000$ s, $d_1 = 5$ cm $= 0.05$ m and $V = 5$ m/s. Let n be the number of water pipes required, Q be the total discharge, q be the discharge through one pipe and v be the volume of the cylindrical container.

$$v = \pi r^2 h = \pi \times 3^2 \times 9 = 254.469 \text{ m}^3$$

$$Q = \frac{v}{t} = \frac{254.469}{3000} = 0.085 \text{ m}^3/\text{s}$$

$$q = \frac{\pi}{4} d_1^2 \times V = \frac{\pi}{4} \times 0.05^2 \times 5 = 0.00982 \text{ m}^3/\text{s}$$

$$n = \frac{Q}{q} = \frac{0.085}{0.00982} \approx \mathbf{9}$$

6.9 ❒ CONTINUITY EQUATION IN DIFFERENTIAL FORM (3-DIMENSIONS)

Consider a three-dimensional space element of dimensions dx, dy and dz in a flow field which has three velocity components, such as u, v and w along x, y and z directions, respectively, through which a fluid flows as shown in Figure 6.7.

The mass flow of fluid that enters the element through surface $ABFE$ in time dt in x-direction is expressed below.

Fluid influx = Density × Velocity in x-direction × Area $ABFE$ × Time

$$\text{Fluid influx} = \rho \times u \times dy\,dz \times dt$$

During the same time interval, the mass of fluid leaving the element through face $DCGH$, a distance dx apart from surface $ABFE$ is given by the following expression.

$$\text{Fluid efflux} = \left[\rho u + \frac{\partial}{\partial x}(\rho u)\,dx\right] dy\,dz\,dt$$

The gain in mass due to flow in x-direction is given by the difference of fluid influx to fluid efflux as given below.

$$\Rightarrow \rho u\,dy\,dz\,dt - \left[\rho u + \frac{\partial}{\partial x}(\rho u)\,dx\right] dy\,dz\,dt = -\frac{\partial}{\partial x}(\rho u)\,dx\,dy\,dz\,dt \quad \text{(i)}$$

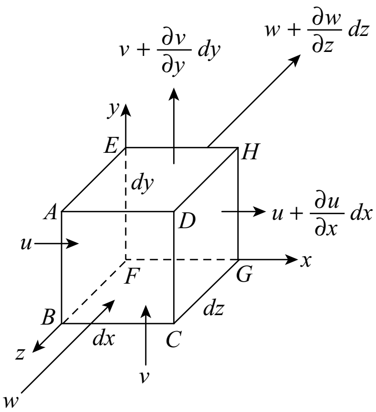

Figure 6.7 *Three-dimensional infinitesimal fluid element*

Similarly, the respective gain in fluid mass due to flow in y and z-direction is expressed as given below.

$$\Rightarrow -\frac{\partial}{\partial y}(\rho v)\, dx\, dy\, dz\, dt \qquad \text{(ii)}$$

$$\Rightarrow -\frac{\partial}{\partial z}(\rho w)\, dx\, dy\, dz\, dt \qquad \text{(iii)}$$

Thus, the total gain in mass fluid in the element in time dt can be obtained by adding the expressions (i), (ii) and (iii) as given below.

$$\Rightarrow -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dx\, dy\, dz\, dt \qquad \text{(iv)}$$

The mass of fluid within the element is given by the following expression.

$$m = \text{Density} \times \text{Volume} = \rho \times dx\, dy\, dz$$

Rate of increase of mass within the element in time dt is given by,

$$\frac{\partial m}{\partial t} dt = \frac{\partial}{\partial t}(\rho\, dx\, dy\, dz)\, dt \qquad \text{(v)}$$

According to the principle of conservation of mass, the total gain in mass equals to the time rate of increase of mass in the element. Thus, by equating the expressions (iv) and (v), we have the following expression.

$$-\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dx\, dy\, dz\, dt = \frac{\partial}{\partial t}(\rho\, dx\, dy\, dz)\, dt$$

By simplification and rearrangement of terms, we get:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \qquad (6.27)$$

Equation (6.27) is the continuity equation in Cartesian coordinates in its most general form which is applicable to any type of flow and for any fluid that is either compressible or incompressible. This equation may be expanded as given below.

$$\frac{\partial \rho}{\partial t} + \left[u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x}\right] + \left[v\frac{\partial \rho}{\partial y} + \rho\frac{\partial v}{\partial y}\right] + \left[w\frac{\partial \rho}{\partial z} + \rho\frac{\partial w}{\partial z}\right] = 0$$

$$\frac{\partial \rho}{\partial t} + \left[u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}\right] + \rho\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] = 0 \qquad (6.28)$$

For a steady flow, $\frac{\partial \rho}{\partial t} = 0$ and for incompressible flow, ρ is constant and thus, Equation (6.28) is derived below.

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0} \qquad (6.29)$$

The Equation (6.29) represents the continuity equation in three-dimensions for a steady and incompressible flow. For a two-dimensional flow, the component $w = 0$ and thus, Equation (6.29) can be written as follows.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (6.30)$$

6.10 ❒ CONTINUITY EQUATION IN CYLINDRICAL POLAR COORDINATES

Consider a point $P(r, \alpha, z)$ in space of a flow field. Let dr, $rd\alpha$ and dz be the small increments and u_r, u_α and u_z be the components of the velocity in the directions r, α and z, respectively, at point P. The general continuity equation can be set up by considering an infinitesimal cylindrical volume element as shown in Figure 6.8 and writing mass flow equations in the radial, tangential and axial directions. In Figure 6.8, we have $AB = EF = DC = GH = dr$, $AE = DG = BF = CH = dz$, $AD = EG = rd\alpha$ and $BC = FH = (r + dr)d\alpha$.

1. **Radial direction ($x - \alpha$ plane):** The mass flow of fluid that enters the element through surface $AEGD$ in time dt in radial direction is given by the following expression.

$$\text{Fluid influx} = \text{Density} \times \text{Velocity in radial direction} \times \text{Area } AEGD \times \text{Time}$$

$$\text{Fluid influx} = \rho \times u_r \times (rd\alpha \times dz) \times dt$$

During the same time interval, the mass of fluid leaving the element through face $BCHF$ at a distance dr apart from the surface $AEGD$ is given by the following expression.

$$\text{Fluid efflux} = \left[\rho u_r + \frac{\partial}{\partial r}(\rho u_r) dr\right](r + dr)\, d\alpha\, dz\, dt$$

The gain in mass due to flow in radial direction is given by,

$$= \text{Fluid influx} - \text{Fluid efflux}$$

$$= \rho u_r (r\, d\alpha\, dz)\, dt - \left[\rho u_r + \frac{\partial}{\partial r}(\rho u_r)\, dr\right](r + dr)\, d\alpha\, dz\, dt$$

$$= \rho u_r r\, d\alpha\, dz\, dt - \left[\rho u_r r\, d\alpha\, dz\, dt + \frac{\partial}{\partial r}(\rho u_r) dr \cdot r\, d\alpha\, dz\, dt + \rho u_r dr\, d\alpha\, dz\, dt\right]$$

$$= -\left[\frac{\partial}{\partial r}(\rho u_r) r + \rho u_r\right] dr\, d\alpha\, dz\, dt \qquad \text{(Neglecting smaller terms)}$$

$$= -\frac{\partial}{\partial r}[(\rho u_r) r]\, dr\, d\alpha\, dz\, dt \qquad \text{(i)}$$

2. **Tangential direction ($r - z$ plane):** The mass flow of fluid that enters the element through surface $ABFE$ in time dt in tangential direction is given by the following expression.

$$\text{Fluid influx} = \rho u_\alpha (dr\, dz)\, dt$$

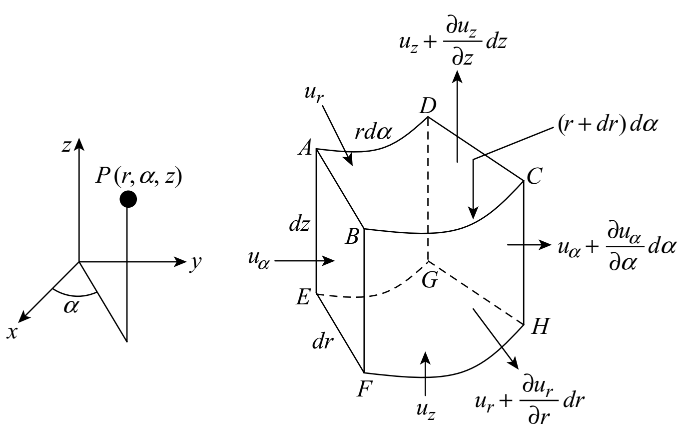

Figure 6.8 *Three-dimensional infinitesimal cylindrical fluid element*

During the same time interval, the mass of fluid leaving the element through face *DGHC*, a distance $r d\alpha$ apart from surface *ABFE* is given by the following expression.

$$\text{Fluid efflux} = \left[\rho u_\alpha + \frac{\partial}{r\partial\alpha}(\rho u_\alpha) r\, d\alpha\right] dr\, dz\, dt$$

The gain in mass due to flow in tangential direction is given by,

$$\Rightarrow \text{Fluid influx} - \text{Fluid efflux}$$

$$\Rightarrow \rho u_\alpha (dr\, dz)\, dt - \left[\rho u_\alpha + \frac{\partial}{r\partial\alpha}(\rho u_\alpha) r d\alpha\right] dr\, dz\, dt$$

$$\Rightarrow \rho u_\alpha (dr\, dz)\, dt - \rho u_\alpha\, dr\, dz\, dt - \frac{\partial}{r\partial\alpha}(\rho u_\alpha) r\, d\alpha\, dr\, dz\, dt$$

$$\Rightarrow -\frac{\partial}{\partial\alpha}(\rho u_\alpha) dr\, d\alpha\, dz\, dt \qquad \text{(ii)}$$

3. **Axial direction ($r - \alpha$ plane):** The mass flow of fluid that enters the element through surface *EFHG* in time *dt* in axial direction is given by the following expression.

$$\text{Fluid influx} = \rho \times u_z \times (dr \times r d\alpha) \times dt$$

During the same time interval, the mass of fluid leaving the element through face *ABCD*, a distance *dz* apart from the surface *EFHG* is given by the following expression.

$$\text{Fluid efflux} = \left[\rho u_z + \frac{\partial}{\partial z}(\rho u_z) dz\right] dr \times r\, d\alpha\, dt$$

The gain in mass due to flow in axial direction is given by,

$$\Rightarrow \text{Fluid influx} - \text{Fluid efflux}$$

$$\Rightarrow \rho u_z (dr \cdot r\, d\alpha) dt - \left[\rho u_z + \frac{\partial}{\partial z}(\rho u_z) dz\right] dr \times r\, d\alpha\, dt$$

$$\Rightarrow \rho u_z (dr \times r d\alpha) dt - \rho u_z (dr \times r d\alpha) dt - \frac{\partial}{\partial z}(\rho u_z)\, dz\, dr \times r\, d\alpha\, dt$$

$$\Rightarrow -r \frac{\partial}{\partial z}(\rho u_z)\, dr\, d\alpha\, dz\, dt \qquad \text{(iii)}$$

Thus, the total gain in mass fluid in the element, in time *dt* can be obtained by adding the expressions (i), (ii) and (iii) as shown below.

$$\Rightarrow -\left[\frac{\partial}{\partial r}[(\rho u_r) r] + \frac{\partial}{\partial\alpha}(\rho u_\alpha) + r\frac{\partial}{\partial z}(\rho u_z)\right] dr\, d\alpha\, dz\, dt \qquad \text{(iv)}$$

The mass of fluid within the element is given by,

$$m = \text{Density} \times \text{Volume} = \rho \times (dz \times dr \times r d\alpha)$$

Rate of increase of mass within the element in time *dt* is given by,

$$\frac{\partial m}{\partial t} dt = \frac{\partial}{\partial t}(\rho dz\, dr \cdot r d\alpha) dt \qquad \text{(v)}$$

According to the principle of conservation of mass, the total gain in mass equals the time rate of increase of mass in the element. Thus, by equating the expressions (iv) and (v), we have the following result.

$$-\left[\frac{\partial}{\partial r}[(\rho u_r) r] + \frac{\partial}{\partial\alpha}(\rho u_\alpha) + r\frac{\partial}{\partial z}(\rho u_z)\right] dr\, d\alpha\, dz\, dt = \frac{\partial}{\partial t}(\rho\, dz\, dr \cdot r\, d\alpha) dt$$

Dividing throughout by r and simplifying, we get:

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho u_r r)+\frac{1}{r}\frac{\partial}{\partial \alpha}(\rho u_\alpha)+\frac{\partial}{\partial z}(\rho u_z)+\frac{\partial \rho}{\partial t}=0 \tag{6.31}$$

The Equation (6.31) represents the continuity in cylindrical polar coordinates in its most general form. This equation is applicable for steady or unsteady flow, uniform or non-uniform flow and compressible or incompressible fluids.

For a steady flow, from Equation (6.31), we get:

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho u_r r)+\frac{1}{r}\frac{\partial}{\partial \alpha}(\rho u_\alpha)+\frac{\partial}{\partial z}(\rho u_z)=0 \tag{6.32}$$

For a two-dimensional steady flow, from Equation (6.31), we get:

$$\frac{\partial}{\partial r}(\rho u_r r)+\frac{\partial}{\partial \alpha}(\rho u_\alpha)=0 \tag{6.33}$$

For a two-dimensional steady and incompressible flow, from Equation (6.31), we get:

$$\boxed{\frac{\partial}{\partial r}(u_r r)+\frac{\partial u_\alpha}{\partial \alpha}=0} \tag{6.34}$$

Example 6.12 Two velocity components are given in the following cases, find the third component of velocity such that they satisfy the continuity equation (i) $u=-x^3+y^2-2z^2$ and $v=x^2y+yz+xy$ (ii) $v=-2y^2$ and $w=-2xyz$.

Solution

(i) $u=-x^3+y^2-2z^2$ and $v=x^2y+yz+xy$

$$\frac{\partial u}{\partial x}=-3x^2,\ \frac{\partial v}{\partial y}=x^2+z+x$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad \text{[Continuity equation]}$$

Substituting the values in the above equation, we get:

$$-3x^2+(x^2+z+x)+\frac{\partial w}{\partial z}=0$$

$$\frac{\partial w}{\partial z}=(2x^2-z-x)$$

$$\partial w=(2x^2-z-x)\partial z$$

Integrating both sides, we get:

$$w=\left(2x^2z-\frac{z^2}{2}-xz\right)+C$$

The constant of integration could be a function of x and y, i.e., $f(x, y)$. Thus, we derive the third component as follows.

$$w=\left(\mathbf{2x^2z-\frac{z^2}{2}-xz}\right)+\boldsymbol{f(x, y)}$$

(ii) $v=-2y^2$ and $w=-2xyz$

$$\frac{\partial v}{\partial y}=-4y,\ \frac{\partial w}{\partial z}=-2xy$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad \text{[Continuity equation]}$$

Substituting the values in the continuity equation, we get:

$$\frac{\partial u}{\partial x} - 4y - 2xy = 0$$

$$\frac{\partial u}{\partial x} = 4y + 2xy$$

$$\partial u = (4y + 2xy)\partial x$$

Integrating both sides, we get:

$$u = (4xy + x^2 y) + C$$

The constant of integration could be a function of y and z, i.e., $f(y, z)$. Thus, we derive the third component as follows.

$$\mathbf{u = (4xy + x^2 y) + f(y, z)}$$

Example 6.13 In a flow field, if $\vec{V} = 2x^3 i - 5x^2 yj - x^2 zk$ is the velocity vector, then find whether it is a possible case of steady incompressible flow. If so, then determine the velocity and acceleration of fluid particle at (3, 1, 3).

Solution

Let $\vec{V} = 2x^3 i - 5x^2 yj - x^2 zk$, $x = 3$, $y = 1$ and $z = 3$.

(i) From the velocity vector equation, we get:

$$u = 2x^3,\ v = -5x^2 y \text{ and } w = -x^2 z$$

$$\frac{\partial u}{\partial x} = \frac{\partial(2x^3)}{\partial x} = 6x^2;\ \frac{\partial u}{\partial y} = \frac{\partial(2x^3)}{\partial y} = 0;\ \frac{\partial u}{\partial z} = \frac{\partial(2x^3)}{\partial z} = 0;\ \frac{\partial u}{\partial t} = \frac{\partial(2x^3)}{\partial t} = 0$$

$$\frac{\partial v}{\partial x} = \frac{\partial(-5x^2 y)}{\partial x} = -10xy;\ \frac{\partial v}{\partial y} = \frac{\partial(-5x^2 y)}{\partial y} = -5x^2;\ \frac{\partial v}{\partial z} = \frac{\partial(-5x^2 y)}{\partial z} = 0;\ \frac{\partial v}{\partial t} = \frac{\partial(-5x^2 y)}{\partial t} = 0$$

$$\frac{\partial w}{\partial x} = \frac{\partial(-x^2 z)}{\partial x} = -2xz;\ \frac{\partial w}{\partial y} = \frac{\partial(-x^2 z)}{\partial y} = 0;\ \frac{\partial w}{\partial z} = \frac{\partial(-x^2 z)}{\partial z} = -x^2;\ \frac{\partial w}{\partial t} = \frac{\partial(-x^2 z)}{\partial t} = 0$$

For a possible case of fluid, the following continuity equation must be satisfied.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Thus $$6x^2 + (-5x^2) + (-x^2) = 0$$

Since the continuity equation is satisfied, the given expression for velocity represents a possible case of steady incompressible flow.

(ii) $u = 2x^3 = 2 \times 3^3 = 54$ units

$$v = -5x^2 y = -5 \times 3^2 \times 1 = -45 \text{ units}$$

$$w = -x^2 z = -3^2 \times 3 = -27 \text{ units}$$

The velocity vector is given by,

$$\vec{V} = ui + vj + wk = \mathbf{54i - 45j - 27k}$$

The resultant velocity is given by,

$$V = \sqrt{u^2 + v^2 + w^2} = \sqrt{54^2 + (-45)^2 + (-27)^2} = \mathbf{75.3 \text{ units}}$$

(iii) $a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$

Thus $$a_x = 54(6x^2) + (-45)(0) + (-27)(0) + 0 = 324x^2$$

$$\therefore a_x = 324x^2 = 324 \times 3^2 = 2916 \text{ units}$$

Since $$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

Thus $$a_y = 54(-10xy) + (-45)(-5x^2) + (-27)(0) + 0 = -540xy + 225x^2$$

$$\therefore a_y = -540xy + 225x^2 = -540 \times 3 \times 1 + 225 \times 3^2 = 405 \text{ units}$$

Since $$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Thus $$a_z = 54(-2xz) + (-45)(0) + (-27)(-x^2) + 0 = -108xz + 27x^2$$

$$\therefore a_z = -108xz + 27x^2 = -108 \times 3 \times 3 + 27 \times 3^2 = -729 \text{ units}$$

Acceleration vector is given by,

$$\vec{a} = a_x i + a_y j + a_z k = \mathbf{2916}\boldsymbol{i} + \mathbf{405}\boldsymbol{j} - \mathbf{729}\boldsymbol{k}$$

The resultant acceleration is given by,

$$a = \sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2} = \sqrt{2916^2 + 405^2 + (-729)^2} = \mathbf{3032.91\ units}$$

Example 6.14 Determine whether the velocity components $u_r = r\cos\alpha$ and $u_\alpha = 2r\sin\alpha$ represent a physically possible flow?

Solution

Let $u_r = r\cos\alpha$ and $u_\alpha = 2r\sin\alpha$.

$$u_r r = r^2 \cos\alpha \qquad \text{[Multiply both sides by } r\text{]}$$

$$\frac{\partial}{\partial r}(u_r r) = \frac{\partial}{\partial r}(r^2 \cos\alpha) = 2r\cos\alpha$$

$$\frac{\partial u_\alpha}{\partial \alpha} = -2r\cos\alpha$$

For physically possible flow, the following continuity equation should be satisfied,

$$\frac{\partial}{\partial r}(u_r r) + \frac{\partial u_\alpha}{\partial \alpha} = 0$$

Thus $$2r\cos\alpha + (-2r\cos\alpha) = 0$$

Since the continuity equation is satisfied and therefore, the given velocity components represent a physically possible flow.

Example 6.15 A pipe converges uniformly from 0.2 m diameter to 0.1 m diameter over 1 m length. The discharge through the pipe is 10 litres per second. (i) Determine the convective acceleration at the middle of the pipe. (ii) If the discharge through the pipe increases uniformly from 10 to 20 litres per second in 20 seconds, then find the total acceleration at the middle of the pipe after 10 seconds.

Solution

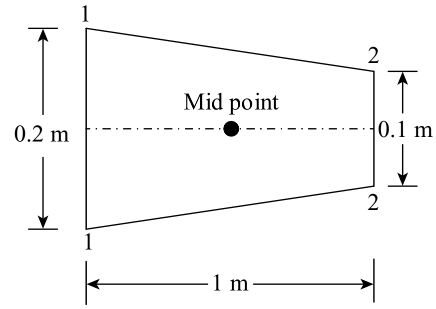

Figure 6.9

Refer Figure 6.9. Let $d_1 = 0.2$ m, $d_2 = 0.1$ m, $l = 1$ m, $Q = Q_1 = 10\, l/\text{s} = 0.01 \text{m}^3/\text{s}$, $Q_2 = 20\, l/\text{s} = 0.02 \text{ m}^3/\text{s}$, $t = 20$ s and $t_1 = 10$ s.

(i) $$u_1 = \frac{Q}{(\pi/4)d_1^2} = \frac{0.01}{(\pi/4)\times 0.2^2} = 0.3183 \text{ m/s}$$

$$u_2 = \frac{Q}{(\pi/4)d_2^2} = \frac{0.01}{(\pi/4)\times 0.1^2} = 1.273 \text{ m/s}$$

Since the cross section of the pipe decreases linearly, the velocity also increases linearly. Therefore, velocity at a distance x can be expressed as given below.

$$u = u_1 + (u_2 - u_1)\times\frac{x}{l} = 0.3183 + (1.273 - 0.3183)\times\frac{x}{1}$$

$$\therefore u = 0.3183 + 0.9547x$$

$$\frac{\partial u}{\partial x} = 0.9547$$

The flow is one-dimensional and thus, $v = 0$ and $w = 0$. Therefore, the convective acceleration can be given from Equation (6.18) and derived as follows.

$$u\frac{\partial u}{\partial x} = (0.3183 + 0.9547x)\times 0.9547$$

The convective acceleration at the middle of the pipe, i.e., at $x = 0.5$ m is given by,

$$u\frac{\partial u}{\partial x} = (0.3183 + 0.9547\times 0.5)\times 0.9547 = \mathbf{0.7596\ m/s^2}$$

(ii) The discharge increases uniformly from $Q_1 = 0.01 \text{ m}^3/\text{s}$ to $Q_2 = 0.02 \text{ m}^3/\text{s}$ in 20 seconds. Thus, the discharge after 10 seconds can be derived as given below.

$$Q = Q_1 + (Q_2 - Q_1)\times\frac{t_1}{t} = 0.01 + (0.02 - 0.01)\times\frac{10}{20} = 0.015 \text{ m}^3/\text{s}$$

$$u_1 = \frac{Q}{(\pi/4)d_1^2} = \frac{0.015}{(\pi/4)\times 0.2^2} = 0.4775 \text{ m/s}$$

$$u_2 = \frac{Q}{(\pi/4)d_2^2} = \frac{0.015}{(\pi/4)\times 0.1^2} = 1.91 \text{ m/s}$$

$$u = u_1 + (u_2 - u_1)\times\frac{x}{l} = 0.4775 + (1.91 - 0.4775)\times\frac{x}{1}$$

$$\therefore u = 0.4775 + 1.4325x$$

$$\frac{\partial u}{\partial x} = 1.4325$$

The convective acceleration can be given from Equation (6.18) and it is derived as given below.

$$u\frac{\partial u}{\partial x} = (0.4775 + 1.4325x) \times 1.4325$$

The convective acceleration at the middle of the pipe, i.e., at $x = 0.5$ m is given as given below.

$$u\frac{\partial u}{\partial x} = (0.4775 + 1.4325 \times 0.5) \times 1.4325 = 1.71 \text{ m/s}^2$$

The diameter at mid-section is derived as follows.

$$d = d_2 - (d_2 - d_1) \times \frac{x}{l} = 0.2 - (0.2 - 0.1) \times \frac{0.5}{1} = 0.15 \text{ m}$$

The local acceleration can be given by Equation (6.17) and it is expressed as follows.

$$\frac{\partial u}{\partial t} = \frac{\text{Change in velocity in time } (\partial t)}{\text{Time } (\partial t)} = \frac{V_2 - V_1}{20} = \frac{1}{20}\left[\frac{Q_2}{A} - \frac{Q_1}{A}\right]$$

Thus $$\frac{\partial u}{\partial t} = \frac{1}{20}\left[\frac{Q_2 - Q_1}{A}\right] = \frac{1}{20}\left[\frac{Q_2 - Q_1}{(\pi/4)d^2}\right] = \frac{1}{20} \times \frac{0.02 - 0.01}{(\pi/4) \times 0.15^2} = 0.0283 \text{ m/s}^2$$

Since total acceleration is the sum of convective acceleration and local acceleration, we get:

$$\text{Total acceleration} = u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 1.71 + 0.0283 = \mathbf{1.7383\ m/s^2}$$

6.11 ❒ TYPES OF MOTIONS OF A FLUID ELEMENT

A fluid particle during its motion can undergo anyone or any combination of the four types of displacements, namely linear translation (or pure translation), linear deformation, angular deformation and rotation as shown in Figure 6.10.

6.11.1 Linear Translation

It is defined as the bodily movement of a fluid element from one position to another in such a way that its new axes are parallel to the original axes. The particle neither rotates nor deformed, i.e., no change in size, shape and orientation of the element as shown in Figure 6.10(a). It may be observed in a parallel uniform flow.

6.11.2 Linear Deformation

In this type of motion, when the fluid element moves, it gets deformed (stretch or shrink) in the linear direction in such a way that the axes of the elements remain parallel but its lengths change as shown in Figure 6.10(b).

6.11.3 Angular Deformation

In this type of motion, the fluid element deforms in such a way that the two axes rotate by the same amount but in opposite directions with respect to the original positions. It is defined as the average change in the angle between the two adjacent sides of a fluid element as illustrated in Figure 6.10(c).

Let $\delta\alpha_1$ be the angular displacement of x-axis and $\delta\alpha_2$ be the angular displacement of y-axis in time δt.

Since $\delta\alpha_1$ is caused by the variation of v along the x-axis, we get:

$$\delta\alpha_1 = \left(\frac{\partial v}{\partial x}\delta x \delta t\right) \times \frac{1}{\delta x}$$

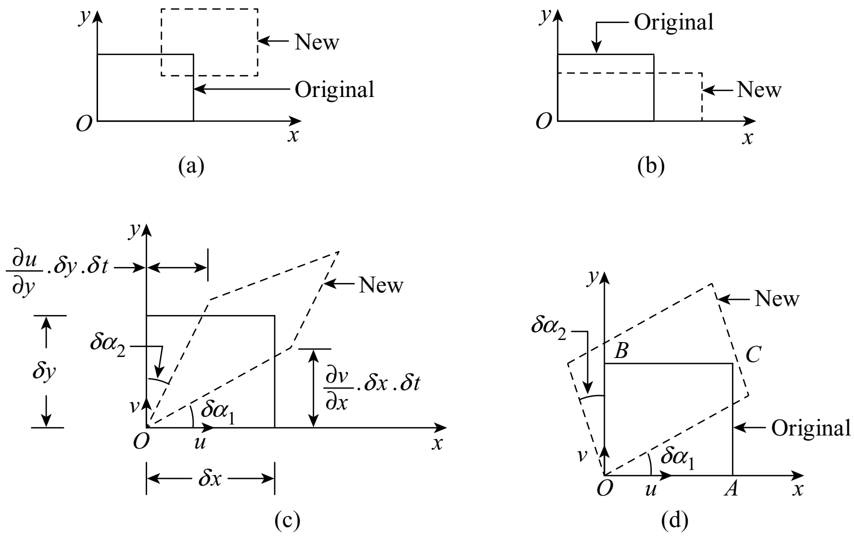

Figure 6.10 *Displacement of fluid particles*

Thus, the angular deformation rate of x-axis (assume $\Delta\alpha_1$) is given by the following expression.

$$\Delta\alpha_1 = \frac{\delta\alpha_1}{\delta t} = \frac{\partial v}{\partial x}$$

Since $\delta\alpha_2$ is caused by the variation of u along the y-axis, we get:

$$\delta\alpha_2 = \left(\frac{\partial u}{\partial y}\delta y \delta t\right) \times \frac{1}{\delta y}$$

Thus, angular deformation rate of y-axis (assume $\Delta\alpha_2$) is given by the following expression.

$$\Delta\alpha_2 = \frac{\delta\alpha_2}{\delta t} = \frac{\partial u}{\partial y}$$

Thus, shear strain rate in $(x - y)$ plane is given by the following expression.

$$\varepsilon_{xy} = \frac{1}{2}(\Delta\alpha_1 + \Delta\alpha_2) = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{6.35a}$$

Similarly, the respective shear strain rate in $(y - z)$ and $(z - x)$ planes are expressed as follows.

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \tag{6.35b}$$

$$\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{6.35c}$$

Direct strain rate or linear strain or dilatancy The direct strain rate or linear strain or dilatancy in x, y and z directions, respectively are as follows.

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \; \varepsilon_{yy} = \frac{\partial v}{\partial y} \text{ and } \varepsilon_{zz} = \frac{\partial w}{\partial z} \tag{6.36}$$

6.11.4 Rotation

In this type of motion, the fluid element moves in such a way that both of its axes rotate in the same direction as shown in Figure 6.10(d). The rotation may be defined as the average of the angular velocities that are perpendicular to the axis of rotation.

The angular velocity of element *OA* about the *z*-axis is given by the following expression

$$\omega_{OA} = \lim_{\delta t \to 0} \frac{\delta \alpha_1}{\delta t} = \frac{\partial v}{\partial x}$$

The angular velocity of element *OB* about the *z*-axis is given by the following expression.

$$\omega_{OB} = \lim_{\delta t \to 0} \frac{\delta \alpha_2}{\delta t} = -\frac{\partial u}{\partial y}$$

The negative sign is introduced because u is negative on the left hand side of the y-axis.

Thus, rotation of the element about z-axis is given by the average of the angular velocities of the line *OA* and the line *OB* as expressed below.

$$\omega_z = \frac{1}{2}\left(\omega_{OA} + \omega_{OB}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{6.37a}$$

Similarly, the rotation about x-axis and y-axis can be respectively expressed as follows.

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \tag{6.37b}$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \tag{6.37c}$$

For irrotational flow, the rotation components ω_x, ω_y and ω_z will be zero. Thus, for the flow to be irrotational, the following conditions must be satisfied.

1. For $\omega_x = 0$: $$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z} \tag{6.38a}$$
2. For $\omega_y = 0$: $$\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} \tag{6.38b}$$
3. For $\omega_z = 0$: $$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \tag{6.38c}$$

In the vector notation, from Equation (6.37), we get:

$$\omega = \frac{1}{2}\left(\omega_x i + \omega_y j + \omega_z k\right) = \frac{1}{2}(\nabla \times V) \tag{6.39}$$

The vector $(\nabla \times V)$ is the curl of velocity vector. Thus, the condition for the flow to be irrotational may be expressed as given below.

$$\text{curl } V = (\nabla \times V) = 0$$

6.11.5 Vorticity

The value of vorticity is generally denoted by ξ (zeta) and its mathematical expression is represented below.

$$\boxed{\xi = (\nabla \times V)} \tag{6.40}$$

From Equation (6.39), we get:

$$\boxed{\xi = 2\omega} \tag{6.41}$$

Thus, from the above expression, it can be noticed that vorticity is equal to twice the value of rotation.

From Equation (6.40), we get:

$$\xi = (\nabla \times V) = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

In vector notation, the expression for vorticity is given by,

$$\xi = \xi_x i + \xi_y j + \xi_z k \tag{6.42}$$

The vorticity components can be separately given by,

$$\xi_x = 2\omega_x = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \tag{6.43a}$$

$$\xi_y = 2\omega_y = \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \tag{6.43b}$$

$$\xi_z = 2\omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{6.43c}$$

For an irrotational flow, the vorticity components ξ_x, ξ_y and ξ_z will be zero.

6.11.6 Circulation

It is defined as the flow along a closed curve, i.e., it represents flow in eddies and vortices. It is denoted by Γ (gamma). Mathematically, circulation is represented as the line integral of the tangential velocity around a closed contour in the flow field.

Let us consider a closed contour with points *P* and *Q* lying on it at a distance *ds* apart and *V* be the velocity inclined at an angle α with the tangent to the contour as shown in Figure 6.11(a). The general expression for circulation is shown below.

$$\Gamma = \int_c V \cos\alpha \, ds \tag{6.44}$$

Considering a rectangular element *ABCD* with sides *dx* and *dy* parallel to *x* -axis and *y*-axis, respectively. The tangential velocities are shown in Figure 6.11(b). Generally, circulation is taken positive in anticlockwise direction. The total circulation for the rectangular element can be obtained by a summation of line integrals around the element in an anticlockwise direction proceeding from corner *A*.

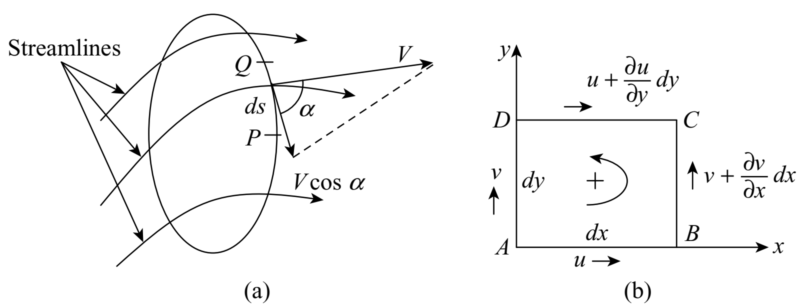

Figure 6.11 *Circulation around (a) closed curve (b) an infinitesimal rectangle*

Thus $$\Gamma = \Gamma_{AB} + \Gamma_{BC} + \Gamma_{CD} + \Gamma_{DA}$$

$$\Gamma = udx + \left(v + \frac{\partial v}{\partial x}dx\right)dy - \left(u + \frac{\partial u}{\partial y}dy\right)dx - vdy$$

$$\Gamma = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy \tag{6.45}$$

Since $$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \xi_z$$

Thus $$\Gamma = \xi_z dxdy$$

$$\therefore \xi_z = \frac{\Gamma}{dxdy} = \frac{\text{Circulation}}{\text{Area}} \tag{6.46}$$

Therefore, vorticity about a normal to the plane in a flow is equal to the circulation per unit area in the given plane. For an irrotational flow, the vorticity is zero and thus, the circulation around any closed path in an irrotational flow will also be zero.

Example 6.16 Examine whether $u = y^3 + 6x - 3x^2 y$ and $v = 3xy^2 - 6y - x^3$ represent a physically possible two-dimensional flow. Also check whether the flow is rotational or irrotational.

Solution

Let $u = y^3 + 6x - 3x^2 y$ and $v = 3xy^2 - 6y - x^3$.

$$\frac{\partial u}{\partial x} = 6 - 6xy,\ \frac{\partial u}{\partial y} = 3y^2 - 3x^2,\ \frac{\partial v}{\partial x} = 3y^2 - 3x^2,\ \frac{\partial v}{\partial y} = 6xy - 6$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{[Continuity equation]}$$

Substituting the values in the continuity equation, we get:

$$(6 - 6xy) + (6xy - 6) = 0$$

Since continuity equation is satisfied, it is a possible case of fluid flow.

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}[(3y^2 - 3x^2) - (3y^2 - 3x^2) = 0$$

Since $\omega_z = 0$ and therefore, the flow is irrotational.

Example 6.17 The velocity components of a three-dimensional incompressible fluid flow are $u = (3x + y + z)t$, $v = (x - 3y + z)t$ and $w = (x + y)t$. State if the flow represented by the given velocity components is a physically possible three-dimensional flow and also state whether the flow is rotational or irrotational.

Solution

Let $u = (3x + y + z)t$, $v = (x - 3y + z)t$ and $w = (x + y)t$.

$$\frac{\partial u}{\partial x} = 3t,\ \frac{\partial v}{\partial y} = -3t,\ \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{[Continuity equation]}$$

Substituting the values in the continuity equation, we get:

$$3t + (-3t) + 0 = 0$$

Since continuity equation is satisfied, it is a possible case of fluid flow.

Now $$(\nabla \times V) = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (3x+y+z)t & (x-3y+z)t & (x+y)t \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(x+y)t - \frac{\partial}{\partial z}(x-3y+z)t\right] i - \left[\frac{\partial}{\partial x}(x+y)t - \frac{\partial}{\partial z}(3x+y+z)t\right] j$$

$$+ \left[\frac{\partial}{\partial x}(x-3y+z)t - \frac{\partial}{\partial y}(3x+y+z)t\right] k$$

$$\therefore (\nabla \times V) = (t-t)\, i - (t-t)\, j + (t-t)\, k = 0$$

Since the curl of velocity vector is zero, the flow is irrotational.

Example 6.18 The velocity components of a two-dimensional incompressible fluid flow are $u = x/(x^2 + y^2)$ and $v = y/(x^2 + y^2)$. State whether the flow is rotational or irrotational.

Solution

Let $u = x/(x^2 + y^2) = x(x^2 + y^2)^{-1}$ and $v = y/(x^2 + y^2) = y(x^2 + y^2)^{-1}$.

Since $$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Thus $$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\{y(x^2 + y^2)^{-1}\} - \frac{\partial}{\partial y}\{x(x^2 + y^2)^{-1}\}\right]$$

$$\therefore \omega_z = \frac{1}{2}\left[-\frac{2xy}{(x^2 + y^2)^2} + \frac{2xy}{(x^2 + y^2)^2}\right] = 0$$

Since the rotation $\omega_z = 0$ and therefore, the flow is irrotational.

Example 6.19 If the velocity components of a two-dimensional incompressible fluid flow are $u = 3x^3$ and $v = -5x^2 y$, then find the shear strain rate and also state whether the flow is rotational or irrotational.

Solution

Let $u = 3x^3$ and $v = -5x^2 y$.

$$\frac{\partial u}{\partial x} = 9x^2,\ \frac{\partial u}{\partial y} = 0,\ \frac{\partial v}{\partial x} = -10xy,\ \frac{\partial v}{\partial y} = -5x^2$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \frac{1}{2}(-10xy + 0) = \mathbf{-5xy}$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}[-10xy - 0] = -5xy$$

Since the rotation $\omega_z \neq 0$ and therefore, the flow is rotational.

Example 6.20 If the velocity vector of a three-dimensional incompressible fluid flow is given by $\vec{V} = (y^2 + z^2)i - (x^2 + z^2)j + (x^2 + y^2)k$, then find the components of rotation at (2, 1, 3).

Solution

$$\vec{V} = (y^2 + z^2)i - (x^2 + z^2)j + (x^2 + y^2)k,\ x = 2,\ y = 1 \text{ and } z = 3.$$

From velocity vector, we get:

$$u = (y^2 + z^2),\ v = -(x^2 + z^2) \text{ and } w = (x^2 + y^2)$$

$$\frac{\partial u}{\partial y} = 2y,\ \frac{\partial u}{\partial z} = 2z,\ \frac{\partial v}{\partial x} = -2x,\ \frac{\partial v}{\partial z} = -2z,\ \frac{\partial w}{\partial x} = 2x,\ \frac{\partial w}{\partial y} = 2y$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}(-2x - 2y) = -(x + y)$$

$$\therefore \omega_z = -(2+1) = \mathbf{-3\ units}$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \frac{1}{2}[2y - (-2z)] = (y + z)$$

$$\therefore \omega_x = (1+3) = \mathbf{4\ units}$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \frac{1}{2}(2z - 2x) = (z - x)$$

$$\therefore \omega_y = (3-2) = \mathbf{1\ units}$$

Example 6.21 If the velocity vector of a two-dimensional incompressible fluid flow is given by $\vec{V} = (2x^2 - z^2)i + (y^2 + 2z^2)j$, then find (i) the third component of velocity and (ii) examine whether the flow is irrotational.

Solution

Let $\vec{V} = (2x^2 - z^2)i + (y^2 + 2z^2)j$, $u = 2x^2 - z^2$ and $v = y^2 + 2z^2$.

(i) $\frac{\partial u}{\partial x} = 4x,\ \frac{\partial v}{\partial y} = 2y$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{[Continuity equation]}$$

Substituting the values in the continuity equation, we get:

$$4x + 2y + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial w}{\partial z} = (-4x - 2y) \Rightarrow \partial w = (-4x - 2y)\partial z$$

Integrating on both sides, we get:

$$w = (-4x - 2y)z + C$$

Neglecting the constant of integration which is a function of x and y, we get:

$$w = \mathbf{(-4x - 2y)z}$$

$$\text{(ii)}\ (\nabla \times V) = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x^2 - z^2 & y^2 + 2z^2 & (-4x-2y)z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-4x-2y)z - \frac{\partial}{\partial z}(y^2+2z^2)\right] i - \left[\frac{\partial}{\partial x}(-4x-2y)z - \frac{\partial}{\partial z}(2x^2-z^2)\right] j + \left[\frac{\partial}{\partial x}(y^2+2z^2) - \frac{\partial}{\partial y}(2x^2-z^2)\right] k$$

$$= (-2z-4z)\, i - [-4z-(-2z)]\ j + (0-0)\ k$$

$$= -6zi + 2zj$$

Since the curl of velocity vector is not zero, the flow is rotational.

Example 6.22 The velocity components of a three-dimensional incompressible fluid flow are $u = xy$, $v = -2yz$ and $w = -yz + z^2$. Examine (i) whether it is a possible case of fluid flow and (ii) whether the flow is rotational or irrotational. Also determine (iii) the angular velocity, (iv) vorticity, (v) shear strain and dilatancy at (1, 2, 3).

Solution

Let $u = xy$, $v = -2yz$ and $w = -yz + z^2$.

(i) $\dfrac{\partial u}{\partial x} = y,\ \dfrac{\partial v}{\partial y} = -2z,\ \dfrac{\partial w}{\partial z} = -y + 2z$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{[Continuity equation]}$$

Substituting the values in the continuity equation, we get:

$$y - 2z - y + 2z = 0$$

Since continuity equation is satisfied, it is a possible case of fluid flow.

$$\text{(ii)}\ (\nabla \times V) = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & -2yz & -yz + z^2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-yz+z^2) - \frac{\partial}{\partial z}(-2yz)\right] i - \left[\frac{\partial}{\partial x}(-yz+z^2) - \frac{\partial}{\partial z}(xy)\right] j + \left[\frac{\partial}{\partial x}(-2yz) - \frac{\partial}{\partial y}(xy)\right] k$$

$$= [-z-(-2y)]\, i - [0-0]\ j + (0-x)\ k$$

$$= (-z+2y)i - xk$$

Since the curl of velocity vector is not zero, the flow is rotational.

(iii) Let $u = xy$, $v = -2yz$ and $w = -yz + z^2$.

$$\frac{\partial w}{\partial y} = -z;\ \frac{\partial v}{\partial z} = -2y;\ \frac{\partial u}{\partial z} = 0;\ \frac{\partial w}{\partial x} = 0;\ \frac{\partial v}{\partial x} = 0;\ \frac{\partial u}{\partial y} = x$$

Since $$\omega = \frac{1}{2}\left(\omega_x i + \omega_y j + \omega_z k\right)$$

or $$\omega = \frac{1}{2}\left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)i + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)j + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)k\right]$$

Thus $$\omega = \frac{1}{2}\left[\{-z-(-2y)\}i + (0-0)j + (0-x)k\right]$$

$$\therefore \boldsymbol{\omega = \frac{1}{2}\left[(-z+2y)i - xk\right]}$$

(iv) Vorticity is given by,

$$\xi = 2\omega = 2\times\frac{1}{2}\left[(-z+2y)i - xk\right]$$

$$\therefore \boldsymbol{\xi = (-z+2y)i - xk}$$

(v) $x = 1,\ y = 2$ and $z = 3$.

Shear strain rate can be given by Equation (6.35) and we get:

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \frac{1}{2}(0+x) = \frac{x}{2} = \frac{1}{2} = \mathbf{0.5}$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = \frac{1}{2}(-z-2y) = \frac{1}{2}(-3-2\times 2) = \mathbf{-3.5}$$

$$\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}(0+0) = \mathbf{0}$$

Dilatancy in x, y and z, directions respectively can be given by Equation (6.36) and we get:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = y = \mathbf{2}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = -2z = -(2\times 3) = \mathbf{-6}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = -y + 2z = -2 + 2\times 3 = \mathbf{4}$$

Example 6.23 If the velocity field is given by $u = 4y - 2x$ and $v = 2y - x$, then determine the circulation and vorticity around the closed curve defined by $x = 2,\ y = 1,\ x = 4$ and $y = 4$.

Solution

Refer Figure 6.12. Let $u = 4y - 2x,\ v = 2y - x,\ x = 2,\ y = 1,\ x = 4$ and $y = 4$.

$$\Gamma = \Gamma_{AB} + \Gamma_{BC} + \Gamma_{CD} + \Gamma_{DA} = \int_{AB} u dx + \int_{BC} v dy + \int_{CD} u dx + \int_{DA} v dy$$

$$\Gamma = \int_2^4 (4y-2x)dx + \int_1^4 (2y-x)dy + \int_4^2 (4y-2x)dx + \int_4^1 (2y-x)dy$$

$$\Gamma = [4xy - x^2]_2^4 + [y^2 - xy]_1^4 + [4xy - x^2]_4^2 + [y^2 - xy]_4^1$$

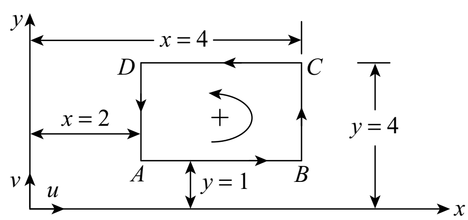

Figure 6.12

For 1st integral $y = 1$, for 2nd integral $x = 4$, for 3rd integral $y = 4$ and for 4th integral $x = 2$.

$$\Gamma = [(4\times4\times1-4^2)-(4\times2\times1-2^2]+[(4^2-4\times4)-(1^2-4\times1)]$$
$$+[(4\times2\times4-2^2)-(4\times4\times4-4^2)]+[(1^2-2\times1)-(4^2-2\times4)]$$
$$\therefore \Gamma = [0-4]+[0-3]+[28-48]+[-1-8] = \mathbf{-36}$$

Area of the rectangle *ABCD* is given by,

$$A = (4-2)\times(4-1) = 6$$

Vorticity, i.e., circulation per unit area is given by,

$$\xi = \frac{\Gamma}{A} = \frac{-36}{6} = \mathbf{-6}$$

6.12 ❐ VELOCITY POTENTIAL AND STREAM FUNCTIONS

6.12.1 Velocity Potential Function

The velocity potential function is defined as a scalar function of space and time such that its derivative with respect to any direction gives the fluid velocity in that direction. It is denoted by the Greek letter ϕ (phi). Mathematically, velocity potential function for steady flow is defined as $\phi = f(x, y, z)$ and the respective expression is as follows.

$$u = \frac{\partial\phi}{\partial x};\ v = \frac{\partial\phi}{\partial y};\ w = \frac{\partial\phi}{\partial z} \tag{6.47}$$

Here, u, v and w, are the components of velocity in x, y and z, directions, respectively. The potential function ϕ decreases along the flow direction and it can also be expressed as shown below.

$$u = -\frac{\partial\phi}{\partial x};\ v = -\frac{\partial\phi}{\partial y};\ w = -\frac{\partial\phi}{\partial z} \tag{6.47a}$$

The lines of constant velocity potential function are called equipotential lines.

In polar coordinates, the velocity component in terms of potential function is expressed as given below.

$$u_r = \frac{\partial\phi}{\partial r},\ u_\alpha = \frac{1}{r}\frac{\partial\phi}{\partial\alpha} \tag{6.47b}$$

For an incompressible steady flow, the continuity equation is given by Equation (6.29) and we get:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0$$

Substituting the values of u, v and w, from Equation (6.47) in the above equation, we get:

$$\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right) = 0$$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} = 0 \tag{6.48}$$

Equation (6.48) is known as Laplace equation and it may be expressed in vector notation as given below.

$$\nabla^2\phi = 0 \tag{6.48a}$$

Any function ϕ that satisfies the Laplace equation will correspond to some case of fluid flow. For a rotational flow, the rotation components are given in terms of the velocity potential by substituting Equation (6.47) in Equation (6.37a, b, c) and it is expressed as given below.

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right) \tag{6.49a}$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \frac{1}{2}\left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right) \tag{6.49b}$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \frac{1}{2}\left(\frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial^2 \phi}{\partial x \partial z}\right) \tag{6.49c}$$

If ϕ is a continuous function, then we have,

$$\frac{\partial^2 \phi}{\partial x\, \partial y} = \frac{\partial^2 \phi}{\partial y\, \partial x};\ \frac{\partial^2 \phi}{\partial y\, \partial z} = \frac{\partial^2 \phi}{\partial z\, \partial y};\ \frac{\partial^2 \phi}{\partial z\, \partial x} = \frac{\partial^2 \phi}{\partial x\, \partial z}$$

According to which $\omega_x = \omega_y = \omega_z = 0$ and thus, the flow is irrotational.

Thus, if the velocity potential ϕ satisfies the Laplace equation, then it represents the possible steady incompressible irrotational flow. The velocity potential exists only for irrotational flows of fluids. Generally, an irrotational flow is known as potential flow.

6.12.2 Stream Function

Stream function is defined as the scalar function of space and time such that its partial derivative with respect to any direction gives the velocity component at right angles to that direction. It is denoted by the Greek letter ψ (psi). Thus, the stream function for a two-dimensional steady flow can be defined as $\psi(x, y)$ and it is mathematically expressed as follows.

$$u = \frac{\partial \psi}{\partial y},\ v = -\frac{\partial \psi}{\partial x} \tag{6.50}$$

The line drawn in a steady flow field along which the stream function is constant is called a streamline.

In cylindrical polar coordinates, we get:

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \alpha},\ u_\alpha = -\frac{\partial \psi}{\partial r} \tag{6.50a}$$

The rotational component in the $x - y$ plane is ω_z and from Equation (6.37a), we get:

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Substituting the values of v and u from Equation (6.50) in the above expression, we get:

$$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)\right]$$

$$\omega_z = \frac{1}{2}\left(-\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}\right) \tag{6.51}$$

For irrotational flow, $\omega_z = 0$ and from Equation (6.51), we get:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Therefore, it is the Laplace equation in ψ.

The existence of ψ signifies a possible case of fluid flow that may be rotational or irrotational. However, when the function ψ satisfies Laplace equation, then it is a possible case of an irrotational flow.

6.12.3 Cauchy–Riemann Equations (Relation between Stream Function and Velocity Potential Function)

In a potential flow, the Cauchy–Riemann equations enable us to find the stream function if velocity potential is known and vice versa. From the definition of stream function and velocity function, derive the following expressions.

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \text{ and } v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Thus
$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \text{ and } \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{6.52}$$

Equation (6.52) is known as Cauchy–Riemann equations for irrotational flow.

6.12.4 Orthogonality of Streamlines and Equipotential Lines

Since ψ is a function of x and y, its total differential is derived as follows.

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy$$

A streamline is given by $\psi = \text{Constant}$, i.e., $d\psi = 0$.

Thus
$$\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0$$

The slope of a streamline is given by,

$$\frac{dy}{dx} = \frac{-(\partial \psi/\partial x)}{(\partial \psi/\partial y)} = \frac{v}{u} \tag{6.53}$$

Since ϕ is a function of x and y, its total differential is given by,

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$

An equipotential line is given by $\phi = \text{Constant}$, i.e., $d\phi = 0$.

Thus
$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$

The slope of an equipotential line is given by,

$$\frac{dy}{dx} = -\frac{(\partial \phi/\partial x)}{(\partial \phi/\partial y)} = -\frac{u}{v} \tag{6.54}$$

The product of the two slopes given by equations (6.53) and (6.54) is expressed as follows.

$$\frac{v}{u} \times \left(-\frac{u}{v}\right) = -1$$

This indicates that the streamline and equipotential lines intersect each other orthogonally at all points of intersection. Thus, the streamlines are normal to equipotential lines.

6.12.5 Flow Net

A flow net is a grid obtained by drawing a series of streamlines and equipotential lines. A flow net drawn for a two-dimensional irrotational flow gives a complete visual picture of the flow pattern. Especially, it helps in analysing the flow problems when mathematical relations for stream function and potential function are not available. The flow nets can be drawn by the following methods, such as (i) analytical method which involves the solution of Laplace equation in ψ and ϕ, (ii) graphical method, (iii) electrical analogy method, (iv) relaxation method and (v) viscous flow analogy method.

Uses of flow nets The flow nets have many advantages, such as (i) it can be used to determine the discharge, (ii) it helps in determining the velocity at any point in the flow field provided the velocity at any reference point is known, (iii) it helps in avoiding separation and points of stagnation in the design of boundary shapes, (iv) it helps in the estimation of pressure distribution and (v) it also helps in making the calculations for the drag force.

Limitations of flow nets The flow nets have certain limitations, such as (i) the analysis of flow net cannot be applied in the region close to the boundary where the viscosity effects are predominant, (ii) it cannot be applied to a sharply diverging flow and (iii) it does not provide any information about wake formation, i.e., the disturbed flow in the rear of a solid body.

Example 6.24 A stream function is given by $\psi = 2x^2 - y^3$. Determine the magnitude and direction of velocity components at the point (2, 1).

Solution

Let $\psi = 2x^2 - y^3$, $x = 2$ and $y = 1$. Let V be the magnitude of velocity.

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}(2x^2 - y^3) = -4x$$

$$\therefore v = -4 \times 2 = -8$$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}(2x^2 - y^3) = -3y^2$$

$$\therefore u = -3 \times 1^2 = -3$$

$$V = \sqrt{u^2 + v^2} = \sqrt{(-3)^2 + (-8)^2} = \mathbf{8.54}$$

$$\tan\alpha = \frac{v}{u} = \frac{8}{3}$$

$$\therefore \alpha = \tan^{-1}\left(\frac{8}{3}\right) = \mathbf{69.44°}$$

Example 6.25 The velocity potential function is given by $\phi = 3x^2 - y^2$. Determine the magnitude of velocity components at the point (2, 3).

Solution

Let $\phi = 3x^2 - y^2$, $x = 2$ and $y = 3$. Let V be the magnitude of velocity.

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}(3x^2 - y^2) = 6x$$

$$\therefore u = 6 \times 2 = 12$$

$$v = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}(3x^2 - y^2) = -2y$$

$$\therefore v = -2 \times 3 = -6$$

$$V = \sqrt{u^2 + v^2} = \sqrt{12^2 + (-6)^2} = \mathbf{13.42}$$

Example 6.26 If the stream function $\psi = \sqrt{2}xy$ describes a flow, then determine the point at which the velocity vector has a magnitude of 2 units and makes an angle of 45°.

Solution

Let $\psi = \sqrt{2}xy$, $V = 2$ and $\alpha = 45°$. Let x and y be the required points.

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}(\sqrt{2}xy) = -\sqrt{2}y$$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}(\sqrt{2}xy) = \sqrt{2}x$$

Since $$\tan\alpha = \frac{v}{u}$$

Thus $$\tan 45° = \frac{-\sqrt{2}y}{\sqrt{2}x}$$

$$\therefore y = -x$$

Since $$V = \sqrt{u^2 + v^2}$$

Thus $$2 = \sqrt{(\sqrt{2}x)^2 + (-\sqrt{2}y)^2}$$

$$2 = \sqrt{(\sqrt{2}x)^2 + (\sqrt{2}x)^2} \qquad [\because y = -x]$$

$$4 = 4x^2$$

$$\therefore x = \mathbf{1}$$

$$y = -x = \mathbf{-1}$$

Example 6.27 Sketch the streamline represented by (i) $\psi = x^2 + y^2$ and (ii) $\psi = x^2 - y^2$. Also find out the velocity and its direction for each case at point (1, 2).

Solution

(i) Let $\psi = x^2 + y^2$, $x = 1$ and $y = 2$.

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}(x^2 + y^2) = -2x$$

$$\therefore v = -2 \times 1 = -2$$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2) = 2y$$

$$\therefore u = 2 \times 2 = 4$$

$$V = \sqrt{u^2 + v^2} = \sqrt{4^2 + (-2)^2} = \mathbf{4.47}$$

$$\tan\alpha = \frac{v}{u} = \frac{-2}{4} = -0.5$$

$$\therefore \alpha = \tan^{-1}(-0.5) = \mathbf{-26.56°}$$

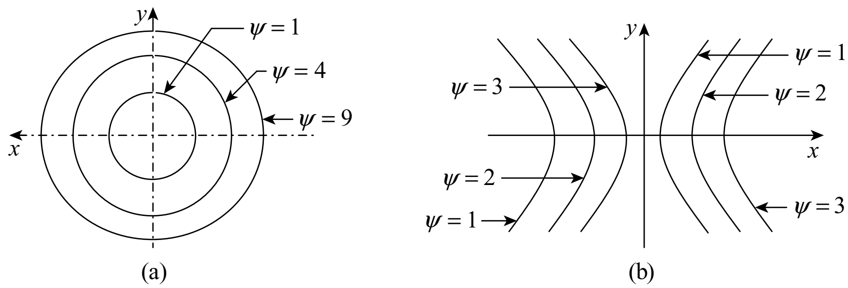

Figure 6.13

If $\psi = x^2 + y^2 = r^2$ and $r = 1, 2, 3, \ldots\ldots$, then $\psi = x^2 + y^2 = 1, 4, 9\ldots.$

Thus, we get concentric circles of different diameters as shown in Figure 6.13(a).

(ii) $\psi = x^2 - y^2$, $x = 1$ and $y = 2$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}(x^2 - y^2) = -2x$$

$$\therefore v = -2 \times 1 = -2$$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}(x^2 - y^2) = -2y$$

$$\therefore u = -2 \times 2 = -4$$

$$V = \sqrt{u^2 + v^2} = \sqrt{(-4)^2 + (-2)^2} = \mathbf{4.47}$$

$$\tan \alpha = \frac{v}{u} = \frac{-2}{-4} = 0.5$$

$$\therefore \alpha = \tan^{-1}(0.5) = \mathbf{26.56°}$$

If $\psi = x^2 - y^2$, then $x = \pm\sqrt{y^2 + \psi}$.

The streamlines are lines of constant ψ and thus, the given equation represents hyperbola, which may be plotted for different values of $\psi = 1, 2, 3\ldots..$ by taking $y = 0, 1, 2, 3\ldots..$ as shown in Figure 6.13(b) and the values are given below in Table 6.1.

Table 6.1

	y	0	1	2	3
$\psi = 1$	$x = \pm\sqrt{y^2 + \psi}$	$x = \pm 1$	$x = \pm\sqrt{2}$	$x = \pm\sqrt{5}$	$x = \pm\sqrt{10}$
$\psi = 2$	$x = \pm\sqrt{y^2 + \psi}$	$x = \pm 2$	$x = \pm\sqrt{3}$	$x = \pm\sqrt{6}$	$x = \pm\sqrt{11}$
$\psi = 3$	$x = \pm\sqrt{y^2 + \psi}$	$x = \pm 3$	$x = \pm\sqrt{4}$	$x = \pm\sqrt{7}$	$x = \pm\sqrt{12}$

Example 6.28 If the velocity potential function is given by $\phi = (y^3/3) + 2x - x^2 y$, then find the velocity components and also show that ϕ represents a possible case of flow.

Solution

Let $$\phi = \frac{y^3}{3} + 2x - x^2 y$$

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y^3}{3} + 2x - x^2 y\right) = \mathbf{2 - 2xy}$$

$$v = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y^3}{3} + 2x - x^2 y\right) = \mathbf{y^2 - x^2}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial}{\partial x}(2 - 2xy) = -2y$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial}{\partial y}(y^2 - x^2) = 2y$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2y + 2y = 0$$

Since Laplace equation is satisfied, ϕ represents a possible case of flow.

Example 6.29 For a two-dimensional flow, $\phi = 3xy$ and $\psi = 1.5(y^2 - x^2)$. Find the velocity components at the points $P(1, 3)$ and $Q(2, 3)$. Also find the discharge between the streamlines passing through the given points.

Solution

Let $\phi = 3xy$, $\psi = 1.5(y^2 - x^2)$, $P(1, 3)$ and $Q(2, 3)$.

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}(3xy) = 3y$$

$$v = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}(3xy) = 3x$$

The velocity components at point $P(1, 3)$ are derived as follows.

$$u = 3 \times 3 = \mathbf{9}$$

$$v = 3 \times 1 = \mathbf{3}$$

The velocity components at point $Q(2, 3)$ are derived as follows.

$$u = 3 \times 3 = \mathbf{9}$$

$$v = 3 \times 2 = \mathbf{6}$$

Now $$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial[1.5(y^2 - x^2)]}{\partial x} = 3x$$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial[1.5(y^2 - x^2)]}{\partial y} = 3y$$

which are same as obtained above.

The value of ψ for streamline passing through $P(1, 3)$ is given by,

$$\psi_1 = 1.5(3^2 - 1^2) = 12$$

The value of ψ for streamline passing through $Q(2, 3)$ is given by,

$$\psi_2 = 1.5(3^2 - 2^2) = 7.5$$

Thus discharge passing between these two streamlines is given by,

$$\psi_1 - \psi_2 = 12 - 7.5 = \mathbf{4.5}$$

Example 6.30 If stream function is $\psi = x^3 - 3xy^2$, then show that the flow is irrotational.

Solution

Let $\psi = x^3 - 3xy^2$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2}{\partial x^2}(x^3 - 3xy^2) = 6x$$

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial^2}{\partial y^2}(x^3 - 3xy^2) = -6x$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 6x - 6x = 0$$

Since ψ satisfies Laplace equation, it is a possible case of an irrotational flow.

Example 6.31 In a two-dimensional incompressible flow, the fluid velocity components are given by $u = x - 4y$ and $v = -y - 4x$. Show that the velocity potential exists and determine its form as well as stream function.

Solution

Let $u = x - 4y$ and $v = -y - 4x$.

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x - 4y) = 1$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(-y - 4x) = -1$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 1 - 1 = 0$$

Since the continuity equation is satisfied, the flow is possible.

Now
$$(\nabla \times V) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x - 4y) & -y - 4x & 0 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-y - 4x)\right] i - \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x - 4y)\right] j$$

$$+ \left[\frac{\partial}{\partial x}(-y - 4x) - \frac{\partial}{\partial y}(x - 4y)\right] k$$

$$= [0 - 0)]\, i - [0 - 0]\, j + (-4 + 4)\, k$$

$$= 0$$

Since the curl of velocity vector is zero, the flow is irrotational and therefore, velocity potential exists.

Since ϕ is a function of x and y, its total differential is given as follows.

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = u\,dx + v\,dy$$

$$d\phi = (x - 4y)dx + (-y - 4x)dy$$

$$d\phi = xdx - ydy - 4d(xy) \quad \text{(i)}$$

Integrating the expression (i), we get:

$$\boldsymbol{\phi = \frac{x^2}{2} - \frac{y^2}{2} - 4xy + C_1}$$

Since ψ is a function of x and y, its total differential is expressed below.

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -vdx + udy$$

$$d\psi = (y + 4x)dx + (x - 4y)dy$$

$$d\psi = 4xdx - 4ydy + d(xy) \quad \text{(ii)}$$

Integrating the expression (ii), we get:

$$\boldsymbol{\psi = 2x^2 - 2y^2 + xy + C_2}$$

Example 6.32 In a two-dimensional flow, the velocity potential is given by $\phi = x(2y - 1)$. Determine the velocity at the point $P(4, 5)$ and also determine the value of stream function at the given point.

Solution

Let $\phi = x(2y - 1)$ and $P(4, 5)$.

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}[x(2y - 1)] = 2y - 1$$

$$v = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}[x(2y - 1)] = 2x$$

At point $P(4, 5)$, the velocity components can be given by,

$$u = 2 \times 5 - 1 = 9$$

$$v = 2 \times 4 = 8$$

Thus, the velocity vector is given in the following expression.

$$\vec{V} = ui + vj = 9i + 8j$$

Resultant velocity is given by,

$$V = \sqrt{u^2 + v^2} = \sqrt{9^2 + 8^2} = \textbf{12.04 units}$$

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -vdx + udy$$

$$d\psi = -2xdx + (2y - 1)dy$$

$$d\psi = -2xdx + 2ydy - dy \quad \text{(i)}$$

Integrating the expression (i) and neglecting constant of integration, we get:

$$\psi = -x^2 + y^2 - y$$

Thus, the stream function at point $P(4, 5)$ is given by,

$$\psi = -4^2 + 5^2 - 5 = \mathbf{4\ units}$$

Example 6.33 In a two-dimensional flow, the stream function is given by $\psi = 2xy$. Show that the flow is irrotational and it also determines the corresponding velocity potential.

Solution

Let $\psi = 2xy$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2}{\partial x^2}(2xy) = 0$$

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial^2}{\partial y^2}(2xy) = 0$$

Thus

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Since ψ satisfies Laplace equation, it is a possible case of an irrotational flow.

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} = 2x$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} = -2y$$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = u\,dx + v\,dy$$

$$d\phi = 2x\,dx - 2y\,dy \tag{i}$$

Integrating the expression (i), we get:

$$\phi = \mathbf{x^2 - y^2 + C_1}$$

Example 6.34 If in a two-dimensional flow, the velocity function is $\phi = -x^2 + y^2$, then (i) determine the velocity components in x and y directions and show that flow is possible and it satisfies the conditions of irrotational flow, (ii) determine stream function and the flow rate between the streamlines (3, 0) and (3, 3) and (iii) also show that the streamlines and potential lines intersect orthogonally at the point (3, 3).

Solution

Let $\phi = -x^2 + y^2$

(i) $u = \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}[-x^2 + y^2] = -2x$

$$v = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}[-x^2 + y^2] = 2y$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(-2x) = -2 \text{ and } \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(2y) = 2$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -2 + 2 = 0$$

Since the continuity equation is satisfied, flow is possible.

Now $$(\nabla \times V) = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -2x & 2y & 0 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(2y)\right] i - \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(-2x)\right] j + \left[\frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial y}(-2x)\right] k$$

$$= [0-0)]\, i - [0-0]\; j + (0+0)\; k = 0$$

Since the curl of velocity vector is zero, the flow is irrotational.

(ii) $d\psi = \dfrac{\partial \psi}{\partial x} dx + \dfrac{\partial \psi}{\partial y} dy = -vdx + udy$

$$d\psi = -2ydx - 2xdy = -2d(xy) \qquad \text{(i)}$$

Integrating the expression (i) and neglecting constant of integration, we get:

$$\boldsymbol{\psi = -2xy}$$

The value of ψ for streamline passing through (3, 0) is given by,

$$\psi_1 = -2(3 \times 0) = 0$$

The value of ψ for streamline passing through (3, 3) is given by,

$$\psi_2 = -2(3 \times 3) = -18$$

Thus, the discharge passing between these two streamlines is given by,

$$\psi_1 - \psi_2 = 0 - (-18) = \textbf{18 units}$$

(iii) The slope of a streamline at point (3, 3) is given by,

$$\frac{dy}{dx} = \frac{v}{u} = \frac{2y}{-2x} = \frac{2 \times 3}{-2 \times 3} = -1$$

The slope of an equipotential line at point (3, 3) is given by,

$$\frac{dy}{dx} = -\frac{u}{v} = -\frac{-2x}{2y} = \frac{2 \times 3}{2 \times 3} = 1$$

The product of the two slopes is given by,

$$= -1 \times 1 = -1$$

This shows that the streamline and equipotential lines intersect each other orthogonally at point (3, 3).

Summary

1. Fluid kinematics deals with the geometry of fluid motion in terms of displacement, velocity and acceleration without considering the forces causing the motion.
2. **Steady flow:** The fluid characteristics do not change with time.
3. **Unsteady flow:** The fluid characteristics change with time.
4. **Uniform flow:** The fluid velocity does not change with location.
5. **Non-uniform flow:** The fluid velocity at any given time changes with location.
6. **Laminar flow:** A smooth flow of one layer of fluid over the adjacent layer.
7. **Turbulent flow:** The fluid particles move in a zigzag manner.
8. **Reynolds number (Re):** The ratio of inertia force to the viscous force.
9. For laminar flow, Re < 2000 and for turbulent flow, Re > 4000.
10. **Incompressible flow:** The density remains constant.
11. **Compressible flow:** The density does not remain constant.
12. **One-dimensional flow:** Velocity is a function of time and one space coordinate.
13. **Two-dimensional flow:** Velocity is a function of time and two space coordinates.
14. **Three-dimensional flow:** Velocity is a function of time and three space coordinates.
15. **Rotational flow:** The fluid particles rotate about their own axis.
16. **Irrotational flow:** The fluid particles do not rotate about their own axis.
17. **Streamline:** An imaginary line drawn through a flowing fluid in such a way that the tangent to it at any point gives the direction of the velocity of flow.
18. **The differential equation for streamlines:** $dx/u = dy/v$.
19. **Stream-tube:** Cylindrical passage formed by a bundle of neighbouring streamlines.
20. **Pathline:** The trace of the path of a single particle over a period of time.
21. **Streakline:** The line traced by a fluid particle through a fixed point in a flow field.
22. **Timeline:** The line formed by a number of adjacent fluid particles in a flow field.
23. **Lagrangian method:** A single particle is followed over the flow field during its course of motion by a moving rectangular coordinate system.
24. **Eulerian method:** A finite volume called control volume is defined through which fluid flows in and out.
25. **The components of acceleration of the fluid particles:**

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t},$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t},$$

and $$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

26. **Local acceleration:** The rate of change of velocity of the fluid particles with respect to time is given by $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}$ and $\frac{\partial w}{\partial t}$.
27. **Convective acceleration:** The rate of change of velocity due to change of position of fluid particles in a flow field is given by,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z};\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z};\ u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

28. The rate of flow (or discharge) is given by $Q = AV$, here A is the area of cross section and V is the average velocity of the liquid.
29. **Continuity equation:** $A_1V_1 = A_2V_2$.
30. **Continuity equation for a steady and incompressible flow:** $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$.
31. **The general continuity equation in cylindrical polar coordinates:**

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho u_r r) + \frac{1}{r}\frac{\partial}{\partial \alpha}(\rho u_\alpha) + \frac{\partial}{\partial z}(\rho u_z) + \frac{\partial \rho}{\partial t} = 0$$

32. The rotation components are given by,

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right);\ \omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right);\ \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

33. **The condition for the flow to be irrotational:** $curl\ V = (\nabla \times V) = 0$.
34. **Vorticity:** $\xi = (\nabla \times V)$, this is equal to twice the rotation.
35. The vorticity components are given by,

$$\xi_x = 2\omega_x = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right);\ \xi_y = 2\omega_y = \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right);$$

$$\xi_z = 2\omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

36. Circulation is defined as the flow along a closed curve.

37. Velocity potential function: $u = \frac{\partial \phi}{\partial x}$; $v = \frac{\partial \phi}{\partial y}$; $w = \frac{\partial \phi}{\partial z}$.

38. Stream function: $\frac{\partial \psi}{\partial x} = -v$; $\frac{\partial \psi}{\partial y} = u$.

39. Cauchy–Riemann equations for irrotational flow: $\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$; $\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$.

40. Flow net: Grid obtained by drawing a series of streamlines and equipotential lines.

Multiple-choice Questions

1. A highly viscous fluid flowing at a low velocity is called
(a) Steady flow.
(b) Uniform flow.
(c) Turbulent flow.
(d) Laminar flow.

2. A streamline
(a) Is defined for uniform flow only.
(b) Is drawn normal to the velocity vector at every point.
(c) Is fixed in space in steady flow.
(d) Is the line connecting the midpoints of flow cross section.

3. If the velocity in a fluid flow does not change with respect to the length of direction of flow, then it is known as
(a) Rotational flow.
(b) Compressible flow.
(c) Uniform flow.
(d) None of the above.

4. Streamlines, streaklines and pathlines are all identical in the case of
(a) Non-uniform flow.
(b) Unsteady flow.
(c) Steady flow.
(d) None of the above.

5. An equipotential line
(a) Has constant dynamic pressure.
(b) Is same as streamline.
(c) Has no velocity component tangent to it
(d) Has no velocity component normal to it.

6. Vorticity is
(a) Thrice the rotation.
(b) Twice the rotation.
(c) 1 to 1.5 times the rotation.
(d) None of the above.

7. The constant rate water flow through a tapering pipe is
(a) Steady non-uniform flow.
(b) Steady uniform flow.
(c) Unsteady non-uniform flow.
(d) All the above.

8. The flow during the opening of a valve is
(a) Rotational.
(b) Uniform.
(c) Unsteady.
(d) None of the above.

9. When velocity potential exists in a flow it means
(a) Flow satisfies the condition of rotational flow.
(b) Vorticity is non-zero.
(c) Flow satisfies the condition of irrotational flow.
(d) All the above.

10. When a stream function exits it means
(a) Flow is steady and incompressible.
(b) Flow is uniform.
(c) Flow is turbulent.
(d) Function represents a possible flow field.

11. A pathline is a
(a) Trace made by a single particle over a period of time.
(b) Path traced by continuously injected tracer at a point.
(c) Direction of a number of particles at the same instant of time.
(d) None of the above.

12. If a stream function exists for a flow and satisfies the Laplace equation, then the flow is
(a) Rotational.
(b) Irrotational.
(c) Irrotational and satisfies the continuity equation.
(d) None of the above.

13. The continuity equation is mathematical representation of the principle of
(a) Conservation of momentum.
(b) Conservation of energy.
(c) Conservation of mass.
(d) All the above.

14. The path traced by a single particle of smoke from a cigarette forms a
(a) Flow line.
(b) Pathline.
(c) Streakline.
(d) All the above.

15. The steady irrotational flow of an incompressible fluid is called
(a) Steady flow.
(b) Uniform flow.
(c) Potential flow.
(d) None of the above.

Review Questions

1. Define the fluid kinematics and velocity of fluid particles.
2. Define and discuss the types of fluid flow.
3. Define and distinguish between streamline, pathline and streakline.
4. What do you mean by stream-tube and a timeline?
5. What are the methods of describing fluid flow? Explain.
6. What do you mean by local and convective accelerations?
7. Define and discuss the tangential and normal accelerations.
8. What do you mean by discharge?
9. Define and obtain an expression for continuity equation in a three-dimensional flow.
10. Derive continuity equation in cylindrical polar coordinates.
11. Discuss the various types of motions of a fluid element.
12. What do you understand by rotation, vorticity and circulation? Explain.
13. Define (i) velocity potential function, (ii) stream function and (iii) Cauchy–Riemann equations. Also give their physical significance.
14. Discuss the orthogonality of streamlines and equipotential lines.
15. What do you mean by flow net? What are their uses and limitations?

Problems

1. If a fluid flow is given by $V = 5x^3 i - 15x^2 yj + tk$, then determine the acceleration components and resultant at a point (1, 2, 3) in the field and at $t = 2$.
[**Ans.** 75, 150, 1, 167.7]

2. Determine the components of acceleration at a point (3, 1, 2) when the flow is described by $V = (y^2 + z^2)i + (x^2 + z^2)j + (x^2 + y^2)k$.
[**Ans.** 66, 70, 56]

3. If a fluid flow is given by $V = 2x^3 i + 3x^2 yj$, then determine whether (i) the flow is steady or unsteady (ii) flow is two-dimensional or three-dimensional? Also determine the velocity, local acceleration and convective acceleration at a point (1, 2, 3) in the field.
[**Ans.** Steady, two-dimensional, 6.324, 0, 0, 12, 42, 43.68]

4. The water flows through a pipeline 1.2 m diameter at a velocity of 3 m/s which bifurcates at a *y*-junction into two branches. The first branch is 0.8 m in diameter and carries 1/3rd of the flow, whereas the 2nd branch carries the water with a velocity of 2.4 m/s. Calculate (i) discharge, (ii) velocity in the 1st branch and (iii) diameter of the 2nd branch.
[**Ans.** 3.39 m^3/s, 2.25 m/s, 1.095 m]

5. Determine the velocity and acceleration at a point (1, 3, 5) and at $t = 1$ when the flow is described by $V = (yz + t)i + (xz - t)j + xyk$.
[**Ans.** 16.76, 101.63]

6. Determine the third velocity component that satisfies the continuity equation when the two velocity components are $u = x^2 + y^2 + z^2$ and $v = xy^2 - yz^2 + xy$.
[**Ans.** $w = -3xz - 2xyz + z^3/3$]

7. Determine the acceleration component when the two velocity components are $u = x/(x^2 + y^2)$ and $v = y/(x^2 + y^2)$.
[**Ans.** $-x/(x^2 + y^2)^2$, $-y/(x^2 + y^2)^2$]

8. Determine the inflow required in m^3/s for filling a rectangular pool which is 10 m × 25 m × 3 m in 2 hours. Also determine the number of hoses required if 50 mm diameter hoses are available and the water velocity in each hose is limited to 2.5 m/s.
[**Ans.** 0.1042 m^3/s, 22]

9. For the velocity vector $V = (6xt + yz^2)i + (3t + xy^2)j + (xy - 2xyz - 6tz)k)$ verify whether the flow exists or not. If it exists, then also determine the resultant acceleration at a point (1, 2, 3) at $t = 1$.
[**Ans.** Exists, 216.52]

10. What is the irrotational velocity field associated with the potential function $\phi = 3x^2 - 3x + 3y^2 + 16t^2 + 12zt$? Does this flow field satisfy the incompressible continuity equation?
[**Ans.** $(6x - 3)i + 6yj + 12tk$, does not satisfy]

11. A 0.25 m diameter pipe carries oil (specific gravity = 0.8) at a velocity of 2 m/s. If at another section, the diameter is 0.2 m, then determine the velocity at this section and the oil mass flow rate.
[**Ans.** 3.12 m/s, 78.4 kg/s]

12. Determine the rotation components when $u = Cyz$, $v = Czx$, $w = Cxy$, where C is a numeric constant. State whether the flow is rotational or irrotational.
[**Ans.** 0, 0, 0, irrotational flow]

13. Determine the velocity components if $\phi = (-xy^3)/3 - x^2 + (x^3 y)/3 + y^2$ and also show that it represents a possible case of flow.

[**Ans.** $u = -(y^3/3) - 2x + x^2 y,\ v = -xy^2 + (x^3/3) + 2y$]

14. If the velocity vector in a two-dimensional flow is given by $V = [(y^3/3) + 2x - x^2 y]i + [(xy^2 - 2y - (x^3/3)]j$, then (i) show that it is a possible case of irrotational flow, (ii) find the stream function and (iii) find the velocity potential function.

[**Ans.** $4xy - x^2y^2 + x^4/12 + y^4/12$,
$(2/3)xy^3 - (2/3)yx^3 + x^2 - y^2$]

15. The velocity vector $V = (a + by - cz)i + (d - bx - ez)j + (f + cx - ey)k$ represents a three-dimensional flow, where a, b, c, d, e, f are the constants. Does it represent an irrotational flow? Also find the vorticity and rotation.

[**Ans.** irrotational flow, $\xi = 2\sqrt{c^2 + b^2}$, $\omega = \sqrt{c^2 + b^2}$]

16. For a stream function $\psi = 3x^2 y + (2 + t)y^2$, determine the velocity field and its value at a point (1, 2, 3) when $t = 2$.

[**Ans.** $V = 19i - 12j$, $\alpha = -32.27°$, 22.47 m/s]

17. If the velocity vector $V = xyi + 2yzj - (yz + z^2)k$ represents a three-dimensional flow, then (i) show that it represents a possible three-dimensional steady incompressible continuous flow, (ii) state whether the flow is rotational or irrotational, if irrotational, then determine at a point (3, 2, 1), (iii) also find angular velocity, vorticity, shear strain and dilatancy.

[**Ans.** three-dimensional steady incompressible flow, rotational flow, $\omega = -(1/2)(5i + 3k)$, $\xi = -(5i + 3k)$, 3/2, 3/2, 0, 2, 2, –4]

18. For a two-dimensional flow, the velocity vector is $V = 5x^3 i - 15x^2 yj$, determine the velocity, acceleration and stream function at a point (2, 4).

[**Ans.** 243.31 m/s, 5366.56 m/s^2, 160 m^3/s]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (d)	**2.** (c)	**3.** (c)	**4.** (c)	**5.** (c)
6. (b)	**7.** (a)	**8.** (c)	**9.** (c)	**10.** (d)
11. (a)	**12.** (c)	**13.** (c)	**14.** (b)	**15.** (c)

Chapter 7

Fluid Dynamics

7.1 ❐ INTRODUCTION

Fluid dynamics deals with fluid motion considering the forces causing the flow. The dynamics of fluid motion is governed by Euler's and Bernoulli's equations. These equations can be derived by Newton's second law of motion which states that the resultant force on any fluid element must be equal to the product of the mass and the acceleration of the element, and the acceleration has the direction of the resultant force. The analysis of fluid flow problems is made by considering a fixed region known as control volume whose size is chosen as per convenience. In this chapter, the derivation of energy equation and momentum equation along with their applications for solving a wide variety of fluid flow problems have been discussed in brief notion. Concepts regarding to kinetic energy correction factor, momentum correction factor and free liquid jet are also discussed in this chapter.

7.2 ❐ ENERGY AND FORCES ACTING ON A FLOWING FLUID

7.2.1 Energy of a Flowing Fluid

The total energy of a fluid remains in various forms. When a fluid flows, it transfers energy from one form to another. Generally, a flowing incompressible fluid possesses three forms of energy, namely potential energy, kinetic energy and pressure energy. In fluid flow studies, it is required to express the energy as the head of fluid in metres.

1. **Potential energy:** The energy possessed by a liquid due to the virtue of its position with respect to the datum line is called potential energy. If m is the mass of liquid at a height z above a datum line, then the potential energy of the liquid at that location is given by P.E. = mgz. Potential energy per unit weight is called potential head and its expression is given below.

$$\text{Potential head } = \frac{mgz}{mg} = z \tag{7.1}$$

2. **Kinetic energy:** The energy possessed by a liquid due to the virtue of its motion is called kinetic energy. If m is the mass of the moving liquid with a velocity V, then kinetic energy of the liquid is given by $\text{K.E.} = (1/2)mV^2$. The kinetic energy per unit weight is called kinetic head and its expression is given below.

$$\text{Kinetic head} = \frac{(1/2)mV^2}{mg} = \frac{V^2}{2g} \tag{7.2}$$

3. **Pressure energy:** The energy created by a liquid in rest when contained in a container is called pressure energy which is also equal to the flow energy (or flow work). The pressure energy per unit weight is called pressure head. If p is the pressure of liquid and w is its weight density, then the pressure head is given below.

$$\text{Pressure head } = \frac{p}{w} = \frac{p}{\rho g} \tag{7.3}$$

4. **Total energy:** The total energy of a flowing liquid in terms of head is given by the summation of potential head, kinetic head and pressure head.

$$\therefore \boxed{\text{Total head } = z + \frac{V^2}{2g} + \frac{p}{\rho g}} \tag{7.4}$$

7.2.2 Forces Acting on a Flowing Fluid

The various forces that may influence the motion of a fluid are (i) gravity force (F_g) which is due to the weight of the fluid, (ii) pressure force (F_p) which is due to the pressure gradient between two points in the direction of flow, (iii) viscous force (F_v) which is due to the viscosity of the flowing fluid, (iv) turbulent force (F_t) which is due to the turbulence of the flow, (v) compressibility force (F_c) which is due to the elastic property of the fluid and (vi) surface tension force (F_s) which is due to the cohesive property of the fluid.

7.3 ❐ EQUATIONS OF MOTION

If a moving fluid element of mass m is influenced by all the above-mentioned forces, then according to Newton's second law of motion, the net force acting on the element in x-direction is given by the following equation of motion.

$$F_x = (F_g)_x + (F_p)_x + (F_v)_x + (F_t)_x + (F_c)_x + (F_s)_x = ma_x \tag{7.5}$$

Similarly, the net force acting in y and z-directions can also be obtained.

In most of the fluid flow problems, the surface tension and compressibility forces are not significant. Thus, after neglecting these two forces, Equation (7.5) can be rewritten as follows.

$$F_x = (F_g)_x + (F_p)_x + (F_v)_x + (F_t)_x = ma_x \tag{7.6}$$

Therefore, Equation (7.6) is known as Reynolds equation of motion. When turbulent forces are neglected as in laminar or viscous flow, then Equation (7.6) can be rewritten as follows.

$$F_x = (F_g)_x + (F_p)_x + (F_v)_x = ma_x \tag{7.7}$$

Therefore, Equation (7.7) is known as Navier–Stokes equation which is useful in the analysis of viscous flow. When viscous forces are neglected as in ideal fluid flow problems, then Equation (7.7) can be rewritten as follows.

$$F_x = (F_g)_x + (F_p)_x = ma_x \tag{7.8}$$

Thus, Equation (7.8) is known as Euler's equation of motion.

7.4 ❐ EULER'S EQUATION OF MOTION

Euler's equation of motion can be derived by applying Newton's second law of motion to a small element of fluid moving along a streamline (Figure 7.1) by considering few assumptions, such as (i) flow is steady, (ii) motion of fluid element is along a streamline and (iii) fluid is ideal (frictionless, i.e., viscosity is zero).

Let dA be the cross-sectional area, ds be the length, ρ be the density, $(\rho gdAds)$ be the weight and α be the angle of the fluid element with the vertical surface.

The body force and pressure force acting at the ends of the fluid element are listed below.

(i) Pressure force pdA in the direction of flow.

(ii) Pressure force $\left(p+\frac{\partial p}{\partial s}ds\right)dA$ in the opposite direction of flow.

(iii) Gravity force due to the weight of the fluid element acting vertically downward whose component in the direction of flow = $\rho gdAds\cos\alpha$.

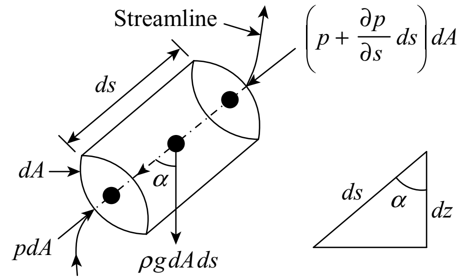

Figure 7.1 *Forces on a fluid element (Euler's equation)*

We know that: Force = Mass × Acceleration [Newton's second law of motion]

$$pdA-\left(p+\frac{\partial p}{\partial s}ds\right)dA-\rho gdAds\cos\alpha=\rho dAds\times a \quad \text{(i)}$$

Since velocity of the fluid element is a function of distance (s) and time (t), i.e., $V=f(s,t)$.

$$dV=\frac{\partial V}{\partial s}ds+\frac{\partial V}{\partial t}dt$$

$$\frac{dV}{dt}=\frac{\partial V}{\partial s}\frac{ds}{dt}+\frac{\partial V}{\partial t}$$

$$a=V\frac{\partial V}{\partial s}+\frac{\partial V}{\partial t} \qquad [\because a=(dV/dt) \text{ and } V=(ds/dt)]$$

For steady flow $(dV/dt)=0$ and we get:

$$a=V\frac{\partial V}{\partial s}$$

Substituting the value of a in expression (i), we get:

$$pdA-\left(p+\frac{\partial p}{\partial s}ds\right)dA-\rho gdAds\cos\alpha=\rho dAds\times V\frac{\partial V}{\partial s}$$

$$-\frac{\partial p}{\partial s}dsdA-\rho gdAds\cos\alpha=\rho dAds\times V\frac{\partial V}{\partial s}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial s}-g\cos\alpha=V\frac{\partial V}{\partial s}$$

For steady flow, p and V are functions of s only and thus, partial differential becomes the total differential.

$$\frac{1}{\rho}\frac{dp}{ds}+g\frac{dz}{ds}+V\frac{dV}{ds}=0 \qquad [\because \cos\alpha=dz/ds]$$

$$\boxed{\frac{dp}{\rho}+gdz+VdV=0} \quad (7.9)$$

Therefore, Equation (7.9) is the Euler's equation of motion.

7.5 ❐ BERNOULLI'S EQUATION

In case of incompressible flow, the given Euler's equation (7.9) can be integrated to obtain Bernoulli's equation. However, Bernoulli's equation can be obtained by few assumptions, such as (i) flow is steady, (ii) motion of fluid element is along a streamline (i.e., flow is one-dimensional), (iii) fluid is ideal (frictionless, i.e., viscosity is zero), (iv) the flow is incompressible (i.e., density of fluid remains constant), (v) the flow is continuous and velocity is uniform, (vi) the flow is irrotational, (vii) only gravity and pressure forces are present and no energy (heat or work) is either added or extracted from the fluid.

By integrating Equation (7.9), we get:

$$\int \frac{dp}{\rho} + \int g dz + \int V dV = \text{Constant}$$

$$\frac{p}{\rho} + gz + \frac{V^2}{2} = \text{Constant}$$

$$\boxed{\frac{p}{\rho g} + \frac{V^2}{2g} + z = \text{Constant}} \tag{7.10}$$

Therefore, Equation (7.10) is known as Bernoulli's equation in which the first term represents the pressure head, second term represents the kinetic head and third term represents the potential head. This equation shows that the sum of the pressure head, kinetic head and datum head in a steady, ideal flow of an incompressible fluid is constant along a streamline at any point of the fluid. The Bernoulli's equation finds its applications in practical designs to estimate pressure and velocity in flow through ducts, venturimeter, orificemeter, etc.

7.6 ❐ BERNOULLI'S EQUATION FOR REAL FLUIDS

The Bernoulli's equation was derived by assuming that the fluid has zero viscosity (i.e., non-viscous or inviscid fluid) and thus, frictionless. Practically, all fluids are real which are more or less viscous and hence, their flow is accompanied by resistance or frictional forces. There are always certain losses of energy in real fluid flows and thereby, energy at the downstream section is less than that at its upstream section. Thus, if h_L represents the loss of energy per unit weight of fluid between the sections 1 and 2, then the Bernoulli's equation for real fluids may be modified as given below.

$$\boxed{\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + h_L} \tag{7.11}$$

The Equation (7.11) states that in a steady flow of real fluid (through a pipe or channel or any passage), the total head at any section (section 1) is equal to that at any subsequent section (section 2) plus the loss of head occurring between the two sections (sections 1 and 2).

7.7 ❐ BERNOULLI'S EQUATION FROM ENERGY EQUATION

Considering various energies of the fluid in a control volume in which entry of the fluid is at section 1 and exit occurs at section 2 as shown in Figure 7.2. Let p_1, V_1, A_1, ρ_1 and z_1 be the pressure, velocity, area, density and height from the datum, respectively at section 1 and p_2, V_2, A_2, ρ_2 and z_2 be the corresponding values at section 2.

The energies per unit weight at section 1 of the control volume during time *dt* are listed below.

(i) Internal energy u_1 is the sum of all microscopic forms of energy.

(ii) Kinetic energy per unit weight is $V_1^2/(2g)$.

(iii) Potential energy per unit weight is z_1.

(iv) Flow energy per unit weight (pressure energy per unit weight) is $p_1/(\rho g)$ as given below at the inlet section.

Since Flow energy/Weight $= \dfrac{\text{Flow work}}{\text{Weight of fluid}}$

$$= \frac{\text{Force} \times \text{Velocity} \times \text{Time}}{\text{Weight density} \times \text{Volume}}$$

Thus Flow energy/Weight $= \dfrac{p_1 A_1 V_1 dt}{w A_1 V_1 dt} = \dfrac{p_1}{w} = \dfrac{p_1}{\rho g}$

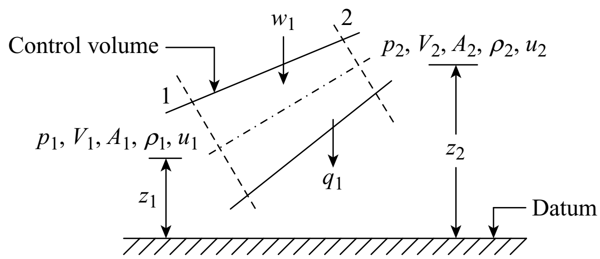

Figure 7.2 *Bernoulli's equation from energy equation*

Similarly, the energies per unit weight can be given at section 2. Let work w_1 be added to the system and heat q_1 be dissipated to the surrounding from the system. Thus, the general energy equation per unit weight of the fluid flowing between two sections is given below.

$$\frac{p_1}{\rho_1 g} + \frac{V_1^2}{2g} + z_1 + u_1 + w_1 = \frac{p_2}{\rho_2 g} + \frac{V_2^2}{2g} + z_2 + u_2 + q_1 \tag{7.12}$$

The Equation (7.12) represents general energy equation which is applicable to steady flow, ideal, real, compressible and incompressible fluids. The Bernoulli's equation can be obtained by making the following assumptions.

(i) No heat transfer across the system boundaries, i.e., $q_1 = 0$.
(ii) No work is done on or by the fluid, i.e., $w_1 = 0$.
(iii) Fluid is ideal and thus, $u_1 = u_2$.
(iv) Fluid is incompressible, i.e., $\rho_1 = \rho_2 = \rho$.

With the above assumptions, the energy equation given by Equation (7.12) becomes,

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

or

$$\frac{p}{\rho g} + \frac{V^2}{2g} + z = \text{Constant} \tag{7.13}$$

This is Bernoulli's equation for steady incompressible and non-viscous flow.

Example 7.1 A pipe of diameter 0.2 m carries oil (specific gravity = 0.85) at the rate of 100 litres per second and the pressure at a point P is 19.62 kN/m^2 (gauge). If the point P is 3 m above the datum line, then determine the total energy at point P in metres of oil.

Solution

Let $d = 0.2$ m, $S_{\text{oil}} = 0.85$, $Q = 100\ l/\text{s} = 0.1\ \text{m}^3/\text{s}$, $p = 19.62\ \text{kN/m}^2$ and $z = 3$ m. Let E be the total energy in terms of metres of oil.

$$\rho = S_{\text{oil}} \rho_w = 0.85 \times 1000 = 850\ \text{kg/m}^3$$

$$A = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 0.2^2 = 0.0314\ \text{m}^2$$

$$V = \frac{Q}{A} = \frac{0.1}{0.0314} = 3.185\ \text{m/s}$$

$$E = \frac{p}{\rho g} + \frac{V^2}{2g} + z = \frac{19.62 \times 10^3}{850 \times 9.81} + \frac{3.185^2}{2 \times 9.81} + 3 = \mathbf{5.87\ m\ of\ oil}$$

Example 7.2 The water is flowing through a tapering pipe having diameters 0.25 m and 0.125 m at sections 1 and 2, respectively. The discharge through the pipe is 40 litres per second. The section 1 is 5 m above the datum and section 2 is 3 m above the datum. If the pressure at section 1 is 0.4 MPa, then determine the intensity of pressure at section 2.

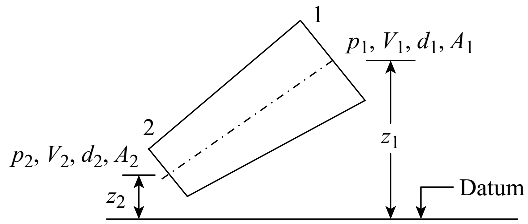

Figure 7.3

Solution

Refer Figure 7.3. Let p_1, V_1, d_1, A_1 and z_1 be the pressure, velocity, diameter, area and height from the datum, respectively, at section 1 and p_2, V_2, d_2, A_2 and z_2 be the corresponding values at section 2. Let $d_1 = 0.25$ m, $d_2 = 0.125$ m, $Q = 40\ l/\text{s} = 0.04\ \text{m}^3/\text{s}$, $z_1 = 5$ m, $z_2 = 3$ m and $p_1 = 0.4$ MPa $= 0.4 \times 10^6$ Pa.

$$A_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} \times 0.25^2 = 0.0491\ \text{m}^2$$

$$V_1 = \frac{Q}{A_1} = \frac{0.04}{0.0491} = 0.815\ \text{m/s}$$

$$A_2 = \frac{\pi}{4} d_2^2 = \frac{\pi}{4} \times 0.125^2 = 0.0123\ \text{m}^2$$

$$V_2 = \frac{Q}{A_2} = \frac{0.04}{0.0123} = 3.252\ \text{m/s}$$

Since $$\frac{p_1}{\rho_w g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho_w g} + \frac{V_2^2}{2g} + z_2 \quad \text{[Bernoulli's equation]}$$

Thus $$\frac{0.4 \times 10^6}{1000 \times 9.81} + \frac{0.815^2}{2 \times 9.81} + 5 = \frac{p_2}{1000 \times 9.81} + \frac{3.252^2}{2 \times 9.81} + 3$$

$$45.81 = \frac{p_2}{9810} + 3.539$$

$$\therefore p_2 = \frac{(45.81 - 3.539) \times 9810}{1000} = \mathbf{414.68\ kPa}$$

Example 7.3 The water is flowing through a pipe having diameters 0.3 m and 0.5 m at the upper and bottom ends, respectively. The intensity of pressures at the upper and bottom ends are 100 kPa and 300 kPa, respectively. If the rate of flow through the pipe is 50 litres per second, then determine the difference in datum head.

Solution

Let p_1, V_1, d_1, A_1 and z_1 be the pressure, velocity, diameter, area and height from the datum, respectively, at the upper end (section 1) and p_2, V_2, d_2, A_2 and z_2 be the corresponding values at the bottom end (section 2). Let $d_1 = 0.3$ m, $d_2 = 0.5$ m, $p_1 = 100$ kPa, $p_2 = 300$ kPa and $Q = 50\ l/\text{s} = 0.05\ \text{m}^3/\text{s}$.

$$A_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} \times 0.3^2 = 0.0707\ \text{m}^2$$

$$V_1 = \frac{Q}{A_1} = \frac{0.05}{0.0707} = 0.7072\ \text{m/s}$$

$$A_2 = \frac{\pi}{4} d_2^2 = \frac{\pi}{4} \times 0.5^2 = 0.19635 \text{ m}^2$$

$$V_2 = \frac{Q}{A_2} = \frac{0.05}{0.19635} = 0.2546 \text{ m/s}$$

Since $$\frac{p_1}{\rho_w g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho_w g} + \frac{V_2^2}{2g} + z_2 \qquad \text{[Bernoulli's equation]}$$

Thus $$\frac{100 \times 10^3}{1000 \times 9.81} + \frac{0.7072^2}{2 \times 9.81} + z_1 = \frac{300 \times 10^3}{1000 \times 9.81} + \frac{0.2546^2}{2 \times 9.81} + z_2$$

$$10.2192 + z_1 = 30.5843 + z_2$$

$$\therefore (z_1 - z_2) = 30.5843 - 10.2192 = \mathbf{20.3651 \text{ m}}$$

Example 7.4 A 200 m long pipe has a slope of 1 in 100 and tapers from 1 m diameter at the high end to 0.5 m diameter at the low end. If the quantity of water flowing through the pipe is 80 litres per second and the pressure at the high end is 70 kPa, then determine the pressure at the low end. Assume that datum passes through the lower end and neglect the losses.

Solution

Refer Figure 7.4 in which sections 1 and 2 denotes the high end and the low end of the pipe, respectively. Let p_1, V_1, d_1, A_1 and z_1 be the pressure, velocity, diameter, area and height from datum, respectively, at the high end and p_2, V_2, d_2, A_2 and z_2 be the corresponding values at the low end of the pipe. Let $l = 200$ m, Slope = 1 in 100, $d_1 = 1$ m, $d_2 = 0.5$ m, $Q = 80\ l/\text{s} = 0.08 \text{ m}^3/\text{s}$, $p_1 = 70$ kPa and $z_2 = 0$.

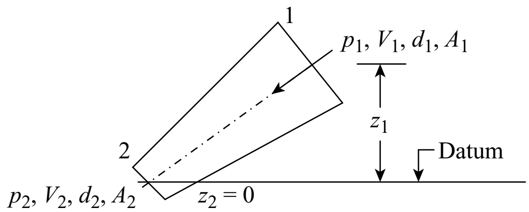

Figure 7.4

$$A_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} \times 1^2 = 0.7854 \text{ m}^2$$

$$V_1 = \frac{Q}{A_1} = \frac{0.08}{0.7854} = 0.102 \text{ m/s}$$

$$A_2 = \frac{\pi}{4} d_2^2 = \frac{\pi}{4} \times 0.5^2 = 0.19635 \text{ m}^2$$

$$V_2 = \frac{Q}{A_2} = \frac{0.08}{0.19635} = 0.4074 \text{ m/s}$$

Since the datum line passes through the centre of the lower end, $z_2 = 0$.

$$z_1 = \frac{1}{100} \times 200 = 2 \text{ m} \qquad [\because \text{Slope} = 1 \text{ in } 100]$$

Since $$\frac{p_1}{\rho_w g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho_w g} + \frac{V_2^2}{2g} + z_2 \qquad \text{[Bernoulli's equation]}$$

$$\frac{70\times10^3}{1000\times9.81}+\frac{0.102^2}{2\times9.81}+2=\frac{p_2}{1000\times9.81}+\frac{0.4074^2}{2\times9.81}+0$$

$$9.1361=\frac{p_2}{9810}+0.0085$$

$$\therefore p_2=\frac{(9.1361-0.0085)\times9810}{1000}=\mathbf{89.542\ kPa}$$

Example 7.5 The diameter of a pipe carrying oil (specific gravity = 0.85) changes from 0.5 m at section 1 to 0.25 m diameter at section 2 which is at 5 m lower level. The pressures at sections 1 and 2 are 60 kPa and 100 kPa, respectively. If the discharge through the pipe is 250 litres per second, then determine the direction of flow and the loss of head.

Solution

Refer Figure 7.4. Let p_1, V_1, d_1, A_1 and z_1 be the pressure, velocity, diameter, area and height from the datum, respectively, at section 1 and p_2, V_2, d_2, A_2 and z_2 be the corresponding values at section 2. Let $S_{\text{oil}}=0.85$, $d_1=0.5$ m, $d_2=0.25$ m, $z_1=5+z_2$, $p_1=60$ kPa, $p_2=100$ kPa and $Q=250\ l/\text{s}=0.25\ \text{m}^3/\text{s}$.

$$A_1=\frac{\pi}{4}d_1^{\,2}=\frac{\pi}{4}\times0.5^2=0.19635\ \text{m}^2$$

$$V_1=\frac{Q}{A_1}=\frac{0.25}{0.1963}=1.273\ \text{m/s}$$

$$A_2=\frac{\pi}{4}d_2^{\,2}=\frac{\pi}{4}\times0.25^2=0.0491\ \text{m}^2$$

$$V_2=\frac{Q}{A_2}=\frac{0.25}{0.0491}=5.092\ \text{m/s}$$

Assume that datum line passes through the centre of the lower end and thus, $z_2=0$.

$$z_1=5+z_2=5+0=5\ \text{m}$$

$$\rho=S_{\text{oil}}\rho_w=0.85\times1000=850\ \text{kg/m}^3$$

Total energy at point 1 is given by,

$$E_1=\frac{p_1}{\rho g}+\frac{V_1^{\,2}}{2g}+z_1=\frac{60\times10^3}{850\times9.81}+\frac{1.273^2}{2\times9.81}+5=12.278\ \text{m}$$

Total energy at point 2 is given by,

$$E_2=\frac{p_2}{\rho g}+\frac{V_2^{\,2}}{2g}+z_2=\frac{100\times10^3}{850\times9.81}+\frac{5.092^2}{2\times9.81}+0=13.314\ \text{m}$$

Since $E_2 > E_1$, flow occurs from section 2 to section 1.

Loss of head between the given points is given by,

$$h_L=E_2-E_1=13.314-12.278=\mathbf{1.036\ m}$$

Example 7.6 A 0.25 m pipe carries water at a velocity of 20 m/s. At points 1 and 2, the measurements of pressure and elevation are 350 kPa, 275 kPa, 30 m and 33 m, respectively. For steady flow, determine the loss of head between the given points.

Solution

Let $d = 0.25$ m, $V = V_1 = V_2 = 20$ m/s, $p_1 = 350$ kPa, $p_2 = 275$ kPa, $z_1 = 30$ m and $z_2 = 33$ m. Let h_L be the loss of head between the points 1 and 2.

Total energy at point 1 is given by,

$$E_1 = \frac{p_1}{\rho_w g} + \frac{V_1^2}{2g} + z_1 = \frac{350 \times 10^3}{1000 \times 9.81} + \frac{20^2}{2 \times 9.81} + 30 = 86.065 \text{ m}$$

Total energy at point 2 is given by,

$$E_2 = \frac{p_2}{\rho_w g} + \frac{V_2^2}{2g} + z_2 = \frac{275 \times 10^3}{1000 \times 9.81} + \frac{20^2}{2 \times 9.81} + 33 = 81.42 \text{ m}$$

$$\therefore h_L = E_1 - E_2 = 86.065 - 81.42 = \mathbf{4.645 \text{ m}}$$

Example 7.7 A vertical conical draft tube of a turbine is 2 m long and is kept vertical with the smaller diameter end facing upwards. The measured pressure head at the smaller end is 2.5 m of water. The loss of head in the tube expressed in metres is $[0.35\,(V_1 - V_2)^2]/(2g)$, where V_1 and V_2 is the velocities at the upper and lower ends, respectively. If the velocities at the upper and lower ends are 4.5 m/s and 1.5 m/s, respectively, then determine the pressure head at the lower end.

Solution

Refer Figure 7.5. Let the datum line passes through the lower end (section 2), p_1, V_1 and z_1 be the pressure, velocity and height from the datum, respectively, at the upper end (section 1), p_2, V_2 and z_2 be the corresponding values at the bottom end (section 2). Let $l = z_1 = 2$ m, $p_1/(\rho_w g) = 2.5$ m, $h_L = [0.35\,(V_1 - V_2)^2]/(2g)$, $V_1 = 4.5$ m/s and $V_2 = 1.5$ m/s.

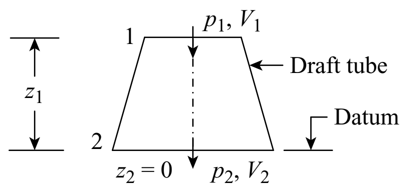

Figure 7.5

Since
$$\frac{p_1}{\rho_w g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho_w g} + \frac{V_2^2}{2g} + z_2 + h_L \qquad \text{[Bernoulli's equation]}$$

$$\frac{p_1}{\rho_w g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho_w g} + \frac{V_2^2}{2g} + z_2 + \frac{0.35\,(V_1 - V_2)^2}{2g}$$

Thus
$$2.5 + \frac{4.5^2}{2 \times 9.81} + 2 = \frac{p_2}{\rho_w g} + \frac{1.5^2}{2 \times 9.81} + 0 + \frac{0.35\,(4.5 - 1.5)^2}{2 \times 9.81}$$

$$5.5321 = \frac{p_2}{\rho_w g} + 0.2752$$

$$\therefore \frac{p_2}{\rho_w g} = 5.5321 - 0.2752 = \mathbf{5.2569 \text{ m of water}}$$

Example 7.8 The diameter and pressure at the inlet of the draft tube is 1 m and 0.45 bar (absolute), respectively. If the discharge of water is 1500 litres per second, pressure at the exit is atmospheric, and the vertical distance between the inlet and outlet is 5.6 m, then determine the exit diameter of the draft tube.

Solution

Refer Figure 7.5. Let p_1, V_1, d_1, A_1 and z_1 be the pressure, velocity, diameter, area and height from the datum, respectively, at the upper inlet end (section 1) and p_2, V_2, d_2, A_2 and z_2 be the corresponding values at the exit of the tube (section 2). Let $d_1 = 1$ m, $p_1 = 0.45$ bar, $Q = 1500\ l/\text{s} = 1.5\ \text{m}^3/\text{s}$, $p_2 = 1.01325$ bar, $z_1 = 5.6$ m and $z_2 = 0$.

$$V_1 = \frac{Q}{A_1} = \frac{Q}{(\pi/4)d_1^2} = \frac{1.5}{(\pi/4)\times 1^2} = 1.91 \text{ m/s}$$

Since $$\frac{p_1}{\rho_w g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho_w g} + \frac{V_2^2}{2g} + z_2 \qquad \text{[Bernoulli's equation]}$$

Thus $$\frac{0.45\times 10^5}{1000\times 9.81} + \frac{1.91^2}{2\times 9.81} + 5.6 = \frac{1.01325\times 10^5}{1000\times 9.81} + \frac{V_2^2}{2\times 9.81} + 0$$

$$10.373 = 10.329 + 0.051 V_2^2$$

$$\therefore V_2 = \sqrt{\frac{10.373 - 10.329}{0.051}} = 0.929 \text{ m/s}$$

Since $$A_1 V_1 = A_2 V_2 \qquad \text{[Continuity equation]}$$

Thus $$\frac{A_2}{A_1} = \frac{d_2^2}{d_1^2} = \frac{V_1}{V_2}$$

$$\frac{d_2^2}{1^2} = \frac{1.91}{0.929}$$

$$\therefore d_2 = \sqrt{\frac{1.91}{0.929}} = \mathbf{1.434\ m}$$

Example 7.9 Gasoline (specific gravity = 0.74) flows upwards in a vertical pipe of length 1.2 m which tapers from 0.4 m to 0.2 m diameter. The mercury differential manometer fitted between the two ends shows a gauge reading of 0.6 mHg. Determine the differential gauge reading in terms of gasoline head and the discharge if losses are neglected.

Solution

Refer Figure 7.6. Let p_1, V_1, d_1, A_1 and z_1 be the pressure, velocity, diameter, area and height from the datum, respectively, at the bottom end (section 1) and p_2, V_2, d_2, A_2 and z_2 be the corresponding values at the exit of the pipe (section 2) and Q be the discharge. Let $S_{\text{gasoline}} = 0.74$, $z_2 = 1.2$ m, $d_1 = 0.4$ m, $d_2 = 0.2$ m and $y = 0.6$ mHg.

Since $$h = \frac{p_1 - p_2}{\rho g} = y \times \left(\frac{S_m}{S_{\text{gasoline}}} - 1 \right)$$

$$\therefore h = \frac{p_1 - p_2}{\rho g} = 0.6 \times \left(\frac{13.6}{0.74} - 1 \right) = \mathbf{10.427\ m\ of\ gasoline}$$

$$V_2 = \frac{A_1 V_1}{A_2} = \frac{d_1^2 V_1}{d_2^2} = \frac{0.4^2 V_1}{0.2^2} = 4V_1 \qquad \text{[From Continuity equation]}$$

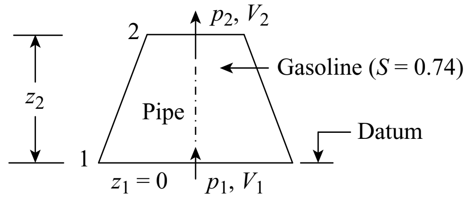

Figure 7.6

Since the datum line passes through the lower end, $z_1 = 0$.

Since $$\left(\frac{p_1}{\rho g} - \frac{p_2}{\rho g} \right) + \left(\frac{V_1^2}{2g} - \frac{V_2^2}{2g} \right) + (z_1 - z_2) = 0 \qquad \text{[Bernoulli's equation]}$$

Thus $$10.427 + \left[\frac{V_1^2}{2g} - \frac{(4V_1)^2}{2g} \right] + (0 - 1.2) = 0$$

$$9.227 - \frac{15V_1^2}{2g} = 0$$

$$\therefore V_1 = \sqrt{\frac{9.227 \times 2g}{15}} = \sqrt{\frac{9.227 \times 2 \times 9.81}{15}} = 3.474 \text{ m/s}$$

$$Q = A_1 V_1 = \frac{\pi}{4} d_1^2 \times V_1 = \frac{\pi}{4} \times 0.4^2 \times 3.474 = \mathbf{0.4365\ m^3/s}$$

Example 7.10 The smaller diameter 0.1 m of a 4 m long pipe at lower level is inclined at an angle of 30° with the horizontal and its large diameter is 0.3 m. If the velocity of water at the smaller diameter section is 2 m/s, then calculate the difference of pressure between the smaller and larger diameter sections of the pipe.

Solution

Refer Figure 7.7. Let the datum passes through the smaller diameter end (section 1), p_1, V_1, d_1, A_1 and z_1 be the pressure, velocity, diameter, area and height from the datum, respectively, at the lower level (section 1) and p_2, V_2, d_2, A_2 and z_2 be the corresponding values at the large diameter of the pipe (section 2). Let $d_1 = 0.1$ m, $l = 4$ m, $\alpha = 30°$, $d_2 = 0.3$ m and $V_1 = 2$ m/s.

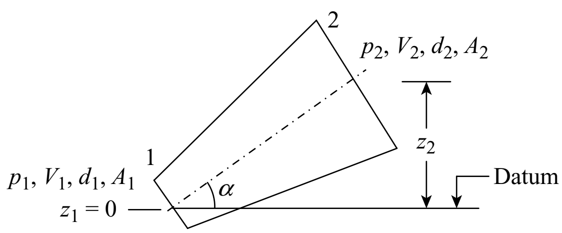

Figure 7.7

$$A_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} \times 0.1^2 = 0.007854 \text{ m}^2$$

$$A_2 = \frac{\pi}{4} d_2^2 = \frac{\pi}{4} \times 0.3^2 = 0.0707 \text{ m}^2$$

$$V_2 = \frac{A_1 V_1}{A_2} = \frac{0.007854 \times 2}{0.0707} = 0.2222 \text{ m/s} \qquad [\because A_1 V_1 = A_2 V_2]$$

Since the datum line passes through the lower end, $z_1 = 0$.

$$z_2 = l \sin\alpha = 4 \sin 30° = 2 \text{ m}$$

$$\left(\frac{p_1}{\rho_w g}-\frac{p_2}{\rho_w g}\right)=\left(\frac{V_2^2}{2g}-\frac{V_1^2}{2g}\right)+(z_2-z_1) \qquad \text{[From Bernoulli's equation]}$$

Thus
$$\left(\frac{p_1-p_2}{\rho_w g}\right)=\left(\frac{0.2222^2}{2\times 9.81}-\frac{2^2}{2\times 9.81}\right)+(2-0)=1.799$$

$$\therefore (p_1-p_2)=1.799\rho_w g=1.799\times 1000\times 9.81=\mathbf{17.65\ kN/m^2}$$

7.8 ❐ PRACTICAL APPLICATIONS OF BERNOULLI'S EQUATION

Bernoulli's equation is widely used in the solution of energy based problems of incompressible fluid flow. Some of the fluid flow measuring devices in which Bernoulli's equation is used are (i) venturimeter, (ii) orificemeter, (iii) pitot tube, (iv) rotameter, (v) siphon and (vi) sluice gate.

7.8.1 Venturimeter

A venturimeter is a device commonly used to measure the flow rate of a fluid flowing through a pipe. It is named after the notable Italian physicist G. B. Venturi (1746–1822). In venturimeter, a pressure difference is created by reducing the cross-sectional area of the flow passage and the measurement of the pressure difference enables the determination of the flow rate through the pipe. A venturimeter consists of the following three parts (Figure 7.8).

(i) **A short converging part** (at the inlet) with a cone angle of about 21° to 22°. The inlet of the venturimeter has the same diameter as that of the pipe, i.e., d_1. The length of convergent cone is kept nearly equal to $2.7\ (d_1-d_2)$, where (d_2) is the throat diameter.

(ii) **Throat**, a short cylindrical region of a constant area. The throat diameter d_2 may vary from $0.33d_1$ to $0.75d_1$ but commonly it is taken as $0.5d_1$.

(iii) **A divergent cone** (or diffuser at exit) with a diverging angle of about 5° to 7°. The divergent cone of the venturimeter is kept longer with a gradual divergence (preferably 6°) to avoid the flow separation. The divergent part is not used for discharge measurement since separation of flow may take place in this portion.

A venturimeter may be fixed horizontally, vertically or in an inclined way in a section of the pipe. A venturimeter fitted to a horizontal pipe is shown in Figure 7.8(a).

Expression for discharge through venturimeter Let d_1 be the diameter, $a_1=(\pi/4)d_1^2$ be the area, V_1 be the velocity of fluid and p_1 be the pressure at section 1 (inlet section) and d_2, $a_2=(\pi/4)d_2^2$, V_2 and p_2 be the corresponding values at section 2 (throat).

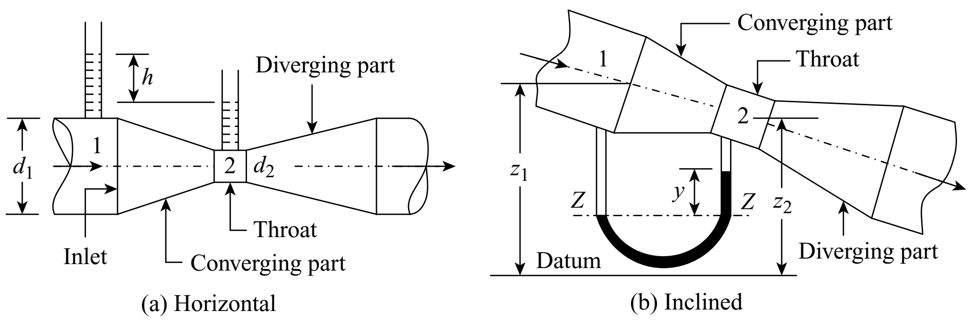

Figure 7.8 *Venturimeter*

Applying Bernoulli's equation at sections 1 and 2 of the horizontal venturimeter shown in Figure 7.8(a), we get the below expression.

$$\frac{p_1}{\rho g}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\rho g}+\frac{V_2^2}{2g}+z_2 \tag{i}$$

$$\frac{p_1}{\rho g}-\frac{p_2}{\rho g}=\frac{V_2^2}{2g}-\frac{V_1^2}{2g} \qquad [\because z_1=z_2]$$

$$\frac{p_1-p_2}{\rho g}=\frac{V_2^2}{2g}-\frac{V_1^2}{2g} \tag{ii}$$

The term $(p_1-p_2)/(\rho g)$ in expression (ii) is the difference in pressure heads between the two sections, which is called venturi head and it is denoted by h.

Thus
$$h=\frac{V_2^2}{2g}-\frac{V_1^2}{2g} \tag{iii}$$

From continuity equation between sections 1 and 2, we get:

$$V_1=\frac{a_2V_2}{a_1} \qquad [\because a_1V_1=a_2V_2]$$

Substituting the value of V_1 in expression (iii), we get:

$$h=\frac{V_2^2}{2g}-\frac{[(a_2V_2)/a_1]^2}{2g}=\frac{V_2^2}{2g}\left[1-\frac{a_2^2}{a_1^2}\right]=\frac{V_2^2}{2g}\left[\frac{a_1^2-a_2^2}{a_1^2}\right]$$

$$V_2^2=\frac{a_1^2\times 2gh}{a_1^2-a_2^2}$$

$$\therefore V_2=\sqrt{\frac{a_1^2\times 2gh}{a_1^2-a_2^2}}=\frac{a_1\sqrt{2gh}}{\sqrt{a_1^2-a_2^2}}$$

If Q_{th} is the theoretical discharge through the pipe, then we get the following expression.

$$Q_{th}=a_2V_2=\frac{a_1a_2\sqrt{2gh}}{\sqrt{a_1^2-a_2^2}} \tag{7.14}$$

In deriving Equation (7.14), the head loss between sections 1 and 2 is not considered. Thus, it gives the discharge under ideal conditions. However, in actual practice there is always some loss of head when fluid flows through the venturimeter, as a result of which the actual discharge will be less than the theoretical discharge. Thus, the expression for actual discharge Q_a is given below.

$$\boxed{Q_a=\frac{C_da_1a_2\sqrt{2gh}}{\sqrt{a_1^2-a_2^2}}} \tag{7.15}$$

Here, C_d is the coefficient of discharge of the venturimeter which is defined as the ratio of actual discharge to the theoretical discharge. Its value is always less than 1 (lies between 0.96 and 0.98) and in general for fluids of low viscosity, it is taken as 0.98. It can also be calculated from the following relation.

$$\boxed{C_d=\sqrt{\frac{h-h_f}{h}}} \tag{7.16}$$

Here, h_f is the friction head loss between the inlet and the throat of the venturimeter.

Value of '*h*' given by differential U-tube manometer

Case I: The differential manometer contains heavier liquid (say mercury) than the liquid flowing through the pipe. Thus, the value of h is given below.

$$\boxed{h = y\left(\frac{S_m}{S} - 1\right)} \tag{7.17}$$

Here, y is the difference of the mercury column (heavier liquid column) in U-tube, S_m is the specific gravity of mercury (heavier liquid) and S is the specific gravity of the liquid flowing through the pipe.

Case II: The differential manometer contains lighter liquid than the liquid flowing through the pipe. Thus, the value of h is given below.

$$\boxed{h = y\left(1 - \frac{S_l}{S}\right)} \tag{7.18}$$

Here, y is the difference of the lighter liquid column in U-tube, S_l is the specific gravity of lighter liquid and S is the specific gravity of the liquid flowing through the pipe.

Inclined venturimeter The Equation (7.15) can also be used for calculating the discharge through inclined or vertical venturimeter. A venturimeter fitted to an inclined pipe is shown in Figure 7.8(b). The value of h for inclined venturimeter in the differential manometer contains a liquid which is heavier than the liquid flowing through the pipe and it is given below.

$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = y\left(\frac{S_m}{S} - 1\right) \tag{7.19}$$

The value of h for inclined venturimeter in which the differential manometer contains a liquid which is lighter than the liquid flowing through the pipe is given below.

$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = y\left(1 - \frac{S_l}{S}\right) \tag{7.20}$$

Example 7.11 A venturimeter has a diameter of 0.16 m at the enlarged end and 0.08 m diameter at the throat. It is fitted in a horizontal pipeline of diameter 0.16 m which carries an oil (specific gravity = 0.85). If the coefficient of discharge of the venturimeter is 0.98 and the difference of pressure head between the enlarged end and the throat recorded by a U-tube is 0.18 mHg, then determine the discharge through the pipe. Take specific gravity of mercury as 13.6.

Solution

Let $d_1 = 0.16$ m, $d_2 = 0.08$ m, $S_{\text{oil}} = 0.85$, $C_d = 0.98$, $y = 0.18$ mHg and $S_m = 13.6$.

$$a_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} \times 0.16^2 = 0.0201 \text{ m}^2$$

$$a_2 = \frac{\pi}{4} d_2^2 = \frac{\pi}{4} \times 0.08^2 = 0.00503 \text{ m}^2$$

$$h = y\left(\frac{S_m}{S_{oil}} - 1\right) = 0.18 \times \left(\frac{13.6}{0.85} - 1\right) = 2.7 \text{ m}$$

Since $$Q_a = \frac{C_d a_1 a_2 \sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}}$$

$$\therefore Q_a = \frac{0.98 \times 0.0201 \times 0.00503 \times \sqrt{2 \times 9.81 \times 2.7}}{\sqrt{0.0201^2 - 0.00503^2}} = \mathbf{0.03706\ m^3/s}$$

Example 7.12 A venturimeter has a diameter of 0.2 m at the inlet and 0.1 m diameter at the throat. It is fitted in a horizontal pipeline to measure the flow of oil of specific gravity 0.82. If 5900 kg of oil is collected in 2 minutes and the difference of levels in the U-tube differential manometer reads 0.185 mHg, then determine the discharge coefficient for the pipe venturimeter. Take specific gravity of mercury as 13.6.

Solution

Let $d_1 = 0.2\text{ m},\ d_2 = 0.1\text{ m},\ S_{\text{oil}} = 0.82,\ m = 5900\text{ kg per 2 min},\ y = 0.185\text{ mHg}$ and $S_m = 13.6$.

$$a_1 = \frac{\pi}{4}d_1^2 = \frac{\pi}{4}\times 0.2^2 = 0.031416\text{ m}^2$$

$$a_2 = \frac{\pi}{4}d_2^2 = \frac{\pi}{4}\times 0.1^2 = 0.007854\text{ m}^2$$

$$h = y\left(\frac{S_m}{S_{\text{oil}}} - 1\right) = 0.185\times\left(\frac{13.6}{0.82} - 1\right) = 2.883\text{ m}$$

$$m = \frac{5900}{2\times 60} = 49.167\text{ kg/s}$$

$$\rho = S_{\text{oil}}\rho_w = 0.82\times 1000 = 820\text{ kg/m}^3$$

$$Q_a = \frac{m}{\rho} = \frac{49.167}{820} = 0.05996\text{ m}^3/\text{s}$$

But
$$Q_a = \frac{C_d a_1 a_2\sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}}$$

Thus
$$0.05996 = \frac{C_d\times 0.031416\times 0.007854\times\sqrt{2\times 9.81\times 2.883}}{\sqrt{0.031416^2 - 0.007854^2}}$$

$$0.05996 = C_d\times 0.061$$

$$\therefore C_d = \frac{0.05996}{0.061} = \mathbf{0.983}$$

Example 7.13 A venturimeter with inlet and throat diameters as 150 mm and 75 mm, respectively is fitted in a horizontal water pipeline to measure the flow. If the pressure at the inlet is 175 kPa and the vacuum pressure at the throat is 275 mmHg, then determine the discharge. Take the values of coefficient of discharge and specific gravity of mercury as 0.97 and 13.6, respectively.

Solution

Let $d_1 = 150\text{ mm} = 0.15\text{ m},\ d_2 = 75\text{ mm} = 0.075\text{ m},\ p_1 = 175\text{ kPa},\ p_2/(\rho g) = 275\text{ mmHg }(vac) = -0.275\text{ mHg},\ C_d = 0.97$ and $S_m = 13.6$.

$$a_1 = \frac{\pi}{4}d_1^2 = \frac{\pi}{4}\times 0.15^2 = 0.01767\text{ m}^2$$

$$a_2 = \frac{\pi}{4}d_2^2 = \frac{\pi}{4}\times 0.075^2 = 0.00442\text{ m}^2$$

$$\frac{p_1}{\rho_w g} = \frac{175 \times 10^3}{1000 \times 9.81} = 17.84 \text{ m of water}$$

$$\frac{p_2}{\rho_w g} = -0.275 \times 13.6 = -3.74 \text{ m of water}$$

Thus $$h = \frac{p_1}{\rho_w g} - \frac{p_2}{\rho_w g} = 17.84 - (-3.74) = 21.58 \text{ m of water}$$

Since $$Q_a = \frac{C_d a_1 a_2 \sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}}$$

$$\therefore Q_a = \frac{0.97 \times 0.01767 \times 0.00442 \times \sqrt{2 \times 9.81 \times 21.58}}{\sqrt{0.01767^2 - 0.00442^2}} = \mathbf{0.09112 \ m^3/s}$$

Example 7.14 A venturimeter having an inlet diameter of 0.3 m is fitted in a horizontal pipeline to measure the flow of water. If the water flow rate through the venturimeter is 0.3 m^3/s and the pressure of water in the pipe is 285 kPa, then determine the least throat diameter of the venturimeter to avoid any cavitation. Take atmospheric pressure as 10.34 m of water and assume that cavitation occurs when the absolute pressure head falls below 2.5 m (abs).

Solution

Let $d_1 = 0.3$ m, $Q = 0.3 \text{ m}^3/\text{s}$, $p_1 = 285$ kPa, $p_{atm}/(\rho_w g) = 10.34$ m of water and $p_2/(\rho_w g) = 2.5$ m (abs).

$$a_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} \times 0.3^2 = 0.0707 \text{ m}^2$$

$$V_1 = \frac{Q}{a_1} = \frac{0.3}{0.0707} = 4.243 \text{ m/s}$$

$$\frac{p_2}{\rho_w g} = 2.5 - 10.34 = -7.84 \text{ m (Gauge)}$$

Since $$\frac{p_1}{\rho_w g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho_w g} + \frac{V_2^2}{2g} \qquad \text{[Bernoulli's equation]}$$

$$\frac{285 \times 10^3}{1000 \times 9.81} + \frac{4.243^2}{2 \times 9.81} = -7.84 + \frac{V_2^2}{2 \times 9.81}$$

$$29.97 = -7.84 + \frac{V_2^2}{19.62}$$

$$\therefore V_2 = \sqrt{(29.97 + 7.84) \times 19.62} = 27.237 \text{ m/s}$$

$$a_2 = \frac{Q}{V_2} = \frac{0.3}{27.237} = 0.011 \text{ m}^2$$

Thus
$$\frac{\pi}{4}d_2^{\,2} = 0.011$$

$$\therefore d_2 = \sqrt{\frac{0.011\times 4}{\pi}} = \mathbf{0.1183\ m}$$

Example 7.15 A venturimeter with an inlet diameter of 0.3 m and throat diameter of 0.1 m is fitted in a horizontal pipeline to measure the flow of water. The pressure intensity of water at the inlet of venturimeter is 120 kPa and the vacuum pressure head at the throat is 300 mmHg. If 3.5% of head is lost in between the inlet and throat, then determine (i) the coefficient of discharge for the venturimeter and (ii) rate of flow through it.

Solution

Let $d_1 = 0.3$ m, $d_2 = 0.1$ m, $p_1 = 120$ kPa, $p_2/(\rho g) = -300$ mmHg $= -0.3$ mHg and $h_f = 3.5\%$.

(i) $$a_1 = \frac{\pi}{4}d_1^{\,2} = \frac{\pi}{4}\times 0.3^2 = 0.0707\ \text{m}^2$$

$$a_2 = \frac{\pi}{4}d_2^{\,2} = \frac{\pi}{4}\times 0.1^2 = 0.007854\ \text{m}^2$$

$$h = \frac{p_1}{\rho_w g} - \frac{p_2}{\rho_w g} = \frac{120\times 10^3}{1000\times 9.81} - (-0.3\times 13.6) = 16.312\ \text{m}$$

$$h_f = 3.5\%\ \text{of}\ h = \frac{3.5}{100}\times 16.312 = 0.571\ \text{m of water}$$

$$C_d = \sqrt{\frac{h-h_f}{h}} = \sqrt{\frac{16.312-0.571}{16.312}} = \mathbf{0.982}$$

(ii) $$Q_a = \frac{C_d a_1 a_2\sqrt{2gh}}{\sqrt{a_1^{\,2}-a_2^{\,2}}}$$

$$\therefore Q_a = \frac{0.982\times 0.0707\times 0.007854\times\sqrt{2\times 9.81\times 16.312}}{\sqrt{0.0707^2-0.007854^2}} = \mathbf{0.1388\ m^3/s}$$

Example 7.16 A venturimeter with an inlet diameter of 0.3 m and throat diameter of 0.15 m is fitted in a horizontal pipeline carrying oil (specific gravity = 0.86) to measure the discharge through the pipe. The venturimeter is connected to a mercury manometer. If the discharge through the venturimeter is 100 litres per second and its coefficient of discharge is 0.98, then determine the reading of mercury manometer head in cm.

Solution

Let $d_1 = 0.3$ m, $d_2 = 0.15$ m, $S_{\text{oil}} = 0.86$, $Q_a = 100\ l/\text{s} = 0.1\ \text{m}^3/\text{s}$ and $C_d = 0.98$.

$$a_1 = \frac{\pi}{4}d_1^{\,2} = \frac{\pi}{4}\times 0.3^2 = 0.0707\text{m}^2$$

$$a_2 = \frac{\pi}{4}d_2^{\,2} = \frac{\pi}{4}\times 0.15^2 = 0.01767\ \text{m}^2$$

Since $$Q_a = \frac{C_d a_1 a_2 \sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}}$$

Thus $$0.1 = \frac{0.98 \times 0.0707 \times 0.01767 \times \sqrt{2 \times 9.81h}}{\sqrt{0.0707^2 - 0.01767^2}}$$

$$0.1 = 0.07922\sqrt{h}$$

$$\therefore h = \left(\frac{0.1}{0.07922}\right)^2 = 1.5934 \text{ m}$$

Also $$h = y\left(\frac{S_m}{S_{\text{oil}}} - 1\right)$$

Thus $$1.5934 = y \times \left(\frac{13.6}{0.86} - 1\right)$$

$$\therefore y = \frac{1.5934}{14.814} = \mathbf{0.1076 \text{ m or } 10.76 \text{ cm of Hg}}$$

Example 7.17 A venturimeter with its axis vertical is used to measure the flow rate of petrol (specific gravity = 0.8) in a vertical pipeline. The inlet and throat diameters of venturimeter are 0.15 m and 0.075 m, respectively. The throat is 0.25 m above the inlet and its coefficient of discharge is 0.97. If the rate of flow through the venturimeter in the upward direction is 0.03 m^3/s, then determine the pressure difference between the inlet and the throat.

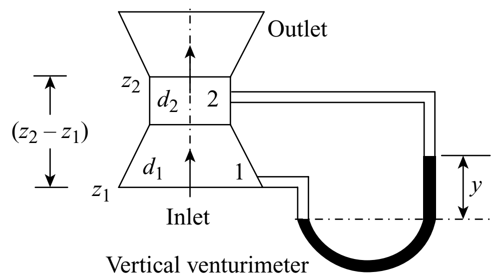

Figure 7.9

Solution

Refer Figure 7.9. Let $S_{\text{petrol}} = 0.8,\ d_1 = 0.15$ m, $d_2 = 0.075$ m, $z_2 - z_1 = 0.25$ m, $C_d = 0.97$ and $Q_a = 0.03$ m^3/s.

$$\rho = S_{\text{petrol}} \rho_w = 0.8 \times 1000 = 800 \text{ kg/m}^3$$

$$a_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} \times 0.15^2 = 0.01767 \text{ m}^2$$

$$a_2 = \frac{\pi}{4} d_2^2 = \frac{\pi}{4} \times 0.075^2 = 0.00442 \text{ m}^2$$

Since $$Q_a = \frac{C_d a_1 a_2 \sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}}$$

Thus $$0.03 = \frac{0.97 \times 0.01767 \times 0.00442 \times \sqrt{2 \times 9.81h}}{\sqrt{0.01767^2 - 0.00442^2}}$$

$$0.03 = 0.019614\sqrt{h}$$

$$\therefore h = \left(\frac{0.03}{0.019614}\right)^2 = 2.3394 \text{ m}$$

But
$$h = \left(\frac{p_1}{\rho g} - \frac{p_2}{\rho g}\right) - (z_2 - z_1)$$

Thus
$$2.3394 = \left(\frac{p_1}{\rho g} - \frac{p_2}{\rho g}\right) - 0.25$$

$$\left(\frac{p_1 - p_2}{\rho g}\right) = 2.3394 + 0.25 = 2.5894$$

Thus
$$(p_1 - p_2) = 2.5894 \rho g$$

$$\therefore (p_1 - p_2) = 2.5894 \times 800 \times 9.81 = \mathbf{20321.61\ N/m^2}$$

Example 7.18 A 0.3 m × 0.15 m venturimeter is provided in a vertical pipeline carrying oil of specific gravity 0.9, where the flow being upward. The difference in elevation of the throat section and entrance section of the venturimeter is 0.3 m. The differential U-tube mercury manometer shows a gauge deflection of 0.25 m. Calculate (i) the discharge of oil and (ii) pressure difference between the entrance section and the throat section. Take the coefficient of discharge of meter as 0.98 and specific gravity of mercury as 13.6.

Solution

Refer Figure 7.9. Let $d_1 = 0.3$ m, $d_2 = 0.15$ m, $S_{\text{oil}} = 0.9$, $z_2 - z_1 = 0.3$ m, $y = 0.25$ m, $C_d = 0.98$ and $S_m = 13.6$.

(i)
$$a_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} \times 0.3^2 = 0.0707 \text{ m}^2$$

$$a_2 = \frac{\pi}{4} d_2^2 = \frac{\pi}{4} \times 0.15^2 = 0.01767 \text{ m}^2$$

$$h = y\left(\frac{S_m}{S_{\text{oil}}} - 1\right) = 0.25 \times \left(\frac{13.6}{0.9} - 1\right) = 3.528 \text{ m}$$

Since
$$Q_a = \frac{C_d a_1 a_2 \sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}}$$

$$\therefore Q_a = \frac{0.98 \times 0.0707 \times 0.01767 \times \sqrt{2 \times 9.81 \times 3.528}}{\sqrt{0.0707^2 - 0.01767^2}} = \mathbf{0.1488\ m^3/s}$$

(ii) $\rho = S_{\text{oil}} \rho_w = 0.9 \times 1000 = 900 \text{ kg/m}^3$

Since
$$h = \left(\frac{p_1}{\rho g} - \frac{p_2}{\rho g}\right) - (z_2 - z_1)$$

Thus
$$3.528 = \left(\frac{p_1}{\rho g} - \frac{p_2}{\rho g}\right) - 0.3$$

$$\left(\frac{p_1 - p_2}{\rho g}\right) = 3.528 + 0.3 = 3.828$$

Thus $$(p_1 - p_2) = 3.828\rho g$$

$$\therefore (p_1 - p_2) = 3.828 \times 900 \times 9.81 = \mathbf{33797.412\ N/m^2}$$

Example 7.19 A venturimeter inclined at 60° to the vertical is fitted to a 300 mm diameter pipe and its 150 mm diameter throat is 1.3 m from the entrance along its length. The gasoline (specific gravity = 0.78) flows upwards at a rate of 230 litres per second. Determine (i) the discharge coefficient of venturimeter if the pressure gauges fitted at the entrance and throat indicate pressures of 150 kPa and 80 kPa, respectively and (ii) if pressure gauges fitted at the entrance and throat of the meter are replaced by a U-tube mercury manometer, then determine the reading in differential mercury column. Take specific gravity of mercury as 13.6.

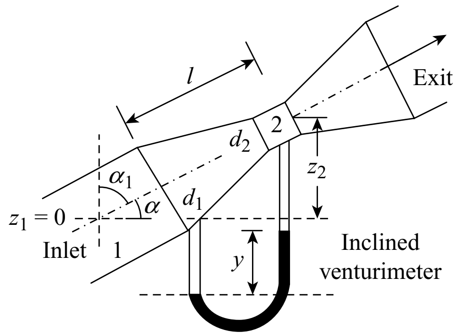

Figure 7.10

Solution

Refer Figure 7.10. Let $\alpha_1 = 60°$, $d_1 = 300$ mm $= 0.3$ m, $d_2 = 150$ mm $=$ 0.15 m, $l = 1.3$ m, $S_{\text{gasoline}} = 0.78$, $Q_a = 230\ l/\text{s} = 0.23\ \text{m}^3/\text{s}$, $p_1 = 150$ kPa, $p_2 = 80$ kPa and $S_m = 13.6$.

(i) $a_1 = \frac{\pi}{4}d_1^2 = \frac{\pi}{4}\times 0.3^2 = 0.0707\ \text{m}^2$

$$a_2 = \frac{\pi}{4}d_2^2 = \frac{\pi}{4}\times 0.15^2 = 0.01767\ \text{m}^2$$

$$\rho = S_{\text{gasoline}}\rho_w = 0.78\times 1000 = 780\ \text{kg/m}^3$$

$$\alpha = 90° - \alpha_1 = 90° - 60° = 30°$$

Assume that the datum line passes through the lower end, where $z_1 = 0$.

$$z_2 = l\sin\alpha = 1.3\sin 30° = 0.65\ \text{m}$$

$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = \left(\frac{150\times 10^3}{780\times 9.81} + 0\right) - \left(\frac{80\times 10^3}{780\times 9.81} + 0.65\right) = 8.5\ \text{m}$$

Since $$Q_a = \frac{C_d a_1 a_2\sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}}$$

$$0.23 = \frac{C_d \times 0.0707\times 0.01767\times\sqrt{2\times 9.81\times 8.5}}{\sqrt{0.0707^2 - 0.01767^2}}$$

$$0.23 = C_d \times 0.2357$$

$$\therefore C_d = \frac{0.23}{0.2357} = \mathbf{0.976}$$

(ii) $h = y\left(\dfrac{S_m}{S_{\text{gasoline}}} - 1\right)$

$$8.5 = y \times \left(\frac{13.6}{0.78} - 1\right)$$

$$\therefore y = \frac{8.5}{16.436} = \mathbf{0.5172\ m\ or\ 51.72\ cm}$$

Example 7.20 A venturimeter of convergent length 0.8 m and throat diameter 0.2 m is fitted in a 0.4 m diameter pipeline carrying oil (specific gravity = 0.82). The venturimeter is inclined at 30° to the horizontal and oil flows upwards. If the U-tube mercury manometer indicates a deflection of 5 cm and the coefficient of discharge of the meter is 0.98, then determine (i) the discharge through the pipe and (ii) if instead of the mercury manometer, pressure gauges are fitted at the entrance and throat of the meter, then determine the pressure gauge reading at the throat when the pressure gauge reading at the entrance is 160 kPa. Take specific gravity of mercury as 13.6.

Solution

Refer Figure 7.10. Let $l = 0.8$ m, $d_2 = 0.2$ m, $d_1 = 0.4$ m, $S_{\text{oil}} = 0.82$, $\alpha = 30°$, $y = 5$ cm $= 0.05$ m, $C_d = 0.98$, $p_1 = 160$ kPa and $S_m = 13.6$.

(i) $a_1 = \dfrac{\pi}{4} d_1^2 = \dfrac{\pi}{4} \times 0.4^2 = 0.1257\ \text{m}^2$

$$a_2 = \frac{\pi}{4} d_2^2 = \frac{\pi}{4} \times 0.2^2 = 0.0314\ \text{m}^2$$

$$h = y\left(\frac{S_m}{S_{\text{oil}}} - 1\right) = 0.05 \times \left(\frac{13.6}{0.82} - 1\right) = 0.78\ \text{m}$$

Since
$$Q_a = \frac{C_d a_1 a_2 \sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}}$$

$$\therefore Q_a = \frac{0.98 \times 0.1257 \times 0.0314 \times \sqrt{2 \times 9.81 \times 0.78}}{\sqrt{0.1257^2 - 0.0314^2}} = \mathbf{0.1243\ m^3/s}$$

(ii) Assume that datum line passes through the lower end, where $z_1 = 0$.

$$z_2 = l \sin\alpha = 0.8 \sin 30° = 0.4\ \text{m}$$

$$\rho = S_{\text{oil}} \rho_w = 0.82 \times 1000 = 820\ \text{kg/m}^3$$

Since
$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right)$$

Thus
$$0.78 = \left(\frac{160 \times 10^3}{820 \times 9.81} + 0\right) - \left(\frac{p_2}{820 \times 9.81} + 0.4\right)$$

$$0.78 = 19.89 - \frac{p_2}{8044.2} - 0.4$$

$$\therefore p_2 = (19.89 - 0.78 - 0.4) \times 8044.2 = \mathbf{150.51\ kPa}$$

Example 7.21 A venturimeter having throat diameter of 0.2 m is fitted in 0.4 m diameter inclined pipe carrying water. An inverted U-tube manometer filled with a liquid of specific gravity 0.72 indicates the difference of pressure between the entrance and throat of meter as 0.28 m. If the loss of head between the entrance and throat is 0.25 times of the kinetic head of the pipe, then determine the rate of flow of water through the pipe.

Solution

Let $d_2 = 0.2$ m, $d_1 = 0.4$ m, $S_l = 0.72$, $y = 0.28$ m and $h_L = 0.25 \times [V_1^2/(2g)]$.

$$a_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} \times 0.4^2 = 0.1257\ \text{m}^2$$

$$a_2 = \frac{\pi}{4} d_2^2 = \frac{\pi}{4} \times 0.2^2 = 0.0314\ \text{m}^2$$

Since
$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = y\left(1 - \frac{S_l}{S}\right)$$

Thus
$$\left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = 0.28 \times \left(1 - \frac{0.72}{1}\right) = 0.0784\ \text{m}$$

$$V_1 = \frac{a_2 V_2}{a_1} = \frac{d_2^2 V_2}{d_1^2} = \frac{0.2^2 V_2}{0.4^2} = \frac{V_2}{4} \quad \text{(From continuity equation)}$$

Thus $V_2 = 4V_1$

$$\left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) + \frac{V_1^2}{2g} - \frac{V_2^2}{2g} = h_L \quad \text{(From Bernoulli's equation)}$$

Thus
$$0.0784 + \frac{V_1^2}{2g} - \frac{(4V_1)^2}{2g} = 0.25 \times \frac{V_1^2}{2g}$$

$$\frac{15.25 V_1^2}{2g} = 0.0784$$

$$\therefore V_1 = \sqrt{\frac{0.0784 \times 2g}{15.25}} = \sqrt{\frac{0.0784 \times 2 \times 9.81}{15.25}} = 0.32\ \text{m/s}$$

$$Q = a_1 V_1 = 0.1257 \times 0.32 = \mathbf{0.040224\ m^3/s}$$

7.8.2 Orificemeter

An orificemeter (or orifice plate) is another simple device which is commonly used for measuring the discharge of a fluid through a pipe. The cost of this device is inexpensive and it requires less space than venturimeter. It also works on the same principle as that of venturimeter. It consists of a flat thin circular plate with a circular sharp edged hole called orifice, which is concentric with the pipe. It is held in the pipeline between two flanges. Generally, the diameter of orifice is kept 0.5 times the diameter of the pipe, but it may vary from 0.4 to 0.8 times the diameter of the pipe. A differential manometer is connected at section 1–1 which is at a distance of 1.5 to 2 times the diameter of the pipe on the upstream from the orifice plate and at section 2–2 which is at a distance of about half the diameter of the orifice on the downstream from the orifice plate as shown in Figure 7.11.

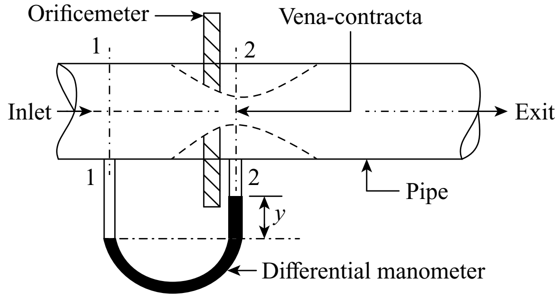

Figure 7.11 *Orificemeter*

Expression for discharge through orificemeter Let a_1 be the area, V_1 be the velocity of fluid and p_1 be the pressure at section 1–1 (inlet section), and a_2, V_2 and p_2 be the corresponding values at section 2–2 (i.e., at vena contracta which is the least cross section of the converging jet) and a_0 be the area of the orifice.

Applying Bernoulli's equation at sections 1–1 and 2–2 of the orificemeter as shown in Figure 7.11, we get the following expression.

$$\frac{p_1}{\rho g}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\rho g}+\frac{V_2^2}{2g}+z_2 \quad \text{(i)}$$

$$\left(\frac{p_1}{\rho g}+z_1\right)-\left(\frac{p_2}{\rho g}+z_2\right)=\frac{V_2^2}{2g}-\frac{V_1^2}{2g}$$

Since
$$h=\left(\frac{p_1}{\rho g}+z_1\right)-\left(\frac{p_2}{\rho g}+z_2\right)$$

Thus
$$h=\frac{V_2^2}{2g}-\frac{V_1^2}{2g}$$

$$V_2^2-V_1^2=2gh$$

$$V_2=\sqrt{2gh+V_1^2} \quad \text{(ii)}$$

The coefficient of contraction (C_c) for the orifice is given by,

$$C_c=\frac{a_2}{a_0}$$

$$a_2=a_0C_c$$

Applying continuity equation between sections 1–1 and 2–2, we get:

$$a_1V_1=a_2V_2$$

$$V_1=\frac{a_2V_2}{a_1}=\frac{C_ca_0V_2}{a_1}$$

Substituting the value of V_1 in expression (ii), we get:

$$V_2=\sqrt{2gh+\frac{C_c^2a_0^2V_2^2}{a_1^2}}$$

$$V_2^2 = 2gh + \frac{C_c^2 a_0^2 V_2^2}{a_1^2}$$

$$V_2^2 \left[1 - \frac{C_c^2 a_0^2}{a_1^2}\right] = 2gh$$

Thus
$$V_2 = \frac{\sqrt{2gh}}{\sqrt{1-(a_0/a_1)^2 C_c^2}}$$

Now
$$Q = a_2 V_2 = a_0 C_c \frac{\sqrt{2gh}}{\sqrt{[1-(a_0/a_1)^2 C_c^2}} \quad \text{(iii)}$$

Let
$$C_d = C_c \frac{\sqrt{1-(a_0/a_1)^2}}{\sqrt{1-(a_0/a_1)^2 C_c^2}}$$

$$C_c = C_d \frac{\sqrt{1-(a_0/a_1)^2 C_c^2}}{\sqrt{1-(a_0/a_1)^2}}$$

Substituting the value of C_c in expression (iii), we get:

$$Q = a_0 C_d \frac{\sqrt{1-(a_0/a_1)^2 C_c^2}}{\sqrt{1-(a_0/a_1)^2}} \times \frac{\sqrt{2gh}}{\sqrt{[1-(a_0/a_1)^2 C_c^2}} = \frac{a_0 C_d \sqrt{2gh}}{\sqrt{1-(a_0/a_1)^2}}$$

$$\therefore \boxed{Q = \frac{C_d a_0 a_1 \sqrt{2gh}}{\sqrt{a_1^2 - a_0^2}}} \quad (7.21)$$

In Equation (7.21), C_d is the coefficient of discharge for orificemeter whose value is much smaller than that for a venturimeter. Its approximate value is 0.67 which depends upon the type of the orifice edge, size of the opening, surface of the plate and the nature of the approaching flow.

The advantage of the orificemeter is that it can be easily manufactured, installed and replaced. It requires very less space but it has very large head loss when compared to other flow meters. Its coefficient of discharge is very low and it is susceptible to inaccuracies resulting from corrosion, erosion and scaling.

Example 7.22 An orificemeter of diameter 0.1 m in a 0.2 m diameter pipe carrying oil (specific gravity = 0.78) has a coefficient of discharge equal to 0.67. If the pressure difference on the two sides of the orifice plate measured by a mercury oil differential manometer is 0.6 mHg, then determine the discharge through the pipe. Take specific gravity of mercury as 13.6.

Solution

Let $d_0 = 0.1$ m, $d_1 = 0.2$ m, $S_{\text{oil}} = 0.78$, $C_d = 0.67$, $y = 0.6$ mHg and $S_m = 13.6$.

$$a_0 = \frac{\pi}{4} d_0^2 = \frac{\pi}{4} \times 0.1^2 = 0.007854 \text{ m}^2$$

$$a_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} \times 0.2^2 = 0.0314 \text{ m}^2$$

$$h = y\left(\frac{S_m}{S_{\text{oil}}} - 1\right) = 0.6 \times \left(\frac{13.6}{0.78} - 1\right) = 9.86 \text{ m}$$

Since
$$Q = \frac{C_d a_0 a_1 \sqrt{2gh}}{\sqrt{a_1^2 - a_0^2}}$$

$$\therefore Q = \frac{0.67 \times 0.007854 \times 0.0314 \times \sqrt{2 \times 9.81 \times 9.86}}{\sqrt{0.0314^2 - 0.007854^2}} = \mathbf{0.0756\ m^3/s}$$

Example 7.23 An orificemeter having an orifice diameter 0.2 m is fitted in a pipe of diameter 0.4 m carrying water. The pressure gauge fitted with upstream and downstream of orificemeter indicates readings of 196.2 kPa and 98.1 kPa, respectively. If the coefficient of discharge of the meter is 0.65, then determine the discharge through the pipe. Also determine the velocity of water in the pipe.

Solution

Let $d_0 = 0.2$ m, $d_1 = 0.4$ m, $p_1 = 196.2$ kPa, $p_2 = 98.1$ kPa and $C_d = 0.65$.

$$a_0 = \frac{\pi}{4} d_0^2 = \frac{\pi}{4} \times 0.2^2 = 0.0314 \text{ m}^2$$

$$a_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} \times 0.4^2 = 0.1257 \text{ m}^2$$

$$h = \frac{p_1}{\rho_w g} - \frac{p_2}{\rho_w g} = \frac{196.2 \times 10^3}{1000 \times 9.81} - \frac{98.1 \times 10^3}{1000 \times 9.81} = 10 \text{ m}$$

Since
$$Q = \frac{C_d a_0 a_1 \sqrt{2gh}}{\sqrt{a_1^2 - a_0^2}}$$

$$\therefore Q = \frac{0.65 \times 0.0314 \times 0.1257 \times \sqrt{2 \times 9.81 \times 10}}{\sqrt{0.1257^2 - 0.0314^2}} = \mathbf{0.2952\ m^3/s}$$

The velocity in the pipe is given by,

$$V_1 = \frac{Q}{a_1} = \frac{0.2952}{0.1257} = \mathbf{2.35\ m/s}$$

Example 7.24 The water flows at the rate of 15 litres per second through a 0.15 m diameter orifice fitted in a 0.3 m diameter pipe. If the coefficient of discharge is 0.65, then determine the difference in pressure head between the upstream section and the vena contracta section.

Solution

Let $Q = 15\ l/\text{s} = 0.015\ \text{m}^3/\text{s}$, $d_0 = 0.15$ m, $d_1 = 0.3$ m and $C_d = 0.65$.

$$a_0 = \frac{\pi}{4} d_0^2 = \frac{\pi}{4} \times 0.15^2 = 0.01767 \text{ m}^2$$

$$a_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} \times 0.3^2 = 0.0707 \text{ m}^2$$

Since
$$Q = \frac{C_d a_0 a_1 \sqrt{2gh}}{\sqrt{a_1^2 - a_0^2}}$$

Thus
$$0.015 = \frac{0.65 \times 0.01767 \times 0.0707 \times \sqrt{2 \times 9.81 \times h}}{\sqrt{0.0707^2 - 0.01767^2}}$$

$$0.015 = 0.052542\sqrt{h}$$

$$\therefore h = \left(\frac{0.015}{0.052542}\right)^2 = \mathbf{0.0815\ m}$$

7.8.3 Pitot Tube

A pitot tube is a simple device used for measuring the velocity of flow at any point in a channel or pipe. It is named in the honour of its inventor Henri de Pitot (1695–1771), a French engineer. In its simplest form, the pitot tube consists of a glass tube bent at right angle and open at both ends. The lower limb known as the body is inserted in the direction of flow, whereas the other end (vertical end) known as stem remains open to atmosphere as shown in Figure 7.12. The liquid enters the tube and rises up in the vertical end to a height *h*.

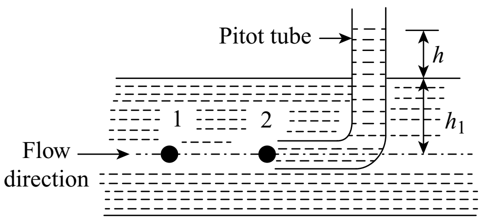

Figure 7.12 *Simple pitot tube*

A pitot tube works on the principle that if the velocity of flow at a point is reduced to zero, then the pressure at that point increases due to the conversion of kinetic energy into pressure energy. Thus, by measuring the increase in pressure energy at this point, the velocity can be measured.

The pressure in the flow far away from the tube is called static pressure which can be measured by fitting a piezometer tube in the static orifice (or a pressure tap). The fluid velocity at the tip of the tube becomes zero and that point is called stagnation point, where the velocity head is converted into pressure head. Therefore, the rise of liquid in the tube represents the sum of the static head and the velocity head. The pressure at the stagnation point is known as stagnation pressure.

Consider two points 1 and 2 at the same level in such a way that point 1 is far away from the tube and point 2 is just at the inlet of the pitot tube (i.e., at the stagnation point). Applying Bernoulli's equation at points 1 and 2, we get:

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$$z_1 = z_2 \text{ and } V_2 = 0$$

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} \tag{i}$$

$$\frac{p_1}{\rho g} = h_1 \text{ and } \frac{p_2}{\rho g} = (h + h_1)$$

From expression (i), we get:

$$h_1 + \frac{V_1^2}{2g} = (h + h_1) \Rightarrow \frac{V_1^2}{2g} = h$$

$$\therefore V_1 = \sqrt{2gh} \tag{7.22}$$

Equation (7.22) gives the value of theoretical velocity and the actual velocity is given below.

$$\therefore \boxed{V_1 = C_v \sqrt{2gh}} \tag{7.23}$$

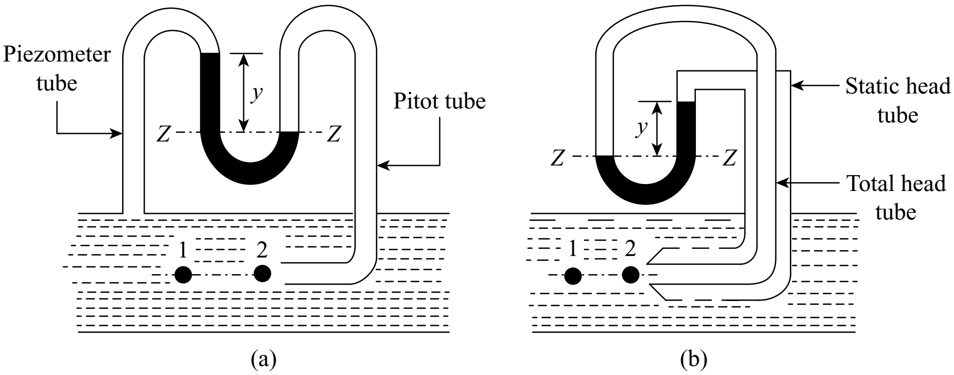

Figure 7.13 *(a) Pitot tube (b) Pitot-static tube*

Here, C_v is the coefficient of the pitot tube whose value will be less than unity and its probable value is about 0.98. However, the actual value can be known by calibration. It can be observed that the liquid level in the vertical limb of the pitot tube rises above the free surface by an amount equal to the velocity head.

The velocity of flow can be measured by inserting a pitot tube as shown in Figure 7.13(a). The dynamic pressure head h can be measured by connecting a differential mercury manometer between the pitot tube and the static orifice for measuring the static pressure as shown in Figure 7.13(a). The following relation is used to determine the velocity head.

$$\boxed{h = y\left(\frac{S_m}{S} - 1\right)} \tag{7.24}$$

Here, y is the difference of mercury level in the U-tube, S_m is the specific gravity of the mercury and S is the specific gravity of the liquid in the pipe.

A pitot-static tube is shown in Figure 7.13(b) which combines the measurement of stagnation and static pressures. The static tube surrounds the total head tube. Two or more holes are drilled in the surface of static head tube (outer tube) in the horizontal portion. The static head tube and the total head tube are connected to a differential manometer to measure the dynamic head h which can be determined by Equation (7.24). The pitot-static tube gives very high accuracies when it is carefully pointed in the direction of flow.

Example 7.25 A pitot tube connected to the limbs of a U-tube mercury manometer is placed in front of the submarine moving horizontally in the sea whose axis lies below the water surface. If the difference in mercury level is observed to be 165 mm and specific gravity of sea water is 1.023, then find the speed of submarine. Take specific gravity of mercury as 13.6 and coefficient of the tube as unity.

Solution

Let $y = 165\text{ mm} = 0.165\text{ m}$, $S_{\text{seawater}} = 1.023$, $S_m = 13.6$ and $C_v = 1$. Let V_1 be the speed of submarine.

$$h = y\left(\frac{S_m}{S_{\text{seawater}}} - 1\right) = 0.165 \times \left(\frac{13.6}{1.023} - 1\right) = 2.03\text{ m}$$

$$V_1 = C_v\sqrt{2gh} = 1 \times \sqrt{2 \times 9.81 \times 2.03} = \mathbf{6.31\ m/s}$$

Example 7.26 One of the orifices of a pitot-static probe placed in the centre of a 0.2 m pipeline points upstream and the other perpendicular to it. When the water flow rate through the pipe is 20 litres per second, the pressure difference between the orifices is 3.5 cm of water. If the mean velocity in the pipe is 0.82 times the central velocity, then determine the coefficient of the probe.

Solution

Let $d = 0.2$ m, $Q = 20\ l/\text{s} = 0.02\text{ m}^3/\text{s}$, $h = 3.5\text{ cm} = 0.035\text{ m}$ and $V = 0.82 \times V_1$.

$$a = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 0.2^2 = 0.0314 \text{ m}^2$$

$$V = \frac{Q}{a} = \frac{0.02}{0.0314} = 0.64 \text{ m/s}$$

$$V_1 = \frac{V}{0.82} = \frac{0.64}{0.82} = 0.78 \text{ m/s}$$

But
$$V_1 = C_v\sqrt{2gh}$$

Thus
$$0.78 = C_v \times \sqrt{2\times 9.81\times 0.035}$$

$$\therefore C_v = \frac{0.78}{\sqrt{2\times 9.81\times 0.035}} = \mathbf{0.94}$$

Example 7.27 A pitot tube placed at the centre of the pipe of 0.35 m diameter pipe indicates stagnation pressure and static vacuum pressure as 10 kPa and 0.11 mHg, respectively. If the coefficient of velocity is 0.975 and the mean velocity of flow is 0.82 times the velocity at the centre of pipe, then find the discharge through the pipe. Take specific gravity of mercury as 13.6.

Solution

Let $d = 0.35$ m, $p_o = 10$ kPa, $h_s = -0.11$ mHg, $C_v = 0.975$, $V = 0.82\times V_1$ and $S_m = 13.6$. Let Q be the discharge through the pipe.

$$a = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 0.35^2 = 0.0962 \text{ m}^2$$

Since
$$h = \frac{p_o}{\rho_w g} - p_s = \frac{p_o}{\rho_w g} - h_s S_m$$

Thus
$$h = \frac{10\times 10^3}{1000\times 9.81} - (-0.11\times 13.6) = 2.5154 \text{ m of water}$$

$$V_1 = C_v\sqrt{2gh} = 0.975\times\sqrt{2\times 9.81\times 2.5154} = 6.85 \text{ m/s}$$

$$V = 0.82\,V_1 = 0.82\times 6.85 = 5.617 \text{ m/s}$$

$$Q = aV = 0.0962\times 5.617 = \mathbf{0.54035\ m^3/s}$$

7.9 ❐ KINETIC ENERGY AND MOMENTUM CORRECTION FACTORS

In deriving the Bernoulli's equation, the velocity distribution across a single stream tube is assumed uniform. However, in the case of flow of real fluids due to viscosity and boundary resistance, the velocity distribution across any cross section area of the flow passage is not uniform. In order to obtain the exact amount of kinetic energy or momentum at a given section, kinetic energy correction factor (α_{cf}) and momentum correction factor (β_{cf}) are to be considered.

7.9.1 Kinetic Energy Correction Factor

It is defined as the ratio of the kinetic energy (K.E.) of flow per second based on actual velocity across a section to the kinetic energy of flow per second based on average velocity across the same section.

Consider the velocity distribution for the flow through a passage as shown in Figure 7.14.

Let dA be the area of cross section of the fluid element, u be the actual velocity, ρ be the mass density of the fluid, V be the average velocity and A be the whole cross section.

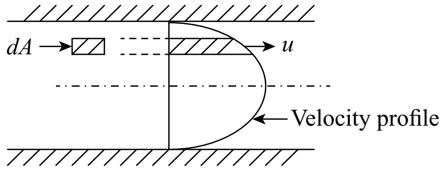

Figure 7.14

The kinetic energy correction factor is given by,

$$\alpha_{cf} = \frac{\int (1/2)(\rho dAu)u^2}{(1/2)(\rho AV)V^2} = \frac{1}{AV^3}\int u^3 dA \quad (7.25)$$

The actual value of α_{cf} depends on the velocity distribution at the flow section. Its numerical value will always be greater than 1. The practical value of α_{cf} for laminar flow in a pipe is 2 and for turbulent flow it varies from 1.03 to 1.06. However, in many fluid mechanics problems, its value is taken as 1 since the velocity is a small percentage of the total head.

7.9.2 Momentum Correction Factor

It is defined as the ratio of momentum of the flow per second based on actual velocity to the momentum of flow per second based on average velocity across a section. The mathematical expression for momentum correction factor is given below.

$$\beta_{cf} = \frac{\int (\rho dAu)u}{(\rho AV)V} = \frac{1}{AV^2}\int u^2 dA \quad (7.26)$$

The actual value of β_{cf} also depends on the velocity distribution at the section. The value of β_{cf} will be equal to 1 if the velocity is uniform over the entire cross section. The practical value of β_{cf} for laminar flow in a pipe is 1.33 and for turbulent flow it varies from 1.02 to 1.05. However, in many fluid mechanics problems, its value is taken as 1, since most of the flow situations are turbulent in nature.

Example 7.28 The velocity profile in a circular pipe is given by $u = u_m[1-(r/R)^2]$, where u is the velocity at any radius r, u_m is the velocity at the pipe axis and R is the radius of the pipe. Determine the average velocity, the energy correction factor and the momentum correction factor.

Solution

Let $u = u_m[1-(r/R)^2]$. Refer Figure 7.15. Let V be the average velocity.

Consider an elementary area dA in the form of a ring of thickness dr at a distance r from the pipe axis and thus, $dA = 2\pi r dr$.

Flow rate through the ring is given by,

$$dQ = dA \times u = 2\pi r dr u$$

Thus, the total flow can be obtained by the integration of above expression as,

$$\int dQ = \int_0^R 2\pi r u dr$$

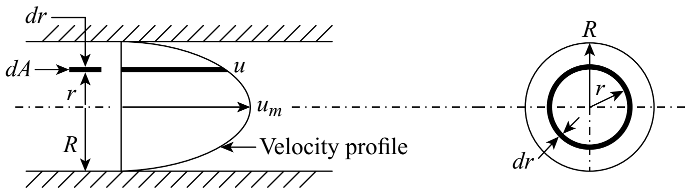

Figure 7.15

Substitute the value of $u = u_m[1-(r/R)^2]$, we get:

$$\int dQ = \int_0^R 2\pi u_m\left[1-\left(\frac{r}{R}\right)^2\right]rdr = 2\pi u_m \int_0^R\left(r-\frac{r^3}{R^2}\right)dr$$

$$Q = 2\pi u_m\left(\frac{r^2}{2}-\frac{r^4}{4R^2}\right)_0^R = 2\pi u_m\left(\frac{R^2}{2}-\frac{R^2}{4}\right) = 2\pi u_m \times \frac{R^2}{4} \qquad \text{(i)}$$

(i) $Q = AV = \pi R^2 \times V$ (ii)

Thus $\pi R^2 V = 2\pi u_m \times \frac{R^2}{4}$ [From (i) and (ii)]

$$\therefore V = \boldsymbol{\frac{u_m}{2}}$$

(ii) $\alpha_{cf} = \frac{1}{AV^3}\int u^3 dA = \frac{1}{AV^3}\int_0^R\left[u_m\left\{1-\left(\frac{r}{R}\right)^2\right\}\right]^3 2\pi r dr$

$$\alpha_{cf} = \frac{2\pi u_m{}^3}{AV^3}\int_0^R\left(r-\frac{3r^3}{R^2}+\frac{3r^5}{R^4}-\frac{r^7}{R^6}\right)dr = \frac{2\pi u_m{}^3}{AV^3}\left[\frac{r^2}{2}-\frac{3r^4}{4R^2}+\frac{3r^6}{6R^4}-\frac{r^8}{8R^6}\right]_0^R$$

Thus $\alpha_{cf} = \frac{2\pi u_m{}^3}{AV^3}\left[\frac{R^2}{2}-\frac{3R^2}{4}+\frac{R^2}{2}-\frac{R^2}{8}\right] = \frac{2\pi u_m{}^3}{AV^3}\frac{R^2}{8}$

Substituting $A = \pi R^2$ and $V = (u_m/2)$, we get:

$$\alpha_{cf} = \frac{2\pi u_m{}^3}{(\pi R^2)(u_m/2)^3}\frac{R^2}{8} = \mathbf{2}$$

(iii) $\beta_{cf} = \frac{1}{AV^2}\int u^2 dA = \frac{1}{AV^2}\int_0^R\left[u_m\left\{1-\left(\frac{r}{R}\right)^2\right\}\right]^2 2\pi r dr$

$$\beta_{cf} = \frac{2\pi u_m{}^2}{AV^2}\int_0^R\left(r-\frac{2r^3}{R^2}+\frac{r^5}{R^4}\right)dr = \frac{2\pi u_m{}^2}{AV^2}\left[\frac{r^2}{2}-\frac{2r^4}{4R^2}+\frac{r^6}{6R^4}\right]_0^R$$

Thus $\beta_{cf} = \frac{2\pi u_m{}^2}{AV^2}\left[\frac{R^2}{2}-\frac{R^2}{2}+\frac{R^2}{6}\right] = \frac{2\pi u_m{}^2}{AV^2}\frac{R^2}{6}$

Substituting $A = \pi R^2$ and $V = (u_m/2)$, we get:

$$\beta_{cf} = \frac{2\pi u_m{}^2}{(\pi R^2)(u_m/2)^2}\frac{R^2}{6} = \frac{4}{3} = \mathbf{1.33}$$

Example 7.29 The velocity distribution for turbulent flow in a circular pipe is given approximately by Prandtl's one-seventh power law as $u = u_m(y/r_o)^{1/7}$, where u is the local velocity of flow at a distance y from the pipe wall, u_m is the maximum velocity at the centreline of the pipe and r_o is the radius of the pipe. Determine the average velocity, the kinetic energy correction factor and the momentum correction factor.

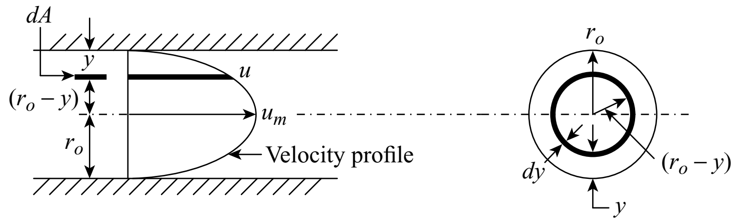

Figure 7.16

Solution

Let $u = u_m(y/r_o)^{1/7}$. Refer Figure 7.16. Let V be the average velocity.

Consider an elementary area dA in the form of a ring of thickness dy at a radius $(r_o - y)$ from the pipe axis and thus, $dA = 2\pi(r_o - y)dy$.

Flow rate through the ring is given by,

$$dQ = dA \times u = 2\pi(r_o - y)dyu$$

Thus, total flow can be obtained by the integration of above expression after substituting the value of $u = u_m(y/r_o)^{1/7}$ as follows.

$$\int dQ = \int_0^{r_o} 2\pi(r_o - y)dyu = \int_0^{r_o} 2\pi u_m \left(\frac{y}{r_o}\right)^{1/7} (r_o - y)dy$$

$$Q = \frac{2\pi u_m}{r_o^{1/7}} \int_0^{r_o} (r_o y^{1/7} - y^{8/7})dy = \frac{2\pi u_m}{r_o^{1/7}} \left(r_o \times \frac{7}{8} y^{8/7} - \frac{7}{15} y^{15/7} \right)_0^{r_o}$$

$$Q = \frac{2\pi u_m}{r_o^{1/7}} \left(\frac{7}{8} r_o^{15/7} - \frac{7}{15} r_o^{15/7} \right) = \frac{2\pi u_m}{r_o^{1/7}} r_o^{15/7} \left(\frac{7}{8} - \frac{7}{15} \right) = 2\pi u_m \left(\frac{49}{120} r_o^2 \right) \quad \text{(i)}$$

(i) $Q = AV = \pi r_o^2 \times V$ (ii)

$$\pi r_o^2 V = 2\pi u_m \left(\frac{49}{120} r_o^2 \right) \qquad \text{[From (i) and (ii)]}$$

$$\therefore V = \frac{2\pi u_m}{\pi r_o^2} \left(\frac{49}{120} r_o^2 \right) = \mathbf{\frac{49}{60} u_m}$$

(ii) $$\alpha_{cf} = \frac{1}{AV^3} \int u^3 dA = \frac{1}{AV^3} \int_0^{r_o} \left[u_m \left(\frac{y}{r_o} \right)^{1/7} \right]^3 2\pi(r_o - y)dy$$

$$\alpha_{cf} = \frac{2\pi u_m^3}{AV^3 r_o^{3/7}} \int_0^{r_o} \left(r_o y^{3/7} - y^{10/7} \right) dy = \frac{2\pi u_m^3}{AV^3 r_o^{3/7}} \left[r_o \times \frac{7}{10} y^{10/7} - \frac{7}{17} y^{17/7} \right]_0^{r_o}$$

Thus $$\alpha_{cf} = \frac{2\pi u_m^{\,3}}{AV^3 r_o^{\,3/7}}\left[\frac{7}{10}r_o^{\,17/7} - \frac{7}{17}r_o^{\,17/7}\right] = \frac{2\pi u_m^{\,3}}{AV^3 r_o^{\,3/7}}\left(\frac{49}{170}r_o^{\,17/7}\right)$$

Substituting $A = \pi r_o^{\,2}$ and $V = (49/60)u_m$, we get:

$$\alpha_{cf} = \frac{2\pi u_m^{\,3}}{(\pi r_o^{\,2})[(49/60)u_m]^3 r_o^{\,3/7}}\left(\frac{49}{170}r_o^{\,17/7}\right) = \mathbf{1.06}$$

(iii) $$\beta_{cf} = \frac{1}{AV^2}\int u^2 dA = \frac{1}{AV^2}\int_0^{r_o}\left[u_m\left(\frac{y}{r_o}\right)^{1/7}\right]^2 2\pi(r_o - y)dy$$

$$\beta_{cf} = \frac{2\pi u_m^{\,2}}{AV^2 r_o^{\,2/7}}\int_0^{r_o} y^{2/7}(r_o - y)\,dy = \frac{2\pi u_m^{\,2}}{AV^2 r_o^{\,2/7}}\left[r_o \times \frac{7}{9}y^{9/7} - \frac{7}{16}y^{16/7}\right]_0^{r_o}$$

Thus $$\beta_{cf} = \frac{2\pi u_m^{\,2}}{AV^2 r_o^{\,2/7}}\left[\frac{7}{9}r_o^{\,16/7} - \frac{7}{16}r_o^{\,16/7}\right] = \frac{2\pi u_m^{\,2}}{AV^2 r_o^{\,2/7}}\left[\frac{49}{144}r_o^{\,16/7}\right]$$

Substituting $A = \pi r_o^{\,2}$ and $V = (49/60)u_m$, we get:

$$\beta_{cf} = \frac{2\pi u_m^{\,2}}{(\pi r_o^{\,2})[(49/60)u_m]^2 r_o^{\,2/7}}\left[\frac{49}{144}r_o^{\,16/7}\right] = \mathbf{1.02}$$

7.10 ❐ FREE LIQUID JET

A jet of liquid coming out from a nozzle in the atmosphere is called a free liquid jet. Under the action of gravity, the liquid jet travels a parabolic path known as trajectory. Consider a jet coming out from a nozzle at point A with a velocity V which makes an angle α with the horizontal as shown in Figure 7.17.

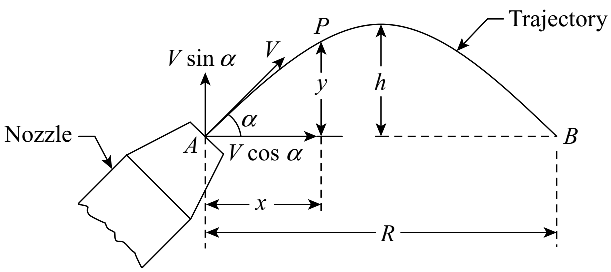

Figure 7.17 *Free liquid jet*

The horizontal and vertical components of the velocity are $V\cos\alpha$ and $V\sin\alpha$, respectively. Let t be the time taken by a liquid particle to reach from point A to point P. The horizontal distance (x) and vertical distance (y) travelled by the liquid particle is respectively expressed below.

$$x = V\cos\alpha \times t \tag{7.27}$$

$$y = V\sin\alpha \times t - \frac{1}{2}gt^2 \tag{7.28}$$

It can be observed from the above expressions that the horizontal component of velocity remains constant but the vertical component is affected by the gravity.

From Equation (7.27), we have $t = x/(V\cos\alpha)$.

Substituting the value of t in Equation (7.28), we get:

$$y = V\sin\alpha \times \frac{x}{V\cos\alpha} - \frac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2 = x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha \tag{7.29}$$

Equation (7.29) is the equation of a parabola and thus, the path travelled by the free liquid jet is parabolic.

1. **Maximum height attained by the jet:** To find the maximum height attained by the jet, we use the following relation.

$$V_2^2 - V_1^2 = -2gh \tag{7.30}$$

As the particle moves upwards and gravity acts downwards, negative sign is used in the above equation.

Here V_1 = Initial vertical component = $V \sin\alpha$

and $V_2 = 0$ at the highest point

Thus $0 - (V \sin\alpha)^2 = -2gh$

$$\therefore \boxed{h = \frac{V^2 \sin^2\alpha}{2g}} \tag{7.31}$$

2. **Time of flight:** It is the time taken by the fluid particle in reaching from point A to point B as shown in Figure 7.17.

Using $y = V \sin\alpha \times t - \frac{1}{2} g t^2$

When the particle reaches at B, $y = 0$ and $t = T$.

Thus $0 = V \sin\alpha \times T - \frac{1}{2} g T^2$

$$\boxed{T = \frac{2V \sin\alpha}{g}} \tag{7.32}$$

Time to reach the highest point is given by,

$$\boxed{T' = \frac{T}{2} = \frac{V \sin\alpha}{g}} \tag{7.33}$$

3. **Horizontal range of the jet:** It is the total horizontal distance travelled by the fluid particle which is denoted by R (Figure 7.17). It is given by the product of horizontal velocity component and the time taken by the particle in reaching from A to B.

Thus $$R = V \cos\alpha \times T$$

$$R = V \cos\alpha \times \frac{2V \sin\alpha}{g} = \frac{V^2 \sin 2\alpha}{g} \tag{7.34}$$

The range will be maximum when $\sin 2\alpha = 1$, i.e., $\sin 2\alpha = \sin 90°$.

Thus $$2\alpha = 90° \text{ or } \alpha = 45°$$

The maximum range is given by,

$$\boxed{R_{max} = \frac{V^2 \sin(2 \times 45°)}{g} = \frac{V^2}{g}} \tag{7.35}$$

Example 7.30 A liquid jet of 30 mm diameter comes out of a nozzle into atmosphere with a velocity of 4.5 m/s at an angle 60° above the horizontal. Determine (i) the equation of trajectory, (ii) maximum height attained by the jet, (iii) horizontal range of the jet and (iv) maximum range. Neglect air friction and assume the jet continuous throughout the trajectory.

Solution

Let $d = 30\text{ mm} = 0.03\text{ m}$, $V = 4.5\text{ m/s}$ and $\alpha = 60°$.

(i) $y = x\tan\alpha - \dfrac{gx^2}{2V^2}\sec^2\alpha$ [Equation of trajectory]

$$y = x\tan 60° - \frac{9.81x^2}{2\times 4.5^2}\sec^2 60° = \mathbf{1.732x - 0.969x^2}$$

(ii) Maximum height attained by the jet is given by,

$$h = \frac{V^2\sin^2\alpha}{2g} = \frac{4.5^2\sin^2 60°}{2\times 9.81} = \mathbf{0.7741\ m}$$

(iii) Horizontal range of the jet is given by,

$$R = \frac{V^2\sin 2\alpha}{g} = \frac{4.5^2\sin 2(60°)}{9.81} = \mathbf{1.79\ m}$$

(iv) Maximum range of the jet is given by,

$$R_{max} = \frac{V^2}{g} = \frac{4.5^2}{9.81} = \mathbf{2.064\ m}$$

Example 7.31 A jet of water produced from a nozzle with a velocity of 16 m/s is projected to the top of a 7 m high wall. If the nozzle is at a distance of 13 m from the wall, then determine its angle of projection with the horizontal.

Solution

Let $V = 16\text{ m/s}$, $y = 7\text{ m}$ and $x = 13\text{ m}$.

Since $$y = x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha \quad \text{[Equation of trajectory]}$$

$$7 = 13\tan\alpha - \frac{9.81\times 13^2}{2\times 16^2}\sec^2\alpha$$

$$7 = 13\tan\alpha - 3.2381\sec^2\alpha$$

$$\sec^2\alpha - 4.015\tan\alpha + 2.162 = 0$$

$$(1+\tan^2\alpha) - 4.015\tan\alpha + 2.162 = 0$$

$$\tan^2\alpha - 4.015\tan\alpha + 3.162 = 0$$

Thus $$\tan\alpha = \frac{4.015 \pm \sqrt{(-4.015)^2 - 4\times 1\times 3.162}}{2\times 1} = 2.9392 \text{ or } 1.0758$$

$$\therefore \alpha = \tan^{-1}(2.9392) = \mathbf{71.21°} \text{ or } \tan^{-1}(1.0758) = \mathbf{47.09°}$$

Example 7.32 A jet of water coming out from a 25 mm diameter nozzle strikes the ground at a horizontal distance of 4.5 m from the nozzle. The nozzle is positioned at a vertical height of 0.6 m from the ground level. If the nozzle is inclined at an angle of 30° with the ground, then determine the discharge from the nozzle.

Solution

Refer Figure 7.18. Let $d = 25$ mm $= 0.025$ m, $x = 4.5$ m, $y = -0.6$ m and $\alpha = 30°$. Let Q be the discharge from the nozzle.

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 0.025^2 = 0.000491\ \text{m}^2$$

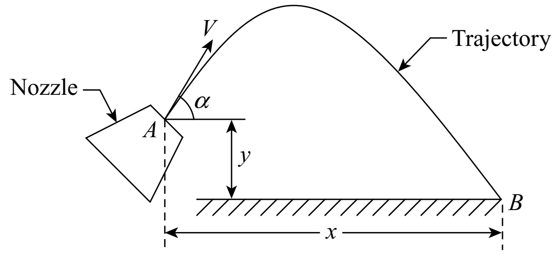

Figure 7.18

Since $y = x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha$ [Equation of trajectory]

Thus
$$-0.6 = 4.5\tan 30° - \frac{9.81\times 4.5^2}{2\times V^2}\sec^2 30°$$

$$-0.6 = 2.6 - \frac{132.435}{V^2}$$

$$\frac{132.435}{V^2} = 3.2$$

$$\therefore V = \sqrt{\frac{132.435}{3.2}} = 6.4332\ \text{m/s}$$

$$Q = AV = 0.000491\times 6.4332 = \mathbf{0.00316\ m^3/s}$$

Example 7.33 Six nozzles each of diameter 30 mm are placed at a height of 1 m from the ground level. All the nozzles are inclined at an angle of 45° to the horizontal and the water jets strike the ground at a horizontal distance of 5.5 m. If the velocity coefficient of nozzles is 0.975, then determine the total discharge from the nozzles and pressure head at them.

Solution

Refer Figure 7.18. Let $n = 6$, $d = 30$ mm $= 0.03$ m, $y = -1$ m, $\alpha = 45°$, $x = 5.5$ m and $C_v = 0.975$. Let h be the pressure head and Q be the total discharge from the nozzles.

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 0.03^2 = 0.00071\ \text{m}^2$$

Since
$$y = x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha \qquad \text{[Equation of trajectory]}$$

Thus
$$-1 = 5.5\tan 45° - \frac{9.81\times 5.5^2}{2V^2}\sec^2 45°$$

$$-1 = 5.5 - \frac{296.7525}{V^2}$$

$$\frac{296.7525}{V^2} = 6.5$$

$$\therefore V = \sqrt{\frac{296.7525}{6.5}} = 6.76\ \text{m/s}$$

$$Q = n \times A \times V = 6 \times 0.00071 \times 6.76 = \mathbf{0.0288\ m^3/s}$$

Since $$V = C_v \sqrt{2gh}$$

Thus $$6.76 = 0.975 \times \sqrt{2 \times 9.81 \times h}$$

$$\therefore h = \left(\frac{6.76}{0.975}\right)^2 \times \frac{1}{2 \times 9.81} = \mathbf{2.45\ m}$$

Example 7.34 A fireman intends to reach a window 25 m above the ground with a water stream from a nozzle at 1 m height and 3 cm in diameter. If the jet discharges water at a rate of 0.019 cubic meters per second, then determine the greatest horizontal distance from the building at which fireman can stand and still reach the water stream upon the window.

Solution

Refer Figure 7.19. Let $y_1 = 25$ m, $y_2 = 1$ m, $y = y_1 - y_2 = 25 - 1 = 24$ m, $d = 3$ cm $= 0.03$ m and $Q = 0.019$ m^3/s. Let x be the greatest horizontal distance from the building.

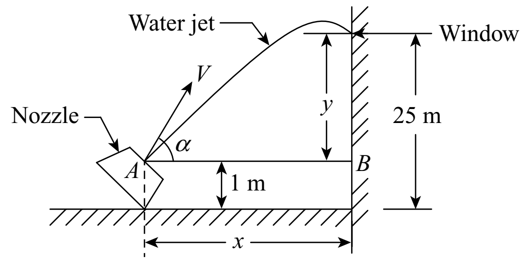

Figure 7.19

$$V = \frac{Q}{A} = \frac{Q}{(\pi/4)d^2} = \frac{0.019}{(\pi/4) \times 0.03^2} = 26.88 \text{ m/s}$$

Since $$y = x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha \qquad \text{[Equation of trajectory]}$$

Thus $$24 = x\tan\alpha - \frac{9.81x^2}{2 \times 26.88^2}\sec^2\alpha$$

$$x\tan\alpha - 0.0068\frac{x^2}{\cos^2\alpha} - 24 = 0 \qquad \text{(i)}$$

Differentiating expression (i) with respect to α, we get:

$$x\sec^2\alpha + \tan\alpha\frac{dx}{d\alpha} - 0.0068 \times \left[x^2\frac{(-2)}{\cos^3\alpha}(-\sin\alpha) + \frac{1}{\cos^2\alpha}2x\frac{dx}{d\alpha}\right] = 0 \qquad \text{(ii)}$$

For maximum value of x, $(dx/d\alpha) = 0$ and thus, expression (ii) becomes,

$$x\sec^2\alpha - 0.0068 \times \left[x^2\frac{(-2)}{\cos^3\alpha}(-\sin\alpha)\right] = 0$$

$$\frac{x}{\cos^2\alpha} - \frac{x}{\cos^2\alpha}(0.0136x\tan\alpha) = 0$$

$$1 - 0.0136x\tan\alpha = 0$$

$$x = \frac{73.53}{\tan\alpha} \qquad \text{(iii)}$$

Substituting expression (iii) in expression (i), we get:

$$\frac{73.53}{\tan\alpha}\tan\alpha - 0.0068\times\left(\frac{73.53}{\tan\alpha}\right)^2\frac{1}{\cos^2\alpha} - 24 = 0$$

$$73.53 - \frac{36.7653}{\sin^2\alpha} - 24 = 0$$

$$\frac{36.7653}{\sin^2\alpha} = 49.53$$

$$\sin\alpha = \sqrt{\frac{36.7653}{49.53}} = 0.86156$$

$$\therefore \alpha = \sin^{-1}(0.86156) = 59.49°$$

Substituting the value of α in expression (iii), we get:

$$x = \frac{73.53}{\tan 59.49°} = \mathbf{43.33\ m}$$

Example 7.35 A fireman intends to reach a window 7.5 m above the ground with a water stream from a nozzle at 1.5 m height and 5 m away from the building. If the water jet discharging from the nozzle is 5 cm in diameter and it strikes the window at a velocity of 13 m/s, then determine the angle of inclination of the nozzle and the amount of water falling on the window.

Solution

Refer Figure 7.19. Let $y_1 = 7.5$ m, $y = y_1 - y_2 = 7.5 - 1.5 = 6$ m, $x = 5$ m, $d = 5$ cm $= 0.05$ m and $V = 13$ m/s. Let α be the angle of inclination and Q be the amount of water falling on the window.

Since $$y = x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha \qquad \text{[Equation of trajectory]}$$

Thus $$6 = 5\tan\alpha - \frac{9.81\times 5^2}{2\times 13^2}(1+\tan^2\alpha)$$

$$6 = 5\tan\alpha - 0.7256 - 0.7256\tan^2\alpha$$

$$0.7256\tan^2\alpha - 5\tan\alpha + 6.7256 = 0$$

$$\tan^2\alpha - 6.891\tan\alpha + 9.269 = 0$$

Thus $$\tan\alpha = \frac{6.891 \pm \sqrt{(-6.891)^2 - 4\times 1\times 9.269}}{2} = 5.0587 \text{ or } 1.8323$$

$$\therefore \alpha = \tan^{-1}(5.0587) = 78.82° \text{ or } \tan^{-1}(1.8323) = \mathbf{61.38°}$$

$$Q = AV = \frac{\pi}{4}d^2\times V = \frac{\pi}{4}\times 0.05^2\times 13 = \mathbf{0.0255\ m^3/s}$$

7.11 ❒ IMPULSE-MOMENTUM EQUATION

The impulse-momentum equation is based on Newton's second law of motion according to which the rate of change of momentum is equal to the applied force and it takes place in the direction of force. Momentum is the product of mass and velocity of the body and it represents the energy of motion stored in a moving body.

Let m be the mass of fluid, V be the velocity of fluid, F be the force acting on the fluid and $a = (dV/dt)$ be the acceleration in the direction of force.

According to Newton's second law of motion, we get:

$$F = m \times a = m \times \frac{dV}{dt}$$

Since m is a constant, it can be taken inside the differential.

$$\therefore F = \frac{d(mV)}{dt} \tag{7.36}$$

Equation (7.36) is known as impulse-momentum principle which can be written as follows.

$$\boxed{F \cdot dt = d(mV)} \tag{7.37}$$

Equation (7.37) is known as impulse-momentum equation in which $F.dt$ is the impulse and $d(mV)$ is the resulting change in momentum in the direction of force. It is a very useful principle for solving several fluid flow problems, such as (i) force on a pipe bend, (ii) force exerted by a fluid jet striking fixed or moving vanes (or blades), (iii) force on propeller vanes and (iv) jet propulsion.

7.11.1 Impulse-Momentum Equation for Steady Flow and Force on a Pipe Bend

Consider a steady flow through a diverging stream tube in a pipe lying in the $x-y$ plane as shown in Figure 7.20(a). Assume the flow as uniform and normal to the inlet and outlet areas.

Let ρ_1, A_1 and V_1 be the density, area and velocity respectively at the entrance and ρ_2, A_2 and V_2 be the corresponding values at the exit. Let α_1 and α_2 be the inclinations with horizontal to the centrelines of the pipe and p_1 and p_2 be the static pressures at the entrance and exit, respectively.

Let the mass of fluid in the region $ABDC$ moves to $A'B'D'C'$ in a short time dt. Let $AA' = ds_1 = V_1 dt$ and $CC' = ds_2 = V_2 dt$. The area $A'B'DC$ is common to both the regions and thus, it will not experience any momentum change. Therefore, the fluid masses in the section $ABB'A'$ and $CDD'C'$ are to be considered. According to the principle of mass conservation, we get the following relation.

$$\text{Fluid mass in } ABB'A' = \text{Fluid mass in } CDD'C'$$

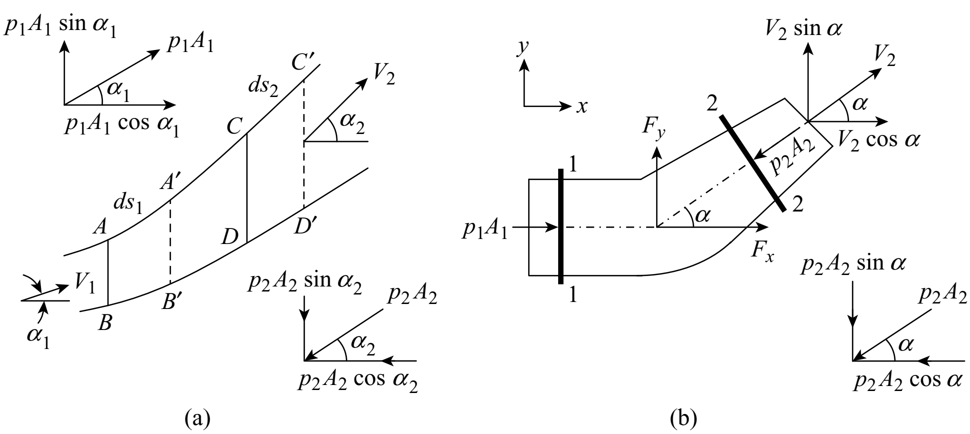

Figure 7.20 *Momentum equation for steady flow and force on a pipe bend*

$$\rho_1 A_1 ds_1 = \rho_2 A_2 ds_2$$

$$\rho_1 A_1 V_1 dt = \rho_2 A_2 V_2 dt$$

$$\text{Momentum of fluid in the region } ABB'A' = (\rho_1 A_1 V_1 dt)V_1$$

$$\text{Momentum of fluid in the region } CDD'C' = (\rho_2 A_2 V_2 dt)V_2$$

$$\therefore \text{Change in momentum } = (\rho_2 A_2 V_2 dt)V_2 - (\rho_1 A_1 V_1 dt)V_1$$

$$A_1 V_1 = A_2 V_2 = Q \qquad \text{[Continuity equation]}$$

$$\rho_1 = \rho_2 = \rho \qquad \text{[Steady incompressible flow]}$$

$$\therefore \text{Change in momentum } = \rho Q(V_2 - V_1)dt$$

Applying impulse-momentum principle, we get:

$$F \cdot dt = \rho Q(V_2 - V_1)dt$$

$$F = \frac{wQ}{g}(V_2 - V_1) \qquad [\because \rho = w/g] \tag{7.38}$$

The quantity $(wQ)/g = \rho Q$ is the mass flow per second and is termed as mass flux.

Components of force (F) along x-axis and y-axis are given by,

$$F_x = \rho Q(V_2 \cos\alpha_2 - V_1 \cos\alpha_1) \tag{7.39a}$$

$$F_y = \rho Q(V_2 \sin\alpha_2 - V_1 \sin\alpha_1) \tag{7.39b}$$

Equations 7.39(a) and 7.39(b) represent the force components exerted by the pipe bend on the fluid mass. Thus, the fluid mass would also exert the same force on the pipe bend but in opposite direction. Therefore, the dynamic force components exerted by the fluid on the pipe bend are given below.

$$F_x = \rho Q(V_1 \cos\alpha_1 - V_2 \cos\alpha_2) \tag{7.40a}$$

$$F_y = \rho Q(V_1 \sin\alpha_1 - V_2 \sin\alpha_2) \tag{7.40b}$$

When considering static pressures at the entrance and exit, the above equations become,

$$F_x = \rho Q(V_1 \cos\alpha_1 - V_2 \cos\alpha_2) + p_1 A_1 \cos\alpha_1 - p_2 A_2 \cos\alpha_2 \tag{7.41a}$$

$$F_y = \rho Q(V_1 \sin\alpha_1 - V_2 \sin\alpha_2) + p_1 A_1 \sin\alpha_1 - p_2 A_2 \sin\alpha_2 \tag{7.41b}$$

Components of force (F) along x-axis and y-axis on a pipe bend shown in Figure 7.20(b) are given by,

$$F_x = \rho Q(V_1 - V_2 \cos\alpha) + p_1 A_1 - p_2 A_2 \cos\alpha \tag{7.41c}$$

$$F_y = \rho Q(-V_2 \sin\alpha) - p_2 A_2 \sin\alpha \tag{7.41d}$$

The resultant force exerted by the fluid and its direction with x-axis is given by,

$$F = \sqrt{F_x{}^2 + F_y{}^2} \tag{7.42}$$

$$\tan\alpha = \frac{F_y}{F_x} \tag{7.43}$$

Example 7.36 A 0.3 m diameter pipe carries water under a head of 20.6 m with a velocity of 4 m/s. If the axis of the pipe turns through 45°, then find the magnitude and direction of the resultant force on the bend.

Solution

Refer Figure 7.21. Let $d_1 = d_2 = 0.3$ m, $p/(\rho_w g) = 20.6$ m, $V_1 = V_2 = 4$ m/s and $\angle = 45°$. Let the subscripts 1 and 2 denote the values at sections 1 and 2, respectively.

$$A_1 = A_2 = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 0.3^2 = 0.0707 \text{ m}^2$$

$$Q = A_1V_1 = 0.0707\times 4 = 0.2828 \text{ m}^3/\text{s}$$

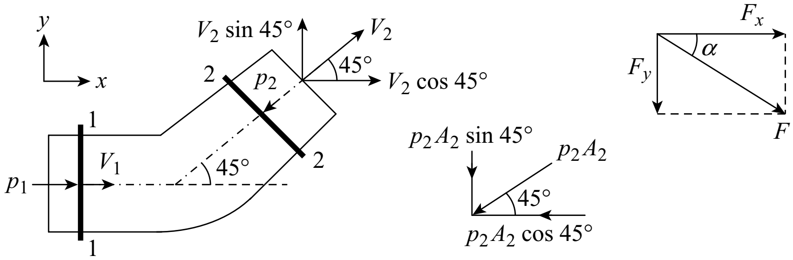

Figure 7.21

Since $$\frac{p}{\rho_w g} = 20.6 \text{ m of water}$$

Thus $$p = p_1 = p_2 = 20.6\rho_w g = \frac{20.6\times 1000\times 9.81}{10^3} = 202.086 \text{ kN/m}^2$$

Force along x-axis is given by,

$$F_x = \rho_w Q[V_1 - V_2\cos 45°] + p_1A_1 - p_2A_2\cos 45°$$

$$F_x = \frac{1000\times 0.2828\times[4-4\cos 45°]}{10^3} + 202.086\times 0.0707 - 202.086\times 0.0707\cos 45°$$

$$\therefore F_x = 4.516 \text{ kN}(\rightarrow)$$

Force along y-axis is given by,

$$F_y = \rho_w Q(0 - V_2\sin 45°) - p_2A_2\sin 45°$$

$$\therefore F_y = \frac{1000\times 0.2828\times(0-4\sin 45°)}{10^3} - 202.086\times 0.0707\sin 45° = -10.9 \text{ kN}(\downarrow)$$

The magnitude of resultant force is given by,

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{4.516^2 + (-10.9)^2} = \mathbf{11.7985 \text{ kN}}$$

The direction of F with x-axis is given by,

$$\alpha = \tan^{-1}\left(\frac{-10.9}{4.516}\right) = \mathbf{-67.49°}$$

Example 7.37 In a system, 260 litres per second of oil of specific gravity 0.9 is flowing in a pipe having a diameter of 0.3 m. If the pipe is bent by 135° and the pressure of oil flowing in the pipe is 400 kPa, then find the magnitude and direction of the resultant force on the bend.

Solution

Refer Figure 7.22. Let $Q = 260\ l/\text{s} = 0.26\ \text{m}^3/\text{s}$, $S_{\text{oil}} = 0.9$, $d = 0.3$ m, $\angle = 135°$ and $p_1 = p_2 = 400$ kPa. Let the subscripts 1 and 2 denote the values at sections 1 and 2, respectively.

$$A_1 = A_2 = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 0.3^2 = 0.0707\ \text{m}^2$$

$$V_1 = V_2 = \frac{Q}{A} = \frac{0.26}{0.0707} = 3.6775\ \text{m/s}$$

$$\rho = S_{\text{oil}}\rho_w = 0.9\times 1000 = 900\ \text{kg/m}^3$$

Force along x-axis is given by,

$$F_x = \rho Q[V_1 - (-V_2\cos 45°)] + p_1A_1 + p_2A_2\cos 45°$$

$$F_x = \frac{900\times 0.26\times[3.6775 + 3.6775\cos 45°]}{10^3} + 400\times 0.0707 + 400\times 0.0707\cos 45°$$

$$\therefore F_x = 49.746\ \text{kN}(\rightarrow)$$

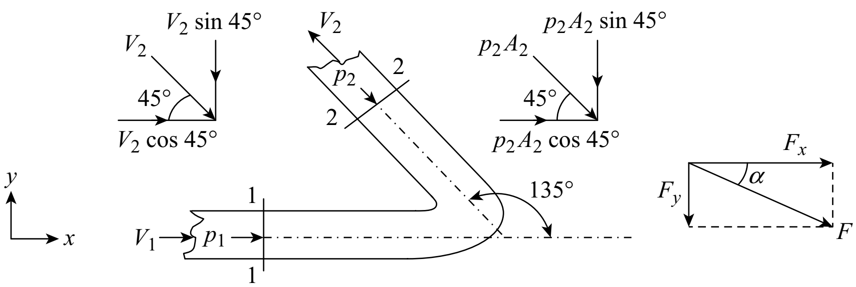

Figure 7.22

Force along y-axis is given by,

$$F_y = \rho Q(0 - V_2\sin 45°) - p_2A_2\sin 45°$$

$$\therefore F_y = \frac{900\times 0.26\times(0 - 3.6775\sin 45°)}{10^3} - 400\times 0.0707\sin 45° = -20.605\ \text{kN}(\downarrow)$$

The magnitude of resultant force is given by,

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{49.746^2 + (-20.605)^2} = \mathbf{53.8445\ kN}$$

The direction of F with x-axis is given by,

$$\alpha = \tan^{-1}\left(\frac{F_y}{F_x}\right) = \tan^{-1}\left(\frac{-20.605}{49.746}\right) = \mathbf{-22.5°}$$

Example 7.38 A 0.2 m diameter to 0.15 m diameter reducing bend fitted in a horizontal oil pipeline deviates from its initial direction to final direction by 60°. The pipe carries oil of specific gravity 0.9 at a rate of 0.3 m^3/s and the pressure at the inlet of the bend is 300 kN/m^2. If 10% of the exit kinetic energy is lost in the bend, then determine the magnitude and direction of the force exerted by the bend.

Solution

Refer Figure 7.23. Let $d_1 = 0.2$ m, $d_2 = 0.15$ m, $\angle = 60°$, $S_{oil} = 0.9$, $Q = 0.3\ m^3/s$, $p_1 = 300\ kN/m^2$ and $h_L = 0.1 \times [V_2^2/(2g)]$. Let the subscripts 1 and 2 denote the values at sections 1 and 2, respectively.

$$A_1 = \frac{\pi}{4}d_1^2 = \frac{\pi}{4}\times 0.2^2 = 0.031416\ m^2$$

$$A_2 = (\pi/4)d_2^2 = (\pi/4)\times 0.15^2 = 0.01767\ m^2$$

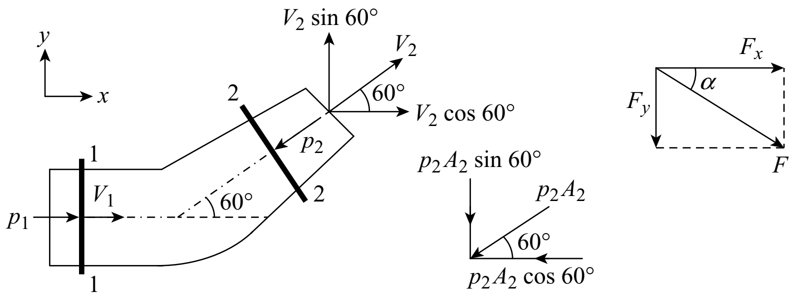

Figure 7.23

$$V_1 = \frac{Q}{A_1} = \frac{0.3}{0.031416} = 9.55\ m/s$$

$$V_2 = \frac{Q}{A_2} = \frac{0.3}{0.01767} = 16.978\ m/s$$

$$h_L = \frac{0.1V_2^2}{2g} = \frac{0.1\times 16.978^2}{2\times 9.81} = 1.4692\ m$$

$$\rho = S_{oil}\rho_w = 0.9\times 1000 = 900\ kg/m^3$$

Since $$\frac{p_1}{\rho g}+\frac{V_1^2}{2g}+z_1 = \frac{p_2}{\rho g}+\frac{V_2^2}{2g}+z_2+h_L \qquad \text{[Bernoulli's equation]}$$

or $$\frac{p_1}{\rho g}+\frac{V_1^2}{2g} = \frac{p_2}{\rho g}+\frac{V_2^2}{2g}+h_L \qquad [\because z_1 = z_2]$$

Thus $$\frac{300\times 10^3}{900\times 9.81}+\frac{9.55^2}{2\times 9.81} = \frac{p_2}{900\times 9.81}+\frac{16.978^2}{2\times 9.81}+1.4692$$

$$38.6274 = \frac{p_2}{8829}+16.161$$

$$\therefore p_2 = \frac{(38.6274-16.161)\times 8829}{10^3} = 198.356\ kN/m^2$$

Force along x-axis is given by,

$$F_x = \rho Q[V_1 - V_2 \cos 60°] + p_1 A_1 - p_2 A_2 \cos 60°$$

$$F_x = \frac{900 \times 0.3 \times [9.55 - 16.978 \cos 60°]}{10^3} + 300 \times 0.031416 - 198.356 \times 0.01767 \cos 60°$$

$$\therefore F_x = 7.9588 \text{ kN}(\rightarrow)$$

Force along y-axis is given by,

$$F_y = \rho Q(0 - V_2 \sin 60°) - p_2 A_2 \sin 60°$$

$$\therefore F_y = \frac{900 \times 0.3 \times [0 - 16.978 \sin 60°]}{10^3} - 198.356 \times 0.01767 \sin 60° = -7.005 \text{ kN}(\downarrow)$$

The magnitude of resultant force is given by,

$$F = \sqrt{F_x{}^2 + F_y{}^2} = \sqrt{7.9588^2 + (-7.005)^2} = \mathbf{10.602\ kN}$$

The direction of F with x-axis is given by,

$$\alpha = \tan^{-1}\left(\frac{F_y}{F_x}\right) = \tan^{-1}\left(\frac{-7.005}{7.9588}\right) = \mathbf{-41.35°}$$

Example 7.39 In a 45° bend, a rectangular air duct of 1 m^2 cross-sectional area is gradually reduced to 0.5 m^2 area. Find the magnitude and direction of force required to hold the duct in position if the velocity of air flow (density = 1.16 kg/m^3) at 1 m^2 section is 10 m/s and pressure is 30 kPa.

Solution

Refer Figure 7.24. Let $\angle = 45°$, $A_1 = 1$ m^2, $A_2 = 0.5$ m^2, $\rho = 1.16$ kg/m^3, $V_1 = 10$ m/s and $p_1 = 30$ kPa. Let the subscripts 1 and 2 denote the values at sections 1 and 2, respectively.

$$V_2 = \frac{A_1 V_1}{A_2} = \frac{1 \times 10}{0.5} = 20 \text{ m/s}$$

Since $$\frac{p_1}{\rho g} + \frac{V_1{}^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2{}^2}{2g} + z_2 \qquad \text{[Bernoulli's equation]}$$

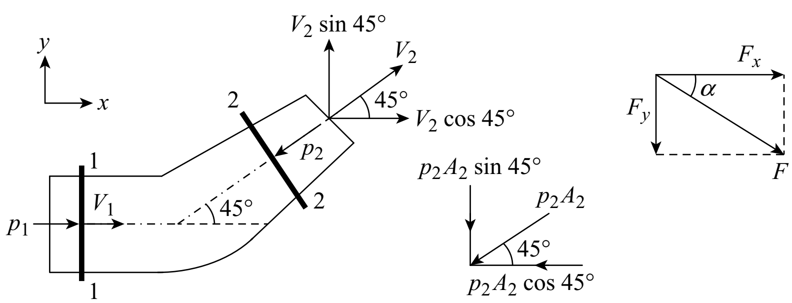

Figure 7.24

Thus $$\frac{30\times10^3}{1.16\times9.81}+\frac{10^2}{2\times9.81}=\frac{p_2}{1.16\times9.81}+\frac{20^2}{2\times9.81} \qquad [\because z_1 = z_2]$$

$$2641.3934=\frac{p_2}{11.3796}+20.3874$$

$$\therefore p_2=\frac{(2641.3934-20.3874)\times11.3796}{10^3}=29.826\text{ kN/m}^2$$

$$Q = A_1V_1 = 1\times10 = 10\text{ m}^3/\text{s}$$

Force along x-axis is given by,

$$F_x = \rho Q[V_1 - V_2\cos45°] + p_1A_1 - p_2A_2\cos45°$$

$$\therefore F_x=\frac{1.16\times10\times[10-20\cos45°]}{10^3}+30\times1-29.826\times0.5\cos45° = 19.407\text{ kN}(\rightarrow)$$

Force along y-axis is given by,

$$F_y = \rho Q(0 - V_2\sin45°) - p_2A_2\sin45°$$

$$\therefore F_y=\frac{1.16\times10\times(0-20\sin45°)}{10^3}-29.826\times0.5\sin45° = -10.709\text{ kN}(\downarrow)$$

The magnitude of resultant force is given by,

$$F=\sqrt{F_x^{\,2}+F_y^{\,2}}=\sqrt{19.407^2+(-10.709)^2}=\mathbf{22.1656\text{ kN}}$$

The direction of F with x-axis is given by,

$$\alpha=\tan^{-1}\left(\frac{F_y}{F_x}\right)=\tan^{-1}\left(\frac{-10.709}{19.407}\right)=\mathbf{-28.89°}$$

Example 7.40 A right angled vertical reducing bend with inlet diameter of 0.3 m and exit diameter of 0.2 m is fitted to a pipe which carries 0.4 m^3/s of water. The volume of the bend is 0.1 m^3 and the pressure at its inlet is 125 kPa. Find the magnitude and direction of force on the bend if water enters the bend at 45° to the horizontal level. Neglect friction and take both inlet and outlet at the same horizontal level.

Solution

Refer Figure 7.25. Let $d_1 = 0.3$ m, $d_2 = 0.2$ m, $Q = 0.4\text{ m}^3/\text{s}$, $v = 0.1\text{ m}^3$, $p_1 = 125$ kPa, and $\angle = 45°$. Let the subscripts 1 and 2 denote the values at sections 1 and 2, respectively.

$$A_1=\frac{\pi}{4}d_1^{\,2}=\frac{\pi}{4}\times0.3^2=0.0707\text{ m}^2$$

$$A_2=\frac{\pi}{4}d_2^{\,2}=\frac{\pi}{4}\times0.2^2=0.0314\text{ m}^2$$

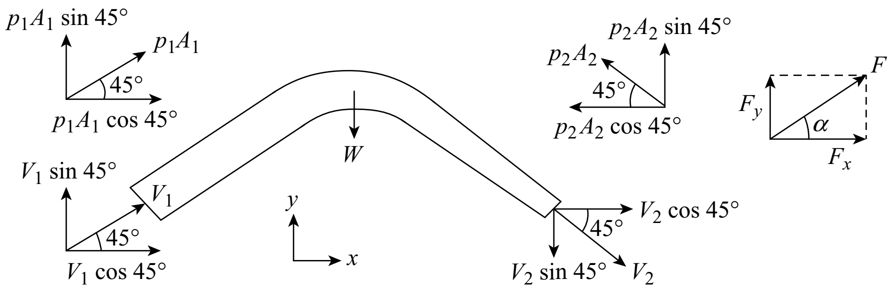

Figure 7.25

$$V_1 = \frac{Q}{A_1} = \frac{0.4}{0.0707} = 5.658 \text{ m/s}$$

$$V_2 = \frac{Q}{A_2} = \frac{0.4}{0.0314} = 12.739 \text{ m/s}$$

Since $$\frac{p_1}{\rho_w g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho_w g} + \frac{V_2^2}{2g} + z_2 \qquad \text{[Bernoulli's equation]}$$

$$\frac{125\times10^3}{1000\times9.81} + \frac{5.658^2}{2\times9.81} = \frac{p_2}{1000\times9.81} + \frac{12.739^2}{2\times9.81} \qquad [\because z_1 = z_2]$$

$$14.374 = \frac{p_2}{9810} + 8.271$$

$$\therefore p_2 = \frac{(14.374 - 8.271)\times 9810}{10^3} = 59.87 \text{ kN/m}^2$$

Force along x-axis is given by,

$$F_x = \rho_w Q[V_1 \cos 45° - V_2 \cos 45°] + p_1 A_1 \cos 45° - p_2 A_2 \cos 45°$$

$$V_1 \cos 45° = 5.658 \cos 45° = 4 \text{ m/s}$$

$$V_2 \cos 45° = 12.739 \cos 45° = 9.01 \text{ m/s}$$

$$p_1 A_1 \cos 45° = 125 \times 0.0707 \cos 45° = 6.249 \text{ kN}$$

$$p_2 A_2 \cos 45° = 59.87 \times 0.0314 \cos 45° = 1.3293 \text{ kN}$$

$$\therefore F_x = \frac{1000\times0.4\times(4-9.01)}{10^3} + 6.249 - 1.3293 = 2.9157 \text{ kN}(\rightarrow)$$

Force along y-axis is given by,

$$F_y = \rho_w Q(V_1 \sin 45° - V_2 \sin 45°) + p_1 A_1 \sin 45° + p_2 A_2 \sin 45° - W$$

$$V_1 \sin 45° = 5.658 \sin 45° = 4 \text{ m/s}$$

$$V_2 \sin 45° = 12.739 \sin 45° = 9.01 \text{ m/s}$$

$$p_1 A_1 \sin 45° = 125 \times 0.0707 \sin 45° = 6.249 \text{ kN}$$

$$p_2 A_2 \sin 45° = 59.87 \times 0.0314 \sin 45° = 1.3293 \text{ kN}$$

$$W = \frac{v \rho_w g}{10^3} = \frac{0.1 \times 1000 \times 9.81}{10^3} = 0.981 \text{ kN}$$

$$\therefore F_y = \frac{1000 \times 0.4 \times (4 - 9.01)}{10^3} + 6.249 + 1.3293 - 0.981 = 4.5933 \text{ kN}(\uparrow)$$

The magnitude of resultant force is given by,

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{2.9157^2 + 4.5933^2} = \mathbf{5.4406 \text{ kN}}$$

The direction of F with x-axis is given by,

$$\alpha = \tan^{-1}\left(\frac{F_y}{F_x}\right) = \tan^{-1}\left(\frac{4.5933}{2.9157}\right) = \mathbf{57.59°}$$

7.12 ❐ MOMENT OF MOMENTUM EQUATION (ANGULAR MOMENTUM PRINCIPLE)

The moment of momentum equation is based on the principle of moment of momentum which states that the resulting torque (T) acting on a rotating fluid is equal to the rate of change of moment of momentum. The moment of the force is called the torque and the moment of momentum is called the angular momentum, where the moments are taken about the axis of rotation.

Let Q be the rate of fluid flow, ρ be the density of the fluid, $m = \rho Q$ be the mass of fluid per unit time, V_1 be the velocity of fluid flow at section 1 (i.e., inlet), V_{w1} is the component of velocity V_1 in tangential direction at section 1 and r_1 be the radius of curvature at section 1. V_2, V_{w2} and r_2 be the corresponding values at section 2 (i.e., exit).

$$\text{Angular momentum per second of fluid at inlet} = \rho Q V_{w1} r_1$$

$$\text{Angular momentum per second of fluid at exit} = \rho Q V_{w2} r_2$$

$$\text{Rate of change of angular momentum} = \rho Q V_{w2} r_2 - \rho Q V_{w1} r_1$$

According to the moment of momentum principle, we get:

$$\text{Resultant torque} = \text{Rate of change of angular momentum}$$

$$\therefore \boxed{T = \rho Q (V_{w2} r_2 - V_{w1} r_1)} \tag{7.44}$$

In a turbine, the fluid exerts torque on the runner, thus Equation (7.44) becomes,

$$T = \rho Q (V_{w1} r_1 - V_{w2} r_2) \tag{7.44a}$$

Equation (7.44) is known as moment of momentum equation. It is used to find the torque exerted by water on the sprinklers and to analyse the flow problems in turbomachines, such as turbines and centrifugal pumps.

Example 7.41 A lawn sprinkler has two similar nozzles of diameter 10 mm each fitted at the ends of rotating arms. One of the nozzles discharges water in vertically upwards direction while the other downwards. The nozzles are at a radial distance of 0.4 m and 0.3 m from the centre of the rotor. If the velocity of flow from each nozzle is 8 m/s, then determine (i) the torque required to hold the arm stationary and (ii) the speed of rotation of the arm neglecting friction.

Solution

Refer Figure 7.26. Let $d = 10$ mm $= 0.01$ m, $r_1 = 0.4$ m, $r_2 = 0.3$ m and $V_1 = V_2 = V = 8$ m/s. Let ω be the angular speed of rotation of the sprinkler and subscripts 1 and 2 denote the values of nozzles 1 and 2, respectively.

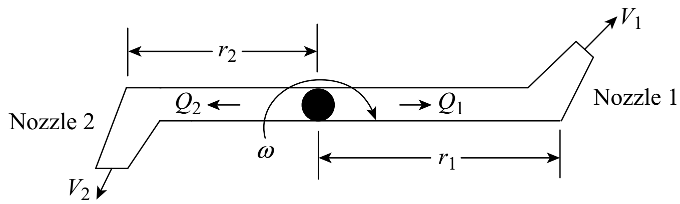

Figure 7.26

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 0.01^2 = 0.0000785 \text{ m}^2$$

$$V_{w1} = V_1 - \omega r_1 = V - \omega r_1$$

$$V_{w2} = V_2 - \omega r_2 = V - \omega r_2$$

$$Q_1 = Q_2 = Q = AV$$

$$T = \rho_w Q_1 V_{w1} r_1 + \rho_w Q_2 V_{w2} r_2 = \rho_w Q \times [V_{w1} r_1 + V_{w2} r_2]$$

$$T = \rho_w AV \times [(V - \omega r_1) r_1 + (V - \omega r_2) r_2] \qquad \text{(i)}$$

(i) Torque required to hold the arm stationary will be equal to torque exerted by water on sprinkler. Substituting $\omega = 0$ and the other values in expression (i), we get the below result.

$$T = 1000 \times 0.0000785 \times 8 \times [(8-0)\times 0.4 + (8-0)\times 0.3]$$

$$\therefore T = \mathbf{3.5168\ Nm}$$

(ii) The sprinkler will rotate freely if the resultant torque on the sprinkler is zero. Thus, substituting $T = 0$ and the other values in expression (i), we get the following result.

$$0 = 1000 \times 0.0000785 \times 8 \times [(8 - 0.4\omega)\times 0.4 + (8 - 0.3\omega)\times 0.3]$$

$$0 = 3.2 - 0.16\omega + 2.4 - 0.09\omega$$

$$0.25\omega = 5.6 \Rightarrow \omega = \frac{5.6}{0.25} = 22.4 \text{ rad/s}$$

Thus $$\frac{2\pi N}{60} = 22.4 \text{ rad/s}$$

$$\therefore N = \frac{22.4 \times 60}{2\pi} = \mathbf{213.9\ rpm}$$

Example 7.42 A lawn sprinkler has two similar nozzles of diameter 5 mm each fitted at the ends of rotating arms. The nozzles are at a radial distance of 0.3 m and 0.2 m from the centre of the rotor which is fed with 0.15 litres of water per second. Both the nozzles have equal discharge of water in vertically downwards direction. Determine (i) the torque required to hold the arm stationary and (ii) the speed of rotation of the arm neglecting friction.

Solution

Refer Figure 7.27. Let $d = 5\text{ mm} = 0.005\text{ m}$, $r_1 = 0.3\text{ m}$, $r_2 = 0.2\text{ m}$ and $Q = 0.15\ l/\text{s} = 0.15\times 10^{-3}\text{ m}^3/\text{s}$. Let ω be the angular speed of rotation of the sprinkler and subscripts 1 and 2 denote the values at nozzles 1 and 2, respectively. The nozzle 1 has a greater radius arm and thus, if the sprinkler is free to rotate, then it will rotate in anticlockwise direction.

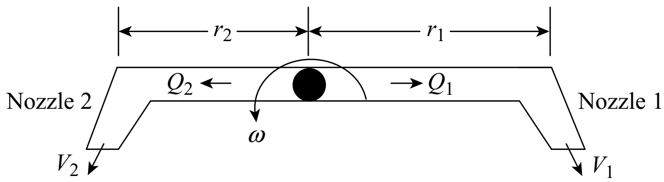

Figure 7.27

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 0.005^2 = 0.0000196\text{ m}^2$$

$$q = \frac{Q}{2} = \frac{0.15\times 10^{-3}}{2} = 0.075\times 10^{-3}\text{ m}^3/\text{s} \qquad [\because Q_1 = Q_2 = q]$$

$$V_1 = V_2 = V = \frac{q}{A} = \frac{0.075\times 10^{-3}}{0.0000196} = 3.83\text{ m/s}$$

$\because V_1$ and ωr_1 are in opposite directions.

Thus $$V_{w1} = V_1 - \omega r_1 = V - \omega r_1$$

$\because V_2$ and ωr_2 are in same direction.

Thus $$V_{w2} = V_2 + \omega r_2 = V + \omega r_2$$

$$T = \rho_w Q_1 V_{w1} r_1 - \rho_w Q_2 V_{w2} r_2 = \rho_w q\times[V_{w1}r_1 - V_{w2}r_2]$$

$$T = \rho_w AV\times[(V-\omega r_1)r_1 - (V+\omega r_2)r_2] \qquad \text{(i)}$$

(i) Torque required for holding the arm stationary will be equal to torque exerted by water on sprinkler. Substituting $\omega = 0$ and the other values in expression (i), we get the following result.

$$T = 1000\times 0.0000196\times 3.83\times[(3.83-0)\times 0.3 - (3.83-0)\times 0.2]$$

$$\therefore T = \mathbf{0.02875\ Nm}$$

(ii) The sprinkler will rotate freely if the resultant torque on the sprinkler is zero. Thus, substituting $T = 0$ and the other values in expression (i), we get the below result.

$$0 = 1000\times 0.0000196\times 3.83\times[(3.83-0.3\omega)\times 0.3 - (3.83+0.2\omega)\times 0.2]$$

$$0 = 1.149 - 0.09\omega - 0.766 - 0.04\omega$$

$$0.13\omega = 0.383 \Rightarrow \omega = \frac{0.383}{0.13} = 2.946 \text{ rad/s}$$

Thus $$\frac{2\pi N}{60} = 2.946$$

$$\therefore N = \frac{2.946 \times 60}{2\pi} = \mathbf{28.13\ rpm}$$

Summary

1. Fluid dynamics deals with the fluid motion considering the forces causing the flow. It is governed by Euler's and Bernoulli's equations.
2. A flowing incompressible fluid possesses potential, kinetic and pressure energies.
3. The forces which influence the motion of a fluid are gravity force (F_g), pressure force (F_p), viscous force (F_v), turbulent force (F_t), compressibility force (F_c) and surface tension force (F_s).
4. **Euler's equation of motion:** $(dp/\rho) + gdz + VdV = 0$.
5. **Bernoulli's equation:** $\frac{p}{\rho g} + \frac{V^2}{2g} + z = \text{Constant}$, i.e., the sum of the pressure head, kinetic head and datum head in a steady, ideal flow of an incompressible fluid is constant along a streamline at any point of the fluid.
6. **Bernoulli's equation for real fluids:** $\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + h_L$, here h_L is the loss of energy per unit weight of fluid between the sections 1 and 2.
7. A venturimeter is used to measure the flow rate of a fluid flowing through a pipe.
8. **Discharge through venturimeter:** $Q_a = (C_d a_1 a_2 \sqrt{2gh}) / \sqrt{a_1^2 - a_2^2}$, here a_1 and a_2 be the area at the inlet and outlet of the venturimeter, respectively, C_d is the coefficient of discharge and h is the difference of pressure head.
9. $h = y[(S_m/S) - 1]$ (For heavier manometric liquid)

   $h = y[1 - (S_l/S)]$ (For lighter manometric liquid)

   Here, y is the difference of the liquid column in U-tube, S_m is the specific gravity of mercury, S_l is the specific gravity of lighter liquid and S is the specific gravity of the liquid flowing through the pipe.
10. Values of h for inclined venturimeter is given by,

    $$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = y\left(\frac{S_m}{S} - 1\right)$$

    (For heavier manometric liquid)

    $$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = y\left(1 - \frac{S_l}{S}\right)$$

    (For lighter manometric liquid)
11. Orificemeter (or orifice plate) is used for measuring the discharge of a fluid through a pipe. Its discharge is given by $Q = (C_d a_0 a_1 \sqrt{2gh}) / \sqrt{a_1^2 - a_0^2}$, here a_1 and a_0 is the area at the inlet and the orifice, respectively and C_d is the coefficient of discharge.
12. A pitot tube is used for measuring the velocity of flow at any point in a pipe. The velocity is given by $V_1 = C_v\sqrt{2gh}$, here C_v is the coefficient of the pitot tube and h is the rise of liquid in the tube above free surface of liquid whose value is given by $h = y[(S_m / S) - 1]$.
13. A pitot-static tube combines the measurement of stagnation and static pressures.
14. **Kinetic energy correction factor:**

    $$\alpha_{cf} = \frac{\text{K.E. per second based on actual velocity}}{\text{K.E. per second based on average velocity}}$$
15. **Momentum correction factor:**

    $$\beta_{cf} = \frac{\text{Momentum per second based on actual velocity}}{\text{Momentum per second based on average velocity}}$$
16. A jet of liquid coming out from a nozzle in atmosphere is called a free liquid jet. The equation of jet is $y = x \tan \alpha - [(gx^2)/2V^2]\sec^2\alpha$, here x and y is horizontal and vertical distances, respectively, V is the jet velocity and α is the jet inclination.

17. **Maximum height attained by the jet:**

$h = (V^2 \sin^2 \alpha)/(2g)$

18. **Time of flight:** $T = (2V \sin\alpha)/g$

19. **Time to reach the highest point:**

$T' = T/2 = (V \sin\alpha)/g$

20. **Horizontal range of the jet:** $R = (V^2 \sin 2\alpha)/g$

21. **Maximum range:** $R_{max} = V^2/g$

22. **Impulse-momentum equation:** $F \cdot dt = d(mV)$, here $F \cdot dt$ is impulse and $d(mV)$ is the resulting change in momentum in the direction of force.

23. Force on a pipe bend in x and y directions respectively are:

$F_x = \rho Q(V_1 \cos\alpha_1 - V_2 \cos\alpha_2) + p_1 A_1 \cos\alpha_1 - p_2 A_2 \cos\alpha_2$

$F_y = \rho Q(V_1 \sin\alpha_1 - V_2 \sin\alpha_2) + p_1 A_1 \sin\alpha_1 - p_2 A_2 \sin\alpha_2$

24. The resultant force exerted by the fluid on the pipe bend and its direction with x axis is given by $F = \sqrt{F_x{}^2 + F_y{}^2}$ and $\tan\alpha = F_y/F_x$.

25. **Moment of momentum equation:** $T = \rho Q(V_{w1} r_1 - V_{w2} r_2)$, here T is the resultant torque, V_{w1} and r_1 is the component of velocity in tangential direction and radius of curvature at section 1, respectively and V_{w2} and r_2 are the corresponding values at section 2.

Multiple-choice Questions

1. The value of momentum correction factor for laminar flow through a pipe is
   (a) 0.3.
   (b) 0.33.
   (c) 1.
   (d) 1.33.

2. The value of kinetic energy correction factor for laminar flow through a pipe is
   (a) 1.
   (b) 2.
   (c) 3.
   (d) 4.

3. The Bernoulli's equation refers to conservation of
   (a) Force.
   (b) Momentum.
   (c) Energy.
   (d) Mass.

4. Bernoulli's equation is applicable to
   (a) Venturimeter.
   (b) Pitot tube.
   (c) Orificemeter.
   (d) All the above.

5. Venturimeter is used to measure
   (a) Fluid velocity.
   (b) Fluid pressure.
   (c) Fluid discharge.
   (d) None of the above.

6. A change in angular momentum of fluid flowing in a curved path results in a
   (a) Change in total energy.
   (b) Change in force.
   (c) Change in pressure.
   (d) Torque.

7. In a venturimeter, the pressure of liquid at throat is
   (a) Equal than at inlet.
   (b) Higher than at inlet.
   (c) Lower than at inlet.
   (d) None of the above.

8. In a venturimeter, the velocity of liquid at throat is
   (a) Equal than at inlet.
   (b) Higher than at inlet.
   (c) Lower than at inlet.
   (d) None of the above.

9. The velocity of liquid flowing through the divergent portion of a venturimeter
   (a) Decreases.
   (b) Increases.
   (c) Remains constant.
   (d) All the above.

10. The total energy represented by Bernoulli's equation, $p/(\rho g) + V^2/(2g) + z = C$ has the units
    (a) Ns/m.
    (b) Nm/m.
    (c) Nm/N.
    (d) None of the above.

11. The value of coefficient of discharge of a venturimeter lies within the range of
    (a) 0.65–0.69.
    (b) 0.75–0.79.
    (c) 0.85–0.89.
    (d) 0.95–0.99.

12. Orificemeter is used to measure
    (a) Pressure.
    (b) Velocity.
    (c) Temperature.
    (d) Discharge.

13. Discharge through a venturimeter varies with venture head (h) as

(a) $h^{2.5}$.

(b) $h^{1.5}$.

(c) $h^{1.0}$.

(d) $h^{0.5}$.

14. The coefficient of discharge of an orificemeter than that of venturimeter is

(a) Much larger.

(b) Equal.

(c) Much smaller.

(d) None of the above.

Review Questions

1. Explain briefly the various heads and forces acting on a flowing fluid. Also give a brief discussion on equations of motion.
2. Derive Euler's equation of motion. Also obtain Bernoulli's equation from it.
3. Give the assumptions, limitations and practical applications of Bernoulli's equation.
4. State and prove Bernoulli's equation and also give its assumptions.
5. What is venturimeter? Explain its working principle. Also obtain an expression for the discharge through it.
6. What is an orificemeter and what are its merits and demerits? Also derive an expression for discharge through it.
7. What is a pitot tube and how will you determine the velocity at any point with the help of it? Also state how it is different than pitot-static tube?
8. Define kinetic energy and momentum correction factors.
9. Define free jet of liquid. Derive an expression for the path travelled by free jet of liquid coming out from a nozzle. Also find expressions for (i) maximum height attained by the jet, (ii) time of flight and (iii) horizontal range of the jet.
10. State and derive impulse-momentum equation for steady flow.
11. What is moment of momentum equation? Also give its applications.
12. Define the coefficient of discharge of venturimeter and coefficient of contraction of orificemeter.

Problems

1. If water flows through a pipe of 50 mm diameter under a pressure of 290 kPa (gauge) with mean velocity of 2 m/s, then determine the total head at 3 m above the datum line.
   [**Ans.** 32.766 m]
2. A pipe carrying oil of specific gravity 0.9 varies in diameter from 0.3 m at section 1 to 0.6 m diameter at section 2 which is 5 m at a higher level. If the pressures at sections 1 and 2 are 100 kPa and 60 kPa, respectively and discharge is 0.3 m^3/s, then determine the loss of head and direction of flow.
   [**Ans.** 0.3895 m, Flow occurs from section 1 to 2]
3. The rate of flow of water through a pipe having diameters 10 cm and 5 cm at sections 1 and 2, respectively, is 25 litres per second. The sections 1 and 2 are 3 m and 2 m above the datum, respectively. Determine the intensity of pressure at section 2 if the pressure at section 1 is 392.4 kPa.
   [**Ans.** 325.94 kPa]
4. The water flow through a conical vertical tube of length 2.5 m is in the downward direction. The velocity of flow at the inlet and outlet ends is 8 m/s and 3 m/s, respectively. If the pressure head at the inlet end is 3 m of liquid and the loss of head in the tube is $0.3[(V_1 - V_2)^2/(2g)]$, where V_1 and V_2 are the velocities at the inlet and outlet ends of the tube, respectively, determine the pressure head at the lower end.
   [**Ans.** 7.92 m of water]
5. The water flows through a taper pipe of length 100 m having diameters 0.6 m at the upper end and 0.3 m at the lower end at the rate of 0.05 m^3/s. If the pressure at the higher level is 196.2 kPa and the pipe has a slope of 1 in 40, then determine the pressure at the lower end.
   [**Ans.** 220.49 kPa]
6. A horizontal venturimeter 20 cm × 10 cm is used to measure the flow of oil (specific gravity = 0.7). (i) Determine the deflection of the oil mercury gauge, if the discharge of the oil is 60 litres per second and the coefficient of discharge is unity. (ii) If the deflection of mercury gauge is 0.2 m, then determine the coefficient of meter.
   [**Ans.** 0.153 m, 0.877]
7. Determine the discharge of water through a horizontal venturimeter with inlet and throat diameters 300 mm and 150 mm, respectively, if the reading of differential manometer connected to the inlet and throat is 250 mmHg and coefficient of discharge of meter is 0.97.
   [**Ans.** 0.13916 m^3/s]

8. A venturimeter of throat diameter 0.1 m is fitted in the pipeline of diameter 0.2 m through which water flows at a rate of 30 litres per second. A differential manometer in the pipeline has an indicator liquid and the manometer reading is 1.16 m. Determine the relative density of the manometer liquid when the venturimeter coefficient is 0.96 and the density of water is 998 kg/m^3.

[**Ans.** 1.65]

9. A horizontal venturimeter with inlet diameter 200 mm and throat diameter 100 mm is used to measure the discharge. If the pressure at the inlet and vacuum pressure at the throat is 180 kPa and 0.35 mHg, respectively and the coefficient of meter is 0.98, then determine its discharge.

[**Ans.** 0.16918 m^3/s]

10. A venturimeter of coefficient 0.96 is to be fitted in a pipe 250 mm diameter, where the pressure head is 7.6 m of flowing liquid and the maximum flow is 0.135 m^3/s. Determine the least diameter of the throat to ensure that the pressure head does not become negative.

[**Ans.** 119.4 mm]

11. A horizontal venturimeter of throat diameter 125 mm is fitted in a water pipeline of diameter 300 mm. The pressure in the pipeline is 140 kPa and the vacuum in the throat is 375 mmHg. If 4% of the differential head is lost between the gauges, then determine the discharge in the pipeline.

[**Ans.** 0.238 m^3/s]

12. A vertical venturimeter carrying a liquid (specific gravity = 0.8) has inlet and throat diameters as 15 cm and 7.5 cm, respectively. The pressure connection at the throat is 15 cm above that at the inlet. If the actual rate of flow is 0.04 m^3/s and the coefficient of meter is 0.96, then determine the pressure difference between inlet and throat of the venturimeter.

[**Ans.** 34.84 kPa]

13. A venturimeter of 0.5 contraction ratio has been fitted in a 0.1 m diameter horizontal pipe. When there is no flow, the head of water on the meter is 3 m (gauge). If the throat pressure is 2 m of water (abs), coefficient of discharge of venturimeter is 0.97 and atmospheric pressure head is 10.3 m of water, then find the discharge.

[**Ans.** 29.31 litres/s]

14. Determine the discharge in the pipeline when the following data are given for an inclined venturimeter, where diameter of the pipeline = 0.4 m, throat diameter = 0.2 m, inclination of the pipeline with the horizontal = 30°, distance between inlet and throat of the venturimeter = 0.6 m, specific gravity of the oil flowing through the pipeline = 0.7, specific gravity of the mercury in U-tube manometer = 13.6, reading of the differential manometer = 5 cm and coefficient of discharge of the venturimeter = 0.98.

[**Ans.** 137.5 litres/s]

15. The difference of mercury level in a differential U-tube manometer connected to the pitot tube is 10 cm. If the coefficient of tube is 0.975 and the specific gravity of oil is 0.82, then find the velocity of flow.

[**Ans.** 5.39 m/s]

16. Determine the discharge of oil for the following data given for an orificemeter, such as diameter of the pipe = 0.25 m, diameter of the orifice = 0.125 m, specific gravity of oil = 0.8, reading of mercury differential manometer = 0.5 m and coefficient of discharge of the meter = 0.65.

[**Ans.** 103.5 litres/s]

17. A jet of water comes out from a 50 mm diameter nozzle and strikes the ground at a horizontal distance of 4 m from the nozzle. The nozzle is positioned at a vertical height of 1.2 m from the ground level. If the nozzle is inclined at an angle of 45° with the ground, then determine the discharge from the nozzle.

[**Ans.** 0.0108 m^3/s]

18. An orifice of diameter 0.3 m is fitted in a pipeline of 0.6 m diameter carrying oil (specific gravity = 0.9). If the manometer indicates 400 mmHg and coefficient of discharge as 0.65, then determine the oil flow rate and the velocity through the pipe.

[**Ans.** 0.4993 m^3/s, 1.77 m/s]

19. A jet of water coming out from a nozzle is inclined at 60° to the horizontal and is held at level 3 m above the ground. If the jet strikes the ground at a horizontal distance of 15 m from the nozzle and air resistance is negligible, then determine (i) the velocity of the jet coming out from the nozzle, (ii) maximum height reached by the jet and (iii) position of the top most point.

[**Ans.** 12.34 m/s, 5.82 m, 6.72 m]

20. Petroleum oil (specific gravity = 0.93 and $\mu = 13\,cP$) flows isothermally through a horizontal 50 mm pipe. A pitot tube is placed at the centre of the pipe which is connected to a U-tube containing water which shows a reading of 100 mm. If the coefficient of tube is 0.98, then find the volumetric flow of oil through the pipe in litres per second.

[**Ans.** 9.95 litres per second]

21. If a nozzle of diameter 25 mm is fitted in a water pipe of diameter 50 mm, then calculate the force exerted by the nozzle on the water flowing through the pipe at the rate of 1250 litres per minute.

[**Ans.** −990.43 N]

22. A fireman intends to reach a window 40 m above the ground with a water jet, issued from a nozzle 3 cm in diameter and discharging 1800 kg per minute. If the nozzle height is 2 m above the ground, then determine the maximum horizontal distance from the building where the fireman can stand and still the water jet reaches the window. Also determine the amount of water falling on the window.

[**Ans.** 140.58 m, 0.0299 m^3/s]

23. An oil of specific gravity 0.86 enters horizontally and gets turned through 45° in clockwise direction in the reducing bend which tapers from 0.4 m diameter at the inlet to 0.2 m diameter at the outlet. If the oil flows at a rate of 400 litres per second, then the pressure of 30 kPa at the inlet section drops to 12 kPa at the exit. Find the magnitude and direction of the resultant force on the bend.

[**Ans.** 3.685 kN, 65.97°]

24. A 0.4 m diameter pipe carries water under a head of 20 m with a velocity of 4 m/s. If the axis of the pipe turns through 45°, then determine the magnitude and direction of the resultant force on the bend.

[**Ans.** 20415.66 N, −67.5°]

25. A horizontal water pipe fitted with a 90° bend of 0.3 m diameter gives discharges 320 litres per second. If the pressure at the inlet and exit of the bend are 245 kPa and 235 kPa, respectively, then determine the resultant force exerted on the bend.

[**Ans.** 26051.22 N, −43.9°]

26. A lawn sprinkler has two similar nozzles of diameter 7.5 mm each fitted at the ends of rotating arms. One of the nozzles discharges water in vertically upwards direction while the other downwards. The nozzles are at a radial distance of 0.2 m and 0.15 m from the centre of the rotor. If the velocity of flow from each nozzle is 12 m/s, then determine (i) the torque required to hold the arm stationary and (ii) the speed of rotation of the arm neglecting friction.

[**Ans.** 2.228 Nm, 641.71 rpm]

27. A horizontal water pipe fitted with a 90° bend reducer. The pressure at the inlet is 210 kPa where its cross-sectional area is 0.012 m^2. If at the exit section, the velocity is 15 m/s, the area is 0.0024 m^2 and the pressure is atmospheric, then determine the resultant force exerted on the bend and its direction.

[**Ans.** 2.683 kN, −11.61°]

28. A lawn sprinkler has two similar nozzles of diameter 3.5 mm each fitted at the ends of rotating arms. The nozzles are at a radial distance of 0.25 m and 0.15 m from the centre of the rotor which is fed with 0.2 litres of water per second. Both the nozzles have equal discharge of water in vertically downward direction. Determine (i) the torque required to hold the arm stationary and (ii) the speed of rotation of the arm neglecting friction.

[**Ans.** 0.10386 Nm, 116.692 rpm]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (d)	**2.** (b)	**3.** (c)	**4.** (d)	**5.** (c)
6. (d)	**7.** (c)	**8.** (b)	**9.** (a)	**10.** (c)
11. (d)	**12.** (d)	**13.** (d)	**14.** (c)	

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Chapter 8

Vortex Flow

8.1 ❐ INTRODUCTION

The flow of a rotating mass of fluid is called vortex flow (i.e., the motion of fluid along a curved path). It is characterized by curved streamline patterns. When fluid flows between curved streamlines, centrifugal forces are set up which are counterbalanced by the pressure force acting in the radial direction. The vortex flow is of two types, namely forced vortex flow and free vortex flow. When a vessel containing a liquid is rotated about a vertical axis at a constant angular velocity, after a small initial adjustment period, the liquid rotates as a solid mass. In such a case, it becomes important to determine the pressure intensity and velocity field. In this chapter, the characteristics of vortex flow, its types, equation of vortex motion and rotation of liquid in a closed cylindrical vessel have been explained in brief context.

8.2 ❐ TYPES OF VORTEX FLOW

The vortex flow is of two types, namely forced vortex flow and free vortex flow which are discussed in the following sections.

8.2.1 Forced Vortex Flow

In a forced vortex flow, the fluid mass is made to rotate by means of some external power source, which exerts a constant torque on the fluid mass. Therefore, this torque induces the whole mass of fluid to rotate at constant angular velocity (ω) as shown in Figure 8.1.

Thus, in a forced vortex flow, a constant external torque is to be applied to the fluid mass resulting in an expenditure of energy. In this flow, the tangential velocity (V) of any fluid particle at a radius (r) from the axis of rotation is given below.

$$\boxed{V = \omega r} \tag{8.1}$$

Thus, angular velocity is given by,

$$\omega = \frac{V}{r} \tag{8.2}$$

A most common example of a forced vortex flow is the motion of a vertical cylinder containing liquid rotated about its central axis with a constant angular velocity. In Figure 8.1, AB shows the free liquid surface before rotation and $A'OB'$ shows the new free

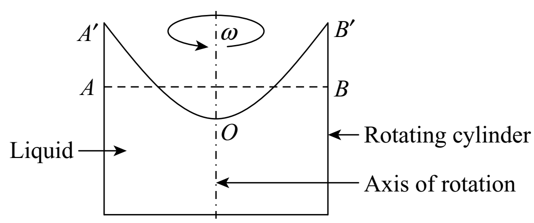

Figure 8.1 *Forced vortex flow*

surface after attaining steady state. Some of the other examples of forced vortex flow are fluid motion inside the runner of a hydraulic turbine or inside the impeller of a centrifugal pump.

8.2.2 Free Vortex Flow

In a free vortex flow, no external torque is required to rotate the fluid mass. The motion may be due to the rotation imparted previously to the fluid particles or due to some internal action (i.e., fluid pressure itself or the gravity force). Some of the examples of free vortex flow are (i) flow of liquid through a hole (or an orifice) provided at the bottom of a vessel or tank, (ii) a whirlpool in a river, (iii) flow of liquid around a circular bend in a pipe, (iv) flow of water in a turbine casing before it enters the guide vanes and (v) flow of liquid in a centrifugal pump casing after it has left the impeller.

Let m be the mass of a fluid particle at a radius r from the axis of rotation and V be its tangential velocity. The rate of change of angular momentum (or moment of momentum) is given below.

$$\frac{\partial}{\partial t}(mVr) \tag{8.3}$$

Since in free vortex flow, no external torque is required to be exerted on the fluid mass. Thus, the rate of change of angular momentum of the flow must be zero and it is given below.

$$\frac{\partial}{\partial t}(mVr) = 0 \tag{8.4}$$

Integrating Equation (8.4), we get:

$$mVr = \text{Constant} = k$$

$$\boxed{Vr = C} \qquad [\because k/m = C] \tag{8.5}$$

The constant C is also known as strength of the vortex. The Equation (8.5) can also be written as given below.

$$V = \frac{C}{r} \tag{8.5a}$$

It can be observed from Equation (8.5a) that the velocity of flow in a free vortex flow varies inversely with the radial distance from the axis of rotation. As $r \to 0$, $V \to \infty$, it mathematically signifies a point of singularity at $r = 0$, which is practically impossible. Thus, the definition of a free vortex flow cannot be extended as $r = 0$. For a free vortex flow, vorticity (ξ) becomes zero. Therefore, free vortex flow is irrotational and hence, it is also referred to as irrotational vortex.

8.2.3 Other Types of Vortex Flow

A vortex flow may also be classified as cylindrical vortex flow and spiral vortex flow.

Cylindrical vortex flow In cylindrical vortex flow, the fluid mass rotates in concentric circles which can be described by concentric circular streamlines.

Spiral vortex flow A spiral vortex flow is a combination of cylindrical vortex flow and radial flow. In a spiral vortex flow, a cylindrical vortex flow is superimposed over the radial flow and the resulting flow is called spiral vortex flow. Thus, in a spiral vortex flow, the mass of fluid either moves spirally outward or spirally inward.

All these types of vortex flow can exist independent of each other. The four types and combination of vortex flow are (i) cylindrical forced vortex, (ii) cylindrical free vortex, (iii) spiral forced vortex and (iv) spiral free vortex.

8.3 ❐ EQUATION OF MOTION FOR A VORTEX FLOW

Consider a fluid element $PQSR$ rotating at a uniform velocity in a horizontal plane about a vertical axis passing through O as shown in Figure 8.2(a). Let the fluid element $PQSR$ of radial thickness dr subtends an angle $\delta\alpha$ at the centre of rotation and is at a radial distance r from the centre. Let dA be the area of the element perpendicular to the radial direction, V be its tangential velocity, $m = \rho dA\, dr$ be its mass, p be the pressure intensity and ρ be the fluid density.

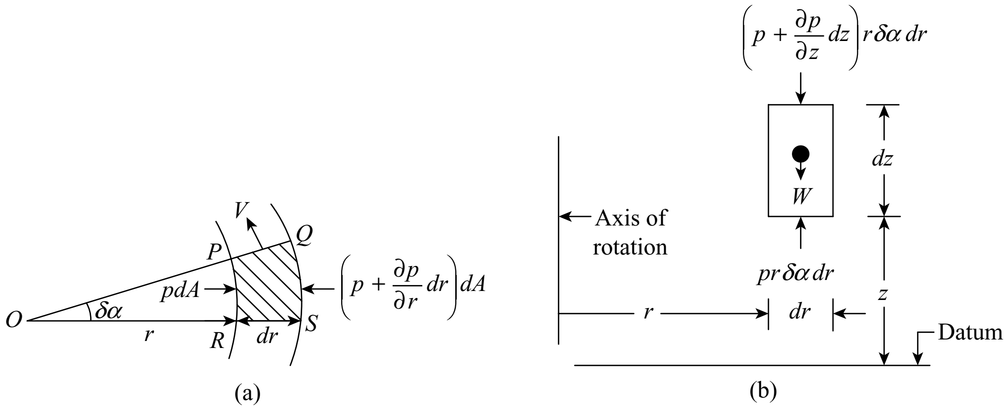

Figure 8.2 *Vortex flow*

$$\text{Pressure force acting on face } PR = p \times dA$$

$$\text{Pressure force acting on face } QS = \left(p + \frac{\partial p}{\partial r} dr\right) \times dA$$

$$\text{Centrifugal force acting on } PQSR = \frac{mV^2}{r} = \rho dA\, dr \times \frac{V^2}{r}$$

There is no variation of pressure in the tangential direction because tangential velocity is constant. However, there is a slight difference in the pressure forces at the two radial faces due to centrifugal force on the fluid element. Thus, by equating the forces in the radial direction, we get the following expression.

$$\left(p + \frac{\partial p}{\partial r} dr\right) dA - pdA = \rho dA dr \frac{V^2}{r}$$

$$\frac{\partial p}{\partial r} dr dA = \rho dA dr \frac{V^2}{r}$$

$$\frac{\partial p}{\partial r} = \rho \frac{V^2}{r} \tag{8.6}$$

Consider the vertical plane of the fluid element as shown in Figure 8.2(b). If there is no acceleration other than gravity, then the following forces will act on the fluid element.

$$\text{Pressure force at the bottom} = p \times r\delta\alpha dr$$

$$\text{Pressure force at the top} = \left(p + \frac{\partial p}{\partial z} dz\right) \times r\delta\alpha dr$$

$$\text{Gravity force acting downwards} = W = mg = (\rho r\delta\alpha\, dr\, dz)g$$

Considering the equilibrium of the fluid element in the vertical direction, we get the following expression.

$$pr\delta\alpha dr - \left(p + \frac{\partial p}{\partial z} dz\right) r\delta\alpha dr = (\rho r\delta\alpha\, dr\, dz)g$$

$$-\frac{\partial p}{\partial z} dz r\delta\alpha dr = \rho g r\delta\alpha\, dr\, dz$$

$$\frac{\partial p}{\partial z} = -\rho g \tag{8.7}$$

It can be seen that p is the function of both r and z. Thus, the total derivative of p is given below.

$$dp = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial z} dz \tag{8.8}$$

Substituting the values from Equations (8.6) and (8.7) in Equation (8.8), we get:

$$\boxed{dp = \rho \frac{V^2}{r} dr - \rho g dz} \tag{8.9}$$

The Equation (8.9) is the fundamental equation for vortex flow. By this equation, the variation of pressure in a vortex flow can be measured.

8.4 ❐ EQUATION OF FORCED VORTEX FLOW

In case of forced vortex flow, the velocity distribution is given by Equation (8.1) as given below.

$$V = \omega r$$

Substituting the value of V given by the above equation in Equation (8.9), we get the following expression.

$$dp = \rho \frac{\omega^2 r^2}{r} dr - \rho g dz \tag{8.10}$$

Considering forced vortex flow in a cylinder subjected to rotation as illustrated in Figure 8.3 in which AB shows the free liquid surface before rotation and $A'OB'$ shows the new free surface after attaining steady state.

Let the points 1 and 2 be at radial distance r_1 and r_2 from the centre of rotation and at elevation z_1 and z_2, respectively. By integrating Equation (8.10) between 1 and 2, we get the following expression.

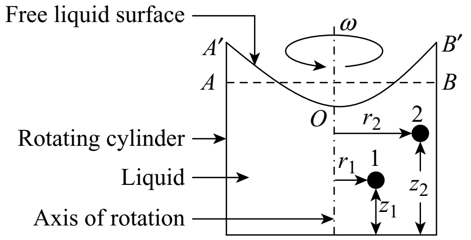

Figure 8.3 *Forced vortex flow*

$$\int_1^2 dp = \rho \omega^2 \int_1^2 r\,dr - \rho g \int_1^2 dz$$

$$(p_2 - p_1) = \frac{1}{2} \rho \omega^2 (r_2^2 - r_1^2) - \rho g (z_2 - z_1)$$

$$(p_2 - p_1) = \frac{\rho}{2} (\omega^2 r_2^2 - \omega^2 r_1^2) - \rho g (z_2 - z_1) \tag{8.11}$$

Since $V_1 = \omega r_1$ and $V_2 = \omega r_2$

$$(p_2 - p_1) = \frac{\rho}{2} (V_2^2 - V_1^2) - \rho g (z_2 - z_1) \tag{8.12}$$

From Equation (8.11) it can be seen that fluid pressure increases with the radial distance from the centre of the vortex.

When the points 1 and 2 lie on the free surface of the liquid, then $p_1 = p_2$ and thus, Equation (8.12) is expressed as follows.

$$0 = \frac{\rho}{2} (V_2^2 - V_1^2) - \rho g (z_2 - z_1)$$

$$z_2 - z_1 = \frac{V_2^2 - V_1^2}{2g} \tag{8.13}$$

When the point 1 lies on the axis of rotation, $r_1 = 0$ and $V_1 = \omega \times 0 = 0$ and let $(z_2 - z_1) = z$.

From Equation (8.13), we get:

$$\boxed{z = \frac{V_2^2}{2g} = \frac{\omega^2 r_2^2}{2g}} \tag{8.14}$$

The Equation (8.14) is an equation of parabola, since z varies with square of r. Therefore, the free liquid surface is a paraboloid of revolution.

Example 8.1 An open cylindrical tank of diameter 1 m and height 2 m contains water up to a depth 1.5 m. If the cylinder rotates about its vertical axis, then what maximum angular velocity can be attained without any spillage?

Solution

Refer Figure 8.4. Let $d = 1$ m, $h_2 = 2$ m and $h_1 = 1.5$ m. Let ω be the angular speed and N be its corresponding speed in rpm. At maximum speed, the water surface will just touch the top of the rim of the cylinder. Let AB be the free liquid surface before rotation and $A'OB'$ shows the new free surface after attaining steady state. Let z be the water surface elevation at the outer edge above vortex O and thus, $OC = z/2$.

$$r = \frac{d}{2} = \frac{1}{2} = 0.5 \text{ m}$$

$$z = 2(h_2 - h_1) = 2(2 - 1.5) = 1 \text{ m}$$

Also $$z = \frac{\omega^2 r^2}{2g}$$

$$1 = \frac{\omega^2 \times 0.5^2}{2 \times 9.81}$$

$$\therefore \omega = \sqrt{\frac{2 \times 9.81}{0.5^2}} = 8.859 \text{ rad/s}$$

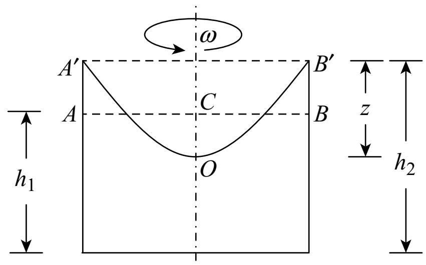

Figure 8.4

Thus $$\frac{2\pi N}{60} = 8.859$$

$$\therefore N = \frac{8.859 \times 60}{2\pi} = \mathbf{84.6\ rpm}$$

Example 8.2 An open cylindrical tank of diameter 0.4 m and height 2 m contains water up to a depth 1.4 m. If the cylinder rotates about its vertical axis at a speed of 240 rpm, then determine the height of the paraboloid formed at the free surface. Also determine the speed of rotation required for the water to start spilling.

Solution

Refer Figure 8.4. Let $d = 0.4$ m, $h_2 = 2$ m, $h_1 = 1.4$ m and $N = 240$ rpm. Let ω be the angular speed and $r = 0.4/2 = 0.2$ m.

(i) $$\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 240}{60} = 25.133 \text{ rad/s}$$

The height of the paraboloid formed at the free surface is given by,

$$z = \frac{\omega^2 r^2}{2g} = \frac{25.133^2 \times 0.2^2}{2 \times 9.81} = \mathbf{1.288\ m}$$

(ii) For the water to spill, the water will just touch the top of the rim of the tank. Thus, the rise of water above the original level becomes equal to $(h_2 - h_1)$, i.e., $(2 - 1.4) = 0.6$ m.

$$\therefore z = 2 \times 0.6 = 1.2 \text{ m}$$

Also
$$z = \frac{\omega^2 r^2}{2g}$$

$$1.2 = \frac{\omega^2 \times 0.2^2}{2 \times 9.81}$$

$$\therefore \omega = \sqrt{\frac{1.2 \times 2 \times 9.81}{0.2^2}} = 24.26 \text{ rad/s}$$

Thus
$$\frac{2\pi N}{60} = 24.26$$

$$\therefore N = \frac{24.26 \times 60}{2\pi} = \mathbf{231.66 \text{ rpm}}$$

Example 8.3 An open cylindrical tank of diameter 0.5 m and height 0.9 m is completely filled with water. It spins about its vertical axis at 104 rpm. Determine the water left in the tank when it reaches to its full speed.

Solution

Refer Figure 8.4. Let $d = 0.5$ m, $h_1 = h_2 = 0.9$ m and $N = 104$ rpm. Let ω be the angular speed and z be the height of the paraboloid formed at the free surface.

$$r = \frac{d}{2} = \frac{0.5}{2} = 0.25 \text{ m}$$

$$\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 104}{60} = 10.891 \text{ rad/s}$$

$$z = \frac{\omega^2 r^2}{2g} = \frac{10.891^2 \times 0.25^2}{2 \times 9.81} = 0.378 \text{ m}$$

Initial volume of water in the tank is given by,

$$v_{\text{initial}} = \pi r^2 h_2 = \pi \times 0.25^2 \times 0.9 = 0.1767 \text{ m}^3$$

Volume of water spilled is equal to volume of paraboloid formed and it is given by,

$$v_{\text{spilled}} = \frac{1}{2}\pi r^2 z = \frac{1}{2} \times \pi \times 0.25^2 \times 0.378 = 0.0371 \text{ m}^3$$

Thus, the volume of water left is given by,

$$v = v_{\text{initial}} - v_{\text{spilled}} = 0.1767 - 0.0371 = \mathbf{0.1396 \text{ m}^3}$$

Example 8.4 An open cylindrical tank of diameter 1 m containing 100 litres of milk is used as a cream separator. Determine the smallest height of the vessel so that it can be rotated at 120 rpm about its vertical axis without any spillage of milk over the sides.

Solution

Refer Figure 8.4. Let $d = 1$ m, $v = 100$ litres $= 0.1$ m^3 and $N = 120$ rpm.

Let ω be the angular speed, h_1 be the height of milk in the vessel and h_2 be the smallest height of the vessel. Let z be the milk surface elevation at the outer edge above vortex O and thus, $OC = z/2$.

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 1^2 = \frac{\pi}{4}\,m^2$$

$$h_1 = \frac{v}{A} = \frac{0.1}{(\pi/4)} = 0.1273 \text{ m}$$

$$\omega = \frac{2\pi N}{60} = \frac{2\times\pi\times 120}{60} = 12.57 \text{ rad/s}$$

$$r = \frac{d}{2} = \frac{1}{2} = 0.5 \text{ m}$$

Since
$$z = 2(h_2 - h_1) = \frac{\omega^2 r^2}{2g}$$

Thus
$$2(h_2 - 0.1273) = \frac{12.57^2 \times 0.5^2}{2\times 9.81}$$

$$\therefore h_2 = \frac{12.57^2 \times 0.5^2}{2\times 2\times 9.81} + 0.1273 = \mathbf{1.134\ m}$$

Example 8.5 An open cylindrical tank of diameter 0.24 m and height 1 m contains water up to a height of 0.74 m. Determine (i) the speed at which the cylinder may be rotated about its vertical axis so that the axial depth becomes zero. Also determine (ii) the difference in total pressure at the sides of the cylinder and (iii) at the bottom of cylinder due to rotation.

Solution

(i) Refer Figure 8.5. Let $D = 0.24$ m, $h_2 = 1$ m and $h_1 = 0.74$ m.

Let ω be the angular speed, N be its corresponding speed in rpm and $R = 0.24/2 = 0.12$ m.

When axial depth become zero, the depth of paraboloid is given by the following expression.

$$z = h_2 = 1 \text{ m}$$

Since
$$z = \frac{\omega^2 R^2}{2g}$$

$$\frac{\omega^2 \times 0.12^2}{2\times 9.81} = 1$$

$$\therefore \omega = \sqrt{\frac{2\times 9.81}{0.12^2}} = 36.912 \text{ rad/s}$$

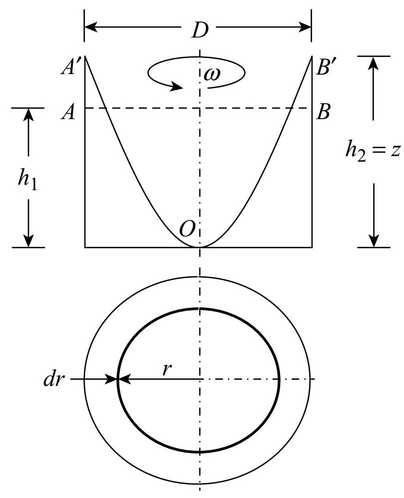

Figure 8.5

Thus $$\frac{2\pi N}{60} = 36.912$$

$$\therefore N = \frac{36.912 \times 60}{2\pi} = \mathbf{352.484\ rpm}$$

(ii) Refer Figure 8.5. Let F_1 and F_2 be the total pressure forces before and after rotation of the cylinder, respectively. Let A be the surface area of the sides of the cylinder in contact with water, h_G be the height of centre of gravity of the above area and ΔF be the increase in pressure force.

Since $$F_1 = \rho_w g A h_G = \rho_w g \times (2\pi R h_1) \times (h_1/2)$$

$$\therefore F_1 = \frac{1000 \times 9.81 \times (2\pi \times 0.12 \times 0.74) \times (0.74/2)}{10^3} = 2.0252\ \text{kN}$$

After rotation, the height of water on the sides of the cylinder becomes equal to the height of the cylinder. Thus, $A = 2\pi R h_2$ and $h_G = h_2/2$.

Since $$F_2 = \rho_w g A h_G = \rho_w g \times (2\pi R h_2) \times (h_2/2)$$

$$\therefore F_2 = \frac{1000 \times 9.81 \times (2\pi \times 0.12 \times 1) \times (1/2)}{10^3} = 3.6983\ \text{kN}$$

$$\Delta F = F_2 - F_1 = 3.6983 - 2.0252 = \mathbf{1.6731\ kN}$$

(iii) Refer Figure 8.5 and the pressure force on the bottom is given by,

$$F_1 = \rho_w g A h_G = \rho_w g \times (\pi/4) D^2 \times h_1$$

$$\therefore F_1 = \frac{1000 \times 9.81 \times (\pi/4) \times 0.24^2 \times 0.74}{10^3} = 0.3284\ \text{kN}$$

Pressure on the bottom after rotation varies as a function of the radial distance. Take an elementary ring at the bottom surface of the cylinder of thickness *dr* at a radial distance *r* from the axis of the cylinder as shown in Figure 8.5.

The force on the elementary ring is given by,

$$dF_2 = \rho_w g \times (2\pi r dr) \times \frac{\omega^2 r^2}{2g} = \pi \rho_w \omega^2 r^3 dr$$

Thus, the total pressure force on the bottom is given by,

$$F_2 = \int_0^R \pi \rho_w \omega^2 r^3 dr = \pi \rho_w \omega^2 \int_0^R r^3 dr = \pi \rho_w \omega^2 \left(\frac{R^4}{4}\right)$$

$$\therefore F_2 = \pi \times 1000 \times 36.912^2 \times \frac{0.12^4}{4} \times \frac{1}{10^3} = 0.2219\ \text{kN}$$

$$\Delta F = F_1 - F_2 = 0.3284 - 0.2219 = \mathbf{0.1065\ kN}$$

Example 8.6 Prove that in case of forced vortex, the rise of liquid level at the ends is equal to the fall of liquid level at the axis of rotation when there is no spillage.

Solution

Refer Figure 8.6. Let d be the diameter of the cylinder, v be the volume of liquid in cylinder, ω be the angular speed, AB be the free liquid surface before rotation and $A'OB'$ shows the new free surface after attaining steady state.

Let z_1 be the fall of liquid at the centre from AB and z_2 be the rise of liquid at the outer edge above AB.

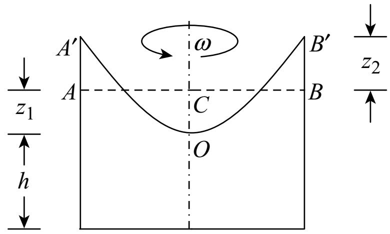

Figure 8.6

$$v = \pi r^2 (h + z_1) \quad \text{(i)}$$

When the cylinder is rotated, the volume of liquid is given by the following relation.

$$v = \text{Volume of cylinder upto } A'B' - \text{Volume of paraboloid}$$

$$v = \pi r^2 (h + z_1 + z_2) - \frac{1}{2}\pi r^2 (z_1 + z_2) = \pi r^2 h + \pi r^2 (z_1 + z_2) - \frac{\pi r^2}{2}(z_1 + z_2)$$

$$v = \pi r^2 h + \frac{\pi r^2}{2}(z_1 + z_2) \quad \text{(ii)}$$

Thus
$$\pi r^2 (h + z_1) = \pi r^2 h + \frac{\pi r^2}{2}(z_1 + z_2) \quad \text{[From (i) and (ii)]}$$

$$\pi r^2 h + \pi r^2 \, z_1 = \pi r^2 h + \frac{\pi r^2}{2} z_1 + \frac{\pi r^2}{2} z_2$$

$$\pi r^2 \, z_1 - \frac{\pi r^2}{2} z_1 = \frac{\pi r^2}{2} z_2$$

$$\frac{\pi r^2 z_1}{2} = \frac{\pi r^2 z_2}{2}$$

$$\therefore \mathbf{z_1 = z_2}$$

Hence, it is proved.

Example 8.7 An open cylindrical vessel of diameter 0.15 m and depth 0.375 m is filled with water up to the top. Determine the volume of water left in the vessel when it is rotated about its vertical axis with a speed of (i) 250 rpm and (ii) 500 rpm.

Solution

Refer Figure 8.7. Let $d = 0.15$ m, $h = 0.375$ m, $N = 250$ rpm and 500 rpm. Let v be the initial volume of water.

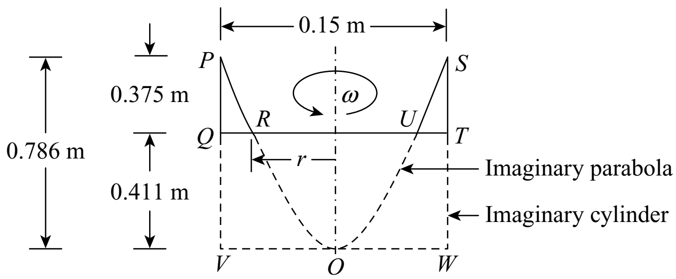

Figure 8.7

(i) $r = \frac{d}{2} = \frac{0.15}{2} = 0.075 \text{ m}$

$$v = \frac{\pi}{4} \times 0.15^2 \times 0.375 = 0.00663 \text{ m}^3$$

$$\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 250}{60} = 26.18 \text{ rad/s}$$

$$z = \frac{\omega^2 r^2}{2g} = \frac{26.18^2 \times 0.075^2}{2 \times 9.81} = 0.1965 \text{ m}$$

Volume of water spilled = Volume of paraboloid = v_1

$$v_1 = \frac{\pi}{4} d^2 \times \frac{z}{2} = \frac{\pi}{4} \times 0.15^2 \times \frac{0.1965}{2} = 0.00174 \text{ m}^3$$

The volume of water left is given by,

$$v_2 = v - v_1 = 0.00663 - 0.00174 = \mathbf{0.00489 \text{ m}^3}$$

(ii) $\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 500}{60} = 52.36 \text{ rad/s}$

$$z = \frac{\omega^2 r^2}{2g} = \frac{52.36^2 \times 0.075^2}{2 \times 9.81} = 0.786 \text{ m}$$

It is known that the height of parabola is greater than the height of the cylinder and thus, the shape of the imaginary parabola will be as shown in Figure 8.7.

The height of imaginary parabola is given by,

$$h_1 = 0.786 - 0.375 = 0.411 \text{ m}$$

The volume of water left in the vessel is given by,

$$v_2 = \text{Volume of portions } PQR \text{ and } STU$$

$$v_2 = v - v_3 (\text{Volume of paraboloid } POS) + v_4 (\text{Volume of paraboloid } ROU)$$

Now $$v = \frac{\pi}{4} \times 0.15^2 \times 0.375 = 0.00663 \text{ m}^3$$

$$v_3 = \frac{\pi}{4} d^2 \times \frac{z}{2} = \frac{\pi}{4} \times 0.15^2 \times \frac{0.786}{2} = 0.006945 \text{ m}^3$$

For imaginary paraboloid,

$$\omega = 52.36 \text{ rad/s and } z = 0.786 - 0.375 = 0.411 \text{ m}$$

Since $$z = \frac{\omega^2 r^2}{2g}$$

$$0.411 = \frac{52.36^2 \times r^2}{2 \times 9.81}$$

$$\therefore r = \sqrt{\frac{0.411 \times 2 \times 9.81}{52.36^2}} = 0.0542 \text{ m}$$

$$v_4 = \frac{1}{2}\pi r^2 z = \frac{1}{2}\pi \times 0.0542^2 \times 0.411 = 0.001896 \text{ m}^3$$

$$\therefore v_2 = v - v_3 + v_4 = 0.00663 - 0.006945 + 0.001896 = \mathbf{0.001581 \text{ m}^3}$$

8.5 ❐ ROTATION OF LIQUID IN A CLOSED CYLINDRICAL VESSEL

A cylindrical vessel of radius R closed at the top containing liquid up to some level (h_1) is shown in Figure 8.8(a). When it is given a rotation (ω_1), the shape of paraboloid of revolution will be as shown in Figure 8.8(b). As the speed of rotation is increased to ω_2, the shape of the paraboloid will be different as illustrated in Figure 8.8(c). In this case, the radius (r) of the parabola at the top of the cylinder and the height of the parabola (z) are not known. These two parameters can be found out by equating the volume of air before rotation (Figure 8.8a) and after rotation (Figure 8.8c).

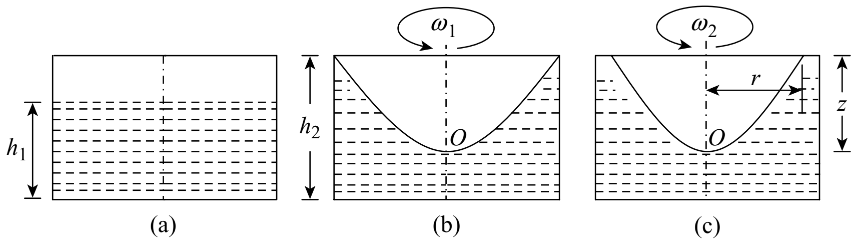

Figure 8.8 *Rotation of liquid in a closed cylinder*

$$\text{Volume of air before rotation} = \pi R^2 (h_2 - h_1) \tag{8.15}$$

$$\text{Volume of air after rotation} = (1/2)\pi r^2 z \tag{8.16}$$

Simplifying the Equations (8.15) and (8.16), we get:

$$\pi R^2 (h_2 - h_1) = \frac{1}{2}\pi r^2 z \tag{8.17}$$

The height of the paraboloid is given by,

$$z = \frac{\omega_2{}^2 r^2}{2g} \tag{8.18}$$

By solving the Equations (8.17) and (8.18), the unknown parameters r and z can be determined.

8.6 ❐ CLOSED CYLINDRICAL ROTATING VESSEL COMPLETELY FILLED WITH A LIQUID

Let a cylindrical vessel of radius R and height h be completely filled with liquid of density ρ is given a rotation ω as shown in Figure 8.9.

The pressure distribution at any radius along a horizontal plane is given by Equation (8.6) and it is given by,

$$\frac{\partial p}{\partial r} = \rho \frac{V^2}{r} = \rho \frac{\omega^2 r^2}{r} = \rho \omega^2 r$$

$$\partial p = \rho \omega^2 r \partial r$$

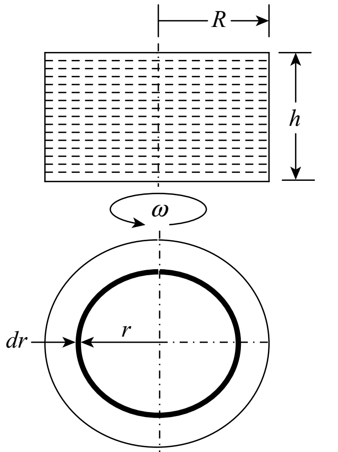

Figure 8.9 *Rotation of a closed cylinder completely filled with a liquid*

Integrating the above expression, we get:

$$\int \partial p = \rho\omega^2 \int r\partial r$$

$$p = \frac{\rho\omega^2 r^2}{2} \tag{8.19}$$

Consider an elementary ring of radius r and width dr on the top of the vessel (Figure 8.9). The pressure force on the ring is given by the product of pressure and area as given below.

$$dF_t = p \times 2\pi r dr = \frac{\rho\omega^2 r^2}{2} \times 2\pi r dr = \rho\omega^2 \pi r^3 dr$$

The total force on the top of the vessel (F_t) can be obtained by integrating the above expression and it is given below.

$$\int dF_t = \rho\omega^2 \pi \int_0^R r^3 dr$$

$$\therefore F_t = \rho\omega^2 \pi \left[\frac{r^4}{4}\right]_0^R = \frac{\rho\omega^2 \pi R^4}{4} \tag{8.20}$$

The total force acting on the bottom of the vessel (F_b) is given by the sum of F_t and weight of liquid in the vessel as represented below.

$$\boxed{F_b = \frac{\rho\omega^2 \pi R^4}{4} + \rho g \pi R^2 h} \tag{8.21}$$

Example 8.8 A cylindrical vessel closed at both ends is 0.25 m in diameter and 1.5 m deep. It is filled with a liquid up to a height of 1 m. Determine (i) the height of paraboloid formed, if it is rotated about its vertical axis at 240 rpm and (ii) speed of rotation of the vessel, when axial depth of liquid is zero.

Solution

(i) Refer Figure 8.10(a) and (b). Let $D = 0.25$ m, $h_2 = 1.5$ m, $h_1 = 1$ m and $N = 240$ rpm. Let r be the radius of the paraboloid and z be its height.

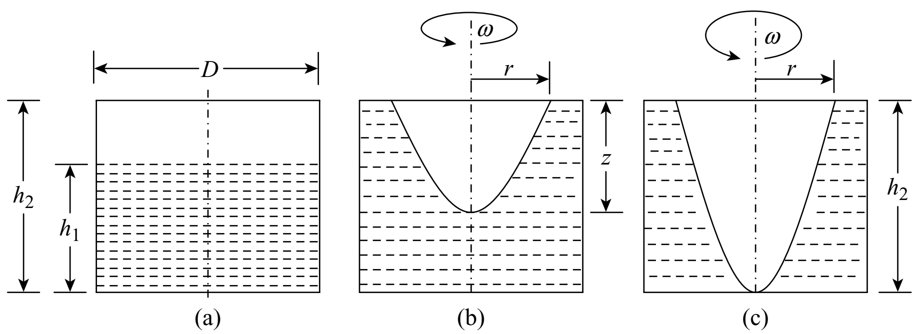

Figure 8.10

$$\omega = \frac{2\pi N}{60} = \frac{2\times\pi\times 240}{60} = 25.133 \text{ rad/s}$$

$$z = \frac{\omega^2 r^2}{2g} = \frac{25.133^2 \times r^2}{2\times 9.81} = 32.195 r^2$$

$$r^2 = \frac{z}{32.195} \qquad \text{(i)}$$

$$R = \frac{D}{2} = \frac{0.25}{2} = 0.125 \text{ m}$$

Volume of air before rotation = Volume of air after rotation

or
$$\pi R^2 (h_2 - h_1) = \frac{1}{2}\pi r^2 z$$

$$\pi \times 0.125^2 \times (1.5 - 1) = \frac{1}{2}\pi r^2 z$$

$$r^2 z = 0.015625 \qquad \text{(ii)}$$

$$\frac{z}{32.195}\times z = 0.015625 \qquad \text{[From (i) and (ii)]}$$

$$z^2 = 0.50305$$

$$\therefore z = \sqrt{0.50305} = \mathbf{0.7093 \text{ m}}$$

(ii) Refer Figure 8.10(c). We know that $z = 1.5$ m. Let r be the radius of the paraboloid and ω be the speed of rotation.

Since
$$z = \frac{\omega^2 r^2}{2g}$$

$$1.5 = \frac{\omega^2 \times r^2}{2\times 9.84}$$

$$\omega^2 r^2 = 29.43 \qquad \text{(i)}$$

Volume of air before rotation = Volume of air after rotation

or
$$\pi R^2 (h_2 - h_1) = \frac{1}{2}\pi r^2 z$$

$$\pi \times 0.125^2 \times (1.5 - 1) = \frac{1}{2}\pi \times r^2 \times 1.5$$

$$r^2 = \frac{0.125^2 \times 0.5 \times 2}{1.5} = 0.01042 \qquad \text{(ii)}$$

$$\omega^2 \times 0.01042 = 29.43 \qquad \text{[From (i) and (ii)]}$$

$$\therefore \omega = \sqrt{\frac{29.43}{0.01042}} = 53.145 \text{ rad/s}$$

Thus
$$\frac{2\pi N}{60} = 53.145$$

$$\therefore N = \frac{53.145 \times 60}{2 \times \pi} = \mathbf{507.5\ rpm}$$

Example 8.9 A cylindrical vessel closed at both ends is 0.24 m in diameter and 0.3 m deep is completely filled with a liquid of specific gravity 0.86. Determine the total pressure force exerted by the liquid on the top and bottom of the vessel when it is rotated about its vertical axis at 300 rpm.

Solution

Let $D = 0.24$ m, $h = 0.3$ m, $S = 0.86$ and $N = 300$ rpm.

$$R = \frac{D}{2} = \frac{0.24}{2} = 0.12 \text{ m}$$

$$\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 300}{60} = 31.416 \text{ rad/s}$$

$$\rho = S\rho_w = 0.86 \times 1000 = 860 \text{ kg/m}^3$$

The total force on the top of the vessel (F_t) is given by,

$$F_t = \frac{1}{4}\rho\omega^2\pi R^4 = \frac{1}{4} \times 860 \times 31.416^2 \times \pi \times 0.12^4 = \mathbf{138.234\ N}$$

The total force acting on the bottom of the vessel (F_b) is given by,

$$F_b = F_t + \text{Weight of liquid in the vessel} = F_t + \rho g \pi R^2 h$$

$$\therefore F_b = 138.234 + 860 \times 9.81\pi \times 0.12^2 \times 0.3 = \mathbf{252.733\ N}$$

Example 8.10 A cylindrical steel vessel closed at both ends is 0.6 m in radius and 2.4 m deep is completely filled with water. The vessel consists of 0.5 cm thick steel plates which can withstand an allowable stress of 3.25×10^5 kPa. Determine the speed of rotation of the vessel at which the water pressure bursts its sides under hoop tension.

Solution

Let $R = 0.6$ m, $h = 2.4$ m, $t = 0.5$ cm $= 0.005$ m and $f_t = 3.25 \times 10^5$ kPa. Let N be the speed of rotation required to burst the vessel.

The maximum permissible pressure is given by,

$$p_{\max} = \frac{f_t t}{R} = \frac{3.25 \times 10^5 \times 10^3 \times 0.005}{0.6} = 2.708 \times 10^6 \text{ N/m}^2$$

The centrifugal pressure head due to rotation is given by,

$$\frac{p}{\rho_w g} = \frac{\omega^2 R^2}{2g}$$

Thus $$p = \frac{\rho_w g\omega^2 R^2}{2g} = \frac{1000 \times 9.81\omega^2 \times 0.6^2}{2 \times 9.81} = 180\omega^2$$

For the cylinder to burst, $p > p_{max}$ and it is given by,

$$180\omega^2 > 2.708 \times 10^6$$

$$\omega > \sqrt{\frac{2.708 \times 10^6}{180}}$$

$$\omega > 122.656 \text{ rad/s}$$

Thus $$\frac{2\pi N}{60} > 122.656$$

$$N > \frac{122.656 \times 60}{2\pi}$$

$$\therefore N > \mathbf{1171.28\ rpm}$$

Example 8.11 A cylindrical steel vessel closed at both ends is 0.1 m in radius and it is completely filled with water. If the vessel rotates at 1240 rpm about its vertical axis, then determine the difference in pressure for a horizontal plane between (i) its circumference and at a radial distance of 0.05 m and (ii) its circumference and the centre.

Solution

Let $R_2 = 0.1$ m, $N = 1240$ rpm and $R_1 = 0.05$ m.

$$\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 1240}{60} = 129.85 \text{ rad/s}$$

(i) The centrifugal pressure head difference due to rotation is given by,

$$\frac{p_2 - p_1}{\rho_w g} = \frac{\omega^2 \times (R_2{}^2 - R_1{}^2)}{2g}$$

$$(p_2 - p_1) = \frac{\rho_w \omega^2 (R_2{}^2 - R_1{}^2)}{2}$$

$$\therefore (p_2 - p_1) = \frac{1000 \times 129.85^2 \times (0.1^2 - 0.05^2)}{2} = \mathbf{63.23\ kN/m^2}$$

(ii) For this case, $R_1 = 0$ and it is given by,

$$\therefore (p_2 - p_1) = \frac{\rho_w \omega^2 R_2{}^2}{2} = \frac{1000 \times 129.85^2 \times 0.1^2}{2} = \mathbf{84.305\ kN/m^2}$$

Example 8.12 A cylindrical shaped vessel closed at both ends contains water up to a height of 0.64 m. The diameter of the vessel is 0.2 m and its length is 1 m. If the vessel rotates at 600 rpm, then determine the area uncovered at the bottom of the vessel.

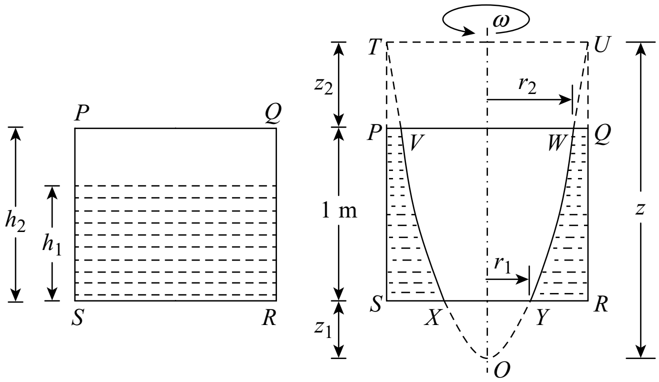

Figure 8.11

Solution

(i) Refer Figure 8.11. Let $h_1 = 0.64$ m, $D = 0.2$ m, $R = 0.2/2 = 0.1$ m, $h_2 = 1$ m and $N = 600$ rpm.

Let $A_{\text{uncovered}}$ be the area uncovered at the base, z_1 and z_2 be the portions of the height of the imaginary parabola as shown in Figure 8.11.

$$\omega = \frac{2\pi N}{60} = \frac{2\times\pi\times 600}{60} = 62.832 \text{ rad/s}$$

If the tank is not closed at the top and it is long enough, then the height of parabola corresponding to $\omega = 62.832$ rad/s is derived as follows.

$$z = \frac{\omega^2 r^2}{2g} = \frac{62.832^2 \times 0.1^2}{2\times 9.81} = 2.0122 \text{ m}$$

$$z = z_1 + 1 + z_2 = 2.0122$$

$$z_1 + z_2 = 1.0122 \qquad \text{(i)}$$

For the paraboloid *VOW*, we get:

$$1 + z_1 = \frac{\omega^2 r_2^2}{2g} = \frac{62.832^2 \times r_2^2}{2\times 9.81} = 201.22 r_2^2$$

$$r_2^2 = \frac{1+z_1}{201.22} \qquad \text{(ii)}$$

For the paraboloid *XOY*, we get:

$$z_1 = \frac{\omega^2 r_1^2}{2g} = \frac{62.832^2 \times r_1^2}{2\times 9.81} = 201.22 r_1^2 \qquad \text{(iii)}$$

Volume of air before rotation = Volume of air after rotation

or
$$\pi R^2 (h_2 - h_1) = \frac{1}{2}\pi r_2^2 (1+z_1) - \frac{1}{2}\pi r_1^2 z_1$$

$$\pi \times 0.1^2 \times (1 - 0.64) = \frac{\pi}{2}[r_2^2(1+z_1) - r_1^2 z_1]$$

Substituting the value of expression (ii) in the above expression, we get:

$$0.01131 = \frac{\pi}{2}\left[\frac{(1+z_1)}{201.22}(1+z_1) - r_1^2 z_1\right] \quad \text{(iv)}$$

Substituting the value of expression (iii) in expression (iv), we get:

$$0.01131 = \frac{\pi}{2}\left[\frac{(1+201.22r_1^2)}{201.22}(1+201.22r_1^2) - r_1^2 \times 201.22r_1^2\right]$$

$$\frac{0.01131 \times 2 \times 201.22}{\pi} = (1+201.22r_1^2)^2 - (201.22)^2 r_1^4$$

$$1.449 = (1+402.44r_1^2 + 40489.49r_1^4) - 40489.49r_1^4$$

$$402.44r_1^2 = 0.449 \Rightarrow r_1^2 = \frac{0.449}{402.44} = 0.001116$$

$$\therefore A_{\text{uncovered}} = \pi r_1^2 = \pi \times 0.001116 = \mathbf{0.003506\ m^2}$$

Example 8.13 A cylindrical shaped vessel closed at both ends contains water up to a height of 0.9 m. The diameter of the vessel is 0.4 m and its length is 1.2 m. If the vessel rotates at 275 rpm and the air above the water surface is at a pressure of 70.2 kPa, then determine the pressure head at the bottom of the vessel (i) at the centre and (ii) at the edge.

Solution

Refer Figure 8.12. Let $h_1 = 0.9$ m, $D = 0.4$ m, $h_2 = 1.2$ m, $N = 275$ rpm and $p = 70.2$ kPa.

Let r_1 and z_1 be the radius and height of the parabola *VOW*, respectively and R and z_2 be the corresponding values of the parabola *TOU*.

$$r_2 = R = \frac{D}{2} = \frac{0.4}{2} = 0.2 \text{ m}$$

$$\omega = \frac{2 \times \pi \times 275}{60} = 28.8 \text{ rad/s}$$

$$z_1 = \frac{\omega^2 r_1^2}{2g} = \frac{28.8^2 \times r_1^2}{2 \times 9.81} = 42.3r_1^2$$

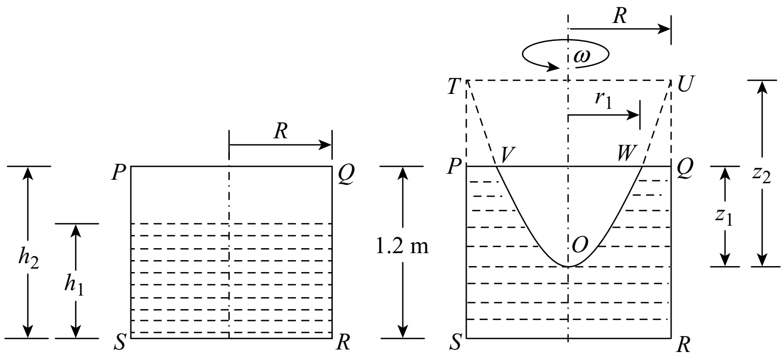

Figure 8.12

$$z_2 = \frac{\omega^2 r_2^2}{2g} = \frac{28.8^2 \times 0.2^2}{2 \times 9.81} = 1.691 \text{ m}$$

Volume of air before rotation = Volume of air after rotation

or
$$\pi R^2 (h_2 - h_1) = \frac{1}{2}\pi r_1^2 z_1$$

$$\pi \times 0.2^2 \times (1.2 - 0.9) = \frac{1}{2}\pi \times r_1^2 \times 42.3 r_1^2$$

$$0.2^2 \times 0.3 \times 2 = 42.3 r_1^4$$

$$\therefore r_1 = \left(\frac{0.2^2 \times 0.3 \times 2}{42.3}\right)^{1/4} = 0.15434 \text{ m}$$

$$z_1 = 42.3 \times 0.15434^2 = 1.01 \text{ m}$$

Head due to air pressure is given by,

$$h = \frac{p}{\rho_w g} = \frac{70.2 \times 10^3}{1000 \times 9.81} = 7.156 \text{ m}$$

(i) Pressure head at the centre of the bottom of the vessel is given by,

$$h_{\text{centre}} = h + (h_2 - z_1) = 7.156 + (1.2 - 1.01) = \mathbf{7.346 \text{ m of water}}$$

(ii) Pressure head at the edge of the bottom of the vessel is given by,

$$h_{\text{edge}} = h + [z_2 + (h_2 - z_1)]$$

$$\therefore h_{\text{edge}} = 7.156 + [1.691 + (1.2 - 1.01)] = \mathbf{9.037 \text{ m of water}}$$

8.7 ❐ EQUATION OF FREE VORTEX FLOW

For free vortex flow, we have Equation (8.5a) as given below.

$$V = \frac{C}{r}$$

Substituting the value of V in Equation (8.9), we get:

$$dp = \rho \frac{(C/r)^2}{r} dr - \rho g\, dz = \rho \frac{C^2}{r^3} dr - \rho g\, dz \qquad (8.22)$$

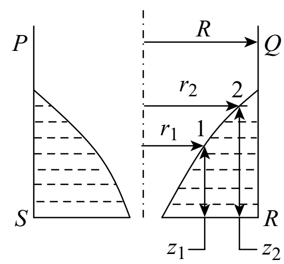

Figure 8.13

Let r_1 and r_2 be the radii of two points 1 and 2, respectively, in the fluid from the central axis and z_1 and z_2 be the corresponding heights from the bottom of the vessel (Figure 8.13).

Integrating Equation (8.22) between the points 1 and 2, we get:

$$\int_1^2 dp = \rho C^2 \int_1^2 \frac{dr}{r^3} - \rho g \int_1^2 dz$$

$$p_2 - p_1 = \rho C^2 \left[\frac{r^{-3+1}}{-3+1}\right]_1^2 - \rho g(z_2 - z_1) = \rho C^2 \left[-\frac{1}{2r^2}\right]_1^2 - \rho g(z_2 - z_1)$$

$$p_2 - p_1 = -\frac{\rho C^2}{2}\left[\frac{1}{r_2^2} - \frac{1}{r_1^2}\right] - \rho g(z_2 - z_1) = \frac{\rho}{2}\left[\frac{C^2}{r_1^2} - \frac{C^2}{r_2^2}\right] - \rho g(z_2 - z_1)$$

$$p_2 - p_1 = \frac{\rho}{2}(V_1^2 - V_2^2) - \rho g(z_2 - z_1)$$

$$\frac{p_2 - p_1}{\rho g} = \frac{V_1^2 - V_2^2}{2g} - (z_2 - z_1)$$

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 \tag{8.23}$$

Therefore, the Equation (8.23) is the Bernoulli's equation which is also applicable to free vortex flow.

Example 8.14 In the free cylindrical vortex water flow at a point 0.2 m radius, the velocity and pressure are 7.5 m/s and 155 kPa, respectively. Determine the pressure at a radius of 0.3 m.

Solution

Let $r_1 = 0.2$ m, $V_1 = 7.5$ m/s, $p_1 = 155$ kPa and $r_2 = 0.3$ m. Let p_2 be the pressure at point 2.

$$V_2 = \frac{V_1 r_1}{r_2} = \frac{7.5 \times 0.2}{0.3} = 5 \text{ m/s} \qquad [\because Vr = C]$$

Since

$$\frac{p_1}{\rho_w g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho_w g} + \frac{V_2^2}{2g} \qquad [\because z_1 = z_2]$$

$$\frac{155 \times 10^3}{1000 \times 9.81} + \frac{7.5^2}{2 \times 9.81} = \frac{p_2}{1000 \times 9.81} + \frac{5^2}{2 \times 9.81}$$

$$18.6672 = \frac{p_2}{9810} + 1.2742$$

$$\therefore p_2 = \frac{(18.6672 - 1.2742) \times 9810}{10^3} = \mathbf{170.6253\ kN/m^2}$$

Example 8.15 At a point which is at a radius of 0.25 m and height 0.125 m, in the free cylindrical vortex fluid flow, the velocity and pressure are 12 m/s and 125 kPa, respectively. If the fluid in air having a density of 1.25 kg/m^3, then determine the pressure at a radius of 0.5 m and at a height of 0.25 m.

Solution

Refer Figure 8.13. Let $r_1 = 0.25$ m, $z_1 = 0.125$ m, $V_1 = 12$ m/s, $p_1 = 125$ kPa, $\rho = 1.25$ kg/m^3, $r_2 = 0.5$ m and $z_2 = 0.25$ m.

Let p_2 be the pressure at point 2.

$$V_2 = \frac{V_1 r_1}{r_2} = \frac{12 \times 0.25}{0.5} = 6 \text{ m/s} \qquad [\because Vr = C]$$

Since
$$\frac{p_1}{\rho g}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\rho g}+\frac{V_2^2}{2g}+z_2$$

$$\frac{125\times10^3}{1.25\times9.81}+\frac{12^2}{2\times9.81}+0.125=\frac{p_2}{1.25\times9.81}+\frac{6^2}{2\times9.81}+0.25$$

$$10201.1444=\frac{p_2}{12.2625}+2.085$$

$$\therefore p_2=\frac{(10201.1444-2.085)\times12.2625}{10^3}=\mathbf{125.066\ kN/m^2}$$

Summary

1. The flow of a rotating mass of fluid is called vortex flow.
2. Forced vortex flow: The fluid mass is made to rotate by means of some external constant torque and thereby, the fluid rotates at constant angular velocity (ω) which is given by the relation $V=\omega r$, here V and r be the tangential velocity and radius, respectively.
3. Free vortex flow: No external torque is required to rotate the fluid mass. The relation between the tangential velocity and radius is given by the relation $Vr = C$, here C is a constant and it is also known as vortex strength.
4. Cylindrical vortex flow: The fluid mass rotates in concentric circles which can be described by concentric circular streamlines.
5. Spiral vortex flow: It is a combination of cylindrical vortex flow and radial flow.
6. Pressure variation in the horizontal plane of a vortex flow: $\partial p/\partial r=\rho(V^2/r)$.
7. Pressure variation in the vertical plane of a vortex flow: $\partial p/\partial z=-\rho g$.
8. Fundamental equation for vortex flow: $dp=\rho(V^2/r)dr-\rho g dz$.
9. In forced vortex flow, the height of parabola is given by $z=V_2^2/2g=(\omega^2 r_2^2)/2g$.
10. The total pressure force on the top of the completely filled vessel rotating about its axis is given by $F_t=(1/4)\rho\omega^2\pi R^4$.
11. The total pressure force on the bottom of the completely filled vessel rotating about its axis is given by $F_b=(1/4)\rho\omega^2\pi R^4+\rho g\pi R^2 h$.
12. The Bernoulli's equation is applicable to free vortex flow.

Multiple-choice Questions

1. Which of the following is an example of free vortex flow?
   (a) A whirlpool in a river.
   (b) Flow of liquid around a circular bend.
   (c) Flow of liquid through a hole provided at the bottom of a container.
   (d) All the above.
2. The flow of liquid through the hole in the bottom of the washbasin is an example of
   (a) Forced vortex flow.
   (b) Free vortex flow.
   (c) Uniform flow.
   (d) Steady flow.
3. For a forced vortex
   (a) Velocity increases with radius.
   (b) Velocity decreases with radius.
   (c) Fluid rotates as a solid mass.
   (d) None of the above.
4. A tornado has greater destructive force near the centre than at its sides because
   (a) Velocity is very low.
   (b) Pressure is very low.
   (c) Pressure is very high.
   (d) Velocity is very high.
5. In case of forced vortex flow, the height of parabola varies with
   (a) Fourth power of radius.
   (b) Cubic power of radius.
   (c) Square of radius.
   (d) Directly proportional to radius.

Review Questions

1. What do you mean by free and forced vortex? Give some practical examples of it.
2. Discuss the various types of vortex flow.
3. Derive the fundamental equation for vortex flow and also obtain an expression for free surface of a forced vortex.
4. Prove that Bernoulli's equation for an ideal fluid also satisfies free vortex flow.
5. Derive expressions for the total pressure force on the top and bottom of a closed cylindrical vessel completely filled with a liquid when rotated about its vertical axis.
6. Prove that in case of forced vortex, the rise of liquid level at the ends is equal to the fall of liquid level at the axis of rotation.
7. Prove that in case of closed cylindrical vessel, the speed at which water touches the top lid is $(2h_1)/h_2$ times the speed at which the air touches the base, where h_1 is the height of air column in the vessel and h_2 is the height of the vessel.

Problems

1. An open cylindrical vessel 200 cm high and 50 cm in diameter is filled with water up to 150 cm height. Determine the rotational speed in rpm of the vessel about its axis so that the water does not spill out.
   [**Ans.** 169.19 rpm]
2. A right circular cylinder of radius *r* and height *h* is open at the top and completely filled with water. Determine its speed of rotation in rpm about its axis so that half of the circular area at the bottom is exposed.
   [**Ans.** $59.82r^{-1}\sqrt{h}$ rpm]
3. A cylindrical vessel 90 cm in diameter and 2 m high open at top is filled with water to a depth of 1.5 m. Find the value of its rotation in rpm about its axis so that water level is raised to the brim.
   [**Ans.** 93.9 rpm]
4. A cylindrical vessel 18 cm in diameter and 120 cm high open at top is filled with water to a depth of 96 cm. Find the value of its maximum rotational speed in rpm about its vertical axis so that no water spills.
   [**Ans.** 325.5 rpm]
5. A cylindrical vessel 10.4 cm in diameter and 25.5 cm high open at top is completely filled with water when at rest. Some amount of water spills out when the vessel is rotated about its vertical axis at a speed of 300.2 rpm. Determine the depth of water in the vessel when it is brought to rest after rotation.
   [**Ans.** 0.1868 m]
6. A cylindrical vessel 100 cm in diameter is used as a cream separator from the milk. If it contains 850 litres of milk, then determine the smallest height of the vessel so that it can be rotated at 125 rpm about its vertical axis without any spillage of milk over the sides.
   [**Ans.** 2.174 m]
7. A cylindrical vessel 20 cm in diameter and 100 cm high open at top is filled with water to a depth of 60 cm. Determine the maximum speed in rpm with which the vessel can be rotated so that water just touches the centre bottom of the vessel. Also determine the pressure forces difference at the bottom and sides of the vessel.
   [**Ans.** 422.94 rpm, 0.3085 kN, 1.972 kN]
8. A cylindrical vessel closed at both the ends is 14 cm in diameter and 100 cm high contains water up to a depth of 64 cm. Determine the height of paraboloid formed, if it is rotated about its vertical axis at a speed of 240 rpm. Also find the speed of rotation of the vessel, when the axial depth of water is zero.
   [**Ans.** 0.3371 m, 711.9 rpm]
9. A closed vertical cylinder 40 cm in diameter and 40 cm in height is completely filled with oil (specific gravity = 0.8). Determine the thrust of oil on the top and bottom of the cylinder when the cylinder is rotated about its vertical axis at a speed of 200 rpm.
   [**Ans.** 0.441 kN, 0.8355 kN]
10. A cylindrical vessel closed at both ends contains water up to a depth of 40 cm. This vessel is 10 cm in radius and 60 cm in height. Determine the height of paraboloid formed when it is rotated about its vertical axis at a speed of 300 rpm. Also find the rotational speed when the paraboloid just touches the centre of the bottom of the vessel.
    [**Ans.** 0.449 m, 401.52 rpm]
11. A cylindrical vessel closed at both ends contains water up to its full depth. This vessel is 30 cm in radius and 50 cm in height. Determine the total pressure of water on each end if it is rotated about its vertical axis at a speed of 200 rpm.
    [**Ans.** 2.789 kN, 4.176 kN]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (d) **2.** (b) **3.** (a) **4.** (d) **5.** (c)

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Chapter **9**

Potential Flow (Ideal Fluid Flow)

9.1 ❒ INTRODUCTION

A flow of a fluid which has no viscosity (i.e., $\mu = 0$) and is incompressible (i.e., ρ = constant) is called potential flow or ideal fluid flow. There is no fluid in nature which behaves as ideal fluid, but water and air may be considered as ideal fluids under certain conditions. Therefore, potential flow is an approximation of flow where density is assumed to be constant and the viscosity effects are negligible. In the absence of viscosity, there cannot be any shear stresses in an ideal flow. It means the only stress at a point in an ideal fluid flow must be pressure and the only stress the fluid may act on the solid boundary must be normal to it at that point. The assumption of zero viscosity simplifies the mathematical equations of fluid motion significantly. In a flow field, both potential function (ϕ) and stream function (ψ) exist. Thus, any flow can be expressed in the form of these two mathematical tools. In potential flow, the flow field can be represented by a potential function ϕ such that it satisfies the Laplace equation. The derivatives of ϕ give the velocities for two-dimensional flows. The potential function exists only if the flow is irrotational, i.e., the viscous effects are absent.

Due to the complex nature of fluid flow, an exact analysis of flow is very difficult and often not required for engineering estimation. Even with computational techniques, it is necessary to proceed by iteration, starting with a good first guess. The first approximation with potential flow analysis is often adequate. Thus, the simple technique of potential flow analysis has considerable importance in the solution of engineering problems.

In this chapter, the important cases of potential flow, namely uniform flow, source flow, sink flow, free vortex flow and superimposed flow are described with the help of ϕ and ψ. The superposition of flows finds many applications in aerodynamics. The analysis of potential flow is restricted to steady, incompressible and two-dimensional flow mostly in the horizontal plane. There are two variables x and y having components of velocity u and v, respectively in the Cartesian coordinates and r and α in the polar coordinates.

9.2 ❒ UNIFORM FLOW

A uniform flow is also termed as free stream flow in which the flow velocity remains constant (or uniform) at any cross section. Figure 9.1(a) and Figure 9.1(b) illustrate the uniform flow parallel to x-axis and y-axis, respectively for which the velocity U remains constant. The velocity U has velocity components u and v along x and y-axis, respectively. Figure 9.1(c) illustrates a uniform flow in which the uniform velocity U is inclined at an angle α to the x-axis. The velocity components in x and y-directions are given by $u = U\cos\alpha$ and $v = U\sin\alpha$.

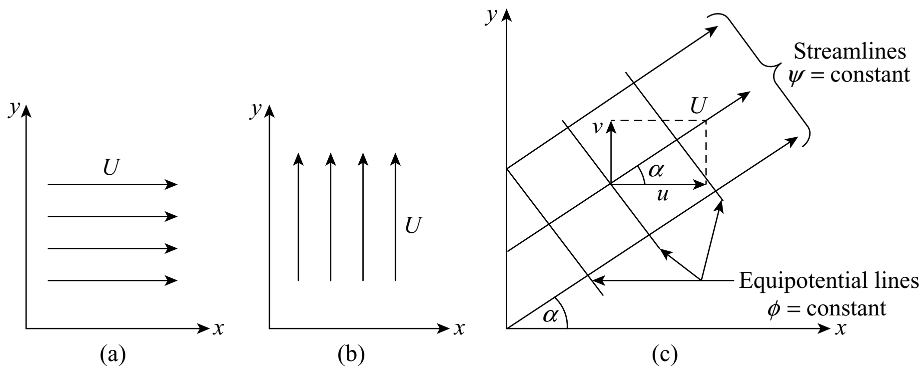


Figure 9.1 *Uniform flow*

Since ψ is a function of x and y, its total differential is given by,

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy$$

Also $\quad \frac{\partial \psi}{\partial x} = -v$ and $\frac{\partial \psi}{\partial y} = u \quad$ [Equation 6.50]

Thus
$$d\psi = -vdx + udy \tag{9.1}$$

Integrating Equation (9.1), we get:

$$\psi = -vx + uy + k_1 \tag{9.2}$$

Here, k_1 is constant of integration. Thus, Equation (9.2) is a family of streamlines (Figure 9.1(c)).

Since ϕ is a function of x and y, its total differential is given by,

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$

Also $\quad u = \frac{\partial \phi}{\partial x}$ and $v = \frac{\partial \phi}{\partial y} \quad$ [Equation 6.47]

Thus
$$d\phi = udx + vdy \tag{9.3}$$

Integrating Equation (9.3), we get:

$$\phi = ux + vy + k_2 \tag{9.4}$$

Here, k_2 is the constant of integration. Equation (9.4) represents a family of potential lines which are perpendicular to the streamlines (due to orthogonality) as shown in Figure 9.1(c).

Case I: Uniform flow parallel to *x*-axis

For uniform flow parallel to the x-axis, the streamlines will be parallel to the x-axis, i.e., $\alpha = 0°$. Thus, the velocity components become $u = U\cos 0° = U$ and $v = U\sin 0° = 0$. Therefore, the Equation (9.2) and (9.4) is written as follows.

$$\psi = Uy + k_1 \tag{9.5}$$

$$\phi = Ux + k_2 \tag{9.6}$$

For determining the constants k_1 and k_2, applying the boundary conditions, we get:

1. At $y = 0$, $\psi = 0$ and thus, $k_1 = 0$.
2. At $x = 0$, $\phi = 0$ and thus $k_2 = 0$.

Therefore, the equations for streamlines and potential lines are respectively given below.

$$\psi = Uy \tag{9.7}$$

$$\phi = Ux \tag{9.8}$$

Equation (9.7) is an equation of streamline parallel to x-axis and at a distance y from the x-axis. In this equation, Uy represents the volume flow rate (thickness of fluid stream is assumed unity) between that streamline and the x-axis, where $\psi = 0$. Thus, the streamlines having the values of ψ_0, ψ_1, ψ_2, ψ_3 and ψ_4 corresponding to $y = 0$, $y = 1$, $y = 2$, $y = 3$ and $y = 4$, respectively, can be plotted as shown in Figure 9.2(a).

Equation (9.8) is an equation of equipotential line (potential line) parallel to y-axis and at a distance x from y-axis. Thus, the potential lines having the values of ϕ_0, ϕ_1, ϕ_2, ϕ_3 and ϕ_4 corresponding to $x = 0$, $x = 1$, $x = 2$, $x = 3$ and $x = 4$, respectively, can be plotted as shown in Figure 9.2(b).

Figure 9.2(c) illustrates the plot of streamlines and equipotential lines for uniform flow parallel to x-axis. The streamlines and potential lines intersect each other at right angles.

The constant of integration do not affect the flow pattern and thus, it will not be considered in the subsequent expressions.

Case II: Uniform flow parallel to *y*-axis

For uniform flow parallel to the y-axis, the streamlines will be parallel to the y-axis, i.e., $\alpha = 90°$. Thus, the velocity components become $u = U\cos 90° = 0$ and $v = U\sin 90° = U$. Therefore, the similar line equations for streamlines and potential lines are respectively given as follows.

$$\psi = -Ux \tag{9.9}$$

$$\phi = Uy \tag{9.10}$$

Thus, the streamlines are parallel to y-axis and equipotential lines are parallel to x-axis. Equation (9.9) has negative sign that indicates that streamlines are in the downward direction. Therefore, the streamlines having the values of ψ_0, ψ_1, ψ_2, ψ_3 and ψ_4 corresponding to $x = 0$, $x = 1$, $x = 2$, $x = 3$ and $x = 4$, respectively, can be plotted as shown in Figure 9.3(a).

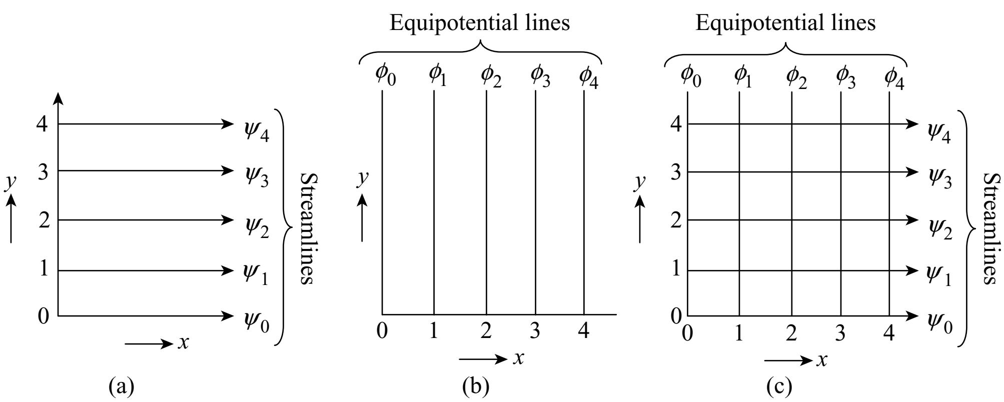


Figure 9.2 *Uniform flow parallel to x-axis*

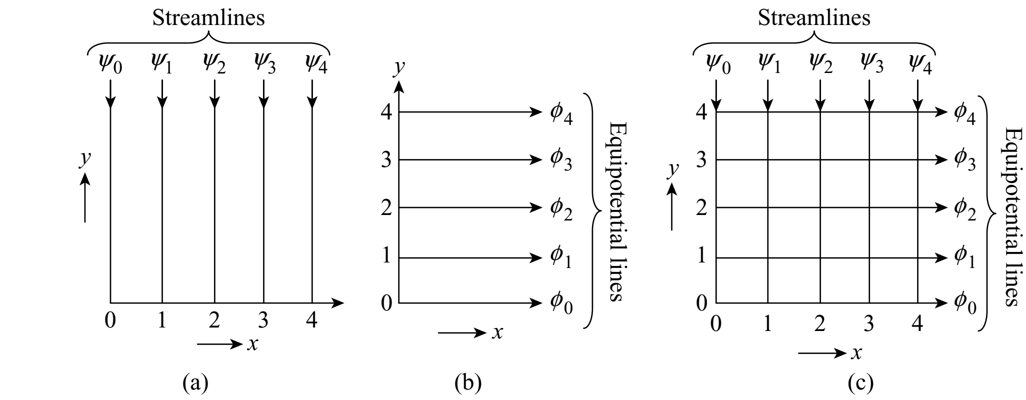


Figure 9.3 *Uniform flow parallel to y-axis*

Similarly, the potential lines having the values of ϕ_0, ϕ_1, ϕ_2, ϕ_3 and ϕ_4 corresponding to $y = 0$, $y = 1$, $y = 2$, $y = 3$ and $y = 4$, respectively, can be plotted as shown in Figure 9.3(b).

Figure 9.3(c) illustrates the plot of streamlines and equipotential lines for uniform flow parallel to *y*-axis. The streamlines and potential lines intersect each other at right angles.

Case III: Uniform flow inclined to *x*-axis

For uniform flow inclined at an angle α to the *x*-axis (Figure 9.1(c)), the streamline and equipotential lines are respectively given as follows.

$$\psi = -vx + uy \tag{9.11}$$

$$\phi = ux + vy \tag{9.12}$$

9.3 ❒ SOURCE FLOW

A source flow can be defined as the flow coming out from a single point and moving radially out in all directions at a constant rate. Figure 9.4(a) shows a practical source flow in which the flow originates from a small hole in one of the two flat parallel plates, a unit distance apart with the fluid particles move radially outwards between the plates.

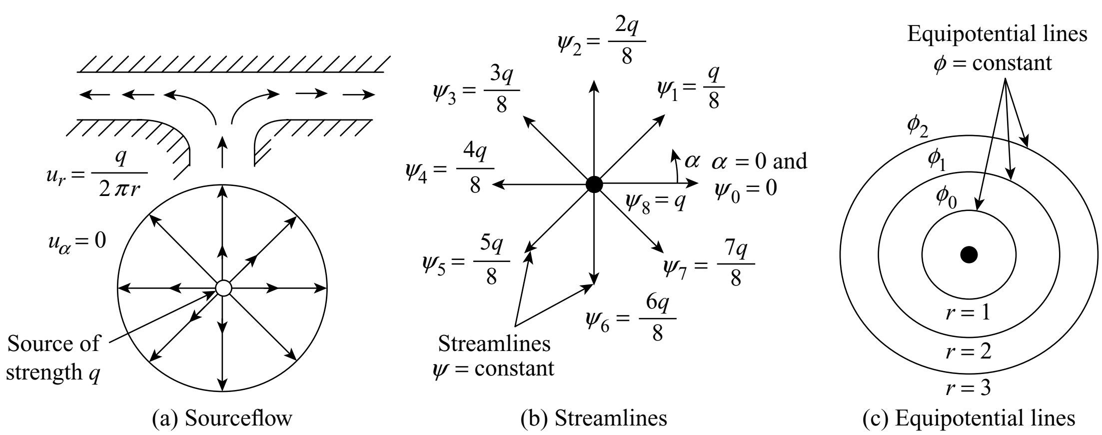


Figure 9.4

Let q be the strength of the source which can be defined as the volume flow rate per unit depth (m^2/s), r be any radius, u_r be the radial velocity component and u_α be the tangential velocity component.

The path of the fluid particle is purely radial and thus, $u_\alpha = 0$. The expression for magnitude of radial velocity at any radius is given below

$$\boxed{u_r = \frac{q}{2\pi r}} \tag{9.13}$$

From Equation (9.13), it can be seen that the radial velocity decreases with the increase of r and at a large distance away from the source it will be nearly equal to zero.

Also $$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \alpha} \qquad \text{[Equation 6.50(a)]}$$

Thus $$\frac{1}{r}\frac{\partial \psi}{\partial \alpha} = \frac{q}{2\pi r}$$

$$\frac{\partial \psi}{\partial \alpha} = \frac{q}{2\pi}$$

Integrating the above equation, we get:

$$\psi = \frac{q\alpha}{2\pi} + k_1$$

If assumed that at $\alpha = 0$, $\psi = 0$, then $k_1 = 0$ and therefore, we get the below expression.

$$\psi = \frac{q\alpha}{2\pi} \tag{9.14}$$

When $\alpha = 2\pi$, $\psi = \frac{q}{2\pi} 2\pi = q$

From Equation (9.13), we have $q = 2\pi r u_r$ and thus, Equation (9.14) is rewritten as follows.

$$\psi = \frac{q\alpha}{2\pi} = \frac{2\pi r u_r \times \alpha}{2\pi} = r u_r \times \alpha = C\alpha$$

Here, C is a constant. Thus, ψ is a function of α and the above expression indicates that the streamlines are radial lines. The streamlines can be plotted by taking different values of α in radians varying from 0 to 2π in Equation (9.14). The different streamlines ψ_0, ψ_1, ψ_2, ψ_3, ψ_4, ψ_5, ψ_6, ψ_7 and ψ_8 shown in Figure 9.4(b) have been plotted by taking $\alpha = 0$, $\alpha = \pi/4$, $\alpha = 2\pi/4$, $\alpha = 3\pi/4$, $\alpha = 4\pi/4$, $\alpha = 5\pi/4$, $\alpha = 6\pi/4$, $\alpha = 7\pi/4$ and $\alpha = 2\pi$, respectively.

In polar coordinates, the velocity component in terms of potential function is given by Equation 6.47(b) as follows.

$$u_r = \frac{\partial \phi}{\partial r}$$

Using Equation (9.13), we get:

$$\frac{\partial \phi}{\partial r} = \frac{q}{2\pi r}$$

Integrating the above equation, we get:

$$\phi = \frac{q}{2\pi} \ln r + k_2$$

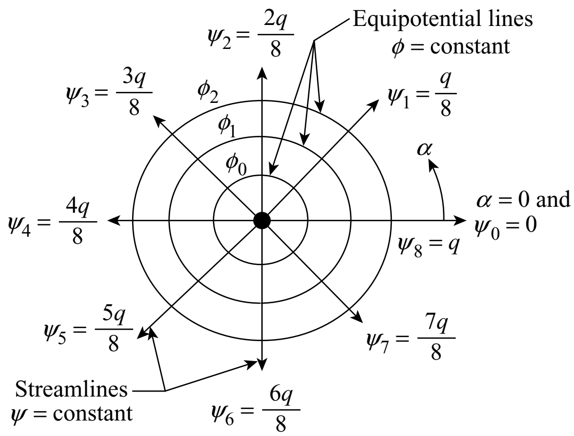


Figure 9.5 *Streamlines and equipotential lines for a source flow*

If assumed that at $r = 1, \phi = 0$, then $k_2 = 0$, therefore, we get:

$$\phi = \frac{q}{2\pi} \ln r \tag{9.14a}$$

or

$$r = e^{2\pi\phi/q}$$

Also

$$\phi = \frac{2\pi r u_r}{2\pi} \times \ln r = r u_r \times \ln r = C \ln r$$

The above expression shows that ϕ is a function of r and the equipotential lines are concentric circles with the centre as origin of the source. Thus, the equipotential lines can be plotted by drawing concentric circles with centre at the origin of the source and radius $r = e^{2\pi\phi/q}$. The different equipotential lines ϕ_0, ϕ_1 and ϕ_2 have been plotted by taking the values of r as 1, 2 and 3, respectively, as shown in Figure 9.4(c).

Figure 9.5 shows the streamlines and equipotential lines for a source flow.

Expression for pressure distribution For deriving an expression for pressure distribution in a source flow considering two points 1 and 2. Let u_r be the velocity and p be the pressure at the point 1 which is at radius r and the corresponding values at the other point (i.e., point 2) which is at a large distance away from the source are $u_r = 0$ and p_∞. Applying Bernoulli's equation, we get the below expression.

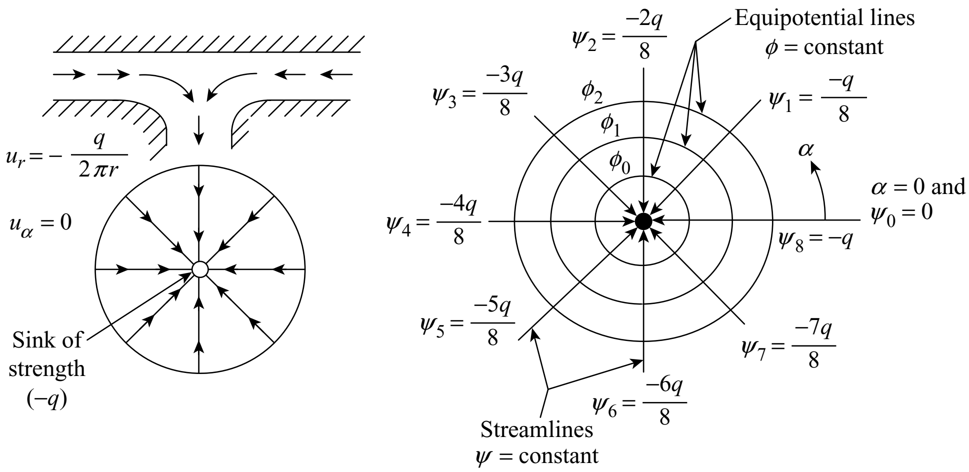
$$\frac{p}{\rho g} + \frac{u_r^2}{2g} = \frac{p_\infty}{\rho g} + \frac{0^2}{2g} \quad [\because z_1 = z_2]$$

$$p - p_\infty = -\frac{\rho u_r^2}{2} = -\frac{\rho}{2} \times \left(\frac{q}{2\pi r}\right)^2 = -\frac{\rho q^2}{8\pi^2 r^2} \tag{9.15}$$

Thus, the pressure increases inversely as the square of the radius from the source.

9.4 ❐ SINK FLOW

A sink flow can be defined as the flow moving radially inwards in a plane towards a point where it disappears at a constant rate. Sink flow is just opposite to the source flow. A practical sink flow between two parallel plates with fluid particles flowing towards the central hole on one of the plates is illustrated in Figure 9.6(a).



(a) Sink flow (b) Streamlines and equipotential lines

Figure 9.6

The pattern of streamlines and equipotential lines of a sink flow are the same as that of a source flow. The strength of a sink flow is taken negative $(-q)$. Thus, the equations for ψ and ϕ can be derived by replacing q in source flow with $-q$. Therefore, for a plane sink flow, we get the below expression.

$$\boxed{u_r = -\frac{q}{2\pi r}} \tag{9.16}$$

The stream function and potential function for a sink flow would be as follows.

$$\psi = -\frac{q\alpha}{2\pi}, \ \phi = -\frac{q}{2\pi}\ln r \text{ and } r = e^{-2\pi\phi/q}$$

Figure 9.6(b) shows the streamlines and equipotential lines for a sink flow. The pressure variation in a plane sink flow is given by the same expression as in a plane source flow given by Equation (9.15).

9.5 ❒ FREE VORTEX FLOW

A free vortex flow is a purely circulatory flow such that the centre of the vortex is singular point with a circulation Γ. Generally, the circulation is taken positive in an anticlockwise direction. In a plane vortex flow, the fluid particles move in concentric circles. While moving round, if the fluid particles do not rotate about their own axis, the flow is known as irrotational free vortex flow. A free vortex is an irrotational flow at all points except at the singular point.

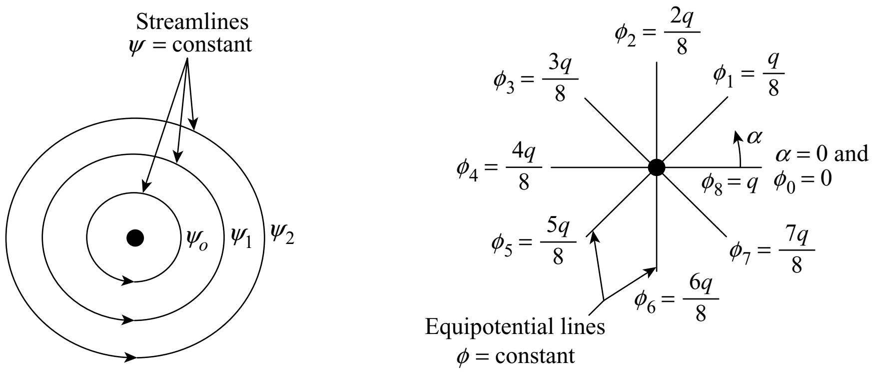
For a free vortex flow (purely circulatory flow), we get the below expression.

$$u_\alpha r = C \tag{i}$$

Here, C is the vortex strength and it is defined as $C = \Gamma/(2\pi)$ and u_α is the tangential velocity at any radius r from the centre. Thus, expression (i) can be rewritten as follows.

$$u_\alpha r = \frac{\Gamma}{2\pi}$$

$$\boxed{\therefore \Gamma = 2\pi r u_\alpha} \tag{9.17}$$



(a) Streamlines (b) Equipotential lines

Figure 9.7

For a free vortex flow, the tangential velocity component and radial velocity component are respectively given as $u_\alpha = \Gamma/2\pi r$ and $u_r = 0$.

Therefore, expression (i) can be written as follows.

$$u_\alpha = \frac{C}{r} \tag{ii}$$

Substituting the value of u_α from Equation (6.50(a)) in expression (ii), we get:

$$-\frac{\partial \psi}{\partial r} = \frac{C}{r}$$

Integrating the above expression, we get:

$$\psi = -C \ln r = -\frac{\Gamma}{2\pi} \ln r \tag{9.18}$$

From Equation (9.18), it can be seen that the stream function is a function of radius (r) and it would be a constant for a given value of r. Thus, streamlines are constant radius lines which are concentric circles with radius $r = e^{-2\pi\psi/\Gamma}$ as shown in Figure 9.7(a).

Now substituting the value of u_α from Equation (6.47(b)) in expression (ii), we get the following expression.

$$\frac{1}{r}\frac{\partial \phi}{\partial \alpha} = \frac{C}{r}$$

$$\frac{\partial \phi}{\partial \alpha} = C$$

Integrating the above expression, we get:

$$\phi = C\alpha = \frac{\Gamma}{2\pi}\alpha \tag{9.19}$$

From Equation (9.19), it can be seen that the velocity potential function is a function of angular displacement (α) and it would be a constant for a given value of α. Thus, equipotential lines are radial lines drawn from the centre as shown in Figure 9.7(b). The pattern of streamlines and equipotential lines are similar to that for the source except that ψ and ϕ lines are interchanged as illustrated in Figure 9.8.

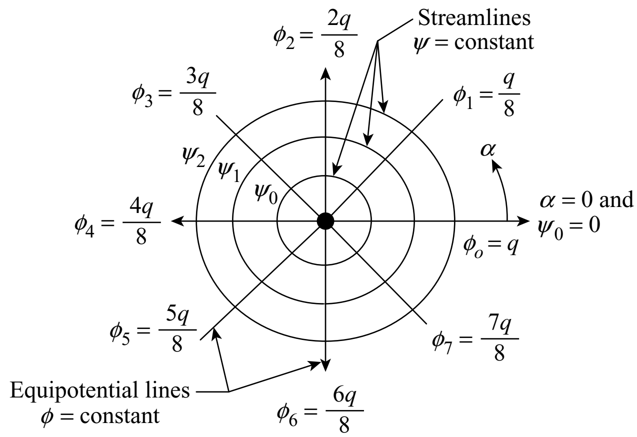


Figure 9.8 *Streamlines and equipotential lines for a free vortex flow*

Example 9.1 A source flow between two flat parallel plates originates from a small hole of diameter 0.05 m in the lower plate, 1.5 m distance apart with the water particles move radially outwards between the plates. If water enters the space between the plates at a rate of 1.2 m^3/s, then determine (i) the strength of the source and velocity at a radius of 0.4 m from the centre of the plate, (ii) pressure at the given location if water pressure at the inlet is 225 kPa and (iii) stream function for the streamlines at 30° and 90° from the streamline with $\psi_o = 0$.

Solution

Let $d_1 = 0.05$ m, $b = 1.5$ m, $Q = 1.2 \text{ m}^3/\text{s}$, $r_2 = 0.4$ m, $p_1 = 225$ kPa, $\alpha_1 = 30°$ and $\alpha_2 = 90°$.

$$r_1 = \frac{d_1}{2} = \frac{0.05}{2} = 0.025 \text{ m}$$

(i) $q = \frac{Q}{b} = \frac{1.2}{1.5} = 0.8 \text{ m}^2/\text{s}$

(ii) $(u_r)_1 = \frac{q}{2\pi r_1} = \frac{0.8}{2\pi \times 0.025} = 5.093$ m/s

$$(u_r)_2 = \frac{q}{2\pi r_2} = \frac{0.8}{2\pi \times 0.4} = 0.3183 \text{ m/s}$$

$$\frac{p_1}{\rho_w g} + \frac{(u_r)_1^2}{2g} + z_1 = \frac{p_2}{\rho_w g} + \frac{(u_r)_2^2}{2g} + z_2 \quad \text{[Bernoulli's equation]}$$

$$\frac{225 \times 10^3}{1000 \times 9.81} + \frac{5.093^2}{2 \times 9.81} = \frac{p_2}{1000 \times 9.81} + \frac{0.3183^2}{2 \times 9.81} \quad [\because z_1 = z_2]$$

$$24.25783 = \frac{p_2}{9810} + 0.005164$$

$$\therefore p_2 = \frac{(24.25783 - 0.005164) \times 9810}{10^3} = \mathbf{237.919\ kPa}$$

(iii) Let ψ_1 and ψ_2 be the stream functions at $\alpha_1 = 30°$ and $\alpha_2 = 90°$, respectively.

$$\psi_1 = \frac{q\alpha_1}{2\pi} = \frac{0.8}{2\pi} \times \frac{30° \times \pi}{180} = \mathbf{0.0667\ m^2/s}$$

$$\psi_2 = \frac{q\alpha_2}{2\pi} = \frac{0.8}{2\pi} \times \frac{90° \times \pi}{180} = \mathbf{0.2\ m^2/s}$$

Example 9.2 If a two-dimensional seawater (specific gravity = 1.02) motion is given by the stream function $\psi = \ln r$, then find the radial pressure gradient at radius $r = 1.2$ m.

Solution

Let $S_{\text{seawater}} = 1.02$, $\psi = \ln r$ and $r = 1.2$ m.

$$\rho = S_{\text{seawater}}\,\rho_w = 1.02 \times 1000 = 1020\ \text{kg/m}^3$$

Since $$\psi = \ln r = -\frac{\Gamma}{2\pi}\ln r \quad \text{[Using Equation (9.18)]}$$

or $$\ln r = -\frac{2\pi r u_\alpha}{2\pi}\ln r$$

Thus $$r u_\alpha = -1 \Rightarrow u_\alpha = -\frac{1}{r}$$

The radial pressure gradient for a vortex motion is given by,

$$\frac{dp}{dr} = \frac{\rho u_\alpha{}^2}{r} = \frac{\rho(-1/r)^2}{r} = \frac{\rho}{r^3} = \frac{1020}{1.2^3} = \mathbf{590.28\ N/m^3}$$

9.6 ❐ SUPERIMPOSED FLOW

For an irrotational flow, both the stream function and the velocity potential function satisfy the Laplace equation, which is a linear partial differential equation. The linearity of the operator implies that if various basic flows are combined together, then the velocity potentials and stream functions can be combined to form new potentials and stream functions. Thus, the principle of superposition helps in the study of complex flows by assuming as if it is made from the superposition of the basic flows for which the potential and stream functions are known. The pressure distribution can be obtained by applying Bernoulli's equation. Therefore, the flow pattern due to a uniform flow, a source flow, a sink flow and a free vortex flow can be superimposed in any linear combination to obtain a resultant flow which closely resembles the flow around bodies. The resultant flow will also be potential flow. In the following sections, some potential flow solutions by superposition are described.

9.6.1 Source and Uniform Flow (Flow Past a Half Body)

Consider a uniform flow of velocity U parallel to the *x*-axis (Figure 9.9(a)) and a source flow of strength q placed at the origin '*O*' (Figure 9.9(b)).

The resulting flow pattern obtained from the combination of a uniform flow and a source flow is shown in Figure 9.10 and it is known as flow past a half body. Let a point $P(x, y)$ lie in the resultant flow field with polar coordinates r and α.

The stream function ψ and the velocity function ϕ for the combined flow is given by,

$$\psi = \psi_{\text{uniform}} + \psi_{\text{source}} = Uy + \frac{q\alpha}{2\pi} = Ur\sin\alpha + \frac{q\alpha}{2\pi} \tag{9.20}$$

$$\phi = \phi_{\text{uniform}} + \phi_{\text{source}} = Ux + \frac{q}{2\pi}\ln r = Ur\cos\alpha + \frac{q}{2\pi}\ln r \tag{9.21}$$

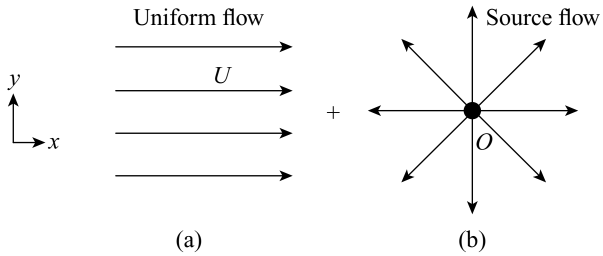


Figure 9.9 *(a) Uniform flow (b) Source flow*

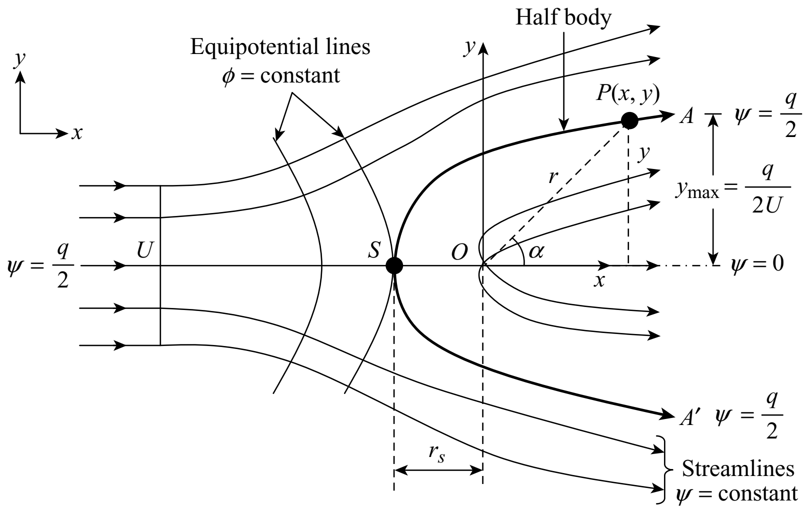


Figure 9.10 *Flow past a half body*

The radial velocity from the source decreases with increase in radial distance from it. The velocity of uniform flow and that due to source are equal and opposite to each other at a distance on the left side of the source from the origin *O* lying on the *x*-axis. Thus, the resultant velocity will be zero at a point known as stagnation point which is denoted by *S*. At the stagnation point (*S*), the radial distance becomes $r = r_s$ and $\alpha = \pi$.

At the stagnation point, the equation for the streamline passing through *S* can be given by substituting $\alpha = \pi$ and $r = r_s$ in Equation (9.20) as follows.

$$\psi_s = Ur_s \sin\pi + \frac{q\pi}{2\pi} = \frac{q}{2} \tag{9.22}$$

The flow particles issued from the source cannot go further to the left of the stagnation point *S*. These are carried along the contour ASA' which separates the source flow from the uniform flow. The curve ASA' can be considered as a solid boundary of a round nosed body, such as an island or bridge pier around which the uniform flow is forced to pass. The contour ASA' is known as the Rankine half body (or plane half body), because it has only the leading point and it trails to infinity at the downstream end. A body is called half body if its surface is not closed at one end, for example, a bullet, spear, etc.

Since no fluid mass crosses a streamline, a streamline is a virtual solid surface. Thus, the composite flow consists of flow over a plane half body (ASA') outside $\psi = q/2$ and the source flow within the plane half body. Therefore, the equation for dividing streamline (Equation of curve ASA') may be obtained by substituting $\psi = q/2$ in Equation (9.20) as follows.

$$Uy + \frac{q\alpha}{2\pi} = \frac{q}{2}$$

$$y = \frac{1}{U}\left[\frac{q}{2} - \frac{q\alpha}{2\pi}\right] = \frac{q[1-(\alpha/\pi)]}{2U} = \frac{q(\pi-\alpha)}{2\pi U} \tag{9.23}$$

Also $$r = \frac{y}{\sin\alpha} = \frac{q(\pi-\alpha)}{2\pi U \sin\alpha}$$

From Equation (9.23), it can be observed that y is maximum when $\alpha = 0$. The principal dimensions of the half body may be obtained from Equation (9.23) as follows.

1. At $\alpha = 0$: $y_{max} = \dfrac{q(\pi-0)}{2\pi U} = \dfrac{q}{2U}$ (Maximum positive ordinate) (9.23a)
2. At $\alpha = \dfrac{\pi}{2}$: $y = \dfrac{q[\pi-(\pi/2)]}{2\pi U} = \dfrac{q}{4U}$ (Upper ordinate at the origin) (9.23b)
3. At $\alpha = \pi$: $y = \dfrac{q(\pi-\pi)}{2\pi U} = 0$ (The leading point) (9.23c)
4. At $\alpha = \dfrac{3\pi}{2}$: $y = \dfrac{q[\pi-(3\pi/2)]}{2\pi U} = -\dfrac{q}{4U}$ (Lower ordinate at the origin) (9.23d)
5. At $\alpha = 2\pi$: $y_{max} = \dfrac{q(\pi-2\pi)}{2\pi U} = -\dfrac{q}{2U}$ (Maximum negative ordinate) (9.23e)

The width of the half body at any point can be given as $b = 2y$.

The velocity at any point P in the flow field is given by,

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\alpha} = \frac{1}{r}\frac{\partial}{\partial\alpha}\left[Ur\sin\alpha + \frac{q\alpha}{2\pi}\right] = U\cos\alpha + \frac{q}{2\pi r} \tag{9.24}$$

$$u_\alpha = -\frac{\partial\psi}{\partial r} = -\frac{\partial}{\partial r}\left[Ur\sin\alpha + \frac{q\alpha}{2\pi}\right] = -U\sin\alpha \tag{9.25}$$

The resultant velocity can be obtained by,

$$V = \sqrt{u_r^2 + u_\alpha^2} \tag{9.26}$$

For determining the point of maximum velocity, put the differential of resultant velocity with respect to α equal to zero, consequently, we obtain $\alpha = \pm 63°$ and the maximum velocity would be equal to $1.25U$.

At the stagnation point, $\alpha = \pi$, $r = r_s$, $u_r = 0$ and $u_\alpha = 0$. Thus, Equation (9.24) can be rewritten as follows.

$$U\cos\pi + \frac{q}{2\pi r_s} = 0 \Rightarrow U = \frac{q}{2\pi r_s}$$

$$\boxed{\therefore r_s = \frac{q}{2\pi U}} \tag{9.27}$$

The position of stagnation point S can be obtained by equating the radial and tangential velocity components given by Equations (9.24) and (9.25), respectively to zero as follows.

$$U\cos\alpha + \frac{q}{2\pi r} = 0 \Rightarrow r\cos\alpha = -\frac{q}{2\pi U}$$

Thus $$x = -\frac{q}{2\pi U} \quad [\because r\cos\alpha = x] \tag{9.27a}$$

Now $$u_\alpha = -U\sin\alpha = 0 \text{ or } \sin\alpha = 0 \text{ or } \alpha = 0, \pi \quad [\because U \neq 0]$$

Thus $$y = r\sin\alpha = r\times 0 = 0 \tag{9.27b}$$

Therefore, the stagnation point $S\{-q/(2\pi U), 0\}$ is the leading point of the Rankine half body.

Let p_∞ be the pressure at infinity, where the velocity is U and p be the pressure at any point P in the flow field, where the velocity is V. Applying Bernoulli's equation, we have the following expressions.

$$\frac{p}{\rho g}+\frac{V^2}{2g}=\frac{p_\infty}{\rho g}+\frac{U^2}{2g} \quad [\because z_1 = z_2]$$

$$\frac{p}{\rho g}-\frac{p_\infty}{\rho g}=\frac{U^2}{2g}-\frac{V^2}{2g}$$

$$p-p_\infty=\frac{\rho}{2}(U^2-V^2)$$

The non-dimensional pressure coefficient (C_p) is given by,

$$C_p=\frac{p-p_\infty}{(1/2)\rho U^2}=\frac{(\rho/2)(U^2-V^2)}{(1/2)\rho U^2}=1-\left(\frac{V}{U}\right)^2 \tag{9.28}$$

Example 9.3 Flow over a plane half body is studied by superimposing a uniform flow at 6 m/s on a source at the origin. If a body has a maximum width of 2.4 m, then determine (i) the coordinates of the stagnation point, (ii) width of the body at the origin and (iii) velocity at a point $(0.5, \pi/2)$.

Solution

Let $U = 6$ m/s, $2y_{max} = 2.4$ m or $y_{max} = 1.2$ m, $r = 0.5$ m and $\alpha = \pi/2$.

Let (x, y) be the coordinates at the stagnation point, b be the width of the body at the origin and V be the resultant velocity at the given point.

Since $$y_{max}=\frac{q}{2U}$$

$$\therefore q = 2Uy_{max} = 2\times 6\times 1.2 = 14.4 \text{ m}^2/\text{s}$$

(i) For the stagnation point using Equations 9.27(a) and 9.27(b), we get:

$$x=-\frac{q}{2\pi U}=-\frac{14.4}{2\pi\times 6}=\mathbf{-0.382\ m}$$

$$y=\mathbf{0}$$

(ii) The value of upper coordinate at the origin (i.e., at $\alpha = \pi/2$) is given by,

$$y=\frac{q}{4U}=\frac{14.4}{4\times 6}=0.6 \text{ m}$$

$$b = 2y = 2\times 0.6 = \mathbf{1.2\ m}$$

(iii) $$u_r = U\cos\alpha+\frac{q}{2\pi r}=6\cos\frac{\pi}{2}+\frac{14.4}{2\pi\times 0.5}=4.584 \text{ m/s}$$

$$u_\alpha = -U\sin\alpha = -6\sin\frac{\pi}{2} = -6 \text{ m/s}$$

$$V=\sqrt{u_r{}^2+u_\alpha{}^2}=\sqrt{4.584^2+(-6)^2}=\mathbf{7.551\ m/s}$$

Example 9.4 Obtain the equation of the dividing streamline for the flow resulting from a superposition of a uniform flow at 10 m/s on a two-dimensional source with a strength of 10 m²/s. Also sketch the flow pattern.

Solution

Let $U = 10$ m/s and $q = 10\ \text{m}^2/\text{s}$.

For source and uniform flow, we get:

$$\psi = Uy + \frac{q\alpha}{2\pi} = 10y + \frac{10\alpha}{2\pi}$$

But
$$\psi = \frac{q}{2} = \frac{10}{2} = 5\ \text{m}^2/\text{s} \quad \text{(Dividing streamline)}$$

Thus
$$5 = 10y + \frac{10\alpha}{2\pi}$$

$$\therefore\ y = \mathbf{0.5} - \frac{\boldsymbol{\alpha}}{\mathbf{2\pi}}$$

The values of y for different values of α from the above equation can be given by,

(i) At $\alpha = 0$, $y = 0.5$ m (Maximum positive ordinate)

(ii) At $\alpha = \frac{\pi}{2}$, $y = 0.25$ m (Upper ordinate at the origin)

(iii) At $\alpha = \pi$, $y = 0$ (The leading point)

(iv) At $\alpha = \frac{3\pi}{2}$, $y = -0.25$ m (Lower ordinate at the origin)

(v) At $\alpha = 2\pi$, $y = -0.5$ m (Maximum negative ordinate)

The horizontal distance of the stagnation point (*S*) from the source towards left is given by,

$$r_s = \frac{q}{2\pi U} = \frac{10}{2\pi \times 10} = \mathbf{0.16\ m}$$

The flow pattern is illustrated in Figure 9.11.

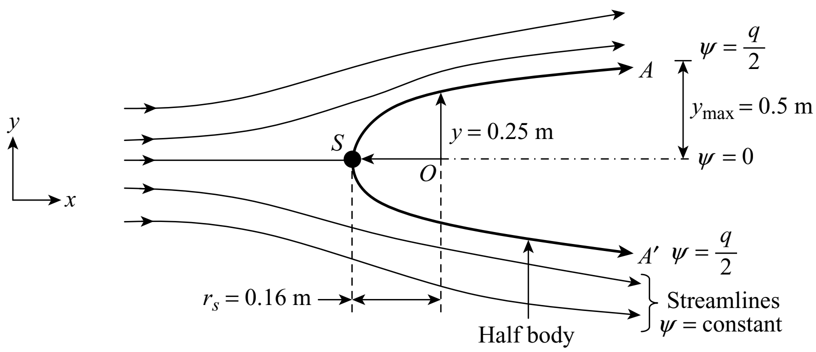


Figure 9.11

Example 9.5 The flow over a plane half body is studied by superimposing a uniform flow at 5 m/s in a source at the origin. If the stagnation point occurs at (–0.5, 0), then determine (i) the strength of the source, (ii) maximum width of the Rankine body and (iii) width of the Rankine body at the source.

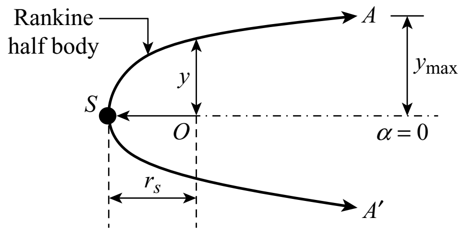


Figure 9.12

Solution

Refer Figure 9.12. Let $U = 5$ m/s and coordinate of stagnation point = (–0.5, 0). This means $r_s = 0.5$ m and stagnation point lies on the x-axis at a distance of 0.5 m towards left of the origin of source.

(i) $\because r_s = \dfrac{q}{2\pi U}$

$$\therefore q = 2\pi U r_s = 2\pi \times 5 \times 0.5 = \mathbf{15.71\ m^2/s}$$

(ii) $y_{max} = \dfrac{q}{2U} = \dfrac{15.71}{2 \times 5} = 1.571$ m

Thus, maximum width of the Rankine body is given by,

$$b = 2y_{max} = 2 \times 1.571 = \mathbf{3.142\ m}$$

(iii) The value of upper ordinate at the origin where source is placed is given by,

$$y = \frac{q}{4U} = \frac{15.71}{4 \times 5} = 0.7855 \text{ m}$$

Thus, the width of the Rankine body at the source is given by,

$$b = 2y = 2 \times 0.7855 = \mathbf{1.571\ m}$$

Example 9.6 A free stream flow with a velocity of 2.4 m/s is flowing over a plane source of strength 16 m^2/s. The free stream flow and the source flow are in the same plane. If a point A is located in the flow field at a distance of 0.5 m and at an angle of 30° to the free stream flow, then determine (i) the stream function at point A, (ii) the resultant velocity of flow at A, (iii) the location of stagnation point from the source and (iv) maximum width of the plane half body.

Solution

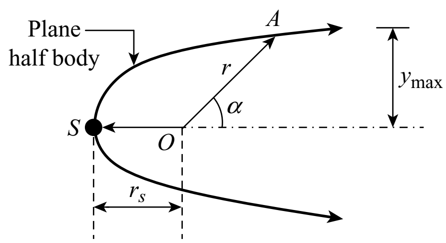


Figure 9.13

Refer Figure 9.13. Let $U = 2.4$ m/s, $q = 16$ m^2/s, $r = 0.5$ m and $\alpha = 30°$.

(i) $\psi = Ur\sin\alpha + \dfrac{q\alpha}{2\pi} = 2.4 \times 0.5 \sin 30° + \dfrac{16}{2\pi} \times \left(30° \times \dfrac{\pi}{180}\right) = \mathbf{1.933\ m^2/s}$

(ii) $u_r = U\cos\alpha + \dfrac{q}{2\pi r} = 2.4\cos 30° + \dfrac{16}{2\pi \times 0.5} = 7.17$ m/s

$$u_\alpha = -U\sin\alpha = -2.4\sin 30° = -1.2 \text{ m/s}$$

$$V = \sqrt{{u_r}^2 + {u_\alpha}^2} = \sqrt{7.17^2 + (-1.2)^2} = \mathbf{7.27\ m/s}$$

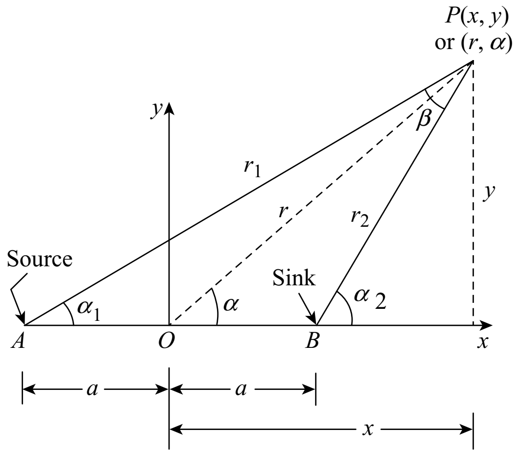
(iii) $r_s = \dfrac{q}{2\pi U} = \dfrac{16}{2\pi \times 2.4} = \mathbf{1.06\ m}$

(iv) $y_{max} = \dfrac{q}{2U} = \dfrac{16}{2 \times 2.4} = 3.333$ m

$$\therefore b = 2y_{max} = 2 \times 3.333 = \mathbf{6.666\ m}$$

9.6.2 Source and Sink Pair

Consider a source and a sink of equal strength placed symmetrically on the x-axis at equal distance a from the origin as shown in Figure 9.14(a).



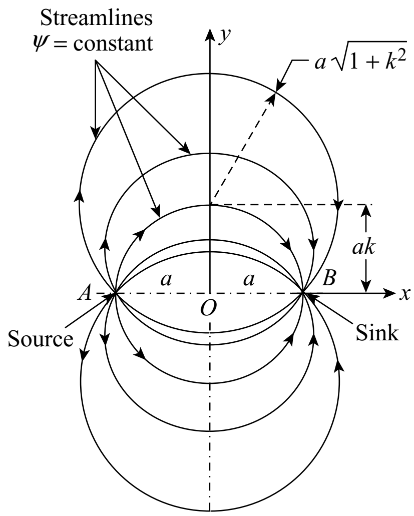


Figure 9.14 *(a) Source-sink pair (b) Streamlines pattern for source-sink pair*

Let the strength of the source and sink placed at A and B be q and $-q$, respectively, (r_1, α_1) and (r_2, α_2) be the positions of any point P in the flow with respect to source and sink, respectively, (r, α) be the cylindrical coordinates of point P and (x, y) be the corresponding Cartesian coordinates with respect to origin O and $\beta = (\alpha_2 - \alpha_1)$ be the angle subtended with respect to r_1 and r_2.

The stream function ψ for the combined flow is given by,

$$\psi = \psi_{\text{source}} + \psi_{\text{sink}} = \frac{q\alpha_1}{2\pi} - \frac{q\alpha_2}{2\pi} = \frac{q}{2\pi}(\alpha_1 - \alpha_2) = -\frac{q}{2\pi}\beta \tag{9.29}$$

For a given streamline ψ = constant, and the angle β as constant (since q is constant), the streamlines of the combined flow field are circles with AB as common chord. Thus, the circles will pass through source A and sink B which has been proved below.

$$\tan\beta = \tan(\alpha_2 - \alpha_1) = \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_1 \tan\alpha_2}$$

$$\tan\alpha_1 = \frac{y}{x+a} \text{ and } \tan\alpha_2 = \frac{y}{x-a}$$

Thus

$$\tan\beta = \frac{[y/(x-a)] - [y/(x+a)]}{1 + [y/(x+a)]\cdot[y/(x-a)]} = \frac{2ay}{x^2 + y^2 - a^2} \tag{9.30}$$

From Equation (9.29), $\beta = -2\pi\psi/q$ and thus, Equation (9.30) is written as follows.

$$\tan\left(\frac{-2\pi\psi}{q}\right) = -\tan\frac{2\pi\psi}{q} = \frac{2ay}{x^2 + y^2 - a^2}$$

$$x^2 + y^2 - a^2 + 2ay\cot\frac{2\pi\psi}{q} = 0$$

or

$$x^2 + y^2 - a^2 + 2ayk = 0, \text{ here } k = \cot(2\pi\psi/q)$$

By adding, subtracting (a^2k^2) and rearranging, we get:

$$x^2 + (y + ak)^2 = a^2(1 + k^2) \tag{9.31}$$

Here, Equation (9.31) represents a circle. The streamlines consist of family of circles starting from the source and ending at sink with the centres $(0, \pm ak)$ on the y-axis having radii $a\sqrt{1+k^2}$ are shown in Figure 9.14(b).

The velocity function ϕ for the combined flow is given by,

$$\phi = \phi_{\text{source}} + \phi_{\text{sink}} = \frac{q}{2\pi}\ln r_1 - \frac{q}{2\pi}\ln r_2 = \frac{q}{2\pi}\ln\left(\frac{r_1}{r_2}\right) \tag{9.32}$$

Equation (9.32) can be transformed into Cartesian coordinates by using the following expression.

$$r_1^2 = y^2 + (x+a)^2 = x^2 + y^2 + a^2 + 2ax$$

$$r_2^2 = y^2 + (x-a)^2 = x^2 + y^2 + a^2 - 2ax$$

Thus

$$\phi = \frac{q}{2\pi}\ln\left(\frac{x^2 + y^2 + a^2 + 2ax}{x^2 + y^2 + a^2 - 2ax}\right)^{1/2} \tag{9.33}$$

$$\frac{x^2 + y^2 + a^2 + 2ax}{x^2 + y^2 + a^2 - 2ax} = e^{\frac{4\pi\phi}{q}}$$

Let

$$e^{\frac{4\pi\phi}{q}} = n$$

Thus

$$x^2 + y^2 + a^2 + 2ax = n(x^2 + y^2 + a^2 - 2ax)$$

$$(x^2 + y^2 + a^2)(n-1) - 2ax(n+1) = 0$$

$$x^2 + y^2 - 2ax\left[\frac{n+1}{n-1}\right] + a^2 = 0 \quad \text{[Dividing by } (n-1)\text{]}$$

or

$$y^2 + \left[x - \left(\frac{n+1}{n-1}\right)a\right]^2 = \left[a\sqrt{\left(\frac{n+1}{n-1}\right)^2 - 1}\right]^2 \tag{9.34}$$

Here, Equation (9.34) represents a circle. The potential lines consisting of family of eccentric non-intersecting circles with centres $[\pm\{(n+1)/(n-1)\}a, 0]$ on the x-axis having radii $a\sqrt{\{(n+1)/(n-1)\}^2 - 1} = (2a\sqrt{n})/(n-1)$ are shown in Figure 9.15.

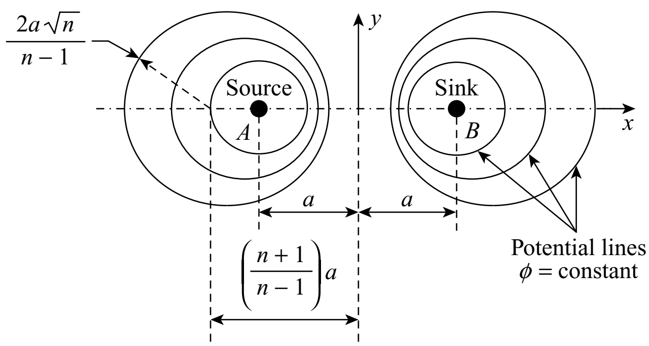


Figure 9.15 *Equipotential lines pattern for source-sink pair*

Example 9.7 A source of strength 7 m^2/s is located at (–1, 0) and a sink of strength 14 m^2/s is located at (1, 0). If the density of the fluid is 1.9 kg/m^3 and the pressure at infinity is negligible, then calculate the velocity, stream function and pressure at point P (1, 1).

Solution

Refer Figure 9.16. Let $q_1 = 7\text{ m}^2/\text{s}$, $A\,(-1, 0)$, $q_2 = 14\text{ m}^2/\text{s}$, $B\,(1, 0)$, $a = 1$, $\rho = 1.9\text{ kg/m}^3$, $p_\infty = 0$, $x = 1$ and $y = 1$.

$$\alpha_1 = \tan^{-1}\left(\frac{y}{x+a}\right) = \tan^{-1}\left(\frac{1}{1+1}\right) = 26.565° = \frac{26.565° \times \pi}{180} = 0.4636 \text{ rad}$$

$$\alpha_2 = \tan^{-1}\left(\frac{y}{x-a}\right) = \tan^{-1}\left(\frac{1}{1-1}\right) = 90° = \frac{90° \times \pi}{180} = \frac{\pi}{2} \text{ rad}$$

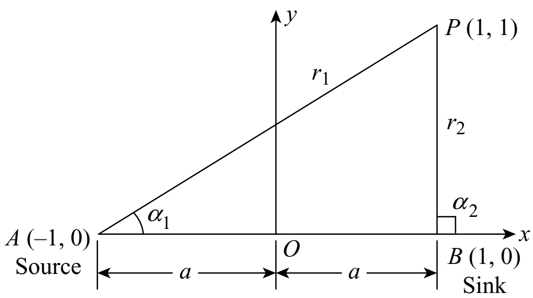


Figure 9.16

Since $\psi = \frac{q_1\alpha_1}{2\pi} - \frac{q_2\alpha_2}{2\pi} = \frac{q_1}{2\pi}\tan^{-1}\left(\frac{y}{x+a}\right) - \frac{q_2}{2\pi}\tan^{-1}\left(\frac{y}{x-a}\right)$

$$\therefore \psi = \frac{7}{2\pi} \times 0.4636 - \frac{14}{2\pi} \times \frac{\pi}{2} = \mathbf{-2.9835\ m^2/s}$$

Now $u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left[\frac{q_1}{2\pi}\tan^{-1}\left(\frac{y}{x+a}\right) - \frac{q_2}{2\pi}\tan^{-1}\left(\frac{y}{x-a}\right)\right]$

$$u = \frac{q_1}{2\pi} \times \frac{1}{1+[y/(x+a)]^2} \times \frac{1}{x+a} - \frac{q_2}{2\pi} \times \frac{1}{1+[y/(x-a)]^2} \times \frac{1}{x-a}$$

Thus
$$u = \frac{q_1}{2\pi} \times \frac{(x+a)}{(x+a)^2 + y^2} - \frac{q_2}{2\pi} \times \frac{(x-a)}{(x-a)^2 + y^2}$$

$$\therefore u = \frac{7}{2\pi} \times \frac{(1+1)}{(1+1)^2 + 1^2} - \frac{14}{2\pi} \times \frac{(1-1)}{(1-1)^2 + 1^2} = 0.4456 \text{ m/s}$$

Now
$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left[\frac{q_1}{2\pi}\tan^{-1}\left(\frac{y}{x+a}\right) - \frac{q_2}{2\pi}\tan^{-1}\left(\frac{y}{x-a}\right)\right]$$

$$v = -\left[\frac{q_1}{2\pi} \times \frac{1}{1+y/(x+a)^2} \times \frac{-y}{(x+a)^2} - \frac{q_2}{2\pi} \times \frac{1}{1+y/(x-a)^2} \times \frac{-y}{(x-a)^2}\right]$$

Thus
$$v = -\left[\frac{q_1}{2\pi} \times \frac{-y}{(x+a)^2 + y^2} - \frac{q_2}{2\pi} \times \frac{-y}{(x-a)^2 + y^2}\right]$$

$$\therefore v = -\left[\frac{7}{2\pi} \times \frac{-1}{(1+1)^2 + 1^2} - \frac{14}{2\pi} \times \frac{-1}{(1-1)^2 + 1^2}\right] = -2.0053 \text{ m/s}$$

$$V = \sqrt{u^2 + v^2} = \sqrt{0.4456^2 + (-2.0053)^2} = \mathbf{2.054\ m/s}$$

The Bernoulli's equation between a point in the uniform flow stream and point P (1, 1) is given by,

$$p_\infty + \frac{1}{2}\rho U_\infty{}^2 = p + \frac{1}{2}\rho V^2 \qquad [\because z_1 = z_2]$$

The velocity of fluid at infinity will be zero, i.e., $U_\infty = 0$.

Thus $$0+0 = p+\frac{1}{2}\rho V^2$$

$$\therefore p = -\frac{1}{2}\rho V^2 = -\frac{1}{2}\times 1.9\times 2.054^2 = \mathbf{-4.008\ N/m^2}$$

9.6.3 Doublet (or Dipole)

A doublet (or dipole) is a special case of a source and sink pair of equal strength q wherein both approach each other in such a way that the distance $2a$ between them approaches zero and the product $2a.q$, also called the doublet strength (μ) remains constant. Thus, the doublet strength is given by the following expression.

$$\mu = 2aq \tag{9.35}$$

Figure 9.17 illustrates the position of the source (at point A) and the sink (at point B) of strength q and $(-q)$, respectively, at a distance $2a$ apart and P is any point in the combined flow field. Let α be the angle made by P at A (source) and $(\alpha+\delta\alpha)$ at B (sink) where $\delta\alpha$ is a very small angle. Let $AP = (r+\delta r)$, BM is the perpendicular on AP, $AM = \delta r$, $PM = PB = r$ and $\angle APB = \delta\alpha$ is very small.

$$BM = r\times\delta\alpha \quad [\because \delta\alpha \text{ is very small}]$$

Also $$BM = 2a\sin\alpha \quad [\text{From right-angled } \Delta BMA]$$

Thus $$r\times\delta\alpha = 2a\sin\alpha$$

$$\therefore \delta\alpha = \frac{2a\sin\alpha}{r} \tag{i}$$

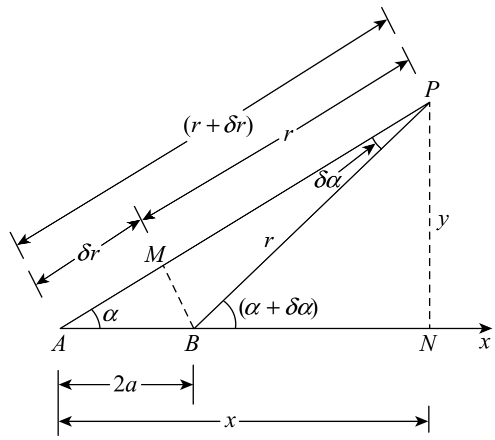


Figure 9.17 *Analysis of a doublet*

The stream function for the doublet is given by,

$$\psi = \frac{q}{2\pi}\alpha - \frac{q}{2\pi}(\alpha+\delta\alpha) = -\frac{q}{2\pi}\delta\alpha = -\frac{q}{2\pi}\times\frac{2a\sin\alpha}{r} \quad [\text{Substitute (i)}]$$

Since $$\mu = 2aq$$

Thus $$\psi = -\frac{\mu}{2\pi}\times\frac{\sin\alpha}{r} \tag{9.36}$$

When $2a$ tends to zero, $\delta\alpha$ becomes very small such that δr vanishes and $AP = r$.

Thus $$\sin\alpha = y/r \text{ and } x^2 + y^2 = r^2$$

Using these values in Equation (9.36), we get:

$$\psi = -\frac{\mu}{2\pi}\times\frac{y}{r\times r} = -\frac{\mu}{2\pi}\times\frac{y}{r^2} = -\frac{\mu}{2\pi}\times\frac{y}{x^2+y^2} \tag{9.37}$$

Rearranging the expression given by Equation (9.37), we get:

$$x^2 + y^2 + \frac{\mu}{2\pi\psi}y = 0 \tag{9.37a}$$

By adding and subtracting with $[\mu/(4\pi\psi)]^2$ and rearranging, we get:

$$x^2 + \left[y + \frac{\mu}{4\pi\psi}\right]^2 = \left(\frac{\mu}{4\pi\psi}\right)^2 \tag{9.38}$$

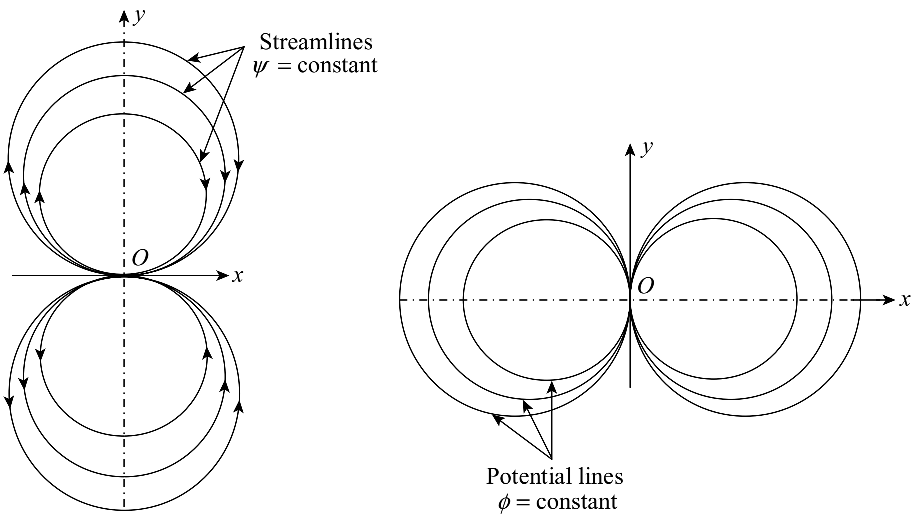


Figure 9.18 *Streamlines and potential lines for a doublet*

Here, Equation (9.38) represents a circle. The streamlines of a doublet are family of circles tangent to the x-axis whose centres $\{0, \pm \mu/(4\pi\psi)\}$ lie on the y-axis having radii $\mu/(4\pi\psi)$ as shown in Figure 9.18(a).

From right-angled triangle BMA, we get:

$$\delta r = 2a \cos \alpha \tag{ii}$$

The potential function ϕ for the combined flow is given by,

$$\phi = \frac{q}{2\pi} \ln(r + \delta r) + \left(-\frac{q}{2\pi}\right) \ln r = \frac{q}{2\pi} \ln(r + \delta r) - \frac{q}{2\pi} \ln r = \frac{q}{2\pi} \ln\left(\frac{r + \delta r}{r}\right)$$

or $$\phi = \frac{q}{2\pi} \ln\left(1 + \frac{\delta r}{r}\right)$$

Using expansion of $\ln(1 + x) = x + x^2/2 + x^3/3 + ...$, the above expression is rewritten as follows.

$$\phi = \frac{q}{2\pi}\left[\frac{\delta r}{r} + \frac{1}{2}\left(\frac{\delta r}{r}\right)^2 + \frac{1}{3}\left(\frac{\delta r}{r}\right)^3 + ..\right]$$

Since $(\delta r/r)$ is a small quantity and neglecting higher powers, we have the below expression.

$$\phi = \frac{q}{2\pi} \times \frac{\delta r}{r}$$

Substituting expression (ii) in the above expression, we get:

$$\phi = \frac{q}{2\pi} \times \frac{2a \cos \alpha}{r} = \frac{2aq}{2\pi} \times \frac{\cos \alpha}{r} = \frac{\mu}{2\pi} \times \frac{\cos \alpha}{r} \tag{9.39}$$

When $2a$ tends to zero, $\delta\alpha$ becomes very small such that δr vanishes and $AP = r$.

Thus $$\cos \alpha = x/r \text{ and } x^2 + y^2 = r^2$$

Using these values in Equation (9.39), we get:

$$\phi = \frac{\mu}{2\pi} \times \frac{x}{r \times r} = \frac{\mu}{2\pi} \times \frac{x}{r^2} = \frac{\mu}{2\pi} \times \frac{x}{(x^2 + y^2)}$$

Rearranging the expression given, we get:

$$x^2 + y^2 - \frac{\mu}{2\pi\phi} x = 0$$

By adding and subtracting $\{\mu/(4\pi\phi)\}^2$ and after rearranging, we get:

$$y^2 + \left[x - \frac{\mu}{4\pi\phi}\right]^2 = \left(\frac{\mu}{4\pi\phi}\right)^2 \quad (9.40)$$

Here, Equation (9.40) represents a circle. The potential lines of a doublet are family of circles tangent to the y-axis whose centres $\{\pm\mu/(4\pi\phi), 0\}$ lie on the x-axis having radii $\mu/(4\pi\phi)$ are shown in Figure 9.18(b).

Example 9.8 If the strength of a doublet is 15 m^2/s, then determine the velocity at point P (1, 2) and the value of stream function passing through it by using both the Cartesian and polar coordinates systems.

Solution

Let $\mu = 15$ m^2/s, $x = 1$ and $y = 2$.

(i) Cartesian coordinate system:

$$\psi = -\frac{\mu}{2\pi} \frac{y}{(x^2 + y^2)} = -\frac{\mu}{2\pi}[y(x^2 + y^2)^{-1}]$$

$$u = \frac{\partial \psi}{\partial y} = \frac{d}{dy}\left[-\frac{\mu}{2\pi}\left\{y(x^2 + y^2)^{-1}\right\}\right] = -\frac{\mu}{2\pi}\frac{d}{dy}\left[y(x^2 + y^2)^{-1}\right]$$

Now $\frac{d}{dy}[y(x^2 + y^2)^{-1}] = \left[y(-1)(x^2 + y^2)^{-2}(2y) + (x^2 + y^2)^{-1}(1)\right]$

or $\frac{d}{dy}[y(x^2 + y^2)^{-1}] = \frac{-2y^2}{(x^2 + y^2)^2} + \frac{1}{(x^2 + y^2)} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$

Thus $u = \frac{\partial \psi}{\partial y} = -\frac{\mu}{2\pi}\left[\frac{x^2 - y^2}{(x^2 + y^2)^2}\right]$

And $v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left[-\frac{\mu}{2\pi}\left\{y(x^2 + y^2)^{-1}\right\}\right] = \frac{\mu}{2\pi}\frac{d}{dx}\left[\frac{y}{x^2 + y^2}\right] = \frac{\mu}{2\pi}\left[\frac{-2xy}{(x^2 + y^2)^2}\right]$

At point $P(1, 2)$ and $\mu = 15$ m^2/s, we get:

$$\psi = -\frac{\mu}{2\pi}\frac{y}{(x^2 + y^2)} = -\frac{15}{2\pi} \times \frac{2}{(1^2 + 2^2)} = \mathbf{-0.955\ m^2/s}$$

$$u = -\frac{15}{2\pi} \times \left[\frac{1^2 - 2^2}{(1^2 + 2^2)^2}\right] = 0.2865 \text{ m/s}$$

$$v = \frac{15}{2\pi} \times \left[\frac{-2 \times 1 \times 2}{(1^2 + 2^2)^2} \right] = -0.382 \text{ m/s}$$

$$V = \sqrt{u^2 + v^2} = \sqrt{0.2865^2 + (-0.382)^2} = \mathbf{0.4775\ m/s}$$

(ii) Polar coordinate system:

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + 2^2} = 2.236$$

$$\sin\alpha = \frac{y}{r} = \frac{2}{2.236} = 0.8944$$

$$\cos\alpha = \frac{x}{r} = \frac{1}{2.236} = 0.4472$$

$$\psi = -\frac{\mu}{2\pi} \times \frac{\sin\alpha}{r} = -\frac{15}{2\pi} \times \frac{0.8944}{2.236} = \mathbf{0.955\ m^2/s}$$

$$\because u_r = \frac{1}{r}\frac{\partial\psi}{\partial\alpha} = \frac{1}{r}\frac{\partial}{\partial\alpha}\left[-\frac{\mu}{2\pi} \times \frac{\sin\alpha}{r} \right] = -\frac{\mu}{2\pi r^2}\cos\alpha$$

$$\therefore u_r = -\frac{15}{2\pi \times 2.236^2} \times 0.4472 = -0.2135 \text{ m/s}$$

$$\because u_\alpha = -\frac{\partial\psi}{\partial r} = -\frac{\partial}{\partial r}\left[-\frac{\mu}{2\pi} \times \frac{\sin\alpha}{r} \right] = -\frac{\mu}{2\pi r^2}\sin\alpha$$

$$\therefore u_\alpha = -\frac{15}{2\pi \times 2.236^2} \times 0.8944 = -0.4271 \text{ m/s}$$

$$V = \sqrt{u_r^2 + u_\alpha^2} = \sqrt{(-0.2135)^2 + (-0.4271)^2} = \mathbf{0.4775\ m/s}$$

Example 9.9 If the relation for a doublet is $x^2 + y^2 - (K/C)y = 0$, then comment on the nature of constants K and C and the form of streamline. Also determine the magnitude and direction of velocity at the point $P(1, 3)$, where $u = 5$ m/s.

Solution

Let $x^2 + y^2 - (K/C)y = 0$, $x = 1$, $y = 3$ and $u = 5$ m/s.

$$x^2 + y^2 - \frac{K}{C}y = 0 \tag{i}$$

The standard equation of streamline for a doublet can be given by Equation (9.37(a)) as follows.

$$x^2 + y^2 + \frac{\mu}{2\pi\psi}y = 0$$

On comparing the above expressions, the following points are noted:

(i) K represents the strength of doublet μ that remains constant for the flow.

(ii) C represents the stream function ψ that remains constant for a streamline.

Thus, expression (i) can be written as follows.

$$x^2 + y^2 + \left(\frac{K}{2C}\right)^2 - \frac{K}{C}y = \left(\frac{K}{2C}\right)^2$$

$$x^2 + \left(y - \frac{K}{2C}\right)^2 = \left(\frac{K}{2C}\right)^2 \quad \text{(ii)}$$

The expression (ii) prescribes the equation of a circle with centre $(0, K/2C)$ and radius $K/2C$. Thus, the streamlines of the doublet are a family of circles with their centres on the y-axis. The equation of the streamline is satisfied at $x = y = 0$. Therefore, all circles pass through the origin.

By partially differentiating expression (ii), we get:

$$2xdx + 2ydy - \frac{K}{C}dy = 0$$

or
$$dy(2y - K/C) = -2xdx$$

Thus
$$\frac{dy}{dx} = \frac{-2x}{(2y - K/C)} \quad \text{(ii)}$$

For the given point $P(1, 3)$, from the expression (i), we get:

$$1^2 + 3^2 - \frac{K}{C} \times 3 = 0 \Rightarrow \frac{K}{C} = \frac{10}{3}$$

From expression (ii), we get:

$$\frac{dy}{dx} = \frac{-2 \times 1}{2 \times 3 - (10/3)} = -0.75$$

Also
$$\frac{dy}{dx} = \frac{v}{u}, \text{ thus } \frac{v}{u} = -0.75$$

$$\therefore v = -0.75u = -0.75 \times 5 = -3.75 \text{ m/s}$$

$$V = \sqrt{u^2 + v^2} = \sqrt{5^2 + (-3.75)^2} = \mathbf{6.25\ m/s}$$

Since
$$\tan \alpha = \frac{dy}{dx}$$

$$\therefore \alpha = \tan^{-1}\frac{dy}{dx} = \tan^{-1}(-0.75) = \mathbf{-36.87°}$$

9.6.4 A Doublet in a Uniform Flow (Flow Past a Circular Cylinder)

A doublet placed in a uniform flow makes an important flow pattern past a circular cylinder (or a Rankine oval of equal axes, i.e., a circle). Figure 9.19(a) shows a uniform flow of velocity U parallel to x-axis and a doublet at the origin with its axis coinciding with x-axis. The resulting flow due to doublet and uniform flow is illustrated in Figure 9.19(b) which is also known as the flow past a Rankine oval of equal axes or flow past a circular cylinder.

The stream function (ψ) for the resultant flow is obtained by adding the stream functions for a uniform flow and a doublet as follows.

$$\psi = \psi_{\text{uniform flow}} + \psi_{\text{doublet}} = Uy + \left(-\frac{\mu}{2\pi}\frac{\sin\alpha}{r}\right)$$

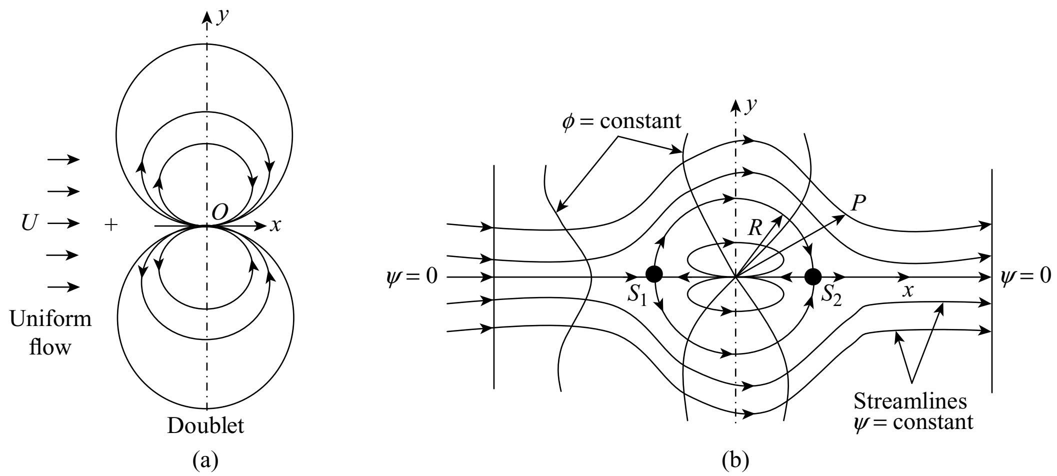


Figure 9.19 *Doublet in a uniform flow*

Since $y = r\sin\alpha$

Thus $$\psi = Ur\sin\alpha - \frac{\mu}{2\pi}\frac{\sin\alpha}{r} = \left(Ur - \frac{\mu}{2\pi r}\right)\sin\alpha \tag{9.41}$$

Similarly, the potential function ϕ for the resulting flow can be given by,

$$\phi = \phi_{\text{uniform flow}} + \phi_{\text{doublet}} = Ux + \frac{\mu}{2\pi}\frac{\cos\alpha}{r}$$

Since $x = r\cos\alpha$

Thus $$\phi = Ur\cos\alpha + \frac{\mu}{2\pi}\frac{\cos\alpha}{r} = \left(Ur + \frac{\mu}{2\pi r}\right)\cos\alpha \tag{9.42}$$

For obtaining the profile of the Rankine oval of equal axes, substituting $\psi = 0$ in Equation (9.41), we get the below expression.

$$\left(Ur - \frac{\mu}{2\pi r}\right)\sin\alpha = 0$$

Thus either $\sin\alpha = 0$ or $\left(Ur - \frac{\mu}{2\pi r}\right) = 0$

(i) If $\sin\alpha = 0$ or $\alpha = 0$ and $\pm\pi$, i.e., a horizontal line through the origin of the doublet.

(ii) If $Ur - \frac{\mu}{2\pi r} = 0$, then $Ur = \frac{\mu}{2\pi r} \Rightarrow r^2 = \frac{\mu}{2\pi U}$

Thus $$r = \sqrt{\frac{\mu}{2\pi U}}$$

Since μ and U are constants, the above expression is also a constant say R.

$$\boxed{r = \sqrt{\frac{\mu}{2\pi U}} = R \text{ or } R^2 = \frac{\mu}{2\pi U}} \tag{9.43}$$

Using Equations (9.43), (9.41) and (9.42), respectively, we get:

$$\psi = \left(Ur - \frac{\mu}{2\pi r} \right) \sin\alpha = U\left(r - \frac{\mu}{2\pi U} \times \frac{1}{r} \right) \sin\alpha = U\left(r - \frac{R^2}{r} \right) \sin\alpha \tag{9.44}$$

$$\phi = \left(Ur + \frac{\mu}{2\pi r} \right) \cos\alpha = U\left(r + \frac{\mu}{2\pi U} \times \frac{1}{r} \right) = U\left(r + \frac{R^2}{r} \right) \cos\alpha \tag{9.45}$$

The streamline $\psi = 0$ comprises a horizontal line through the origin of the doublet and a circle with centre at the origin with a radius $R = \sqrt{\mu/(2\pi U)}$. The streamline $\psi = 0$ has two stagnation point S_1 and S_2. At S_1, it splits into two streams that flow along the circle with radius R. The two streams meet again at S_2 and the flow continues in the downward direction. Apparently, the flow field due to the doublet lies entirely in the circle, whereas the uniform flow occurs outside the circle and does not mingle with the doublet flow.

The velocity components at any point $P(r, \alpha)$ in the flow field in polar coordinates can be determined with the help of stream function as follows.

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \alpha} = \frac{1}{r}\frac{\partial}{\partial \alpha}\left[U\left(r - \frac{R^2}{r} \right) \sin\alpha \right] = U\left(1 - \frac{R^2}{r^2} \right) \cos\alpha \tag{9.46}$$

$$u_\alpha = -\frac{\partial \psi}{\partial r} = -\frac{\partial}{\partial r}\left[U\left(r - \frac{R^2}{r} \right) \sin\alpha \right] = -U\left(1 + \frac{R^2}{r^2} \right) \sin\alpha \tag{9.47}$$

The resultant velocity is given by,

$$V = \sqrt{u_r^2 + u_\alpha^2}$$

At the surface of the cylindrical body, $r = R$ and thus, from Equations (9.46) and (9.47), we get the below expression.

$$\boxed{u_r = 0}$$

$$\boxed{u_\alpha = -2U \sin\alpha} \tag{9.48}$$

In Equation (9.48), the negative sign indicates the clockwise direction of tangential velocity (u_α) at that point. At the surface of the cylindrical body, maximum velocity for u_α occurs at $\alpha = 90°$ and $270°$. Thus, the magnitude of the maximum velocity will be equal to $-2U$. When $\alpha = 0°$ and $180°$, $u_\alpha = 0$ and hence, $V = 0$. This corresponds to the stagnation points S_1 and S_2 on the surface of the cylinder.

Let U be the velocity and p_∞ be the pressure at a point in the uniform flow far away from the cylinder and the corresponding values at a point anywhere in the flow field are V and p, respectively.

Applying Bernoulli's equation, we get:

$$\frac{p_\infty}{\rho g} + \frac{U^2}{2g} = \frac{p}{\rho g} + \frac{V^2}{2g} \qquad [\because z_1 = z_2]$$

$$\frac{p}{\rho g} - \frac{p_\infty}{\rho g} = \frac{U^2}{2g} - \frac{V^2}{2g}$$

$$\frac{p - p_\infty}{\rho} = \frac{U^2}{2}\left[1 - \left(\frac{V}{U} \right)^2 \right]$$

$$\frac{p-p_\infty}{(1/2)\rho U^2}=1-\left(\frac{V}{U}\right)^2$$

The pressure coefficient is given by,

$$C_p=\frac{p-p_\infty}{(1/2)\rho U^2}=1-\left(\frac{V}{U}\right)^2$$

At the surface of the cylinder, $u_r = 0$ and $V = u_\alpha = -2U\sin\alpha$ and therefore, we get the below expression.

$$C_p=1-\left(\frac{-2U\sin\alpha}{U}\right)^2=1-4\sin^2\alpha \tag{9.49}$$

The values of C_p for different values of α calculated from Equation (9.49) are summarized below in Table 9.1.

Table 9.1 *Variation of pressure coefficient for different angles*

S. no.	α	C_p	Remarks
1	0°	1	Stagnation point
2	30°	0	Zero pressure
3	90°	–3	Least pressure
4	150°	0	Zero pressure
5	180°	1	Stagnation point

The variation of pressure coefficient along the surface of the cylinder for different angles is illustrated in Figure 9.20(a) and the pressure distribution on the cylinder surface is illustrated in Figure 9.20(b).

As the velocity distribution is same at the top and bottom sides of the cylinder, the pressure distribution will also be the same on both sides. Therefore, there is no upward force due to pressure difference. The positive pressure acts normal to the surface and towards the surface of the cylinder, whereas the negative pressure acts normal and away from the surface of the cylinder.

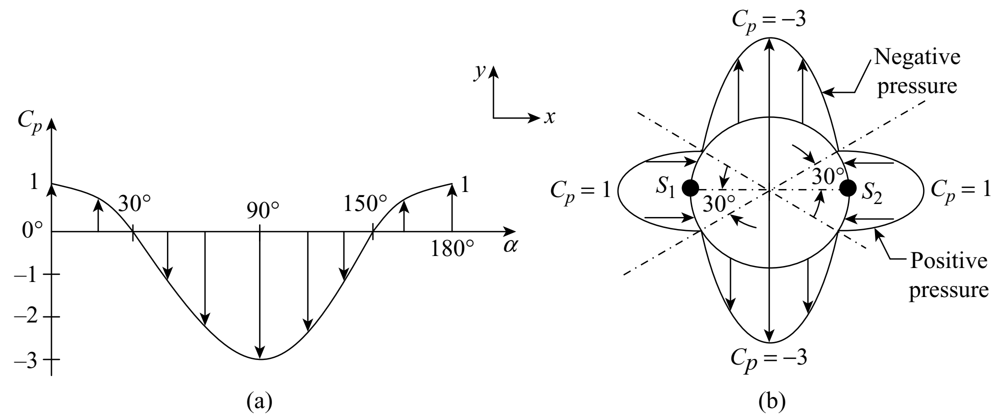


Figure 9.20 *Pressure distribution over the surface of a cylinder*

Example 9.10 A doublet of strength 20 m^2/s is in the line of the uniform flow having a velocity of 15 m/s. Determine the stream function and the resultant velocity in the flow field at a point $P(0.5, 30°)$.

Solution

Let $\mu = 20\ \text{m}^2/\text{s},\ U = 15\ \text{m/s},\ r = 0.5\ \text{m}$ and $\alpha = 30°$.

$$R = \sqrt{\frac{\mu}{2\pi U}} = \sqrt{\frac{20}{2\pi \times 15}} = 0.4607\ \text{m}$$

$$\psi = U\left[r - \frac{R^2}{r}\right]\sin\alpha = 15\left(0.5 - \frac{0.4607^2}{0.5}\right)\sin 30° = \mathbf{0.5663\ m^2/s}$$

$$u_r = U\left[1 - \frac{R^2}{r^2}\right]\cos\alpha = 15\left(1 - \frac{0.4607^2}{0.5^2}\right)\cos 30° = 1.962\ \text{m/s}$$

$$u_\alpha = -U\left[1 + \frac{R^2}{r^2}\right]\sin\alpha = -15\left(1 + \frac{0.4607^2}{0.5^2}\right)\sin 30° = -13.867\ \text{m/s}$$

$$V = \sqrt{u_r^2 + u_\alpha^2} = \sqrt{1.962^2 + (-13.867)^2} = \mathbf{14\ m/s}$$

Example 9.11 If a doublet of strength 20 m^2/s lies in the line of a uniform flow of 12 m/s, then what is the shape of the Rankine body formed? Determine the dimensions of the Rankine body and the values of the tangential and radial velocity components of the surface at a point 30° from x-axis. Also determine the maximum velocity on the Rankine body and its location.

Solution

Let $\mu = 20\ \text{m}^2/\text{s},\ U = 12\ \text{m/s}$ and $\alpha = 30°$.

When a doublet lies in the line of a uniform flow, **the shape of the Rankine body be cylindrical** with radius $R = \sqrt{\mu/2\pi U}$.

$$R = \sqrt{\frac{\mu}{2\pi U}} = \sqrt{\frac{20}{2\pi \times 12}} = \mathbf{0.515\ m}$$

The value of stream function (ψ) on the surface of Rankine body will be zero, i.e., $\boldsymbol{\psi = 0}$.

The values of radial velocity (u_r) and tangential velocity (u_α) on the surface of the Rankine body when $\alpha = 30°$ will be given by Equation (9.48) as given below.

$$u_r = 0 \text{ and } u_\alpha = -2U\sin\alpha = -2 \times 12 \sin 30° = \mathbf{-12\ m/s}$$

The resultant velocity at the surface of the body is given by,

$$V = \sqrt{u_r^2 + u_\alpha^2} = \sqrt{(0)^2 + (-12)^2} = \mathbf{12\ m/s}$$

The resultant velocity at any point on the Rankine body is given by $u_\alpha = -2U\sin\alpha$ and it will be maximum at $\alpha = 90°$.

$$\therefore \text{Maximum velocity} = -2U\sin 90° = -2U = -2 \times 12 = \mathbf{-24\ m/s}$$

9.6.5 Source, Sink and Uniform Flow (Flow Past a Rankine Oval Body)

A uniform flow parallel to x-axis combined with a source-sink pair of equal strength located on the x-axis with the origin of coordinates midway between them results in a flow past a Rankine oval body. Let U be the velocity of uniform flow along x-axis, q be the strength of the source (placed at point A), $-q$ be the strength of the sink (placed at point B) and $2a$ be the distance between the source and the sink. Figure 9.21 shows a uniform flow of velocity U parallel to x-axis and a source-sink pair of equal strength.

The resultant flow also known as flow past a Rankine oval body is illustrated in Figure 9.22. Let P be any point in the combined flow field that makes the angles α_1 and α_2 with source and sink along x-axis, respectively.

The stream function for the resulting flow can be obtained by adding the stream functions for a uniform flow, source and a sink as given below.

$$\psi = \psi_{\text{uniform flow}} + \psi_{\text{source}} + \psi_{\text{sink}} = Uy + \frac{q\alpha_1}{2\pi} - \frac{q\alpha_2}{2\pi}$$

or

$$\psi = Ur\sin\alpha + \frac{q}{2\pi}(\alpha_1 - \alpha_2) \quad [\because y = r\sin\alpha] \tag{9.50}$$

Similarly, the potential function for the resulting flow can be given by,

$$\phi = \phi_{\text{uniform flow}} + \phi_{\text{source}} + \phi_{\text{sink}} = Ux + \frac{q}{2\pi}\ln r_1 - \frac{q}{2\pi}\ln r_2$$

or

$$\phi = Ur\cos\alpha + \frac{q}{2\pi}\ln\frac{r_1}{r_2} \quad [\because x = r\cos\alpha] \tag{9.51}$$

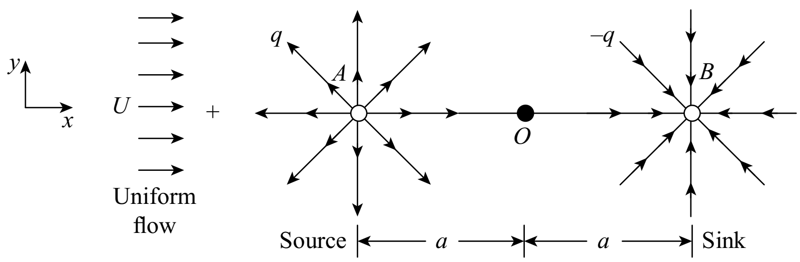


Figure 9.21 *Superposition of uniform flow over source-sink pair*

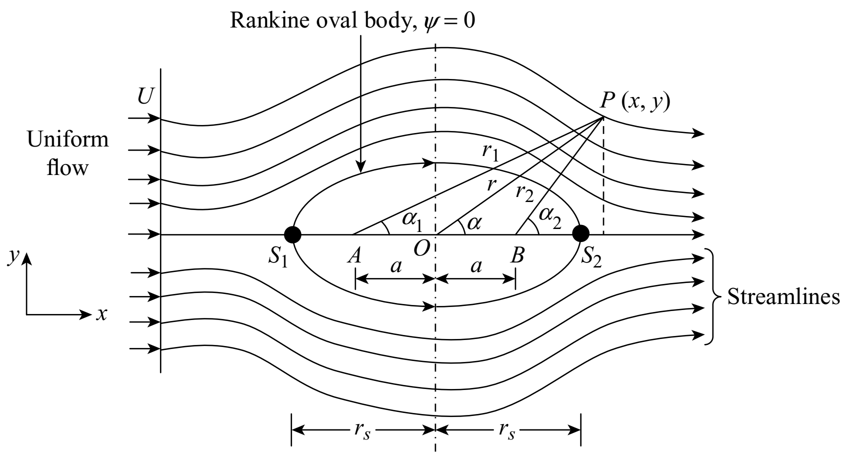


Figure 9.22 *Flow past a Rankine oval body*

The streamlines make exactly similar pattern to the flow past an oval body. There are two stagnation points S_1 and S_2. The stagnation point S_1 lies to the left of the source, whereas S_2 lies to the right of the sink. Let r_s be the distance of the stagnation points from the origin O. At the stagnation point S_1, the velocity of uniform flow is U, the velocity from source is $q/\{2\pi(r_s - a)\}$ and the velocity from sink is $-q/\{2\pi(r_s + a)\}$. At stagnation point S_1, the velocity due to uniform flow is in positive x-direction, whereas due to source and sink it is in the negative x-direction. At the stagnation points, the resultant velocity will be zero. Therefore, we get the following expressions.

$$U - \frac{q}{2\pi(r_s - a)} - \frac{(-q)}{2\pi(r_s + a)} = 0$$

$$U = \frac{q}{2\pi}\left[\frac{1}{(r_s - a)} - \frac{1}{(r_s + a)}\right] = \frac{q}{2\pi}\left[\frac{(r_s + a) - (r_s - a)}{(r_s - a)(r_s + a)}\right] = \frac{q}{2\pi}\frac{2a}{(r_s^2 - a^2)}$$

$$r_s^2 - a^2 = \frac{qa}{\pi U}$$

$$r_s^2 = a^2 + \frac{qa}{\pi U} = a^2\left(1 + \frac{q}{\pi a U}\right)$$

$$\boxed{\therefore r_s = a\sqrt{\left(1 + \frac{q}{\pi a U}\right)}} \quad (9.52)$$

Equation (9.52) locates the position of stagnation points on the x-axis. A streamline passing through the stagnation point has a zero velocity and thus, it can be replaced by a solid body having an oval shape as shown in Figure 9.22.

The flow field due to source and sink lies within the Rankine oval and the uniform flow outside the body does not mingle with the source-sink flow. The equation for the streamlines forming the surface of the oval can be obtained by substituting the upstream stagnation point $S_1(r_s, \pi)$ at which $\alpha_1 = \alpha_2 = \pi$ in Equation (9.50). Thus, we get the below expression.

$$\psi = Ur\sin\pi + \frac{q}{2\pi}(\pi - \pi) = 0 \quad (9.53)$$

Thus, the surface of the oval body is a streamline having stream function equal to zero and it can be prescribed by the following equation.

$$\psi = 0 = Ur\sin\alpha + \frac{q}{2\pi}(\alpha_1 - \alpha_2) \quad \text{[Equation (9.50) and (9.53)]}$$

Thus
$$r = \frac{q}{2\pi}\frac{(\alpha_2 - \alpha_1)}{U\sin\alpha} \quad (9.54)$$

For determining the maximum width (height) of the oval body, y_m shown in Figure 9.23, substituting $\alpha_1 = \beta$, $\alpha_2 = (\pi - \beta)$ and $\alpha = \pi/2$ in Equation (9.54), we have the below expression.

$$y_m = \frac{q}{2\pi}\frac{(\pi - \beta - \beta)}{U\sin(\pi/2)} = \frac{q(\pi - 2\beta)}{2\pi U}$$

Thus
$$\beta = \frac{\pi}{2} - \frac{\pi U y_m}{q}$$

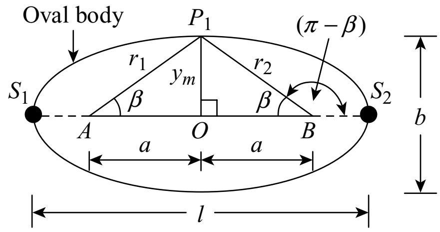


Figure 9.23 *Maximum height of the oval body*

Also from right-angled triangle P_1OA (Figure 9.23), we get the below expression.

$$y_m = a\tan\beta$$

Substituting the value of β in the above expression, we get:

$$y_m = a\tan\left[\frac{\pi}{2} - \frac{\pi U y_m}{q}\right] = a\cot\frac{\pi U y_m}{q} \tag{9.55}$$

From Equation (9.55), the value of y_m can be obtained by hit and trial method.

The length (l) and breadth (b) of the Rankine oval can be obtained by the following relations.

$$\boxed{l = 2r_s = 2a\sqrt{\left(1 + \frac{q}{\pi a U}\right)}} \tag{9.56}$$

$$\boxed{b = 2y_m = 2a\cot\frac{\pi U y_m}{q}} \tag{9.57}$$

Example 9.12 A source and sink of strength 20 m^2/s are located at a distance of 2 m. If a uniform flow of 8 m/s parallel to the line joining the source-sink pair is superimposed, then find the length of the Rankine oval body formed and the distance of the stagnation points from the source. Also determine the width of the Rankine body and its profile equation.

Solution

Let $q = 20\ \text{m}^2/\text{s}$, $2a = 2$ m or $a = 1$ m and $U = 8$ m/s.

$$r_s = a\sqrt{1 + \frac{q}{\pi a U}} = 1\times\sqrt{1 + \frac{20}{\pi\times 1\times 8}} = 1.34 \text{ m}$$

Distance of stagnation points from the source are given by,

S_1 is at a distance $= r_s - a = 1.34 - 1 = \mathbf{0.34\ m}$

S_2 is at a distance $= r_s + a = 1.34 + 1 = \mathbf{2.34\ m}$

Length of the Rankine oval is given by,

$$l = 2r_s = 2\times 1.34 = \mathbf{2.68\ m}$$

$$y_m = a\cot\frac{\pi U y_m}{q} = a\cot\left(\frac{\pi\times 8\times y_m}{20}\times\frac{180}{\pi}\right) = a\cot(72 y_m)^\circ$$

Let L.H.S. $= y_m = 0.742$ m [Hit and trial method]

Then R.H.S. $= a\cot(72\times 0.742) = 0.742$ m

Thus L.H.S. = R.H.S. $= y_m = 0.742$ m

$$b = 2y_m = 2\times 0.742 = \mathbf{1.484\ m}$$

Profile equation of the Rankine body is given by,

$$r = \frac{q}{2\pi}\frac{(\alpha_2 - \alpha_1)}{U\sin\alpha} = \frac{20}{2\pi}\times\frac{(\alpha_2 - \alpha_1)}{8\sin\alpha} = \frac{\mathbf{0.398(\alpha_2 - \alpha_1)}}{\mathbf{\sin\alpha}}$$

9.6.6 Doublet, Free Vortex and Uniform Flow (Flow Past a Cylinder with Circulation)

When a uniform flow parallel to x-axis combined with a doublet and a clockwise irrotational vortex is at origin, the resultant flow will be equivalent to the flow past a rotating (spinning) cylinder. Figure 9.24(a) shows a uniform flow of velocity U parallel to x-axis to which a doublet and a clockwise irrotational vortex at origin are superimposed. The resulting flow pattern (flow past a cylinder with circulation) shown in Figure 9.24(b) is asymmetrical about x-axis and the extent of asymmetry is related to the transverse force acting on the dividing streamline.

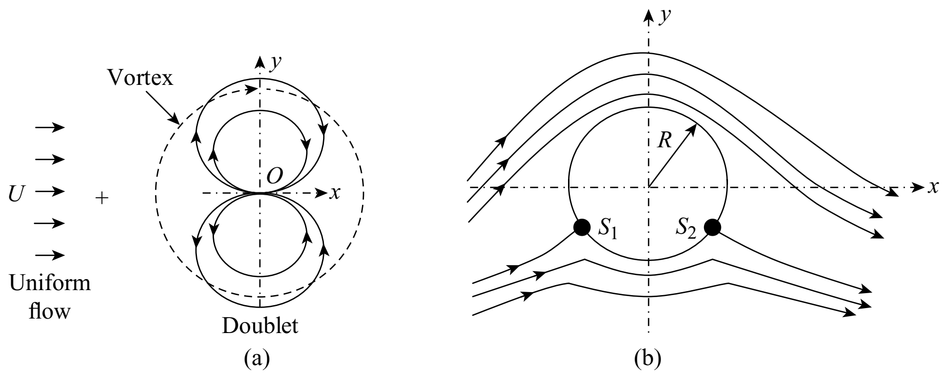


Figure 9.24 *Flow past a cylinder with circulation*

The stream function for the resulting flow can be obtained by adding the stream functions for a doublet in uniform flow given by Equation (9.44) and the clockwise free vortex flow at the origin of the doublet is given below.

$$\psi = U\left(r - \frac{R^2}{r}\right)\sin\alpha + \frac{\Gamma}{2\pi}\ln r \tag{9.58}$$

Similarly, the potential function for the resulting flow can be obtained by adding the potential functions for a doublet in uniform flow given by Equation (9.45) and the clockwise free vortex flow is given as follows.

$$\phi = U\left(r + \frac{R^2}{r}\right)\cos\alpha - \frac{\Gamma}{2\pi}\alpha \tag{9.59}$$

The dividing streamline consists of a circle of radius $R = \sqrt{\mu/(2\pi U)}$.

The velocity components at any point in the flow field are given by,

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\alpha} = \frac{\partial\phi}{\partial r} = U\left(1 - \frac{R^2}{r^2}\right)\cos\alpha \tag{9.60}$$

$$u_\alpha = -\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial\phi}{\partial\alpha} = -U\left(1 + \frac{R^2}{r^2}\right)\sin\alpha - \frac{\Gamma}{2\pi r} \tag{9.61}$$

The resultant velocity is given by,

$$V = \sqrt{{u_r}^2 + {u_\alpha}^2}$$

On the surface of the cylinder, $r = R$ and thus, Equations (9.60) and (9.61) is rewritten as follows.

$$u_r = U\left(1 - \frac{R^2}{R^2}\right)\cos\alpha = 0 \tag{9.62}$$

$$u_\alpha = -U\left(1 + \frac{R^2}{R^2}\right)\sin\alpha - \frac{\Gamma}{2\pi R} = -\left(2U\sin\alpha + \frac{\Gamma}{2\pi R}\right) \tag{9.63}$$

Since dividing streamline is a circle at any point on the dividing streamline (i.e., at the surface of the cylinder), $r = R$ and $u_r = 0$, because no flow can cross a streamline. Therefore, the tangential component of velocity (u_α) is a measure of the flow along the circular streamlines, i.e., $u_\alpha = V$. Therefore, Equation (9.63) can be written as follows.

$$u_\alpha = V = -\left(2U\sin\alpha + \frac{\Gamma}{2\pi R}\right) \tag{9.64}$$

At the stagnation points, the velocity components become zero, i.e., $u_r = 0$ and $u_\alpha = 0$.

Thus $$-\left(2U\sin\alpha + \frac{\Gamma}{2\pi R}\right) = 0$$

$$\boxed{\therefore \sin\alpha = -\frac{\Gamma}{4\pi RU}} \tag{9.65}$$

Thus, the location of stagnation points is given by $r = R$ and $\sin\alpha = -\Gamma/(4\pi RU)$.

The ratio $\Gamma/(4\pi RU)$ may have any value from 0 to more than 1. Thus, there will be two, one or no stagnation points on the cylinder surface corresponding to $|\Gamma|$. These conditions closely resemble the flow around a circular cylinder when the cylinder is rotated at different speeds. Thus, the following may be the cases.

Case I: When $|\Gamma| = 0$, the free vortex will be absent and $\sin\alpha = 0$ which corresponds to the stagnation points S_1 and S_2 located at $\alpha = 0$ and $\alpha = \pi$, respectively, as shown in Figure 9.25(a). This case corresponds to no rotation of the cylinder.

Case II: When $|\Gamma| < 4\pi RU$, the strength of the free vortex will be small, the value of $\sin\alpha$ lies between 0 (when $0 < \alpha < -\pi/2$) and -1 (when $\pi < \alpha < 3\pi/2$) and the stagnation points are located at S_1 and S_2 as shown in Figure 9.25(b). This case corresponds to a small rotational speed of the cylinder.

Case III: When $|\Gamma| = 4\pi RU$, the strength of the free vortex will be just equal to the critical product $4\pi RU$, the value of $\sin\alpha = -1$ which corresponds to a single stagnation point S at $\alpha = -\pi/2$ or $3\pi/2$, i.e., at the bottom of the dividing streamline as shown in Figure 9.25(c). This case corresponds to a critical rotational speed of the cylinder.

Case IV: When $|\Gamma| > 4\pi RU$, the strength of the free vortex will be large, the value of $\sin\alpha > 1$, i.e., α is imaginary. Thus, the stagnation point S will not be on the circular streamline but it will be located below it as shown in Figure 9.25(d). This case corresponds to a high rotational speed of the cylinder.

Let p_∞ be the pressure at infinity where the velocity is U and p be the pressure at the surface of the cylinder where the velocity is V. Applying Bernoulli's equation, we have the following expressions.

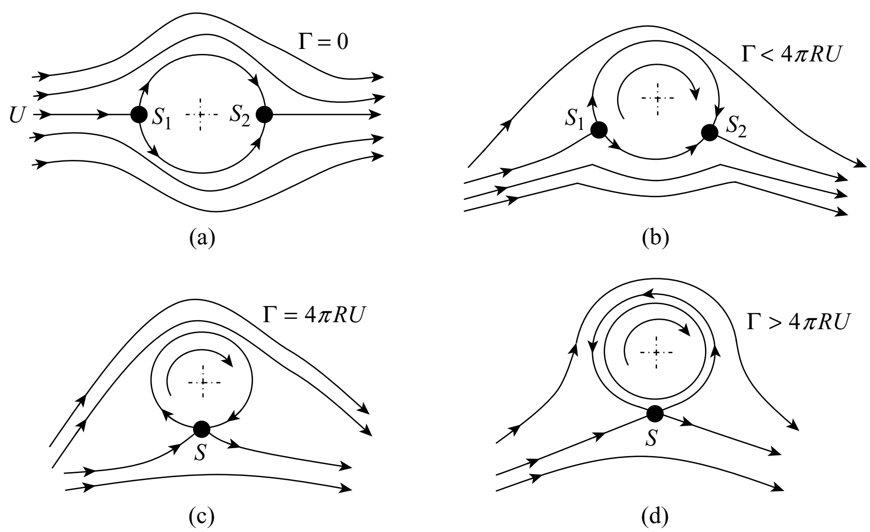


Figure 9.25 *Flow pattern over a rotating circular cylinder*

$$\frac{p}{\rho g}+\frac{V^2}{2g}=\frac{p_\infty}{\rho g}+\frac{U^2}{2g} \quad [\because z_1 = z_2]$$

$$\frac{p}{\rho g}-\frac{p_\infty}{\rho g}=\frac{U^2}{2g}-\frac{V^2}{2g}$$

$$p-p_\infty=\frac{1}{2}\rho U^2\left[1-\left(\frac{V}{U}\right)^2\right]$$

But
$$\frac{V}{U}=-\left(2\sin\alpha+\frac{\Gamma}{2\pi UR}\right) \quad \text{[From Equation (9.64)]}$$

Thus
$$p-p_\infty=\frac{1}{2}\rho U^2\left[1-\left(2\sin\alpha+\frac{\Gamma}{2\pi UR}\right)^2\right] \tag{9.66}$$

When $\alpha=\pi/2$, $\sin\alpha=1$ and p will become pressure at the top, i.e., p_{top} and thus, Equation (9.66) can be rewritten as follows.

$$p_{\text{top}}-p_\infty=\frac{1}{2}\rho U^2[1-(2+k)^2]\text{, here } k=\Gamma/(2\pi UR)$$

$$p_{\text{top}}-p_\infty=-\frac{1}{2}\rho U^2[3+4k+k^2] \tag{9.67a}$$

Similarly, when $\alpha=-\pi/2$, $\sin\alpha=-1$ and p will become pressure at the bottom, i.e., p_{bottom} and thus, Equation (9.66) can be written as follows.

$$p_{\text{bottom}}-p_\infty=-\frac{1}{2}\rho U^2[3-4k+k^2] \tag{9.67b}$$

Subtracting Equation (9.67(b)) from Equation (9.67(a)) and rearranging, we get:

$$\frac{p_{\text{top}}-p_{\text{bottom}}}{(1/2)\rho U^2}=-8k=-8\times\frac{\Gamma}{2\pi UR}=-\frac{4\Gamma}{\pi UR} \tag{9.68}$$

This pressure difference between the top and bottom sides causes an upward force (lift force). This phenomenon of lift generation by a spinning cylinder in a uniform flow is known as Magnus effect. The expression for amount of lift generated per unit length of the cylinder is given below.

$$\boxed{\text{Lift}=\rho U\Gamma} \tag{9.69}$$

From Equation (9.69), it can be seen that the lift force is independent of radius. The derivation of this expression is given in Chapter 16.

Example 9.13 (i) If a circular cylinder of diameter 1 m rotates at 300 rpm in a uniform stream of 10 m/s, then locate the stagnation point. (ii) Also determine the minimum rotational speed for detached stagnation points.

Solution

Let $D=1$ m, $N=300$ rpm and $U=10$ m/s.

(i) $u_\alpha=\dfrac{\pi DN}{60}=\dfrac{\pi\times1\times300}{60}=15.71$ m/s

$$\Gamma=2\pi Ru_\alpha=2\pi\times\frac{1}{2}\times15.71=49.354\text{ m}^2/\text{s}$$

For the stagnation points, we get:

$$\sin\alpha = -\frac{\Gamma}{4\pi RU} = -\frac{49.354}{4\pi\times(1/2)\times 10} = -0.7855$$

$$\therefore \alpha = \sin^{-1}(-0.7855) = \mathbf{-51.77° \text{ and } -128.23°}$$

(ii) The stagnation points coincide and are at the verge of detachment when,

$$\sin\alpha = -\frac{\Gamma}{4\pi RU} = -1 \text{ at } \alpha = -\frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$\sin\left(-\frac{\pi}{2}\right) = -\frac{\Gamma}{4\pi RU}$$

Thus $\Gamma = 4\pi RU$

$$2\pi R u_\alpha = 4\pi RU \quad [\because \Gamma = 2\pi R u_\alpha]$$

Thus $u_\alpha = 2U$

$$\therefore u_\alpha = 2\times 10 = 20 \text{ m/s}$$

But $\dfrac{\pi DN}{60} = 20$

$$\therefore N = \frac{20\times 60}{\pi D} = \frac{20\times 60}{\pi\times 1} = \mathbf{381.972\ rpm}$$

Example 9.14 If a long circular cylinder of diameter 1 m lies in an air stream ($\rho = 1.2$ kg/m^3) of velocity 70 m/s and there is an additional flow around the cylinder with clockwise circulation of 420 m^2/s, then calculate the followings by neglecting viscous and compressibility effects if any (i) maximum velocity due to air stream alone on the cylinder, (ii) velocity at the cylinder surface due to circulation alone, (iii) maximum velocity on the surface of the cylinder, (iv) location of stagnation points, (v) difference of pressures between top and bottom sides of the cylinder and (vi) lift force on the cylinder per units its length.

Solution

Let $D = 1$ m, $\rho = 1.2$ kg/m^3, $U = 70$ m/s and $\Gamma = 420$ m^2/s.

The velocity on the surface of the cylinder is given by,

$$u_\alpha = V = -\left(2U\sin\alpha + \frac{\Gamma}{2\pi R}\right)$$

(i) Due to air stream alone, $V = -2U\sin\alpha$, which is maximum at $\alpha = \pi/2$ or $\sin\alpha = 1$, i.e., at the top of the cylinder. Thus, the expression for maximum velocity is given below.

$$V_{max} = -2U = -2\times 70 = \mathbf{-140\ m/s}$$

(ii) Velocity due to air circulation alone is given by,

$$V = -\frac{\Gamma}{2\pi R} = -\frac{420}{2\pi\times(1/2)} = \mathbf{-133.69\ m/s}$$

(iii) Maximum velocity due to air stream and circulation can be obtained by adding the velocity due to air stream alone and due to circulation alone as follows.

$$V_{max} = (-140) + (-133.69) = \mathbf{-273.69\ m/s}$$

(iv) For the stagnation points, we get:

$$\sin\alpha = -\frac{\Gamma}{4\pi RU} = -\frac{420}{4\pi\times(1/2)\times 70} = -0.955$$

$$\therefore \alpha = \sin^{-1}(-0.955) = \mathbf{-72.75^\circ \text{ and } -107.25^\circ}$$

(v) Pressure difference between the top and bottom sides of the cylinder can be obtained by using Equation (9.67) as given below.

$$p_{\text{top}} - p_{\text{bottom}} = -\frac{1}{2}\rho U^2 \times \frac{4\Gamma}{\pi UR}$$

$$\therefore (p_{\text{top}} - p_{\text{bottom}}) = -\frac{1}{2}\times 1.2\times 70^2 \times \frac{4\times 420}{\pi\times 70\times(1/2)} = \mathbf{-44919.89\ N/m^2}$$

The negative sign indicates the lift force.

(vi) Lift force generated per unit length of the cylinder is given by,

$$\text{Lift} = \rho U\Gamma = \frac{1.2\times 70\times 420}{10^3} = \mathbf{35.28\ kN/m}$$

Summary

1. **Potential flow:** A fluid flow with $\mu = 0$ and $\rho = \text{Constant}$.
2. **Uniform flow:** The flow velocity remains constant at any cross section.
3. **For uniform flow equations for streamline and potential lines:** (i) Parallel to x-axis is $\psi = Uy$ and $\phi = Ux$ (ii) parallel to y-axis is $\psi = -Ux$ and $\phi = Uy$.
4. **Source flow:** The flow coming out from a single point and moving radially out in all directions at a constant rate. The equations for streamline and potential line are $\psi = C\theta$ and $\phi = C\ln r$.
5. **Sink flow:** The flow moving radially inwards in a plane towards a point where it disappears at a constant rate. The sink flow is just opposite to the source flow.
6. **Free vortex flow:** A purely circulatory flow such that the centre of the vortex is singular point with a circulation Γ. The equations for stream and potential functions are $\psi = -(\Gamma/2\pi)\ln r$ and $\phi = (\Gamma/2\pi)\alpha$.
7. The stream function ψ and the velocity function ϕ for the flow past a half body is $\psi = \frac{q}{2\pi}(\alpha_1 - \alpha_2) = -\frac{q}{2\pi}\beta$ and $\phi = Ur\cos\alpha + \frac{q}{2\pi}\ln r$.
8. The velocity components for the flow past a half body is $u_r = U\cos\alpha + q/(2\pi r)$ and $u_\alpha = -U\sin\alpha$.
9. At a stagnation point, all components of velocity are zero.
10. The stream function ψ and the velocity function ϕ for the source-sink pair is $\psi = \frac{q}{2\pi}(\alpha_1 - \alpha_2) = -\frac{q}{2\pi}\beta$ and $\phi = \frac{q}{2\pi}\ln\frac{r_1}{r_2}$.
11. A doublet (or dipole) is a special case of a source and sink pair of equal strength q wherein both approach each other in such a way that the distance $2a$ between them approaches zero and the product $2a\cdot q$, also called the doublet strength μ remains constant. Thus, the doublet strength is given by $\mu = 2a\cdot q$.
12. The stream function ψ and the velocity function ϕ for a doublet is $\psi = -\frac{\mu}{2\pi}\times\frac{r\sin\alpha}{r^2} = -\frac{\mu}{2\pi}\frac{\sin\alpha}{r}$ and $\phi = \frac{\mu}{2\pi}\frac{\cos\alpha}{r}$.
13. A doublet placed in a uniform flow makes an important flow pattern past a circular cylinder (or a Rankine oval of equal axes, i.e., a circle).
14. The stream function ψ and the velocity function ϕ for the flow past a circular cylinder is $\psi = \left(Ur - \frac{\mu}{2\pi r}\right)\sin\alpha$ and $\phi = \left(Ur + \frac{\mu}{2\pi r}\right)\cos\alpha$.
15. The velocity components at any point in the flow field for the flow past a circular cylinder is $u_r = U\left(1 - \frac{R^2}{r^2}\right)\cos\alpha$ and $u_\alpha = -U\left(1 + \frac{R^2}{r^2}\right)\sin\alpha$.

16. A uniform flow parallel to x-axis combined with a source-sink pair of equal strength located on the x-axis with the origin of coordinates midway between them results in a flow past a Rankine oval body.

17. The value of ψ and ϕ for the flow past a Rankine oval body is $\psi = Ur\sin\alpha + \frac{q}{2\pi}(\alpha_1 - \alpha_2)$ and $\phi = Ur\cos\alpha + \frac{q}{2\pi}\ln\frac{r_1}{r_2}$ and the position of stagnation points on the x-axis is $r_s = a\sqrt{1+q/(\pi a U)}$.

18. The maximum width of flow past a Rankine oval body is $y_m = a\cot(\pi U y_m/q)$.

Multiple-choice Questions

1. A stationary tornado is similar to which one of the following composite flow?
(a) Free vortex and sink flow.
(b) Free vortex and uniform flow.
(c) Source and sink flow.
(d) All the above.

2. The velocity potential flow for a source flow changes with respect to radial distance r from the source point as
(a) r
(b) $\ln r$
(c) $1/r$
(d) None of the above.

3. The potential lines in a doublet are
(a) Concentric circles with centre on the y-axis.
(b) Circles tangent to the x-axis.
(c) Circles tangent to the y-axis.
(d) None of the above.

4. Flow pattern around a nosed bridge pier can be represented by
(a) A doublet in a uniform flow.
(b) A source and sink of equal strength.
(c) A source and sink in a uniform flow.
(d) A source in a uniform flow.

5. Flow over an elliptical body may be idealized by superimposing a uniform flow and
(a) a doublet.
(b) a source.
(c) a free vortex.
(d) a source and sink pair.

6. The dividing streamline for a uniform flow superimposed over a two-dimensional doublet is
(a) an ellipse. (b) a sphere.
(c) a circle. (d) a straight line.

7. The stagnation points for uniform flow past a cylinder coincide when circulation of the cylinder is
(a) Equal to $4\pi RU$.
(b) Zero.
(c) Less than zero.
(d) None of the above.

8. The stagnation points in a uniform flow around a circular cylinder are located at
(a) 0° and π.
(b) 0° and $\pi/2$.
(c) $\pi/2$ and π.
(d) None of the above

Review Questions

1. What do you mean by uniform flow? Obtain the expressions for stream function and velocity function for uniform flow of an ideal fluid parallel to (i) x-axis, (ii) y-axis and (iii) inclined to x-axis. Also plot the streamlines and equipotential lines.

2. Define source flow. Obtain expressions for stream function, velocity function and pressure distribution for source flow. Also plot the streamlines and equipotential lines.

3. Define free vortex flow. Obtain the expressions for stream function and velocity function. Also plot the streamlines and equipotential lines.

4. What do you mean by superimposed flow? Explain how the contour of a half body is obtained.

5. Differentiate between a source and sink with sketches.

6. Derive expressions and draw streamlines and equipotential lines for a source-sink pair of equal strength q kept at a distance a from the origin.

7. What do you mean by a doublet and the strength of a doublet? How is it possible to obtain the flow pattern of a doublet?

8. Explain flow past a circular cylinder with a neat sketch. Obtain expressions for stream and velocity potential functions. Also show the pressure variations.

9. Discuss and explain the flow past a Rankine oval body with neat sketches. Also derive expressions for stream function and dimensions of the body.

10. Explain and sketch the flow pattern of an ideal fluid flow past a cylinder with circulation. Also derive the expressions for stream function, velocity function and velocity components.

Problems

1. Two discs one over the other are placed in a horizontal plane. The water enters at the centre of the lower disc and flows radially outward from a source of strength 0.72 m^2/s. If at a radius of 4 cm the pressure is 150 kPa, then determine (i) the pressure at a radius of 0.4 m and (ii) stream function at angles of 30° and 90° if $\psi = 0$ at $\alpha = 0°$.

[**Ans.** 154.065 kPa, 0.06 m^2/s, 0.18 m^2/s]

2. If a cyclone is approximated as a free vortex flow for which the wind velocity at a distance of 5 km from the centre of the cyclone is 40 km/h, then determine the pressure gradient at that point. Also determine the difference in pressure over a radial distance of 2.5 km from that point towards the centre of cyclone. Take density of air as 1.2 kg/m^3.

[**Ans.** 0.0296 Pa/m, 222.18 Pa]

3. A free stream flow of velocity 2.5 m/s is superimposed over a plane source of strength 20 m^2/s. Both flows are in the same plane and a point A lies in the flow field. If the distance of the point A from the source is 0.4 m and it is at an angle of $(\pi/6)$ to the free stream flow, then determine (i) the stream function and resultant velocity at the given point and (ii) location of stagnation point from the source.

[**Ans.** 2.17 m^2/s, 10.2 m/s, 1.273 m]

4. A uniform flow at 5 m/s is superimposed on a source placed at the origin. If the stagnation point occurs at (–0.32, 0), then determine the strength of source, maximum width of the Rankine half body and the flow velocity at a point $(0.5, \pi/2)$ in the flow field.

[**Ans.** 10 m^2/s, 2 m, 5.93 m/s]

5. A uniform flow with a velocity of 3 m/s is superimposed over a source placed at the origin. If the stagnation point lies at (–0.4, 0), then find (i) the strength of the source, (ii) maximum width of the Rankine body and (iii) other principal dimensions of the Rankine body.

[**Ans.** 7.54 m^2/s, 1.26 m, 2.52 m, 0.628 m, 1.256 m]

6. The discharge of 25 m^3/s waste from a plant into a 10 m deep river flowing at 0.2 m/s may be modelled as a two-dimensional source spanning across the river depth. If the discharge is located in the middle of the river, then find the extent of the critical region where the aquatic animals will be unaffected.

[**Ans.** 1.99 m]

7. Determine the velocity and stream function at point (1, 1) for a source-sink pair with the source located at (–1, 0) and the sink located at (1, 0). The strengths of the source and the sink are 5 m^2/s and 10 m^2/s, respectively. Also determine the pressure if the density of the fluid is 2 kg/m^3 and the pressure at infinity is zero.

[**Ans.** 0.318 m/s, 1.432 m/s, 1.466 m/s, –2.149 Pa]

8. A plane half body is to be designed for the nose of a solid strut 76 mm wide placed in an infinite two-dimensional air stream of velocity 10 m/s and density 1.22 kg/m^3. Determine (i) the strength of the source, (ii) gap between the stagnation point and the source and (iii) pressure difference between the stagnation point on the strut where it is 38 mm wide and the resultant velocity.

[**Ans.** 0.76 m^2/s, 0.0121 m, 85.8 Pa, 11.86 m/s]

9. A source flow of strength 10 m^2/s is superimposed on a uniform flow of velocity 2 m/s. Determine the location of the stagnation point, maximum width of the half body and width of the half body at source.

[**Ans.** (–0.7958, 0), 5 m, 2.5 m]

10. State about the pattern of the streamlines if for a two-dimensional doublet the streamline equation is $x^2 + y^2 - (k/c)$ $y = 0$. Also find the magnitude and direction of velocity at point $P(3, 5)$, where $u = 4$ m/s.

[**Ans.** 8.5 m/s, –61.93°]

11. The strength of a doublet is 20 m^2/s. Determine the velocity at a point (2, 1) located in the field of the doublet and the value of stream function passing through it.

[**Ans.** 0.64 m/s, –0.64 m^2/s]

12. If a point $P(0.5, 1)$ is located in the flow field of a doublet of strength 10 m^2/s, then determine the velocity at the given point. Also find the value of stream function.

[**Ans.** 1.273 m/s, –1.273 m^2/s]

13. A 40 mm diameter cylinder is immersed in a fluid stream of 1 m/s uniform velocity. Determine the radial and tangential velocity components on a streamline at a point where radius is 40 mm and the angle is 120° measured from the positive x-axis. Also determine the resultant velocity at the given point. Assume the flow to be ideal.

[**Ans.** –0.375 m/s, –1.08 m/s, 1.143 m/s]

14. A uniform flow of 5 m/s is flowing over a doublet of strength 7.5 m^2/s. If the doublet is in the line of the uniform flow and the polar coordinates of a point in the flow field are 0.5 m and 30°, then determine the streamline function and the resultant velocity at the given point.

[**Ans.** 0.056 m^2/s, 4.884 m/s]

15. If a doublet of strength 25.5 m^2/s lies in the line of a uniform flow of 14.5 m/s, then what is the shape of the Rankine body formed? Also find the dimensions of the Rankine body and the value of the tangential and radial velocities of the surface at a point 30° from the *x*-axis.

[**Ans.** Cylindrical shape, 0.529 m, 0, 0, –14.5 m/s]

16. A source and sink of strength 25 m^2/s are located at a distance of 1.5 m. If a uniform flow of 10 m/s parallel to the line joining the source-sink pair is superimposed, then find the length of the Rankine oval body formed and the distance of the stagnation points from the source.

[**Ans.** 2.16 m, 0.33 m, 1.83 m]

17. If a circular cylinder of diameter 0.5 m rotates at 600 rpm in a uniform stream of 15 m/s, then locate the stagnation point. Also determine the minimum rotational speed for detached stagnation points.

[**Ans.** –31.5º, –148.5°, 1145.9 rpm]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (a)	**2.** (b)	**3.** (c)	**4.** (d)	**5.** (d)
6. (c)	**7.** (a)	**8.** (a)		

Chapter 10

Flow Through Orifices and Mouthpieces

10.1 ❐ INTRODUCTION

An orifice is an opening of any cross section, which may be circular, triangular, square or rectangular, provided in the walls or bottom of the tank (or vessel) through which the fluid is discharged. The top edge of the orifice always lies below the free surface of the liquid in the container. Usually, the orifice is used to measure the rate of flow of fluid.

A mouthpiece is an attachment in the form of a short length of tube or pipe. It is two to three times its diameter in length and is fitted to a circular opening or orifice of the same diameter provided in a tank or vessel containing fluid, through which fluid is discharged. It is also used for measuring the rate of flow of a fluid from a tank or reservoir. Mouthpieces are designed to improve the coefficient of discharge of orifices which leads to increase the amount of discharge. In this chapter, orifice and mouthpieces are described in detail.

10.2 ❐ CLASSIFICATION OF ORIFICES

The orifices may be classified on the basis of their shape, size, shape of the upstream edges and the discharge conditions are discussed below.

1. According to the shape, the orifices may be classified as circular, rectangular, square and triangular orifices depending upon its cross-sectional area. Mostly, the circular and rectangular orifices are used.
2. According to the size, the orifices may be classified as small and large orifices. If the head of the liquid from the centre of the orifice is more than five times the depth of orifice, then it is called small orifice and if the head of liquid is less than five times the depth of the orifice, then it is known as large orifice.
3. According to the shape of the upstream edge, the orifices may be classified as sharp-edged orifices and bell-mouthed orifices (or orifices with round corners). Due to minimum frictional effects, a sharp-edged orifice is considered as a standard orifice and it is most commonly used for the purpose of discharge measurement.
4. According to the discharge conditions, the orifices may be classified as orifices discharging free orifices and submerged orifices (or drowned orifices). A free discharging orifice discharges into the atmosphere, whereas a submerged orifice discharges into another tank of fluid. The submerged orifices (or drowned orifices) may be further classified as fully submerged orifices and partially submerged orifices.

10.3 ❒ FLOW THROUGH AN ORIFICE

Figure 10.1 illustrates a small circular orifice with sharp edge in one of the side wall of a tank and it discharges freely into the atmosphere.

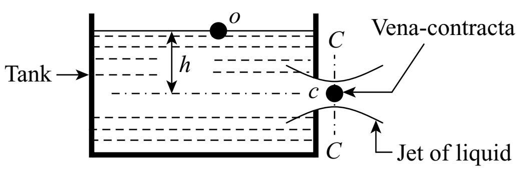


Figure 10.1 *A circular sharp-edged orifice discharging free*

Let h be the head of the liquid above the centreline of the orifice. As the liquid flows through the orifice, it forms a jet of liquid and attains a parallel form at an approximate distance of half the diameter of the orifice. The section C–C of the jet, at which the streamlines are straight, parallel to each other and perpendicular to the plane of the orifice and the jet has minimum cross-sectional area is known as vena contracta (Latin word meaning contracted vein or jet). Beyond this section C–C, the jet diverges and bends in the downward direction due to gravity.

Assuming (i) flow through the orifice is steady under a constant head h and (ii) loss of energy due to flow of liquid through the orifice is zero. Consider two points 'o' and 'c' as shown in Figure 10.1. The point 'o' is at the free surface of the liquid in the tank and the point 'c' is at the centre of the vena contracta. Applying Bernoulli's equation between the points 'o' and 'c', we get the below expression.

$$\frac{p_o}{\rho g}+\frac{V_o^2}{2g}+z_o=\frac{p_c}{\rho g}+\frac{V_c^2}{2g}+z_c$$

However, $p_o = p_c = p_{\text{atm}}$, $z_o = (z_c + h)$, $V_o = 0$ ($\because V_o << V_c$) and $V_c = V_{\text{th}}$, here V_{th} is the theoretical velocity (since losses are neglected).

Thus
$$\frac{p_{\text{atm}}}{\rho g}+\frac{0^2}{2g}+(z_c+h)=\frac{p_{\text{atm}}}{\rho g}+\frac{V_{\text{th}}^2}{2g}+z_c$$

$$h=\frac{V_{\text{th}}^2}{2g} \tag{10.1}$$

$$\boxed{\therefore V_{\text{th}}=\sqrt{2gh}} \tag{10.2}$$

The Equation (10.2) is also known as Torricelli's equation in honour of Evangelista Torricelli who deduced this equation in 1643. It gives the theoretical velocity (V_{th}) or ideal velocity of flow through an orifice which equals the velocity of free fall from the surface of the tank. Due to viscous effects, the actual velocity of flow through an orifice will always be less than the theoretical velocity given by Equation (10.2).

10.4 ❒ HYDRAULIC COEFFICIENTS (COEFFICIENTS FOR AN ORIFICE)

The hydraulic coefficients, namely coefficient of velocity, coefficient of contraction, coefficient of discharge and coefficient of resistance are discussed below. The orifices are calibrated with the help of hydraulic coefficients for using them as a precise discharge measuring device.

Coefficient of velocity (C_v) It is defined as the ratio of the actual velocity of the jet at vena contracta (V) to the theoretical (ideal) velocity of the jet (V_{th}). It is denoted by C_v and it is mathematically expressed as follows.

$$\boxed{C_v=\frac{V}{V_{\text{th}}}=\frac{V}{\sqrt{2gh}}} \tag{10.3}$$

$$V=C_v\sqrt{2gh} \tag{10.4}$$

The difference between the theoretical and the actual velocities of the jet at vena contracta is mainly due to friction at the orifice. The value of C_v varies from 0.95 to 0.99 for different orifices, depending upon the shape and size of the orifice and on the head of liquid under which flow occurs. However, the average value of C_v is assumed as 0.98 for sharp-edged orifices discharging water and other liquids of similar viscosity.

For determining the loss of head h_f through an orifice, applying the Bernoulli's equation between the points 'o' and 'c' (Figure 10.1), we get the below expression.

$$\frac{p_o}{\rho g}+\frac{V_o^2}{2g}+z_o=\frac{p_c}{\rho g}+\frac{V_c^2}{2g}+z_c+h_f$$

However, $p_o = p_c = p_{\text{atm}}$, $z_o=(z_c+h)$, $V_o=0$ ($\because V_o << V_c$) and $V_c = V$, here V is the actual velocity (since losses are considered).

Thus
$$\frac{p_{\text{atm}}}{\rho g}+\frac{0^2}{2g}+(z_c+h)=\frac{p_{\text{atm}}}{\rho g}+\frac{V^2}{2g}+z_c+h_f$$

$$h=\frac{V^2}{2g}+h_f$$

$$h_f=h-\frac{V^2}{2g}=h\left(1-\frac{V^2}{2gh}\right)=h(1-C_v^2) \tag{10.5}$$

$$h_f=\frac{V^2}{2g}\left(\frac{2gh}{V^2}-1\right)=\frac{V^2}{2g}\left(\frac{1}{C_v^2}-1\right) \tag{10.6}$$

Coefficient of contraction (C_c) It is defined as the ratio of the area of the jet at vena contracta (a_c) to the area of the orifice (a). It is denoted by C_c and it is mathematically expressed as follows.

$$\boxed{C_c=\frac{a_c}{a}} \tag{10.7}$$

$$a_c=C_c\times a \tag{10.8}$$

The value of C_c for orifices varies from 0.61 to 0.69 and its average value is assumed as 0.64.

Coefficient of discharge (C_d) It is defined as the ratio of the actual discharge from an orifice (Q) to its theoretical discharge (Q_{th}). It is denoted by C_d and it is mathematically expressed as follows.

$$\boxed{C_d=\frac{Q}{Q_{\text{th}}}=\frac{V}{V_{\text{th}}}\times\frac{a_c}{a}=C_v\times C_c} \tag{10.9}$$

Equation (10.9) shows that the coefficient of discharge of an orifice is the product of coefficient of velocity and coefficient of contraction. The value of C_d varies from 0.62 to 0.65 and its average value is taken as 0.62.

Coefficient of resistance (C_r) It is defined as the ratio of the loss of kinetic energy (K.E.) as the liquid flows through an orifice to the actual kinetic energy possessed by the flowing liquid. It is denoted by C_r and it is mathematically expressed as follows.

$$C_r=\frac{\text{Loss of K.E.}}{\text{Actual K.E.}}=\frac{\text{Theoretical K.E.}-\text{Actual K.E.}}{\text{Actual K.E.}}=\frac{\text{Loss of head}}{\text{Actual head}}$$

$$\text{Theoretical K.E./Weight}=\frac{V_{\text{th}}^2}{2g}=\frac{(\sqrt{2gh})^2}{2g}=h$$

$$\text{Actual K.E./Weight}=\frac{V^2}{2g}=\frac{(C_v\sqrt{2gh})^2}{2g}=hC_v^2$$

Thus $$C_r = \frac{h - hC_v^2}{hC_v^2} = \left(\frac{1}{C_v^2} - 1\right) \tag{10.10}$$

The coefficient of resistance C_r accounts for the loss of energy during flow of liquid through an orifice and its value can be determined from Equation (10.10) by knowing the value of C_v.

Example 10.1 The actual velocity at the vena contracta of a water jet coming out from an orifice of diameter 4 cm is 9.5 m/s. If the orifice is working under a head of 5 m and the measured discharge is 0.0078 m^3/s, then determine (i) the coefficient of velocity, (ii) coefficient of discharge, (iii) coefficient of contraction and (iv) coefficient of resistance.

Solution

Let $d = 4\text{ cm} = 0.04\text{ m}$, $V = 9.5\text{ m/s}$, $h = 5\text{ m}$ and $Q = 0.0078\text{ m}^3/\text{s}$.

$$a = \frac{\pi}{4}d^2 = \frac{\pi}{4} \times 0.04^2 = 0.001257\text{ m}^2$$

$$V_{\text{th}} = \sqrt{2 \times 9.81 \times 5} = 9.9\text{ m/s}$$

$$Q_{\text{th}} = aV_{\text{th}} = 0.001257 \times 9.9 = 0.01244\text{ m}^3/\text{s}$$

(i) $C_v = \dfrac{V}{V_{\text{th}}} = \dfrac{9.5}{9.9} = \mathbf{0.96}$

(ii) $C_d = \dfrac{Q}{Q_{\text{th}}} = \dfrac{0.0078}{0.01244} = \mathbf{0.627}$

(iii) $C_c = \dfrac{C_d}{C_v} = \dfrac{0.627}{0.96} = \mathbf{0.653}$

(iv) $C_r = \left(\dfrac{1}{C_v^2} - 1\right) = \left(\dfrac{1}{0.96^2} - 1\right) = \mathbf{0.0851}$

Example 10.2 If an orifice of diameter 5 cm discharges water under a head of 10 m, coefficient of discharge is 0.62 and coefficient of velocity is 0.97, then determine (i) the actual discharge and (ii) actual velocity of the jet at vena contracta.

Solution

Let $d = 5\text{ cm} = 0.05\text{ m}$, $h = 10\text{ m}$, $C_d = 0.62$ and $C_v = 0.97$.

$$a = \frac{\pi}{4}d^2 = \frac{\pi}{4} \times 0.05^2 = 0.0019635\text{ m}^2$$

$$V_{\text{th}} = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 10} = 14.01\text{ m/s}$$

$$Q_{\text{th}} = aV_{\text{th}} = 0.0019635 \times 14.01 = 0.02751\text{ m}^3/\text{s}$$

(i) $Q = C_d Q_{\text{th}} = 0.62 \times 0.02751 = \mathbf{0.01706\text{ m}^3/\text{s}}$

(ii) $V = C_v V_{\text{th}} = 0.97 \times 14.01 = \mathbf{13.59\text{ m/s}}$

10.5 ❒ EXPERIMENTAL DETERMINATION OF HYDRAULIC COEFFICIENTS

There are many methods by which the value of each of the hydraulic coefficients of an orifice can be determined. The commonly adopted methods are discussed below.

10.5.1 Determination of Coefficient of Velocity (C_v)

Consider a tank provided with a small orifice discharging water under a constant head h which is maintained by a constant supply (Figure 10.2). Let a liquid particle takes t time to travel along the jet from vena contracta (section C–C) to position P.

Let x be the horizontal distance travelled by the particle in time t, y be the vertical distance between the centre of the vena contracta and point P, V be the actual velocity of the jet at vena contracta and V_{th} be the theoretical velocity of the jet.

The horizontal and vertical distances are respectively given below.

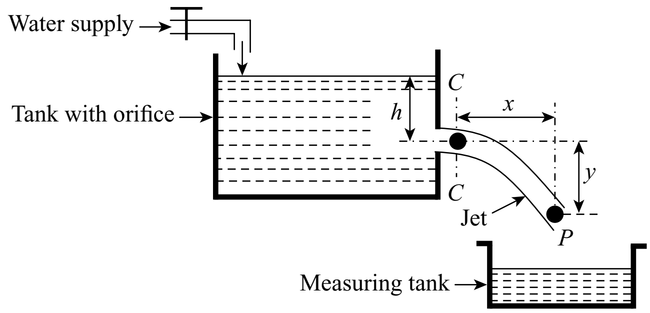


Figure 10.2 *Determination of hydraulic coefficients*

$$x = Vt \quad \text{(i)}$$

$$y = \frac{1}{2}gt^2 \quad \text{(ii)}$$

Substituting $t = (x/V)$ from expression (i) in expression (ii), we get:

$$y = \frac{1}{2}g\left(\frac{x}{V}\right)^2 \Rightarrow V^2 = \frac{gx^2}{2y}$$

Thus $$V = \sqrt{\frac{gx^2}{2y}}$$

$$V_{\text{th}} = \sqrt{2gh}$$

$$C_v = \frac{V}{V_{\text{th}}} = \frac{\sqrt{(gx^2)/(2y)}}{\sqrt{2gh}} = \sqrt{\frac{x^2}{4yh}} = \frac{x}{\sqrt{4yh}} \quad (10.11)$$

10.5.2 Determination of Coefficient of Discharge (C_d)

The water flowing through the orifice of area a under a constant head h is collected in a measuring tank as shown in Figure 10.2. The rise of water level in the measuring tank in a specific time t is recorded. The expression for actual discharge through the orifice is given below.

$$Q = \frac{\text{Area of measuring tank} \times \text{Rise of water level}}{\text{Time}}$$

$$Q_{\text{th}} = a\sqrt{2gh}$$

$$C_d = \frac{Q}{Q_{\text{th}}} = \frac{Q}{a\sqrt{2gh}} \quad (10.12)$$

$$\boxed{\therefore Q = C_d a\sqrt{2gh}} \quad (10.13)$$

10.5.3 Determination of Coefficient of Contraction (C_c)

After knowing the values of C_v and C_d, the coefficient of contraction (C_c) can be determined from the following relation.

$$C_d = C_v \times C_c$$

$$\boxed{\therefore C_c = \frac{C_d}{C_v}} \tag{10.14}$$

Example 10.3 The water discharged through an orifice of diameter 10 cm working under a head of 10 m is collected in a circular tank of diameter 1.4 m. If the water level in the measuring tank rises by 0.9 m in 20 seconds and the coordinates of a point on the water jet measured from vena contracta are 4.5 m horizontal and 0.6 m vertical, then determine (i) the coefficient of velocity, (ii) coefficient of discharge and (iii) coefficient of contraction.

Solution

Let $d = 10 \text{ cm} = 0.1 \text{ m}$, $h = 10 \text{ m}$, $d_1 = 1.4 \text{ m}$, $h_1 = 0.9 \text{ m}$, $t = 20 \text{ s}$, $x = 4.5 \text{ m}$ and $y = 0.6 \text{ m}$.

$$a = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 0.1^2 = 0.007854 \text{ m}^2$$

$$V_{\text{th}} = \sqrt{2gh} = \sqrt{2\times 9.81\times 10} = 14.01 \text{ m/s}$$

$$Q_{\text{th}} = aV_{\text{th}} = 0.007854\times 14.01 = 0.11003 \text{ m}^3/\text{s}$$

$$a_1 = \frac{\pi}{4}d_1^2 = \frac{\pi}{4}\times 1.4^2 = 1.5394 \text{ m}^2$$

$$Q = \frac{a_1 h_1}{t} = \frac{1.5394\times 0.9}{20} = 0.0693 \text{ m}^3/\text{s}$$

(i) $C_d = \dfrac{Q}{Q_{\text{th}}} = \dfrac{0.0693}{0.11003} = \mathbf{0.63}$

(ii) $C_v = \dfrac{x}{\sqrt{4yh}} = \dfrac{4.5}{\sqrt{4\times 0.6\times 10}} = \mathbf{0.9185}$

(iii) $C_c = \dfrac{C_d}{C_v} = \dfrac{0.63}{0.9185} = \mathbf{0.686}$

Example 10.4 A tank has two identical orifices in one of its vertical side. The upper orifice is 2.5 m below the water surface and the lower one is 4.5 m below the water surface. If the coefficient of velocity for both the orifices is 0.97, then find the point at which the two jets intersect.

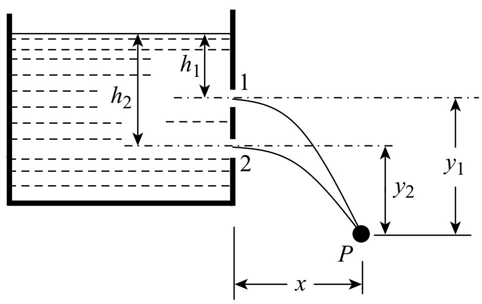


Figure 10.3

Solution

Refer Figure 10.3. Let $h_1 = 2.5$ m, $h_2 = 4.5$ m and $C_{v1} = C_{v2} = 0.97$.

Let x be the horizontal coordinate of the point of intersection of P, y_1 and y_2 be the vertical distance of P from orifice 1 and 2, respectively.

$$C_{v1} = \frac{x}{\sqrt{4y_1h_1}} = \frac{x}{\sqrt{4y_1\times 2.5}} \tag{i}$$

$$C_{v2} = \frac{x}{\sqrt{4y_2h_2}} = \frac{x}{\sqrt{4y_2\times 4.5}} \tag{ii}$$

Thus $$\frac{x}{\sqrt{4y_1 \times 2.5}} = \frac{x}{\sqrt{4y_2 \times 4.5}} \qquad [\because C_{v1} = C_{v2}]$$

$$\therefore y_1 = 1.8y_2 \tag{iii}$$

Since $$y_1 - y_2 = h_2 - h_1$$

Thus $$y_1 - y_2 = 4.5 - 2.5 = 2 \tag{iv}$$

Solving expressions (iii) and (iv), we get:

$$y_2 = 2.5$$

Substituting the values of $y_2 = 2.5$ and C_{v2} in expression (ii), we get:

$$0.97 = \frac{x}{\sqrt{4 \times 2.5 \times 4.5}}$$

$$\therefore x = 0.97 \times \sqrt{4 \times 2.5 \times 4.5} = \mathbf{6.507\ m}$$

Example 10.5 An orifice of diameter 11 cm discharges 50 litres of water per second while working under a constant head of 3.7 m. If a flat plate held normal to the jet downstream from the orifice needs a force of 400 N to resist the impact of jet, then determine the hydraulic coefficients.

Solution

Let $d = 11\text{ cm} = 0.11\text{ m}$, $Q = 50\ l/\text{s} = 50 \times 10^{-3}\ \text{m}^3/\text{s}$, $h = 3.7$ m and $F = 400$ N.

Since $$F = \rho_w QV \qquad \text{[Momentum equation]}$$

Thus $$400 = 1000 \times 50 \times 10^{-3} \times V$$

$$\therefore V = \frac{400}{50} = 8\ \text{m/s}$$

$$a = \frac{\pi}{4}d^2 = \frac{\pi}{4} \times 0.11^2 = 0.009503\ \text{m}^2$$

$$V_{\text{th}} = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 3.7} = 8.52\ \text{m/s}$$

$$Q_{\text{th}} = aV_{\text{th}} = 0.009503 \times 8.52 = 0.080965\ \text{m}^3/\text{s}$$

(i) $C_d = \dfrac{Q}{Q_{\text{th}}} = \dfrac{50 \times 10^{-3}}{0.080965} = \mathbf{0.6175}$

(ii) $C_v = \dfrac{V}{V_{\text{th}}} = \dfrac{8}{8.52} = \mathbf{0.939}$

(iii) $C_c = \dfrac{C_d}{C_v} = \dfrac{0.6175}{0.939} = \mathbf{0.6576}$

Example 10.6 An orifice of diameter 4 cm fitted in the vertical side of a tank discharges water under a constant head of 2 m. If the head loss in the orifice is 0.1 m and the coefficient of contraction is 0.64, then determine (i) the theoretical discharge through the orifice, (ii) coefficient of velocity and the coefficient of discharge, (iii) actual discharge through the orifice and (iv) location of the point of impact of the jet on a horizontal plane located 0.4 m below the centre of the orifice.

Solution

Let $d = 4$ cm $= 0.04$ m, $h = 2$ m, $h_f = 0.1$ m, $C_c = 0.64$ and $y = 0.4$ m.

(i) $V_{th} = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 2} = 6.2642$ m/s

$$V_{actual} = \sqrt{2g(h - h_f)} = \sqrt{2 \times 9.81 \times (2 - 0.1)} = 6.1056 \text{ m/s}$$

$$a = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 0.04^2 = 0.001257 \text{ m}^2$$

$$Q_{th} = aV_{th} = 0.001257 \times 6.2642 = \mathbf{0.0078741\ m^3/s}$$

(ii) $C_v = \dfrac{V_{actual}}{V_{th}} = \dfrac{6.1056}{6.2642} = \mathbf{0.9747}$

$$C_d = C_c C_v = 0.64 \times 0.9747 = \mathbf{0.6238}$$

(iii) $Q = C_d Q_{th} = 0.6238 \times 0.0078741 = \mathbf{4.912 \times 10^{-3}\ m^3/s}$

(iv) $x = C_v \sqrt{4yh} = 0.9747 \times \sqrt{4 \times 0.4 \times 2} = \mathbf{1.7436\ m}$

Example 10.7 A closed tank contains water up to a height of 1.64 m and the upper part of the tank is filled with air at a pressure of 82 kPa above atmospheric pressure. Determine the rate of flow of water through an orifice of diameter 12 cm fitted at the bottom. Take coefficient of discharge for the orifice as 0.62.

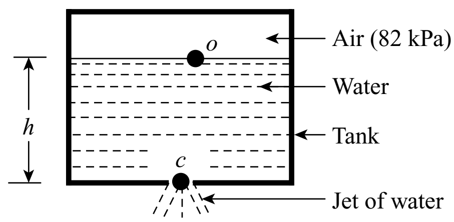


Figure 10.4

Solution

Refer Figure 10.4. Let $h = 1.64$ m, $p_o = p_{atm} + 82$ kPa, $d = 12$ cm $= 0.12$ m and $C_d = 0.62$.

Applying Bernoulli's equation to the water surface (i.e., point o) and the outlet of the orifice (i.e., point c), we get the below expression.

$$\frac{p_o}{\rho_w g} + \frac{V_o^2}{2g} + z_0 = \frac{p_c}{\rho_w g} + \frac{V_c^2}{2g} + z_c \qquad \text{(i)}$$

$$z_o = z_c + h = z_c + 1.64, V_o = 0 \text{ and } p_c/(\rho_w g) = p_{atm} = 10.3 \text{ m of water}$$

$$\frac{p_o}{\rho_w g} = p_{atm} + 82 \text{ kpa} = 10.3 + \frac{82 \times 10^3}{1000 \times 9.81} = 18.659 \text{ m of water}$$

Substituting the above values in expression (i), we get:

$$18.659 + 0 + (z_c + 1.64) = 10.3 + \frac{V_c^2}{2g} + z_c$$

$$\frac{V_c^2}{2g} = 9.999$$

$$\therefore V_c = \sqrt{2g \times 9.999} = \sqrt{2 \times 9.81 \times 9.999} = 14.01 \text{ m/s}$$

$$a = \frac{\pi}{4}d^2 = \frac{\pi}{4} \times 0.12^2 = 0.01131\ \text{m}^2$$

$$Q = C_d a V_c = 0.62 \times 0.01131 \times 14.01 = \mathbf{0.098241\ m^3/s}$$

Example 10.8 If the head loss in the flow through an orifice of diameter 2.5 cm working under a certain head is 8 cm of water and the velocity of water jet is 3.5 m/s, then determine (i) the head on the orifice causing flow, (ii) coefficient of velocity and (iii) diameter of the jet. Take coefficient of discharge of the orifice as 0.62.

Solution

Let $d = 2.5\ \text{cm} = 0.025\ \text{m},\ h_f = 8\ \text{cm} = 0.08\ \text{m},\ V = 3.5\ \text{m/s}$ and $C_d = 0.62.$

Let d_1 be the jet diameter.

(i) $h = \dfrac{V^2}{2g} + h_f = \dfrac{3.5^2}{2 \times 9.81} + 0.08 = \mathbf{0.7044\ m}$

(ii) $C_v = \dfrac{V}{\sqrt{2gh}} = \dfrac{3.5}{\sqrt{2 \times 9.81 \times 0.7044}} = \mathbf{0.9415}$

(iii) $C_c = \dfrac{C_d}{C_v} = \dfrac{0.62}{0.9415} = \mathbf{0.6585}$

Also
$$C_c = \frac{a_c}{a} = \frac{(\pi/4)d_1^2}{(\pi/4)d^2} = \frac{d_1^2}{d^2}$$

$$\therefore d_1 = \sqrt{C_c d^2} = \sqrt{0.6585 \times 0.025^2} = \mathbf{0.0203\ m\ or\ 2.03\ cm}$$

Example 10.9 A tank of height 4 m full of water is placed on the ground. There is a small orifice in its vertical side with its centre at depth h metres below the free surface of water in the tank. Determine the value of h so that the water jet strikes the ground at the maximum distance from the tank. Also determine the maximum value of the horizontal distance if the value of coefficient of velocity is 0.98.

Solution

Refer Figure 10.5. Let $H = 4$ m and $C_v = 0.98$.

Let V be the velocity, t be the time and g be the acceleration due to gravity. The horizontal and vertical distances are respectively given below.

$$x = Vt \qquad \text{(i)}$$

$$y = \frac{1}{2}gt^2 \qquad \text{(ii)}$$

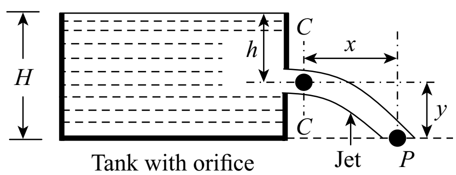


Figure 10.5

Substituting $t = (x/V)$ from expression (i) in expression (ii), we get:

$$y = \frac{1}{2}g\left(\frac{x}{V}\right)^2 \Rightarrow x^2 = \frac{2V^2 y}{g}$$

Substituting $V = C_v\sqrt{2gh}$ and $y = (H - h)$, we get:

$$x^2 = \frac{2(C_v\sqrt{2gh})^2(H-h)}{g}$$

Thus $$x = 2C_v\sqrt{h(H-h)} \quad \text{(iii)}$$

The horizontal distance will be maximum when $h(H-h)$ is maximum and the expression is given below.

$$\frac{d}{dh}[h(H-h)] = 0$$

$$H - 2h = 0$$

Thus $$h = H/2$$

$$\therefore h = \frac{4}{2} = \mathbf{2\ m}$$

Thus, the maximum horizontal distance can be obtained by expression (iii) as follows.

$$x_{max} = 2 \times 0.98 \times \sqrt{2 \times (4-2)} = \mathbf{3.92\ m}$$

10.6 ❐ DISCHARGE THROUGH A LARGE RECTANGULAR ORIFICE

In the case of a large orifice (head of liquid above the centre of orifice is less than five times the depth of orifice), the velocity of the jet cannot be considered constant over the entire cross section of the jet. Thus, the discharge is given by Equation (10.13) as $Q = C_d a\sqrt{2gh}$ for small orifice cannot be used for large orifices. Consider a large rectangular orifice on one side of a tank discharging freely into atmosphere under a constant head maintained by a constant supply (Figure 10.6).

Let h_1 be the height of liquid above the upper edge of the orifice, h_2 be the height of liquid above the lower edge of the orifice, $d = (h_2 - h_1)$ be the depth of the orifice, b be the width of the orifice and C_d be the coefficient of discharge.

Consider an elementary horizontal strip of depth dh at a depth h below the free surface of the liquid. The expression for discharge through the strip is given below.

$$dQ = C_d \times \text{Area of strip} \times \text{Velocity} = C_d \times b\,dh \times \sqrt{2gh} \quad \text{(i)}$$

The total discharge through the orifice can be obtained by integrating expression (i) between the limits h_1 and h_2 as follows.

$$Q = C_d b\sqrt{2g}\int_{h_1}^{h_2} h^{1/2}\,dh = C_d b\sqrt{2g} \times \left[\frac{h^{3/2}}{3/2}\right]_{h_1}^{h_2}$$

Thus $$Q = \frac{2}{3}C_d b\sqrt{2g} \times [h_2^{3/2} - h_1^{3/2}] \quad (10.15)$$

Example 10.10 A rectangular orifice of width 2 m and depth 1.5 m fitted to tank discharges water. If the coefficient of discharge of the orifice is 0.62 and the water level in the tank is 3.5 m above the top edge of the orifice, then determine the discharge through it.

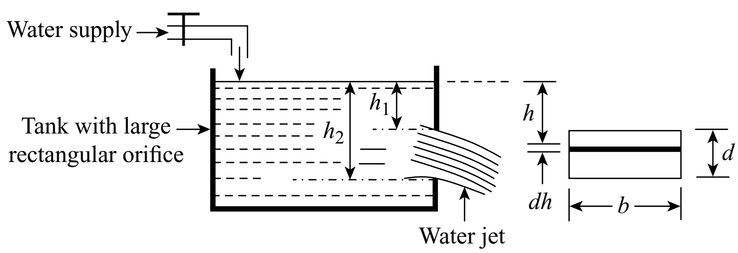


Figure 10.6 *Large rectangular orifice*

Solution

Refer Figure 10.6. Let $b = 2$ m, $d = 1.5$ m, $C_d = 0.62$ and $h_1 = 3.5$ m.

$$h_2 = h_1 + d = 3.5 + 1.5 = 5 \text{ m}$$

Since
$$Q = \frac{2}{3} C_d b \sqrt{2g} \times [h_2^{3/2} - h_1^{3/2}]$$

$$\therefore Q = \frac{2}{3} \times 0.62 \times 2 \times \sqrt{2 \times 9.81} \times [5^{3/2} - 3.5^{3/2}] = \mathbf{16.9625\ m^3/s}$$

10.7 ❒ DISCHARGE THROUGH SUBMERGED ORIFICES

At the discharge end, the orifices may be fully submerged or partially submerged and it is discussed in the following sections.

10.7.1 Fully Submerged Orifice (or Totally Drowned Orifice)

If an orifice has its whole of the outlet side submerged under liquid so that it discharges a jet of liquid into the liquid of same kind, then it is known as fully submerged orifice or totally drowned orifice (Figure 10.7).

Let h_1 be the height of liquid above the upper edge of the orifice, h_2 be the height of liquid above the lower edge of the orifice, $d = (h_2 - h_1)$ be the depth of the orifice, h be the difference in liquid levels in the two tanks, b be the width of the orifice, bd be the area of the orifice and C_d be the coefficient of discharge.

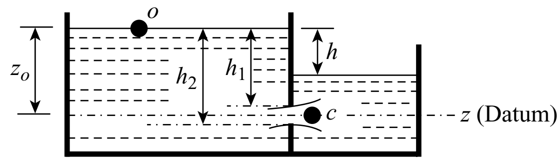


Figure 10.7 *Fully submerged orifice*

Consider two points '*o*' and '*c*' as shown in Figure 10.7. The point '*o*' is at the free surface of the liquid in the tank and point '*c*' is at the centre of the vena contracta. Applying Bernoulli's equation between the points '*o*' and '*c*', we get the below expression.

$$\frac{p_o}{\rho g} + \frac{V_o^2}{2g} + z_o = \frac{p}{\rho g} + \frac{V^2}{2g} + z \qquad \text{(i)}$$

Now
$$p_o = p_{atm} = 0 \text{ (assumed) and } V_o = 0.$$

$$z_o = h_1 + \frac{h_2 - h_1}{2} = \frac{h_1 + h_2}{2} \text{ and } \frac{p}{\rho g} = (z_o - h) = \frac{h_1 + h_2}{2} - h$$

Substituting the above values in expression (i), we get:

$$0 + 0 + \frac{h_1 + h_2}{2} = \left(\frac{h_1 + h_2}{2} - h\right) + \frac{V^2}{2g} + 0$$

$$h = \frac{V^2}{2g}$$

$$\therefore V = \sqrt{2gh}$$

Discharge through the orifice is given by,

$$Q = C_d \times \text{Area of orifice} \times \text{Velocity} = C_d \times b(h_2 - h_1) \times \sqrt{2gh} \qquad (10.16)$$

$$\boxed{Q = C_d \times b \times d \times \sqrt{2gh}} \qquad (10.16a)$$

Example 10.11 Determine the discharge through a fully submerged orifice of width 1.2 m and depth 1 m. If the difference in water levels on both sides of the orifice is 2 m, then take coefficient of discharge of the orifice as 0.62.

Solution

Refer Figure 10.7. Let $b = 1.2$ m, $d = 1$ m, $h = 2$ m and $C_d = 0.62$.

$$Q = C_d bd\sqrt{2gh} = 0.62 \times 1.2 \times 1 \times \sqrt{2 \times 9.81 \times 2} = \mathbf{4.6605\ m^3/s}$$

10.7.2 Partially Submerged Orifice

If the outlet side of an orifice is only partly submerged under liquid, then it is known as partially submerged orifice or partially drowned orifice (Figure 10.8). In partially submerged orifice, its upper portion behaves as an orifice discharging free, whereas the lower portion behaves as a submerged orifice. Only large orifice can behave as a partially submerged orifice. The discharge through a partially submerged orifice may be determined by individually calculating the discharge through the free and submerged portions and then adding together the two discharges produced.

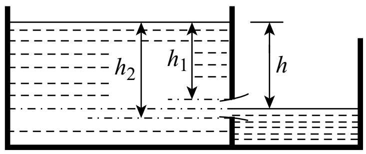


Figure 10.8 *Partially submerged orifice*

Discharge through the submerged portion of the orifice is given by Equation (10.16) as follows.

$$Q_1 = C_d b(h_2 - h)\sqrt{2gh}$$

Discharge through the free portion of the orifice is given by,

$$Q_2 = \frac{2}{3} C_d b\sqrt{2g} \times [h^{3/2} - h_1^{3/2}]$$

Thus, the expression for total discharge through the partially submerged orifice is given below.

$$Q = Q_1 + Q_2 = C_d b(h_2 - h)\sqrt{2gh} + \frac{2}{3} C_d b\sqrt{2g} \times [h^{3/2} - h_1^{3/2}] \quad (10.17)$$

Example 10.12 A large tank is provided with a rectangular orifice of width 1.5 m and depth 2 m in one of its sides. The water level in one side of the orifice is 1.8 m above the top edge of the orifice while on the other side it is 0.9 m below the top edge. If the coefficient of discharge of the orifice is 0.62, then determine the discharge through the orifice.

Solution

Refer Figure 10.9. Let $b = 1.5$ m, $d = 2$ m, $h_1 = 1.8$ m, $y = 0.9$ m and $C_d = 0.62$.

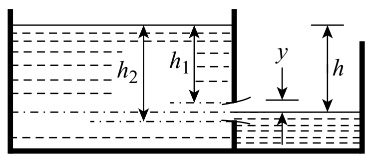


Figure 10.9

$$h = h_1 + y = 1.8 + 0.9 = 2.7\ \text{m}$$

$$h_2 = h_1 + d = 1.8 + 2 = 3.8\ \text{m}$$

Since $$Q = C_d b(h_2 - h)\sqrt{2gh} + \frac{2}{3} C_d b\sqrt{2g} \times [h^{3/2} - h_1^{3/2}]$$

$$Q = 0.62 \times 1.5 \times (3.8 - 2.7) \times \sqrt{2 \times 9.81 \times 2.7} + \frac{2}{3} \times 0.62 \times 1.5 \times \sqrt{2 \times 9.81} \times [2.7^{3/2} - 1.8^{3/2}]$$

$$\therefore Q = 7.446 + 5.552 = \mathbf{12.998\ m^3/s}$$

10.8 ❒ TIME OF EMPTYING A TANK THROUGH AN ORIFICE

Usually, tanks containing liquid are emptied through an orifice and it becomes important to know their time of emptying. In the following sections, time of emptying of the following tanks, namely (i) vertical tank of uniform cross section, (i.e., rectangular, square or cylindrical), (ii) hemispherical tank and (iii) circular horizontal tank is discussed.

10.8.1 Time of Emptying Vertical Tank of Uniform Cross Section

Consider a tank of uniform cross-sectional area containing some liquid. The tank is provided with an orifice at its bottom as shown in Figure 10.10. Let A be the cross-sectional area of the tank, h_1 be the initial height of liquid, h_2 be the final height of liquid, a be the cross-sectional area of the orifice, T be the time in seconds for the liquid to fall from height h_1 to h_2.

Assuming that at some instant, the height of liquid in the tank is h and the level decreases by dh in a small interval of time dT. Let dq be the discharge through the orifice per second.

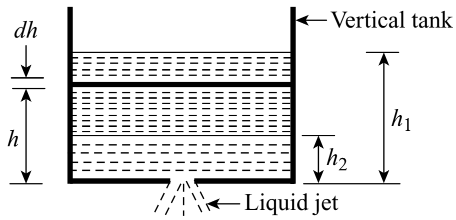


Figure 10.10 *Vertical tank with an orifice at its bottom*

Thus $\quad dq = C_d \times \text{Area of orifice} \times \text{Theoretical velocity} = C_d \times a \times \sqrt{2gh}$

Discharge through the orifice in time dT is given by,

$$dQ = dq \times dT = C_d a\sqrt{2gh} \times dT \qquad \text{(i)}$$

Volume of liquid that has passed the tank in time dT is given by,

$$dQ = \text{Area of liquid surface} \times \text{Fall in liquid level} = -A \times dh \qquad \text{(ii)}$$

The negative sign has been taken in expression (ii) since liquid level falls with time. Thus, equating the expressions (i) and (ii), we get the following expression.

$$C_d a\sqrt{2gh}\ dT = -Adh$$

$$dT = \frac{-Adh}{C_d a\sqrt{2gh}} \qquad \text{(iii)}$$

Therefore, the time for the liquid level to fall from h_1 to h_2 can be calculated by integrating the expression (iii) between the limits h_1 and h_2 as follows.

$$T = \int_{h_1}^{h_2} \frac{-A}{C_d a\sqrt{2gh}}\, dh = \frac{-A}{C_d a\sqrt{2g}} \int_{h_1}^{h_2} h^{-1/2} dh$$

$$T = \frac{-A}{C_d a\sqrt{2g}} \left[\frac{h^{1/2}}{1/2}\right]_{h_1}^{h_2} = \frac{-2A[\sqrt{h_2} - \sqrt{h_1}]}{C_d a\sqrt{2g}} = \frac{2A[\sqrt{h_1} - \sqrt{h_2}]}{C_d a\sqrt{2g}} \qquad (10.18)$$

For emptying the tank completely, $h_2 = 0$ and therefore, Equation (10.18) is written as follows.

$$\boxed{T = \frac{2A\sqrt{h_1}}{C_d a\sqrt{2g}}} \qquad (10.19)$$

In Equations (10.18) and (10.19), the area of the tank A for rectangular, square or circular in shape will remain constant with depth and will be given by lb, b^2 or $(\pi/4)d^2$, respectively.

Example 10.13 A circular tank of diameter 1.4 m containing water up to a height of 4.2 m is provided with an orifice of diameter 3 cm at its bottom. If the coefficient of discharge for the orifice is given as 0.62, then determine the height of water above the orifice after 84 seconds.

Solution

Let $D = 1.4$ m, $h_1 = 4.2$ m, $d = 3$ cm $= 0.03$ m, $C_d = 0.62$ and $T = 84$ s. Let h_2 be the height of water above the orifice after 84 seconds.

$$A = \frac{\pi}{4}D^2 = \frac{\pi}{4}\times 1.4^2 = 1.5394 \text{ m}^2$$

$$a = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 0.03^2 = 0.000707 \text{ m}^2$$

Since
$$T = \frac{2A[\sqrt{h_1} - \sqrt{h_2}]}{C_d a\sqrt{2g}}$$

Thus
$$84 = \frac{2\times 1.5394\times[\sqrt{4.2} - \sqrt{h_2}]}{0.62\times 0.000707\times\sqrt{2\times 9.81}}$$

$$0.053 = [2.05 - \sqrt{h_2}]$$

$$\therefore h_2 = (2.05 - 0.053)^2 = \mathbf{3.988\ m}$$

Example 10.14 A circular tank of diameter 2 m containing water up to a height of 3 m is provided with an orifice of diameter 0.3 m at its bottom. If the coefficient of discharge for the orifice is given as 0.6, then determine the time taken by water, (i) to fall from 3 m to 1 m and (ii) for completely emptying the tank.

Solution

Let $D = 2$ m, $h_1 = 3$ m, $d = 0.3$ m, $C_d = 0.6$ and $h_2 = 1$ m.

$$A = \frac{\pi}{4}D^2 = \frac{\pi}{4}\times 2^2 = 3.1416 \text{ m}^2$$

$$a = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 0.3^2 = 0.0707 \text{ m}^2$$

(i) $$T = \frac{2A[\sqrt{h_1} - \sqrt{h_2}]}{C_d a\sqrt{2g}} = \frac{2\times 3.1416\times[\sqrt{3} - \sqrt{1}]}{0.6\times 0.0707\times\sqrt{2\times 9.81}} = \mathbf{24.48\ s}$$

(ii) $$T = \frac{2A\sqrt{h_1}}{C_d a\sqrt{2g}} = \frac{2\times 3.1416\times\sqrt{3}}{0.6\times 0.0707\times\sqrt{2\times 9.81}} = \mathbf{57.92\ s}$$

Example 10.15 A rectangular vessel with vertical sides is provided with an orifice in one of its side. If the head over the orifice is lowered from 5 m to 2 m in a certain time, then find the constant head at the orifice so that same amount of water is discharged through the orifice at the same time.

Solution

Let $h_1 = 5$ m and $h_2 = 2$ m. Let h be the constant head over the orifice and v be the volume of water leaving the vessel in time T.

$$v = A(h_1 - h_2), \text{ also } v = C_d a\sqrt{2gh}\times T$$

Thus
$$A(h_1 - h_2) = C_d a\sqrt{2gh}\times T$$

Substituting Equation (10.18) in the above expression, we get:

$$A(h_1 - h_2) = C_d a\sqrt{2gh} \times \frac{2A[\sqrt{h_1} - \sqrt{h_2}]}{C_d a\sqrt{2g}}$$

$$(h_1 - h_2) = 2\sqrt{h}[\sqrt{h_1} - \sqrt{h_2}]$$

$$(\sqrt{h_1} - \sqrt{h_2})(\sqrt{h_1} + \sqrt{h_2}) = 2\sqrt{h}[\sqrt{h_1} - \sqrt{h_2}]$$

Thus $$h = \left[\frac{\sqrt{h_1} + \sqrt{h_2}}{2}\right]^2$$

$$\therefore h = \left[\frac{\sqrt{5} + \sqrt{2}}{2}\right]^2 = \mathbf{3.331\ m}$$

10.8.2 Time of Emptying Hemispherical Tank

Consider a hemispherical tank of radius R containing some liquid and provided with an orifice of area a at its bottom as shown in Figure 10.11. In this case, the cross-sectional area A is not constant. As the level of liquid decreases, the area decreases. Let h_1 be the initial height of liquid, h_2 be the final height of liquid and T be the time in seconds for the liquid to fall from height h_1 to h_2.

Assuming that at some instant, the height of liquid in the tank is h, r is the radius of the liquid surface and the level decreases by dh in a small interval of time dT. Let dq be the discharge through the orifice per second.

Thus $dq = C_d \times \text{Area of orifice} \times \text{Theoretical velocity} = C_d \times a \times \sqrt{2gh}$

Discharge through the orifice in time dT is given by,

$$dQ = dq \times dT = C_d a\sqrt{2gh} \times dT \qquad \text{(i)}$$

Volume of liquid that has passed the tank in time dT is given by,

$$dQ = \text{Area of liquid surface} \times \text{Fall in liquid level} = -A \times dh = -\pi r^2 dh \qquad \text{(ii)}$$

The negative sign has been taken in expression (ii) since liquid level falls with time. Thus, by equating expressions (i) and (ii), we get the below expression.

$$C_d a\sqrt{2gh}\ dT = -\pi r^2 dh$$

$$dT = \frac{-\pi r^2 dh}{C_d a\sqrt{2gh}} \qquad \text{(iii)}$$

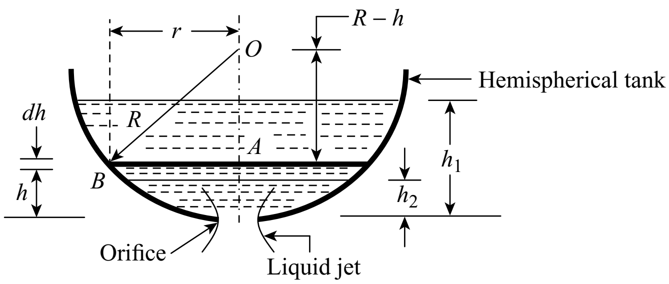


Figure 10.11 *Hemispherical tank with an orifice at its bottom*

Now $OB = R$ and $OA = (R - h)$

$$r^2 = R^2 - (R-h)^2 = 2Rh - h^2 \qquad [\because r^2 = AB^2 = OB^2 - OA^2]$$

Thus, expression (iii) is written as follows.

$$dT = \frac{-\pi(2Rh - h^2)dh}{C_d a\sqrt{2gh}} = \frac{-\pi}{C_d a\sqrt{2g}}(2Rh^{1/2} - h^{3/2})dh \tag{iv}$$

Therefore, the time for the liquid level to fall from h_1 to h_2 can be calculated by integrating expression (iv) between the limits h_1 and h_2 as follows.

$$T = \frac{-\pi}{C_d a\sqrt{2g}}\int_{h_1}^{h_2}(2Rh^{1/2} - h^{3/2})dh = \frac{-\pi}{C_d a\sqrt{2g}}\left[2R\frac{h^{3/2}}{(3/2)} - \frac{h^{5/2}}{(5/2)}\right]_{h_1}^{h_2}$$

$$T = \frac{-\pi}{C_d a\sqrt{2g}}\left[\frac{4}{3}R(h_2^{3/2} - h_1^{3/2}) - \frac{2}{5}(h_2^{5/2} - h_1^{5/2})\right]$$

$$\boxed{\therefore T = \frac{\pi}{C_d a\sqrt{2g}}\left[\frac{4}{3}R(h_1^{3/2} - h_2^{3/2}) - \frac{2}{5}(h_1^{5/2} - h_2^{5/2})\right]} \tag{10.20}$$

For emptying the tank completely, $h_2 = 0$ and therefore, Equation (10.20) is written as follows.

$$T = \frac{\pi}{C_d a\sqrt{2g}}\left[\frac{4}{3}Rh_1^{3/2} - \frac{2}{5}h_1^{5/2}\right] \tag{10.21}$$

Example 10.16 A hemispherical tank of diameter 5 m containing water to a depth of 2 m is provided with an orifice of diameter 50 mm at its bottom. If the coefficient of discharge of the orifice is 0.62, then determine the time required by the water (i) to fall from 2 m to 1 m and (ii) for completely emptying the tank.

Solution

Let $D = 5$ m, $h_1 = 2$ m, $d = 50$ mm $= 0.05$ m and $C_d = 0.62$.

$$a = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 0.05^2 = 0.0019635 \text{ m}^2$$

(i) $h_2 = 1$ m

Since $T = \dfrac{\pi}{C_d a\sqrt{2g}}\left[\dfrac{4}{3}R(h_1^{3/2} - h_2^{3/2}) - \dfrac{2}{5}(h_1^{5/2} - h_2^{5/2})\right]$

Thus $T = \dfrac{\pi}{0.62\times 0.0019635\times\sqrt{2\times 9.81}}\times\left[\dfrac{4}{3}\times\dfrac{5}{2}(2^{3/2} - 1^{3/2}) - \dfrac{2}{5}(2^{5/2} - 1^{5/2})\right]$

$\therefore T =$ **2465.61 s**

(ii) $T = \dfrac{\pi}{C_d a\sqrt{2g}}\left[\dfrac{4}{3}Rh_1^{3/2} - \dfrac{2}{5}h_1^{5/2}\right]$

Thus $T = \dfrac{\pi}{0.62\times 0.0019635\times\sqrt{2\times 9.81}}\times\left[\dfrac{4}{3}\times\dfrac{5}{2}\times 2^{3/2} - \dfrac{2}{5}\times 2^{5/2}\right]$

$\therefore T =$ **4174.6 s**

Example 10.17 A tank has an upper cylindrical portion of diameter 2 m and height 3 m with hemispherical base. If the tank is full of water, then determine the time required to empty it through an orifice of diameter 40 mm at its bottom. Take coefficient of discharge for the orifice as 0.62.

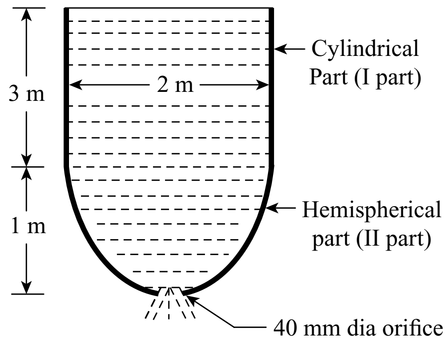


Figure 10.12

Solution

Refer Figure 10.12. Let $D = 2$ m, $h = 3$ m, $d = 40$ mm $= 0.04$ m and $C_d = 0.62$.

$$a = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 0.04^2 = 0.001257 \text{ m}^2$$

$$A = \frac{\pi}{4}D^2 = \frac{\pi}{4}\times 2^2 = 3.1416 \text{ m}^2$$

The problem is split into two parts, namely Ist part (cylindrical part) and IInd part (hemispherical part). Let T_1 be the time required to empty the Ist part and T_2 be the time required to empty the IInd part. Thus, the total time becomes $T = T_1 + T_2$.

(i) For Ist part (cylindrical part): $h_1 = 3+1 = 4$ m and $h_2 = 1$ m.

$$T_1 = \frac{2A[\sqrt{h_1} - \sqrt{h_2}]}{C_d a\sqrt{2g}} = \frac{2\times 3.1416\times[\sqrt{4} - \sqrt{1}]}{0.62\times 0.001257\times\sqrt{2\times 9.81}} = 1820.14 \text{ s}$$

(ii) For IInd part (spherical part): $h_1 = 1$ m, $h_2 = 0$ and $R = 2/2 = 1$ m.

Since
$$T_2 = \frac{\pi}{C_d a\sqrt{2g}}\left[\frac{4}{3}Rh_1^{3/2} - \frac{2}{5}h_1^{5/2}\right]$$

$$\therefore T_2 = \frac{\pi}{0.62\times 0.001257\times\sqrt{2\times 9.81}}\times\left[\frac{4}{3}\times 1\times 1^{3/2} - \frac{2}{5}\times 1^{5/2}\right] = 849.4 \text{ s}$$

$$T = T_1 + T_2 = 1820.14 + 849.4 = \mathbf{2669.54\ s}$$

10.8.3 Time of Emptying a Circular Horizontal Tank

Consider a circular horizontal tank of radius R and length L containing some liquid which is provided with an orifice of area a at its bottom as shown in Figure 10.13. Let h_1 be the initial height of liquid, h_2 be the final height of liquid and T be the time in seconds for the liquid to fall from height h_1 to h_2.

Assuming that at some instant, the height of liquid in the tank is h, r is the radius of the liquid surface and the level decreases by dh in a small interval of time dT. Let dq be the discharge through the orifice per second.

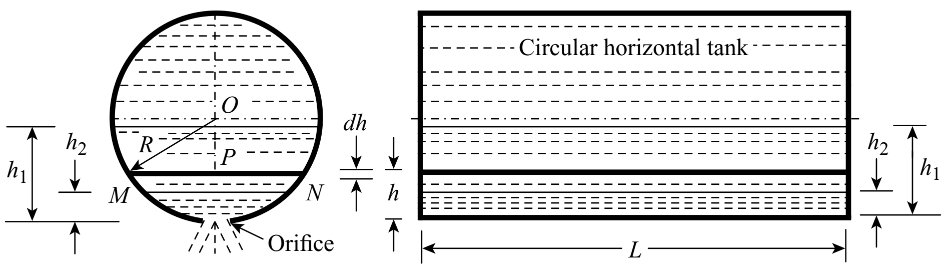


Figure 10.13 *Circular horizontal tank with an orifice at its bottom*

Thus $$dq = C_d \times \text{Area of orifice} \times \text{Theoretical velocity} = C_d \times a \times \sqrt{2gh}$$

Discharge through the orifice in time dT is given by,

$$dQ = dq \times dT = C_d a\sqrt{2gh} \times dT \tag{i}$$

Volume of liquid that has passed the tank in time dT is given by,

$$dQ = \text{Area of liquid surface} \times \text{Fall in liquid level} = -A \times dh \tag{ii}$$

The negative sign is taken in expression (ii), since the liquid level falls with time.

But $$A = L \times MN = L \times (2MP) = L \times [2\sqrt{OM^2 - OP^2}]$$

$$A = 2L\sqrt{R^2 - (R-h)^2} = 2L\sqrt{(2Rh - h^2)}$$

Thus, expression (ii) is written as follows.

$$dQ = -2L\sqrt{(2Rh - h^2)} \times dh \tag{iii}$$

Equating expressions (i) and (iii), we get:

$$C_d a\sqrt{2gh} \times dT = -2L\sqrt{(2Rh - h^2)} \times dh$$

$$dT = \frac{-2L\sqrt{(2Rh - h^2)}\; dh}{C_d a\sqrt{2gh}} \tag{iv}$$

Therefore, the time for the liquid level to fall from h_1 to h_2 can be calculated by integrating expression (iv) between the limits h_1 and h_2 as follows.

$$T = \frac{-2L}{C_d a\sqrt{2g}} \int_{h_1}^{h_2} (2R-h)^{1/2}\, dh = \frac{-2L}{C_d a\sqrt{2g}} \left[\frac{(2R-h)^{3/2}}{(3/2)} (-1) \right]_{h_1}^{h_2}$$

$$\boxed{T = \frac{4L}{3C_d a\sqrt{2g}} \left[(2R-h_2)^{3/2} - (2R-h_1)^{3/2} \right]} \tag{10.22}$$

For emptying the tank completely, $h_2 = 0$ and therefore, Equation (10.22) is written as follows.

$$T = \frac{4L}{3C_d a\sqrt{2g}} \left[(2R)^{3/2} - (2R-h_1)^{3/2} \right] \tag{10.23}$$

Further if the tank is half at the commencement and is to be completely emptied, then substituting $h_1 = R$ in Equation (10.23), we get the below expression.

$$T = \frac{4L}{3C_d a\sqrt{2g}} \left[(2R)^{3/2} - (2R-R)^{3/2} \right] = 0.55 \frac{LR^{3/2}}{C_d a} \tag{10.24}$$

Example 10.18 A horizontal boiler drum of diameter 2 m and length 4 m is provided with an orifice of diameter 5 cm at its bottom. If the drum contains water up to a height of 1.2 m and the discharge coefficient of the orifice is 0.6, then determine the time taken to empty it.

Solution

Let $D = 2$ m, $L = 4$ m, $d = 5$ cm $= 0.05$ m, $h_1 = 1.2$ m and $C_d = 0.6$.

$$a = \frac{\pi}{4}d^2 = \frac{\pi}{4} \times 0.05^2 = 0.0019635 \text{ m}^2$$

Since $$T = \frac{4L}{3C_d a\sqrt{2g}}\left[(2R)^{3/2} - (2R - h_1)^{3/2}\right]$$

Thus $$T = \frac{4 \times 4}{3 \times 0.6 \times 0.0019635 \times \sqrt{2 \times 9.81}} \times \left[(2 \times 1)^{3/2} - (2 \times 1 - 1.2)^{3/2}\right]$$

$$\therefore T = \mathbf{2159.45 \text{ s}}$$

10.9 ❐ CLASSIFICATION OF MOUTHPIECES

A mouthpiece is an attachment in the form of a short length of tube or pipe. It is two to three times its diameter in length and is fitted to the tank containing fluid, through which the fluid is discharged. The types of mouthpieces are classified as follows.

1. According to the position of the mouthpiece.
   (i) **External mouthpiece:** If the pipe projects outwards from the wall of the reservoir, then it is called an external mouthpiece.
   (ii) **Internal mouthpiece:** If the pipe projects inside the tank (i.e., fixed internally), then it is called an internal mouthpiece. The internal mouthpiece is also known as Borda's mouthpiece or reentrant mouthpiece.
2. According to the shape of the mouthpiece.
   (i) **Cylindrical mouthpiece:** If the mouthpiece has uniform circular cross section (i.e., its flow area remains uniform from the inlet to outlet), then it is called cylindrical mouthpiece.
   (ii) **Convergent mouthpiece:** If the flow area of the mouthpiece decreases from its inlet to outlet, then it is called convergent mouthpiece.
   (iii) **Divergent mouthpiece:** If the flow area of the mouthpiece increases from its inlet to outlet, then it is called divergent mouthpiece.
   (iv) **Convergent-divergent mouthpiece:** If the flow area initially decreases and attains a minimum value and then increases, then it is called convergent-divergent mouthpiece.
3. The classification based on the nature of discharge is only for the internal mouthpiece.
   (i) **Mouthpiece running free:** If the jet after contraction does not touch the sides of the orifice, then it is called mouthpiece running free.
   (ii) **Mouthpiece running full:** If the jet after contraction expands and fills the whole mouthpiece, then it is called mouthpiece running full.

10.10 ❐ FLOW THROUGH AN EXTERNAL MOUTHPIECE

A tank having an external cylindrical mouthpiece (small tube of length two to three times its diameter) is shown in Figure 10.14. The jet of liquid entering the mouthpiece contracts to form vena contracta at section C–C and beyond this section it again fills the whole mouthpiece.

Let h be the height of liquid above the centreline of mouthpiece, a be the area of mouthpiece, a_c be the area of flow at vena contracta, V be the velocity of liquid at mouthpiece outlet (1–1 section), V_c be the velocity of liquid at the vena contracta (C–C section), h_L be the loss of head due to sudden enlargement from C–C section to 1–1 section and $C_c = (a_c/a)$ be the coefficient of contraction.

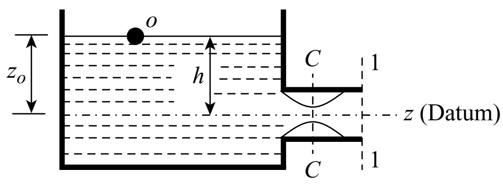


Figure 10.14 *External cylindrical mouthpiece*

$$h_L = \frac{(V_c - V)^2}{2g}$$

Applying continuity equation between sections C–C and 1–1, we get:

$$a_c V_c = aV$$

$$V_c = \frac{aV}{a_c} = \frac{V}{(a_c / a)} = \frac{V}{C_c} = \frac{V}{0.62} \qquad \text{[Taking } C_c = 0.62\text{]} \tag{10.25}$$

Thus $$h_L = \frac{[(V / 0.62) - V]^2}{2g} = \frac{0.375V^2}{2g}$$

Applying Bernoulli's equation between the points 'o' and 1–1, we get:

$$\frac{p_o}{\rho g} + \frac{V_o^2}{2g} + z_o = \frac{p}{\rho g} + \frac{V^2}{2g} + z + h_L$$

However, $p_o = p = p_{\text{atm}} = 0$ (assumed), $V_o = 0$ and $z = 0$.

Thus $$0 + 0 + h = 0 + \frac{V^2}{2g} + 0 + \frac{0.375V^2}{2g}$$

$$h = 1.375\frac{V^2}{2g} \tag{10.26}$$

$$V = \sqrt{\frac{2gh}{1.375}} = 0.855\sqrt{2gh} \tag{10.27}$$

The theoretical velocity of liquid at the outlet is given by Equation (10.2) as follows.

$$V_{\text{th}} = \sqrt{2gh}$$

Therefore, the expression for coefficient of velocity for the mouthpiece is given below.

$$C_v = \frac{\text{Actual velocity}}{\text{Theoretical velocity}} = \frac{0.855\sqrt{2gh}}{\sqrt{2gh}} = 0.855 \tag{10.28}$$

Since the area of the jet of liquid at the outlet is equal to the area of mouthpiece at the outlet, $C_c = 1$.

Thus, the expression for coefficient of discharge for mouthpiece is given below.

$$C_d = C_c \times C_v = 1 \times 0.855 = 0.855$$

It is observed that the value of C_d for a mouthpiece is higher than that for an orifice. Therefore, the discharge through a mouthpiece is more when compared to an orifice and the expression is given below.

$$\boxed{Q = C_d a\sqrt{2gh} = 0.855a\sqrt{2gh}} \tag{10.29}$$

Expression for pressure head at vena contracta Applying Bernoulli's equation between the points 'o' and C–C, we get the following expression.

$$\frac{p_o}{\rho g} + \frac{V_o^2}{2g} + z_o = \frac{p_c}{\rho g} + \frac{V_c^2}{2g} + z$$

However, $p_o = p_{\text{atm}} = p_a, V_o = 0, z_o = h$ and $z = 0$, here p_{atm} is the atmospheric pressure.

Thus $$\frac{p_a}{\rho g}+0+h=\frac{p_c}{\rho g}+\frac{V_c^2}{2g}+0$$

$$\frac{p_c}{\rho g}=\frac{p_a}{\rho g}+h-\frac{1}{0.62^2}\times\frac{V^2}{2g} \qquad [\because V_c = V/0.62]$$

Since $$\frac{V^2}{2g}=\frac{h}{1.375} \qquad \text{[From Equation (10.26)]}$$

Thus $$\frac{p_c}{\rho g}=\frac{p_a}{\rho g}+h-\frac{1}{0.62^2}\times\frac{h}{1.375}$$

$$\frac{p_c}{\rho g}=\frac{p_a}{\rho g}-0.89h \tag{10.30}$$

However, $p_c/(\rho g)=h_c$ and $p_a/(\rho g)=h_a$ and thus, Equation (10.30) is written as follows.

$$\boxed{h_c = h_a - 0.89h} \tag{10.31}$$

Equation (10.31) shows that the pressure head at vena contracta is less than the atmospheric pressure head by an amount equal to 0.89 times the head under which the mouthpiece works.

Example 10.19 Determine the increase in discharge by the addition of an external cylindrical mouthpiece to a circular orifice of the same diameter. Take the coefficients of contraction and velocity for the sharp-edged orifice as 0.62 and 0.98, respectively. If the separation occurs at 2.5 m of water and the barometric pressure is 10.336 m of water, then determine the limiting conditions of head under which the mouthpiece runs full and provide your comments.

Solution

Let $C_c = 0.62$, $C_v = 0.98$, $h_c = 2.5$ m of water and $h_a = 10.336$ m of water.

$$C_d = C_c C_v = 0.62\times 0.98 = 0.6076$$

Discharge through the orifice is given by,

$$Q_o = C_d a\sqrt{2gh} = 0.6076a\sqrt{2gh} \tag{i}$$

Discharge through the mouthpiece is given by Equation (10.29) as follows.

$$Q_m = C_d a\sqrt{2gh} = 0.855a\sqrt{2gh} \tag{ii}$$

Thus, percentage increase in discharge ($\%Q_i$) is given by,

$$\%Q_i = \left(\frac{Q_m - Q_o}{Q_o}\right)\times 100$$

$$\therefore \%Q_i = \left(\frac{0.855a\sqrt{2gh}-0.6076a\sqrt{2gh}}{0.6076a\sqrt{2gh}}\right)\times 100 = \mathbf{40.72\%}$$

Since $$h_c = h_a - 0.89h$$

$$\therefore h = \frac{h_a - h_c}{0.89} = \frac{10.336-2.5}{0.89} = \mathbf{8.804\ m\ of\ water}$$

The mouthpiece will not run full if head over the mouthpiece exceeds 8.804 m of water.

Example 10.20 An external cylindrical mouthpiece of diameter 150 mm fitted to a tank discharges water under a constant head of 7.5 m. If the coefficient of contraction at vena contracta is 0.62, coefficient of velocity is 0.99 and atmospheric pressure head is 10.34 m of water, then determine the discharge through the mouthpiece. Also determine the absolute pressure head of water at vena contracta.

Solution

Let $d = 150 \text{ mm} = 0.15 \text{ m},\ h = 7.5 \text{ m},\ C_c = 0.62,\ C_v = 0.99$ and $h_a = 10.34$ m of water.

$$C_d = C_c C_v = 0.62 \times 0.99 = 0.6138$$

$$a = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 0.15^2 = 0.01767 \text{ m}^2$$

$$Q = 0.855 a\sqrt{2gh} = 0.855 \times 0.01767 \times \sqrt{2 \times 9.81 \times 7.5} = \mathbf{0.1833\ m^3/s}$$

$$h_c = h_a - 0.89h = 10.34 - 0.89 \times 7.5 = \mathbf{3.665\ m\ (abs)}$$

10.11 ❐ FLOW THROUGH A CONVERGENT-DIVERGENT MOUTHPIECE

Figure 10.15 shows a tank fitted with an external convergent-divergent mouthpiece (also known as Bell mouthpiece) whose sectional area initially converges up to vena contracta and then diverges gradually. Since there is no sudden enlargement of the jet, the loss of head due to sudden enlargement gets eliminated. Thus, C_v will be unity. Since C_c is also unity as the area of the jet at the outlet is equal to that of the mouthpiece, the value of C_d will also be unity.

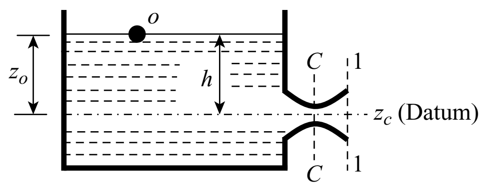


Figure 10.15 *Convergent-divergent mouthpiece*

Let h be the height of liquid above the centreline of mouthpiece, a_c be the area of flow at vena contracta (C–C section), a_1 be the area of mouthpiece at its outlet (1–1 section), V_c be the velocity of liquid at the vena contracta, V_1 be the velocity of liquid at the mouthpiece outlet, h_a be the atmospheric pressure head, h_c be the absolute pressure head at vena contracta and Q be the discharge through the mouthpiece.

Let the suffix 'o' denotes the free liquid surface, 'c' denotes the vena contracta section, and 1 denotes the mouthpiece outlet.

Applying Bernoulli's equation between sections 'o' and 'C–C', we get:

$$\frac{p_o}{\rho g} + \frac{V_o^2}{2g} + z_o = \frac{p_c}{\rho g} + \frac{V_c^2}{2g} + z_c$$

However, $p_o = p_{\text{atm}},\ p_o/(\rho g) = h_a,\ V_o = 0,\ z_o = h,\ p_c/(\rho g) = h_c$ and $z_c = 0$.

Thus
$$h_a + 0 + h = h_c + \frac{V_c^2}{2g} + 0$$

$$\frac{V_c^2}{2g} = (h_a + h - h_c)$$

$$\therefore V_c = \sqrt{2g(h_a + h - h_c)}$$

Applying Bernoulli's equation between sections 'C–C' and '1–1', we get:

$$\frac{p_c}{\rho g} + \frac{V_c^2}{2g} + z_c = \frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1$$

However, $p_c/(\rho g) = h_c$, $V_c^2/(2g) = (h_a + h - h_c)$, $p_1 = p_{\text{atm}}$, $p_1/(\rho g) = h_a$ and $z_1 = z_c$.

Thus $$h_c + (h_a + h - h_c) + 0 = h_a + \frac{V_1^2}{2g} + 0$$

$$\frac{V_1^2}{2g} = h \Rightarrow V_1 = \sqrt{2gh}$$

Applying continuity equation between sections '*C–C*' and, '1–1', we get:

$$a_c V_c = a_1 V_1$$

$$\frac{a_1}{a_c} = \frac{V_c}{V_1}$$

$$\frac{a_1}{a_c} = \frac{\sqrt{2g(h_a + h - h_c)}}{\sqrt{2gh}} = \sqrt{\frac{h_a}{h} + 1 - \frac{h_c}{h}}$$

$$\boxed{\therefore \frac{a_1}{a_c} = \sqrt{1 + \left(\frac{h_a - h_c}{h}\right)}} \tag{10.32}$$

$$Q = a_c\sqrt{2gh} \tag{10.33}$$

The pressure at the vena contracta section (also called throat) should not fall below the vapour pressure of the liquid to avoid cavitation. If the flowing liquid is water, then the limiting value of the suction pressure at vena contracta $(h_a - h_c) = 10.3 - 2.5 = 7.8$ m. Thus, the maximum value of the areas ratio given by Equation (10.32) is written as follows.

$$\boxed{\frac{a_1}{a_c} = \sqrt{1 + \frac{7.8}{h}}} \tag{10.34}$$

Example 10.21 A convergent-divergent mouthpiece is fitted into the side of a water tank. The losses in the divergent part of the mouthpiece are equivalent to 0.15 times the velocity head at the exit and there are no losses in its convergent part. Determine the throat and exit diameters of the mouthpiece to discharge 4.5 litres per second of water for a head of 1.6 m above the centreline of the mouthpiece, if the minimum absolute pressure at the throat is 2.44 m for a barometric pressure of 10.34 m of water.

Solution

Refer Figure 10.15. Let $h_L = 0.15[V_1^2/(2g)]$, $Q = 4.5\ l/\text{s} = 0.0045\ \text{m}^3/\text{s}$, $h = 1.6$ m, $h_c = 2.44$ m and $h_a = 10.34$ m.

Let d_c and d_1 be the throat and exit diameters, respectively.

Applying Bernoulli's equation between the free surface and vena contracta (throat), we get:

$$h_a + h = h_c + \frac{V_c^2}{2g}$$

Thus $$10.34 + 1.6 = 2.44 + \frac{V_c^2}{2g}$$

$$V_c^2 = (10.34 + 1.6 - 2.44) \times 2g$$

$$\therefore V_c = \sqrt{9.5 \times 2 \times 9.81} = 13.6525 \text{ m/s}$$

Since $$Q = a_c V_c = \frac{\pi}{4} d_c^2 V_c$$

Thus $$0.0045 = \frac{\pi}{4} d_c^2 \times 13.6525$$

$$\therefore d_c = \sqrt{\frac{0.0045 \times 4}{\pi \times 13.6525}} = \mathbf{0.0205\ m} \text{ or } \mathbf{20.5\ mm}$$

Applying Bernoulli's equation between the free surface and exit of mouthpiece, we get:

$$h_a + h = h_a + \frac{V_1^2}{2g} + 0.15 \frac{V_1^2}{2g}$$

Thus $$V_1 = \sqrt{\frac{2gh}{1.15}} = \sqrt{\frac{2 \times 9.81 \times 1.6}{1.15}} = 5.225 \text{ m/s}$$

Since $$Q = a_1 V_1 = \frac{\pi}{4} d_1^2 V_1$$

$$\therefore d_1 = \sqrt{\frac{4Q}{\pi V_1}} = \sqrt{\frac{4 \times 0.0045}{\pi \times 5.225}} = \mathbf{0.03311\ m} \text{ or } \mathbf{33.11\ mm}$$

Example 10.22 Water is being discharged under a constant head of 5 m through a convergent-divergent mouthpiece with a throat diameter 50 mm. If the minimum absolute pressure at the throat is 2.5 m for a barometric pressure of 10.3 m of water, then determine the maximum outlet diameter to avoid separation of water flow. Also determine the discharge through the mouthpiece.

Solution

Let $h = 5$ m, $d_c = 50$ mm $= 0.05$ m, $h_c = 2.5$ m and $h_a = 10.3$ m.

$$a_c = \frac{\pi}{4} d_c^2 = \frac{\pi}{4} \times 0.05^2 = 0.0019635 \text{ m}^2$$

Since $$\frac{a_1}{a_c} = \sqrt{1 + \left(\frac{h_a - h_c}{h}\right)}$$

Thus $$\frac{(\pi/4) d_1^2}{(\pi/4) d_c^2} = \sqrt{1 + \left(\frac{10.3 - 2.5}{5}\right)}$$

Thus $$\left(\frac{d_1}{d_c}\right)^2 = 1.6$$

$$\therefore d_1 = \sqrt{1.6}\, d_c = \sqrt{1.6} \times 0.05 = \mathbf{0.06324\ m} \text{ or } \mathbf{63.24\ mm}$$

$$Q = a_c \sqrt{2gh} = 0.0019635 \times \sqrt{2 \times 9.81 \times 5} = \mathbf{0.01945\ m^3/s}$$

10.12 ❐ FLOW THROUGH AN INTERNAL MOUTHPIECE (REENTRANT OR BORDA'S MOUTHPIECE)

A short cylindrical tube attached to a circular orifice in the side wall of a reservoir or tank such that it projects inwardly is called an internal mouthpiece. It is also known as reentrant or Borda's mouthpiece. Depending upon the length of the mouthpiece, it is said to be running free or running full which are discussed below.

10.12.1 Borda's Mouthpiece Running Free

If the length of the tube is equal to the diameter and the jet of liquid leaves without touching the sides of the tube, then it is known as Borda's mouthpiece running free (Figure 10.16(a)).

Let h be the height of liquid above the centreline of mouthpiece, a_c be the area of contracted jet in the mouthpiece, a be the area of mouthpiece, V_c be the velocity of the contracted jet and Q be the discharge through the mouthpiece. Let the suffix 'o' denotes the free liquid surface and 'c' denotes the section C–C.

Applying Bernoulli's equation between sections 'o' and 'C–C', we get:

$$\frac{p_o}{\rho g}+\frac{V_o^2}{2g}+z_o=\frac{p_c}{\rho g}+\frac{V_c^2}{2g}+z_c$$

But $$p_o = p_c = p_{atm} = 0 \text{ (assumed)}, V_o = 0, z_o = h \text{ and } z_c = 0.$$

Thus $$0+0+h=0+\frac{V_c^2}{2g}+0$$

$$\therefore V_c=\sqrt{2gh}$$

As per momentum equation, the expression for force on the mouthpiece is given below.

$$\text{Force} = \text{Rate of change of momentum}$$

$$\text{Pressure} \times \text{Area of orifice} = \text{Mass flowing per second} \times \text{Change of velocity}$$

$$(\rho gh)a = (\rho a_c V_c)(V_c - 0)$$

$$\rho gha = \rho a_c V_c^2 \qquad \text{(i)}$$

$$gha = a_c(\sqrt{2gh})^2 \qquad [\because V_c = \sqrt{2gh}]$$

$$\frac{a_c}{a}=\frac{1}{2}=0.5$$

$$\therefore C_c = 0.5$$

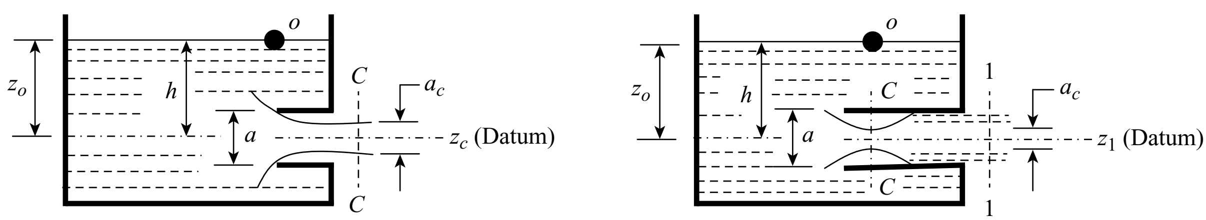
$$C_v = 1 \qquad [\because \text{No head loss}]$$

$$\therefore C_d = C_v C_c = 1\times 0.5 = 0.5$$

Thus, the expression for discharge through a Borda's mouthpiece running free is given below.

$$\boxed{Q = C_d a\sqrt{2gh} = 0.5a\sqrt{2gh}} \qquad (10.35)$$

However, if some loss of energy is considered, then $V_c = C_v\sqrt{2gh}$. Thus, from expression (i), the coefficient of contraction may be obtained as follows.



(a) Running free (b) Running full

Figure 10.16 *Borda's mouthpiece*

$$\rho gha = \rho\, a_c (C_v \sqrt{2gh})^2 \qquad [\because V_c = C_v\sqrt{2gh}]$$

$$\frac{a_c}{a} = \frac{1}{2C_v^2}$$

$$\boxed{\therefore C_c = \frac{a_c}{a} = \frac{1}{2C_v^2}} \qquad (10.36)$$

10.12.2 Borda's Mouthpiece Running Full

If the length of the tube is about three times its diameter and the jet comes out with its diameter equal to the diameter of the mouthpiece at the exit, then it is known as Borda's mouthpiece running full (Figure 10.16(b)).

Let h be the height of liquid above the centreline of mouthpiece, a_c be the area of flow at C–C section (i.e., vena contracta), a be the area of mouthpiece, V_c be the velocity of the liquid jet at C–C section, Q be the discharge through the mouthpiece and h_L be the loss of head due to sudden enlargement from C–C section to 1–1 section. Let the suffix 'o' denotes the free liquid surface and 'c' denotes the section C–C.

Applying continuity equation between sections C–C and 1–1, we get:

$$a_c V_c = a_1 V_1$$

$$V_c = \frac{V_1}{(a_c / a_1)} = \frac{V_1}{C_c}$$

Since the flow pattern at the entrance section of the mouthpiece is same as that for a running free condition, taking $C_c = 0.5$. Therefore, the above expression is written as follows.

$$V_c = \frac{V_1}{0.5} = 2V_1$$

$$h_L = \frac{(V_c - V_1)^2}{2g} = \frac{(2V_1 - V_1)^2}{2g} = \frac{V_1^2}{2g} \qquad (10.37)$$

Applying Bernoulli's equation between sections 'o' and '1–1', we get:

$$\frac{p_o}{\rho g} + \frac{V_o^2}{2g} + z_o = \frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 + h_L$$

But $\quad p_o = p_1 = 0, V_o = 0, z_o = h, z_1 = 0$ and $h_L = V_1^2/(2g)$.

Thus $$0 + 0 + h = 0 + \frac{V_1^2}{2g} + 0 + \frac{V_1^2}{2g}$$

$$2V_1^2 = 2gh \Rightarrow V_1 = \sqrt{gh}$$

The theoretical velocity of liquid at the outlet is given by Equation (10.2) as follows.

$$V_{\text{th}} = \sqrt{2gh}$$

$$C_v = \frac{V_1}{V_{\text{th}}} = \frac{\sqrt{gh}}{\sqrt{2gh}} = \frac{1}{\sqrt{2}} = 0.707$$

Since the area of the jet at exit equals the area of the mouthpiece, we get the below expression.

$$C_c = 1$$

$$\therefore C_d = C_v C_c = 0.707 \times 1 = 0.707$$

Thus, the expression for discharge through a Borda's mouthpiece running full is given below.

$$\boxed{Q = C_d a\sqrt{2gh} = 0.707a\sqrt{2gh}} \tag{10.38}$$

Example 10.23 Determine the diameter of the Borda's mouthpiece working under a head of 5 m and discharging water at a rate of 25 litres per second. Also determine the percentage increase in discharge if the mouthpiece is made to run full by increasing its length.

Solution

Let $h = 5$ m and $Q = 25\ l/\text{s} = 0.025\ \text{m}^3/\text{s}$. Let d be the diameter of the mouthpiece.

The discharge through a Borda's mouthpiece running free is given by Equation (10.35) as follows.

$$Q = 0.5a\sqrt{2gh} = 0.5 \times \frac{\pi}{4} d^2 \times \sqrt{2gh}$$

$$\therefore d = \left[\frac{4Q}{0.5\pi\sqrt{2gh}}\right]^{1/2} = \left[\frac{4 \times 0.025}{0.5\pi \times \sqrt{2 \times 9.81 \times 5}}\right]^{1/2} = \mathbf{0.0802\ m}$$

The discharge through a Borda's mouthpiece running full is given by Equation (10.38) as follows.

$$Q' = 0.707a\sqrt{2gh} = 0.707 \times \frac{\pi}{4} d^2 \times \sqrt{2gh}$$

$$\therefore Q' = 0.707 \times \frac{\pi}{4} \times 0.0802^2 \times \sqrt{2 \times 9.81 \times 5} = 0.0354\ \text{m}^3/\text{s}$$

The percentage increase in discharge ($\%Q_i$) is given by,

$$\%Q_i = \left(\frac{Q' - Q}{Q}\right) \times 100 = \left(\frac{0.0354 - 0.025}{0.025}\right) \times 100 = \mathbf{41.6\%}$$

Summary

1. Orifice is a small opening in the walls or the bottom of the tank through which the fluid is discharged. It is used to measure the rate of flow of fluid.
2. Mouthpiece is a short length of tube fitted to the orifice in a tank, through which fluid is discharged. It is also used for measuring the rate of flow of a fluid.
3. Theoretical velocity of jet of water (Torricelli's equation): $V = \sqrt{2gh}$.
4. The coefficient of velocity (C_v) is defined as the ratio of the actual velocity of the jet at vena contracta (V) to the theoretical (ideal) velocity of the jet (V_{th}).
5. The coefficient of contraction (C_c) is defined as the ratio of the area of the jet at vena contracta (a_c) to the area of the orifice (a).
6. The coefficient of discharge (C_d) is defined as the ratio of the actual discharge from an orifice (Q) to its theoretical discharge (Q_{th}).
7. The coefficient of resistance (C_r) is defined as the ratio of the loss of kinetic energy as liquid flows through an orifice to the actual kinetic energy possessed by the flowing liquid.
8. $C_v = x/\sqrt{4yh}$, here x and y are the horizontal and vertical coordinates, respectively, of any point of jet of water from vena contracta.

9. The coefficient of discharge is $C_d = C_v C_c$.
10. In case of large orifice, the head of liquid above the centre of orifice is less than five times the depth of orifice.
11. Discharge through large rectangular orifice is $Q = (2/3)C_d b\sqrt{2g} \times [h_2^{3/2} - h_1^{3/2}]$, here b is the width of the orifice, h_1 is the height of liquid above the upper edge of the orifice and h_2 is the height of liquid above the lower edge of the orifice.
12. Discharge through fully submerged orifice is $Q = C_d b(h_2 - h_1)\sqrt{2gh}$, here h is the difference in liquid levels in the two tanks.
13. Discharge through partially submerged orifice is given by,

$$Q = C_d b(h_2 - h)\sqrt{2gh} + (2/3)C_d b\sqrt{2g} \times [h^{3/2} - h_1^{3/2}]$$

14. The time (T) for emptying vertical tank is $T = [2A(\sqrt{h_1} - \sqrt{h_2})]/(C_d a\sqrt{2g})$ and for emptying the tank completely is $T = (2A\sqrt{h_1})/(C_d a\sqrt{2g})$, here A is the area of tank, a is the area of orifice, h_1 and h_2 are the initial and final heights of liquid in the tank, respectively.
15. Time for emptying hemispherical tank is given by,

$$T = \frac{\pi}{C_d a\sqrt{2g}}\left[\frac{4}{3}R(h_1^{3/2} - h_2^{3/2}) - \frac{2}{5}(h_1^{5/2} - h_2^{5/2})\right]$$

Time for completely emptying the tank is given by,

$$T = \frac{\pi}{C_d a\sqrt{2g}}\left[\frac{4}{3}Rh_1^{3/2} - \frac{2}{5}h_1^{5/2}\right]$$

Here, h_1 and h_2 be the initial and final heights of liquid.

16. Time for emptying a circular horizontal tank is given by,

$$T = \frac{4L}{3C_d a\sqrt{2g}}\left[(2R)^{3/2} - (2R - h_1)^{3/2}\right]$$

Here, R and L are the radius and length of the tank, respectively and h_1 and h_2 be the initial and final heights of liquid.

When the tank is half full at the commencement and is to be completely emptied,

$$T = 0.55[(LR^{3/2})/(C_d a)]$$

17. Discharge through an external mouthpiece is $Q = 0.855a\sqrt{2gh}$.
18. Pressure head (h_c) at vena contracta of an external mouthpiece is $h_c = h_a - 0.89h$, here h is the height of liquid above the centreline of mouthpiece and h_a is the atmospheric pressure head.
19. Ratio of areas at exit and at vena contracta of a convergent-divergent mouthpiece is given by,

$$a_1/a_c = \sqrt{1 + [(h_a - h_c)/h]}$$

20. Discharge through a Borda's mouthpiece running free is $Q = 0.5a\sqrt{2gh}$.
21. C_c for Borda's mouthpiece running free if some loss of energy is considered:

$$C_c = a_c/a = 1/(2C_v^2)$$

22. Discharge through a Borda's mouthpiece running full is $Q = 0.707a\sqrt{2gh}$.

Multiple-choice Questions

1. The coefficient of contraction (C_c) in terms of areas at the vena contracta (a_c) and exit (a) is
   (a) $C_c = a/a_c$.
   (b) $C_c = a \times a_c$.
   (c) $C_c = a_c/a$.
   (d) None of the above.
2. The average value of coefficient of velocity is
   (a) 0.9.
   (b) 0.8.
   (c) 0.98.
   (d) 0.63.
3. The average value of coefficient of discharge for small sharp-edged orifice varies between
   (a) 0.9 to 0.98.
   (b) 0.8 to 0.88.
   (c) 0.62 to 0.65.
   (d) 1 to 1.2.
4. The Borda's mouthpiece is said to be running free if its length is
   (a) Less than 3 times of its diameter.
   (b) Equals its diameter.
   (c) More than double its length.
   (d) None of the above.
5. Which of the following mouthpiece has maximum coefficient of discharge?
   (a) Convergent-divergent.
   (b) External.
   (c) Internal.
   (d) None of the above.
6. For a Borda's mouthpiece running full, the value of coefficient of discharge is equal to
   (a) 0.507.
   (b) 0.607.
   (c) 0.707.
   (d) 0.807.
7. A mouthpiece cannot be used under very large heads due to
   (a) Cavitation at vena contracta.
   (b) Small coefficient of discharge.
   (c) Uneconomical operation and short life.
   (d) None of the above.

Review Questions

1. What do you mean by an orifice and a mouthpiece and how are they classified?
2. Derive the expression for Torricelli's equation.
3. Define (i) coefficient of velocity, (ii) coefficient of contraction, (iii) coefficient of discharge and (iv) coefficient of resistance.
4. What are hydraulic coefficients and how are these determined experimentally?
5. Differentiate between a large and a small orifice. Derive an expression for discharge through a large rectangular orifice.
6. Differentiate between a wholly submerged orifice and a partially submerged orifice. Also derive an expression for discharge through a wholly submerged orifice.
7. Obtain an expression for discharge through a partially submerged orifice.
8. Derive an expression for time of emptying a vertical tank of uniform cross section.
9. Obtain an expression for time of emptying hemispherical tank.
10. Obtain an expression for time of emptying a circular horizontal tank.
11. Derive an expression for discharge through an external cylindrical mouthpiece. Also obtain an expression for absolute pressure head at its vena contracta.
12. What is the difference between external mouthpiece and reentrant mouthpiece?
13. Derive expressions for discharge through (i) Borda's mouthpiece running free and (ii) Borda's mouthpiece running full.

Problems

1. An orifice of diameter 2.5 cm discharges water under a head of 5 m. If the measured discharge through the orifice is 3 litres per second and the diameter of the jet at vena contracta is measured as 2 cm, then calculate the coefficients of discharge, velocity and contraction of the orifice.
   [**Ans.** 0.617, 0.64, 0.964]
2. The water flows through an orifice of diameter 5 cm under a constant head of 3 m in a rectangular measuring tank 1.7 m × 2 m. The rise of water level in the tank is registered as 170 mm in one minute. Determine the coefficient of discharge.
   [**Ans.** 0.64]
3. A short tube is connected to a water tank to produce a vertical jet. If the water head is 4.5 m above the outlet of the tube and coefficient of velocity is 0.94, then determine the height to which the jet rises.
   [**Ans.** 3.98 m]
4. A tank contains water up to a depth of 3.2 m and the upper part of the tank is filled with air at a pressure of 0.4 bar above atmospheric pressure. Determine the discharge through an orifice of diameter 50 mm fitted at the bottom. Neglect the losses and take coefficient of discharge as 0.62.
   [**Ans.** 0.01455 m^3/s]
5. A water jet comes out from a sharp-edged vertical orifice under a constant head of 50 mm. At a certain point, the horizontal and vertical coordinates measured from the vena contracta are 100 mm and 52 mm, respectively. Calculate the value of coefficient of velocity. Also calculate the value of coefficient of contraction if the coefficient of discharge is 0.62.
   [**Ans.** 0.98, 0.633]
6. A large vessel has a sharp-edged circular orifice of diameter 34.411 mm at a depth of 3 m below a constant water level. The jet issues horizontally and in a horizontal distance of 240 cm, it falls by 53 cm. If the water discharge was measured as 4.3 litres per second, then determine the coefficients of velocity, contraction and discharge for the orifice.
   [**Ans.** 0.952, 0.603, 0.633]
7. Two orifices are fitted on the same side of a vessel containing water to a height of h units. One orifice is situated at a depth of h_1 units from the free surface of water while the other is situated at a height of h_1 units from the bottom of the vessel. If the coefficients of velocity for both orifices are same, then show that the jets strike the ground at the same horizontal distance from the vessel. Also find the horizontal distance if $h = 6$ m, $h_1 = 2.4$ m and $C_v = 0.97$.
   [**Ans.** 5.702 m]
8. The head of water over an orifice of diameter 10 cm is 12.5 m. The water issued from the orifice is collected in a rectangular tank of size 2 m × 0.95 m and the rise of water level was observed 1.2 m in 30 seconds. Determine the coefficient of discharge.
   [**Ans.** 0.62]
9. A tank has two identical orifices A and B located on the same side at depths h_1 and h_2, respectively below the free surface of water in the tank. If the coefficients of velocity of both orifices are same, then prove that their point of intersection is h_2 below orifice A and h_1 below orifice B. Also find the horizontal distance of the point of intersection from the plane of orifices.
   [**Ans.** $2C_v\sqrt{h_1 h_2}$]

10. A tank has two orifices, namely A of diameter d_1 and B of diameter d_2 which are located on the opposite sides at depths h_1 and $(h_1/2)$, respectively, below the free surface of water in the tank. Determine the diameter d_2 in terms of d_1 when the net horizontal force on the tank is zero.

[**Ans.** $d_2 = \sqrt{2}\, d_1$]

11. A vessel standing on the ground containing water up to a height of H metres has an orifice in its wall at a depth of h metres from the free surface of water. Determine the relation between H and h so that the jet of water strikes the ground at a maximum distance from the tank. Also determine the maximum horizontal distance.

[**Ans.** $h = H/2$, $x_{max} = C_v H$]

12. A nozzle of diameter 3 cm discharges 760 litres of water per minute working under a head of 65 m. If the diameter of the jet is 2.55 cm, then determine the values of coefficients of contraction, discharge and velocity. Also determine the loss of head due to friction.

[**Ans.** 0.7225, 0.502, 0.695, 33.603 m]

13. A horizontal pipe of diameter 10 cm is fitted with a nozzle of diameter 5 cm at its discharge end. If the water flow rate through the nozzle is 0.02 m^3/s and the pressure at the base of the nozzle is 60 kPa, then determine the coefficient of discharge of the nozzle.

[**Ans.** 0.9]

14. Determine the discharge through a vertical rectangular orifice of width 0.6 m and depth 1 m , if the top edge of the orifice is 1.2 m below the water surface in the tank. Also determine the percentage error in the determined discharge, if the orifice is treated as a small orifice. Take coefficient of discharge as 0.62.

[**Ans.** 2.1405 m^3/s, 0.368%]

15. Determine the discharge through a fully submerged orifice of width 2 m and depth 1 m , if the difference of water levels on both the sides of the orifice is 3 m and the coefficient of discharge is 0.6.

[**Ans.** 9.21 m^3/s]

16. A large tank is fitted with a rectangular orifice of width 2 m and depth 1.2 m in one of its sides. The water level on one side of the orifice is 3 m above the top edge of the orifice, whereas on the other side of the orifice, the water level is 0.5 m below its top edge. If the coefficient of discharge of the orifice is 0.6, then determine the discharge through it.

[**Ans.** 11.751 m^3/s]

17. A rectangular orifice of width 1.5 m and depth 1 m is provided in one side of a large vessel. The water level in one side of the orifice is 2 m above the top edge of the orifice, while on the other side of the orifice; the water level is 0.4 m below its top edge. Determine the discharge through the orifice if the coefficient of discharge is 0.62.

[**Ans.** 6.27 m^3/s]

18. A hemispherical cistern of diameter 10 m full of water is provided with a 6 cm diameter sharp-edged orifice at its bottom. If the coefficient of discharge of the orifice is 0.6, then determine the time required to lower the level in the cistern by 2 m.

[**Ans.** 9928.4 s]

19. A vertical cylindrical water tank of diameter 3 m has a hemispherical portion at its bottom and its cylindrical portion of height is 4 m. It is fitted with an orifice of diameter 15 cm at its bottom having coefficient of discharge as 0.6. If the given tank is full of water, then determine the time taken to empty it.

[**Ans.** 509.34 s]

20. A swimming pool of length 12 m and width 7 m holds water to a depth of 2 m. If the water is discharged through an opening of area 0.2 m^2 at the bottom of the pool having coefficient of discharge as 0.6, then determine the time required to empty the tank.

[**Ans.** 446.98 s]

21. A horizontal boiler drum of diameter 3 m and length 10 m contains water to a height of 2.5 m. If it is fitted with an orifice of diameter 15 cm at its bottom with coefficient of discharge of 0.62, then determine the time taken for emptying the drum.

[**Ans.** 1328.32 s]

22. An external cylindrical mouthpiece is provided into the vertical side of a tank containing water up to a height of 3 m above the centreline of the mouthpiece. If the diameter of the mouthpiece is 3 cm, then find the discharge through it.

[**Ans.** 0.00464 m^3/s]

23. Water is discharging under a constant head of 2 m through a convergent-divergent mouthpiece with a throat diameter 30 mm. If the minimum absolute pressure at the throat is 2.5 m for a barometric pressure of 10.3 m of water, then determine the maximum outlet diameter to avoid separation of water flow. Also determine the discharge through the mouthpiece.

[**Ans.** 44.63 mm, 0.00443 m^3/s]

24. A convergent-divergent mouthpiece fitted to the side of a tank discharges water at a rate of 0.005 m^3/s working under a constant head of 2.2 m. Determine the throat and exit diameters if the separation pressure is 2.5 m, the head loss in the divergent portion of the mouthpiece is ($1/10^{th}$) of the kinetic head at its outlet and atmospheric pressure is 10.3 m of water.

[**Ans.** 21.32 mm, 31.88 mm]

25. The water is flowing through a convergent-divergent mouthpiece of diameter at convergence 30 mm working under a head of 3.2 m. If the maximum vacuum pressure is 8 m of water, then determine the maximum diameter of divergence to avoid the separation of water flow.

[**Ans.** 0.041 m]

26. An internal mouthpiece of diameter 4 cm discharges water under a constant head of 5 m. Determine the discharge through the mouthpiece when (i) the mouthpiece is running free and (ii) the mouthpiece is running full.

[**Ans.** 0.00622 m^3/s, 0.0088 m^3/s]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (c) **2.** (c) **3.** (c) **4.** (b) **5.** (a) **6.** (c) **7.** (a)

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Chapter 11

Flow Over Notches and Weirs

11.1 ❐ INTRODUCTION

A notch may be defined as an opening provided in the side of a tank (or reservoir) such that the liquid surface in the tank is below the top edge of the opening. Generally, notches are made of metallic plates and are used for measuring the rate of flow of liquid from a tank or in a channel.

A weir is a concrete or masonry structure built across a river (stream or open channel) through which the liquid flows. It is used for measuring the rate of flow of water in rivers or streams. The weir is similar to a small dam constructed across a river with a sharp edge at the top and water flows through its entire length.

The sheet of water flowing through a notch or over a weir is known as the nappe (meaning sheet) or vein. The bottom edge of a notch or the top of the weir through which water flows is known as sill or crest. The height of the crest above the bottom of the tank is known as the crest height. The head under which the notch or weir discharges water is measured from the crest to the free water surface. In this chapter, the concepts regarding notches and weirs are discussed in a brief context.

11.2 ❐ COMPARISON BETWEEN A NOTCH AND A WEIR

Notch	Weir
A notch is a cut or a pass made in a metallic sheet.	A weir is a concrete or masonry structure in the form of a wall or a dam.
The edges of a notch are thin and sharp.	Weirs are fairly wide and have rough crests.
A notch is of lesser dimensions and measures small quantity of discharge.	A weir is of larger dimensions and measures large quantity of discharge.
The ratio between the head over the sill and the length of sill is more than in a weir.	The ratio between the head over the sill and the length of sill is less than in a notch.

11.3 ❐ CLASSIFICATIONS OF NOTCHES AND WEIRS

11.3.1 Classification of Notches

1. Generally, the notches are classified according to the shape of the opening (Figure 11.1), they are namely (i) rectangular notch, (ii) triangular notch (or V-notch), (iii) trapezoidal notch and (iv) stepped notch.

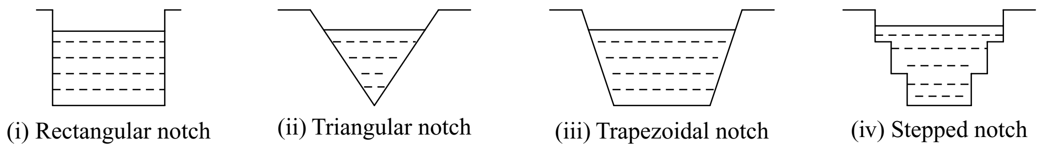

Figure 11.1 *Types of notches*

2. The notches may also be classified according to the effect of the sides on the nappe emerging from a notch as (i) notch with end contraction and (ii) notch without end contraction (or suppressed notch).

   If the sides of a notch cause the contraction of nappe, then it is said to be notch with end contraction, whereas if there is no contraction of the nappe due to the sides, then it is known as notch without end contraction.

11.3.2 Classification of Weirs

The weirs are usually classified according to its physical characteristics and they are listed below.

1. According to the shape of the opening: (i) Rectangular weir, (ii) triangular weir and (iii) trapezoidal weir (Cipolletti weir).
2. According to the shape of the crest: (i) Narrow-crested weir, (ii) broad-crested weir, (iii) sharp-edged weir and (iv) Ogee-shaped weir.
3. According to the effect of the sides on the issuing nappe: (i) Weir with end contraction and (ii) weir without end contraction.
4. According to the discharge conditions: (i) Freely discharging weir and (ii) submerged (or drowned) weir.

11.4 ❐ DISCHARGE OVER A RECTANGULAR NOTCH OR WEIR

Consider a rectangular notch or weir provided in a channel carrying water as shown in Figure 11.2(a). Let H be the height of water over the crest, L be the length of notch or weir, z be the height of weir crest above the bottom and C_d be the coefficient of discharge. Consider an elementary horizontal strip of water of thickness dh and length L at a depth h below the free surface of water as shown in Figure 11.2(b). The discharge dQ through the strip is given below.

$$dQ = C_d \times \text{Area of strip} \times \text{Theoretical velocity} = C_d \times Ldh \times \sqrt{2gh}$$

Total discharge can be determined by integrating the above expression and the expression is as follows.

$$Q = \int_0^H C_d Ldh\sqrt{2gh} = C_d L\sqrt{2g}\int_0^H h^{1/2}dh \tag{11.1}$$

$$\boxed{\therefore Q = C_d L\sqrt{2g}\left[\frac{h^{3/2}}{3/2}\right]_0^H = \frac{2}{3}C_d L\sqrt{2g}H^{3/2}} \tag{11.2}$$

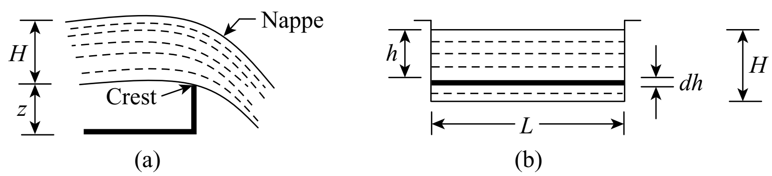

Figure 11.2 *Discharge from a rectangular notch or weir*

11.4.1 Effect on Discharge Due to Error in Measurement of Head

Let $k = (2/3)C_d L\sqrt{2g}$, then Equation (11.2) is written as follows.

$$Q = k \times H^{3/2} \tag{11.3}$$

From Equation (11.3), it can be seen that a slight error in the measurement of head H will reflect upon the accuracy of the discharge. In order to establish a relationship between error in the reading of head and its effect on the computed discharge, differentiating Equation (11.3), we get the below expression.

$$dQ = k \times \frac{3}{2} \times H^{1/2} dH \tag{11.4}$$

Dividing Equation (11.4) by Equation (11.3), we get:

$$\boxed{\frac{dQ}{Q} = \frac{k \times (3/2) \times H^{1/2} dH}{k \times H^{3/2}} = \frac{3}{2}\frac{dH}{H}} \tag{11.5}$$

Here, dQ is the error in discharge and dH is the error in the measurement of the head.

Equation (11.5) shows that an error of 1% in measuring H will produce 1.5% error in the computed Q over a rectangular weir or notch.

11.4.2 Velocity of Approach

In deriving Equation (11.2), the velocity of approach V_a has not been considered. The velocity of approach may be defined as the velocity with which water approaches the weir or notch before it passes over it. The approach velocity generates an additional head, $h_a = V_a^2/(2g)$. Thus, the limits of integration for Equation (11.1) becomes h_a to $(H + h_a)$ instead of 0 to H. Therefore, the discharge given by Equation (11.1) over a rectangular weir or notch is given below.

$$Q = \int_{h_a}^{H+h_a} C_d L dh\sqrt{2gh} = \frac{2}{3}C_d L\sqrt{2g} \times [(H + h_a)^{3/2} - h_a^{3/2}] \tag{11.6}$$

It is pertinent to mention here that Equations (11.2) and (11.6) are applicable only for suppressed weir or notch (i.e., the weirs in which the crest length is equal to the width of the channel) where there is no end contraction.

Example 11.1 Determine the length of a rectangular notch which discharges 250 litres of water per second working under a head of 0.75 m. Take coefficient of discharge as 0.6.

Solution

Let $Q = 250\ l/\text{s} = 0.25\ \text{m}^3/\text{s}$, $H = 0.75$ m and $C_d = 0.6$.

Since
$$Q = \frac{2}{3}C_d L\sqrt{2g}H^{3/2}$$

Thus
$$0.25 = \frac{2}{3} \times 0.6 \times L \times \sqrt{2 \times 9.81} \times 0.75^{3/2}$$

$$\therefore L = \frac{0.25 \times 3}{2 \times 0.6 \times \sqrt{2 \times 9.81} \times 0.75^{3/2}} = \mathbf{0.21724\ m}$$

Example 11.2 A rectangular weir of length 7.5 m built across a rectangular channel discharges 1850 litres per second with a coefficient of discharge of 0.62. If the maximum depth of water on the upstream side of the weir is 2.2 m, then determine the height of the weir.

Solution

Let $L = 7.5$ m, $Q = 1850\ l/\text{s} = 1.85\ \text{m}^3/\text{s}$, $C_d = 0.62$ and $(H + z) = 2.2$ m. Let H be the height of water over the crest and z be the height of the weir.

Since $$Q = \frac{2}{3} C_d L \sqrt{2g} H^{3/2}$$

Thus $$1.85 = \frac{2}{3} \times 0.62 \times 7.5 \times \sqrt{2 \times 9.81} \times H^{3/2}$$

$$\therefore H = \left[\frac{1.85 \times 3}{2 \times 0.62 \times 7.5 \times \sqrt{2 \times 9.81}} \right]^{2/3} = 0.2628 \text{ m}$$

$$\because (H + z) = 2.2 \text{ m} \Rightarrow z = (2.2 - H) \text{ m}$$

$$\therefore z = 2.2 - 0.2628 = \mathbf{1.9372\ m}$$

Example 11.3 If the discharge through a rectangular notch is 30 cubic metres per minute and the head of water is half the width of the notch, then determine the width of the notch. Take coefficient of discharge as 0.6.

Solution

Let $Q = 30 \text{ m}^3/\text{min} = 0.5 \text{ m}^3/\text{s}$, $H = L/2$ and $C_d = 0.6$.

Since $$Q = \frac{2}{3} C_d L \sqrt{2g} H^{3/2}$$

$$0.5 = \frac{2}{3} \times 0.6 \times L \times \sqrt{2 \times 9.81} \times \left(\frac{L}{2} \right)^{3/2}$$

$$0.5 = 0.62642 \times L^{5/2}$$

$$\therefore L = \left[\frac{0.5}{0.62642} \right]^{2/5} = \mathbf{0.9138\ m}$$

Example 11.4 A rectangular notch of length 0.3 m is used for measuring a discharge of 50 litres per second. An error of 1 mm was made in measuring the head over the notch. Determine the percentage error in the discharge if coefficient of discharge is 0.62. Also find the percentage accuracy.

Solution

Let $L = 0.3$ m, $Q = 50\ l/\text{s} = 0.05 \text{ m}^3/\text{s}$, $dH = 1 \text{ mm} = 0.001$ m and $C_d = 0.62$.

Since $$Q = \frac{2}{3} C_d L \sqrt{2g} H^{3/2}$$

$$0.05 = \frac{2}{3} \times 0.62 \times 0.3 \times \sqrt{2 \times 9.81} \times H^{3/2}$$

$$\therefore H = \left[\frac{0.05 \times 3}{2 \times 0.62 \times 0.3 \times \sqrt{2 \times 9.81}} \right]^{2/3} = 0.2024 \text{ m}$$

Since $$\frac{dQ}{Q} = \frac{3}{2} \frac{dH}{H}$$

$$\therefore \frac{dQ}{Q} = \frac{3}{2} \times \frac{0.001}{0.2024} = \mathbf{0.0074 \text{ or } 0.74\%}$$

$$\text{Percentage accuracy} = 100 - 0.74 = \mathbf{99.26\%}$$

Example 11.5 The head of water over a rectangular weir of length 75 m is 1.3 m. If the velocity of approach is 0.6 m/s and the coefficient of discharge is 0.62, then determine the discharge over the weir.

Solution

Let $L = 75$ m, $H = 1.3$ m, $V_a = 0.6$ m/s and $C_d = 0.62$.

The head due to velocity of approach is given by,

$$h_a = \frac{V_a^2}{2g} = \frac{0.6^2}{2\times 9.81} = 0.01835 \text{ m}$$

Since $$Q = \frac{2}{3}C_d L\sqrt{2g}\times[(H+h_a)^{3/2} - h_a^{3/2}]$$

Thus $$Q = \frac{2}{3}\times 0.62\times 75\times\sqrt{2\times 9.81}\times[(1.3+0.01835)^{3/2} - 0.01835^{3/2}]$$

$$\therefore Q = \mathbf{207.512\ m^3/s}$$

Example 11.6 The maximum flow through a rectangular flume of width 1.75 m and depth 1.25 m is 1.6 m^3/s. It is proposed to install a suppressed sharp-crested rectangular weir across the flume to measure the flow. Determine the maximum height at which the weir crest can be placed in order that water may not overflow the sides of the flume (i) when the velocity of approach is neglected and (ii) when the velocity of approach is considered. Take coefficient of discharge as 0.6.

Solution

Let $L = 1.75$ m, $d = (z+H) = 1.25$ m, $Q = 1.6$ m^3/s and $C_d = 0.6$, where z is the height of the weir crest above the bottom of the flume.

(i) $Q = \frac{2}{3}C_d L\sqrt{2g}H^{3/2}$

Thus $$1.6 = \frac{2}{3}\times 0.6\times 1.75\times\sqrt{2\times 9.81}\times H^{3/2}$$

$$\therefore H = \left[\frac{1.6\times 3}{2\times 0.6\times 1.75\times\sqrt{2\times 9.81}}\right]^{2/3} = 0.6433 \text{ m}$$

Since $$z + H = 1.25 \text{ m}$$

$$\therefore z = 1.25 - H = 1.25 - 0.6433 = \mathbf{0.6067\ m}$$

(ii) $V_a = \frac{Q}{Ld} = \frac{1.6}{1.75\times 1.25} = 0.7314$ m/s

$$h_a = \frac{V_a^2}{2g} = \frac{0.7314^2}{2\times 9.81} = 0.02726 \text{ m}$$

Since $$Q = \frac{2}{3}C_d L\sqrt{2g}\times[(H+h_a)^{3/2} - h_a^{3/2}]$$

$$1.6 = \frac{2}{3}\times 0.6\times 1.75\times\sqrt{2\times 9.81}\times[(H+0.02726)^{3/2} - 0.02726^{3/2}]$$

$$\therefore H = \left[\frac{1.6\times 3}{2\times 0.6\times 1.75\times\sqrt{2\times 9.81}} + 0.02726^{3/2}\right]^{2/3} - 0.02726 = 0.6198 \text{ m}$$

$$\because (z+H)=1.25 \text{ m} \Rightarrow z=(1.25-H) \text{ m}$$

$$\therefore z = 1.25 - 0.6198 = \mathbf{0.6302\ m}$$

Example 11.7 A suppressed rectangular weir is built across a channel of width 0.8 m . If the head over the crest is 0.4 m and the weir crest is 0.5 m above the bed of the channel, then determine the discharge over the weir. Consider the velocity of approach and take coefficient of discharge as 0.6.

Solution

Let $L = 0.8$ m, $H = 0.4$ m, $z = 0.5$ m and $C_d = 0.6$.

$$d = z + H = 0.5 + 0.4 = 0.9 \text{ m}$$

Since $$Q = \frac{2}{3} C_d L \sqrt{2g} H^{3/2}$$

$$\therefore Q = \frac{2}{3} \times 0.6 \times 0.8 \times \sqrt{2 \times 9.81} \times 0.4^{3/2} = 0.3586 \text{ m}^3/\text{s}$$

$$V_a = \frac{Q}{Ld} = \frac{0.3586}{0.8 \times 0.9} = 0.498 \text{ m/s}$$

$$h_a = \frac{V_a^{\,2}}{2g} = \frac{0.498^2}{2 \times 9.81} = 0.01264 \text{ m}$$

Since $$Q = \frac{2}{3} C_d L \sqrt{2g} \times [(H + h_a)^{3/2} - h_a^{\,3/2}]$$

Thus $$Q = \frac{2}{3} \times 0.6 \times 0.8 \times \sqrt{2 \times 9.81} \times [(0.4 + 0.01264)^{3/2} - 0.01264^{3/2}]$$

$$\therefore Q = \mathbf{0.3737\ m^3/s}$$

11.5 ❐ EMPIRICAL FORMULAE FOR DISCHARGE OVER RECTANGULAR WEIRS

11.5.1 Francis's Formula

The effect of end contraction for the discharge over a rectangular weir was studied by J. B. Francis. It was found that the end contraction reduces the effective length of the crest of weir by $0.1 \times H$ times, where H is the head over the weir (Figure 11.3). Therefore, the actual discharge decreases.

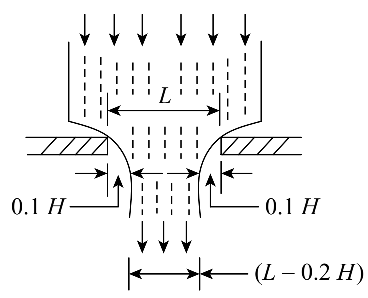

Figure 11.3 *Rectangular weir with end contractions*

If there are n end contractions, then the effective length of the weir is given below.

$$L_{\text{effective}} = (L - 0.1nH), \text{ here } L \text{ is the length of weir}$$

Thus $$Q = \frac{2}{3} C_d (L - 0.1nH) \sqrt{2g} H^{3/2} \tag{11.7}$$

When $C_d = 0.623$ and $g = 9.81$ m/s^2, Equation (11.7) is written as follows.

$$Q = 1.84(L - 0.1nH)H^{3/2} \tag{11.8}$$

When velocity of approach is also considered then Equation (11.8) becomes,

$$Q = 1.84[L - 0.1n(H + h_a)][(H + h_a)^{3/2} - h_a^{\,3/2}] \tag{11.9}$$

For a rectangular weir, there are two end contractions (i.e., $n = 2$) and thus, from Equation (11.7), we get the below expression.

$$Q = \frac{2}{3} C_d (L - 0.2H)\sqrt{2g} H^{3/2} \qquad (11.10)$$

When $C_d = 0.623$ and $g = 9.81$ m/s^2, then Equation (11.10) becomes,

$$\boxed{Q = 1.84(L - 0.2H)H^{3/2}} \qquad (11.11)$$

When end contractions are suppressed, i.e., $n = 0$, then Equation (11.8) becomes,

$$Q = 1.84LH^{3/2} \qquad (11.12)$$

When velocity of approach is also considered, then Equation (11.12) becomes,

$$\boxed{Q = 1.84L[(H + h_a)^{3/2} - h_a{}^{3/2}]} \qquad (11.13)$$

Here $$\boxed{h_a = \frac{V_a{}^2}{2g}}$$

11.5.2 Bazin's Formula

Bazin found that the value of C_d varies with the head H over the crest of the weir and he proposed the following formula for discharge over suppressed weirs.

$$\boxed{Q = mL\sqrt{2g} H^{3/2}} \qquad (11.14)$$

Here $$\boxed{m = 0.405 + \frac{0.003}{H}}$$

When velocity of approach is considered, then we get the below expression.

$$Q = m_1 L\sqrt{2g}(H + h_a)^{3/2} \qquad (11.15)$$

Here $$m_1 = 0.405 + \frac{0.003}{H + h_a}$$

11.5.3 Rehbock's Formula

The following empirical formula was proposed by T. Rehbock for discharge over suppressed rectangular weirs.

$$Q = mL\sqrt{2g} \times (H + 0.0011)^{3/2} \qquad (11.16)$$

Here, $m = \left[0.403 + \frac{0.053(H + 0.0011)}{z}\right]$ and z is the height of weir.

Example 11.8 Determine (i) the discharge over a rectangular weir of length 1 m under a constant head of 80 cm by using Francis's and Bazin's formulae when the end contractions are suppressed. (ii) Also determine the discharge using Francis's formula when the end contractions are considered.

Solution

Let $L = 1$ m and $H = 80$ cm $= 0.8$ m.

(i) $Q = 1.84LH^{3/2}$ [Francis's formula]

$\therefore Q = 1.84 \times 1 \times 0.8^{3/2} =$ **1.3166 m³/s**

Now $$Q = mL\sqrt{2g}H^{3/2} \quad \text{[Bazin's formula]}$$

$$m = 0.405 + \frac{0.003}{H} = 0.405 + \frac{0.003}{0.8} = 0.40875$$

$$\therefore Q = 0.40875 \times 1 \times \sqrt{2 \times 9.81} \times 0.8^{3/2} = \mathbf{1.2955\ m^3/s}$$

(ii) Francis's formula, when end contractions are considered is given below.

$$Q = 1.84(L - 0.2H)H^{3/2}$$

$$\therefore Q = 1.84 \times (1 - 0.2 \times 0.8) \times 0.8^{3/2} = \mathbf{1.10594\ m^3/s}$$

Example 11.9 A stream approaching waterfall of 10 m is measured by a rectangular weir. The length of the weir crest is 2 m and head over the weir is found to be 0.4 m. The velocity of approach is 1 m/s and it increases the head by $1.2 \times [V_a^{\ 2}/(2g)]$. Determine (i) the discharge by using Francis formula if the end contractions are suppressed. (ii) Also determine the power developed from the waterfall if 76% of its energy is utilized.

Solution

Let $H_1 = 10$ m, $L = 2$ m, $H = 0.4$ m, $V_a = 1$ m/s, $h_a = 1.2 \times [V_a^{\ 2}/(2g)]$ and $\eta = 0.76$.

$$h_a = \frac{1.2V_a^{\ 2}}{2g} = \frac{1.2 \times 1^2}{2 \times 9.81} = 0.0612 \text{ m}$$

$$Q = 1.84L[(H + h_a)^{3/2} - h_a^{\ 3/2}] \quad \text{[Francis's formula]}$$

$$\therefore Q = 1.84 \times 2 \times [(0.4 + 0.0612)^{3/2} - 0.0612^{3/2}] = \mathbf{1.097\ m^3/s}$$

Power developed from the waterfall is given by,

$$P = \frac{\eta \rho_w g Q H_1}{1000} = \frac{0.76 \times 1000 \times 9.81 \times 1.097 \times 10}{1000} = \mathbf{81.788\ kW}$$

Example 11.10 A weir of length 48 m is divided into 10 equal bays by vertical posts each of width 0.5 m. Calculate the discharge over the weir by using Francis's formula if the head over the crest of the weir is 1.4 m and the velocity of approach is 2.5 m/s.

Solution

Refer Figure 11.4. Let $L_1 = 48$ m, number of bays = 10, width of each post = 0.5 m, $H = 1.4$ m and $V_a = 2.5$ m/s.

$$L = 48 - 9 \times 0.5 = 43.5 \text{ m}$$

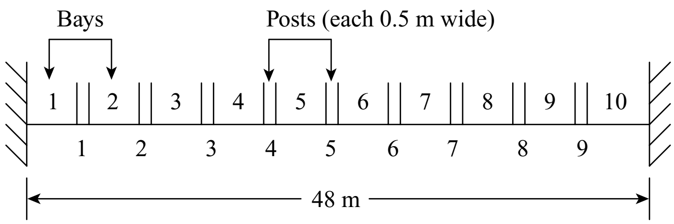

Figure 11.4

Since each bay has two end contractions, we get the following value.

$$n = 2 \times 10 = 20$$

$$h_a = \frac{V_a^2}{2g} = \frac{2.5^2}{2 \times 9.81} = 0.3185 \text{ m}$$

Since $$Q = 1.84[L - 0.1n(H + h_a)][(H + h_a)^{3/2} - h_a^{3/2}]$$

Thus $$Q = 1.84 \times [43.5 - 0.1 \times 20 \times (1.4 + 0.3185)][(1.4 + 0.3185)^{3/2} - 0.3185^{3/2}]$$

$$\therefore Q = \mathbf{152.8176 \text{ m}^3/\text{s}}$$

Example 11.11 In a catchment area, the daily record of rainfall is measured as 1.5×10^5 m^3. Eighty per cent of the rainwater reaches the storage reservoir and then passes over a rectangular weir. If the water level over the weir crests does not rise more than 0.5 m, then determine the length of weir using Bazin's formula.

Solution

Let $Q_1 = 1.5 \times 10^5 \text{ m}^3/\text{day}$, $Q = 80\%$ of Q_1 and $H = 0.5$ m.

$$Q_1 = \frac{1.5 \times 10^5}{24 \times 3600} = 1.7361 \text{ m}^3/\text{s}$$

Thus $$Q = \frac{80}{100} \times 1.7361 = 1.3889 \text{ m}^3/\text{s}$$

Since $$Q = mL\sqrt{2g} \times H^{3/2}$$

$$m = 0.405 + \frac{0.003}{H} = 0.405 + \frac{0.003}{0.5} = 0.411$$

Thus $$1.3889 = 0.411 \times L \times \sqrt{2 \times 9.81} \times 0.5^{3/2}$$

$$\therefore L = \frac{1.3889}{0.411 \times \sqrt{2 \times 9.81} \times 0.5^{3/2}} = \mathbf{2.1579 \text{ m}}$$

Example 11.12 A 20 m long weir is divided into 12 equal bays by vertical posts each of width 0.4 m. Calculate the discharge over the weir using Francis's formula if the head over the crest of the weir is 1.2 m.

Solution

Let $L_1 = 20$ m, number of bays $= 12$, width of each post $= 0.4$ m and $H = 1.2$ m.

$$L = 20 - 11 \times 0.4 = 15.6 \text{ m}$$

Since each bay has two end contractions, we get the following value.

$$n = 2 \times 12 = 24$$

Since $$Q = 1.84(L - 0.1nH)H^{3/2}$$

$$\therefore Q = 1.84 \times (15.6 - 0.1 \times 24 \times 1.2) \times 1.2^{3/2} = \mathbf{30.7664 \text{ m}^3/\text{s}}$$

Example 11.13 A rectangular weir is divided into N number of openings each of span 10 m. If discharge over the weir is 971.73 m^3/s and the velocity of approach is 2 m/s, then determine the number of openings so that the head of water does not exceed 3 m over the weir.

Solution

Let Number of openings = N, $L = 10$ m, $Q_{total} = 971.73\ m^3/s$, $V_a = 2$ m/s and $H = 3$ m.

$$h_a = \frac{V_a^2}{2g} = \frac{2^2}{2\times 9.81} = 0.2039 \text{ m}$$

Let Q be the discharge through each opening which can be given from Equation (11.9) by taking $n = 2$ (because each opening has two number of end contractions) as given below.

$$Q = 1.84[L - 0.1\times 2\times (H + h_a)][(H + h_a)^{3/2} - h_a^{3/2}]$$

Thus

$$Q = 1.84\times[10 - 0.1\times 2\times(3 + 0.2039)][(3 + 0.2039)^{3/2} - 0.2039^{3/2}]$$

$$\therefore Q = 97.173\ m^3/s$$

$$N = \frac{Q_{total}}{Q} = \frac{971.73}{97.173} = \mathbf{10}$$

Example 11.14 A rectangular weir of width 5 m has no end contractions. If the head over the crest is 0.5 m and the crest is 1.3 m above the bed level of the channel, then determine the discharge using Rehbock's formula.

Solution

Let $L = 5$ m, $H = 0.5$ m and $z = 1.3$ m.

Since

$$Q = mL\sqrt{2g}(H + 0.0011)^{3/2}$$

$$m = \left[0.403 + \frac{0.053(H + 0.0011)}{z}\right] = \left[0.403 + \frac{0.053\times(0.5 + 0.0011)}{1.3}\right] = 0.42343$$

$$\therefore Q = 0.42343\times 5\times\sqrt{2\times 9.81}\times(0.5 + 0.0011)^{3/2} = \mathbf{3.3265\ m^3/s}$$

11.6 ❒ DISCHARGE OVER A TRIANGULAR NOTCH OR WEIR

A triangular notch (or weir) has a V-shaped opening, thus it is also known as a V-notch (Figure 11.5). Generally, it is preferred over a rectangular notch when the discharge is to be measured at varying heads. This is due to the reason that the coefficient of discharge in case of V-notch remains fairly constant for different heads, whereas it does not remain constant in a rectangular notch.

The discharge of water flowing over a triangular notch or weir may be computed by using the expression as derived below.

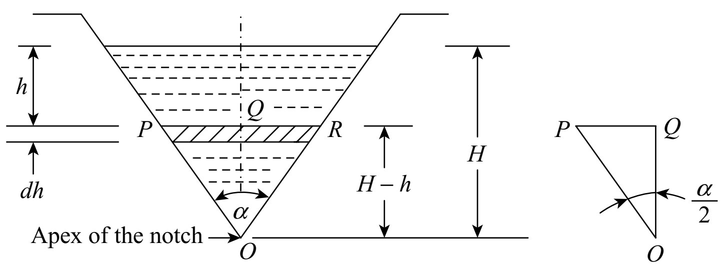

Figure 11.5 *Discharge from a triangular notch or weir*

Let H be the height of water over the crest of V-notch having vertex angle α and C_d be the coefficient of discharge. Consider an elementary horizontal strip of water of thickness dh and length L at a depth h below the free surface of water as shown in Figure 11.5.

$$\text{Width of strip} = 2PQ = 2 \times OQ \tan(\alpha/2) = 2(H-h)\tan(\alpha/2)$$

$$\text{Area of strip} = \text{Width of strip} \times \text{Thickness of strip} = 2(H-h)\tan(\alpha/2)dh$$

The discharge (dQ) through the strip is given by,

$$dQ = C_d \times \text{Area of strip} \times \text{Theoretical velocity}$$

Thus
$$dQ = C_d\left[2(H-h)\tan(\alpha/2)dh\right]\sqrt{2gh}$$

The total discharge can be determined by integrating the above expression and it is given as follows.

$$Q = \int_0^H C_d\left[2(H-h)\tan\frac{\alpha}{2}dh\right]\sqrt{2gh} = 2C_d\sqrt{2g}\tan\frac{\alpha}{2}\int_0^H (H-h)h^{1/2}dh$$

$$Q = 2C_d\sqrt{2g}\tan\frac{\alpha}{2}\left[H \times \frac{h^{3/2}}{3/2} - \frac{h^{5/2}}{5/2}\right]_0^H$$

$$Q = 2C_d\sqrt{2g}\tan\frac{\alpha}{2}\left[\frac{2}{3}H \times H^{3/2} - \frac{2}{5}H^{5/2}\right]$$

$$\boxed{\therefore Q = 2C_d\sqrt{2g}\tan\frac{\alpha}{2}\left[\frac{4}{15}H^{5/2}\right] = \frac{8}{15}C_d\sqrt{2g}\tan\frac{\alpha}{2}H^{5/2}} \tag{11.17}$$

For a right-angled V-notch, $\alpha = 90°$ and if $C_d = 0.6$, then Equation (11.17) is written as follows.

$$\boxed{Q = \frac{8}{15} \times 0.6 \times \sqrt{2 \times 9.81} \times \tan\frac{90°}{2} \times H^{5/2} = 1.417H^{5/2}} \tag{11.18}$$

11.6.1 Effect on Discharge Due to Error in Measurement of Head

Let $k = (8/15)C_d\sqrt{2g}\tan(\alpha/2)$, then Equation (11.17) is written as follows.

$$Q = k \times H^{5/2} \tag{11.19}$$

Differentiating Equation (11.19), we get:

$$dQ = k \times (5/2) \times H^{3/2}dH \tag{11.20}$$

Dividing Equation (11.20) by Equation (11.19), we get:

$$\boxed{\frac{dQ}{Q} = \frac{k \times (5/2) \times H^{3/2}dH}{k \times H^{5/2}} = \frac{5}{2}\frac{dH}{H}} \tag{11.21}$$

Equation (11.21) shows that an error of 1% in measuring H will produce 2.5% error in the computed Q over a triangular weir or notch.

11.6.2 Advantages of a Triangular Notch (or Weir) Over a Rectangular Notch (or Weir)

The following are the advantages of a triangular notch (or weir) over a rectangular notch (or weir).

1. The coefficient of discharge for a triangular weir or notch is fairly constant for all the heads. Thus, it is preferred over a rectangular notch when the discharge is to be measured at varying heads.

2. Even with a low discharge, the head over the crest of a triangular weir or notch is comparatively large and therefore, it can be measured accurately. Thus, a triangular weir or notch is very useful for measuring low discharge.
3. Ventilation of air is not required in a triangular weir or notch.

Example 11.15 A right-angled V-notch weir provided in a 2.5 m wide rectangular channel measures water discharge of 0.3 cumec. If the maximum depth of water does not exceed 1 m, then determine the position of the apex of the notch from the bed of the channel. Assume coefficient of discharge as 0.6.

Solution

Let $\alpha = 90°$, $L = 2.5$ m, $Q = 0.3\ \text{m}^3/\text{s}$, $(z + H) = 1$ m and $C_d = 0.6$, here z is the height of apex of the notch from the bed of the channel.

Let H be the height of water over the triangular notch.

Since $$Q = \frac{8}{15} C_d \sqrt{2g} \tan\frac{\alpha}{2} H^{5/2}$$

Thus $$0.3 = \frac{8}{15} \times 0.6 \times \sqrt{2 \times 9.81} \times \tan\frac{90°}{2} \times H^{5/2}$$

$$\therefore H = \left[\frac{0.3 \times 15}{8 \times 0.6 \times \sqrt{19.62} \times \tan 45°}\right]^{2/5} = 0.53734 \text{ m}$$

Since $$z + H = 1 \text{ m}$$

$$\therefore z = 1 - H = 1 - 0.53734 = \mathbf{0.46266\ m}$$

Example 11.16 The water flows over a rectangular weir of width 1.5 m at a depth of 0.25 m and afterwards, it passes through a triangular right-angled weir. If the coefficients of discharge for the rectangular and triangular weir are 0.623 and 0.6, respectively, then determine the depth of water over the triangular weir.

Solution

The subscripts 1 and 2 denote the values for rectangular and triangular weirs, respectively. $L_1 = 1.5$ m, $H_1 = 0.25$ m, $\alpha = 90°$, $C_{d1} = 0.623$ and $C_{d2} = 0.6$.

Let Q be the discharge through both the weirs and H_2 be the depth of water over the triangular weir.

Since $$Q = (2/3) C_{d1} L_1 \sqrt{2g} \times H_1^{3/2} \qquad \text{[Rectangular weir]}$$

$$\therefore Q = \frac{2}{3} \times 0.623 \times 1.5 \times \sqrt{2 \times 9.81} \times 0.25^{3/2} = 0.345 \text{ m}^3/\text{s}$$

Since $$Q = \frac{8}{15} C_{d2} \sqrt{2g} \tan\frac{\alpha}{2} H_2^{5/2} \qquad \text{[Triangular weir]}$$

$$0.345 = \frac{8}{15} \times 0.6 \times \sqrt{2 \times 9.81} \times \tan\frac{90°}{2} \times H_2^{5/2}$$

$$\therefore H_2 = \left[\frac{0.345 \times 15}{8 \times 0.6 \times \sqrt{19.62} \times \tan 45°}\right]^{2/5} = \mathbf{0.56824\ m}$$

Example 11.17 The actual discharge by a right-angled V-notch with coefficient of discharge as 0.62 is known to be 0.044 m^3/s. However, in an experiment, the discharge is found to be 0.04 m^3/s. Determine the error by assuming that this discrepancy is due to an error in measuring head above the sill.

Solution

Let $\alpha = 90°$, $C_d = 0.62$, $Q = 0.044\ \text{m}^3/\text{s}$ and $Q_{\text{measured}} = 0.04\ \text{m}^3/\text{s}$.

Let H be the actual head over the notch and dH be the error in measurement of head. The actual head over the notch is given by,

$$Q = \frac{8}{15} C_d \sqrt{2g} \tan\frac{\alpha}{2} H^{5/2}$$

Thus
$$0.044 = \frac{8}{15} \times 0.62 \times \sqrt{2 \times 9.81} \times \tan\frac{90°}{2} \times H^{5/2}$$

$$\therefore H = \left[\frac{0.044 \times 15}{8 \times 0.62 \times \sqrt{19.62} \times \tan 45°}\right]^{2/5} = 0.2461\ \text{m}$$

Since
$$\frac{dQ}{Q} = \frac{5}{2}\frac{dH}{H}$$

Thus
$$dH = \frac{2}{5} \times \frac{dQ}{Q} \times H = \frac{2}{5} \times \frac{(Q - Q_{\text{measured}})}{Q} \times H$$

$$\therefore dH = \frac{2}{5} \times \left(\frac{0.044 - 0.04}{0.044}\right) \times 0.2461 = \mathbf{0.00895\ m\ or\ 8.95\ mm}$$

Example 11.18 A rectangular notch of length 1 m and height 40 cm discharges water. If the same quantity of water is allowed to flow over a right-angled V-notch, then determine the height to which water will rise above the apex of the notch. Take coefficient of discharge for both notches as 0.623.

Solution

Let $L = 1$ m, $H = 40$ cm $= 0.4$ m, $\alpha = 90°$ and $C_d = 0.623$.

Let H_1 be the height to which water will rise in the right-angled V-notch.

Since
$$Q = (2/3) C_d L \sqrt{2g} \times H^{3/2} \qquad \text{[Rectangular notch]}$$

$$\therefore Q = \frac{2}{3} \times 0.623 \times 1 \times \sqrt{2 \times 9.81} \times 0.4^{3/2} = 0.46541\ \text{m}^3/\text{s}$$

Since
$$Q = \frac{8}{15} C_d \sqrt{2g} \tan\frac{\alpha}{2} H_1^{5/2} \qquad \text{[Triangular notch]}$$

$$0.46541 = \frac{8}{15} \times 0.623 \times \sqrt{2 \times 9.81} \times \tan\frac{90°}{2} \times H_1^{5/2}$$

$$\therefore H_1 = \left[\frac{0.46541 \times 15}{8 \times 0.623 \times \sqrt{19.62} \times \tan 45^o}\right]^{2/5} = \mathbf{0.631\ m}$$

Example 11.19 The head and discharge over the V-notch having coefficient of discharge as 0.62 is 80 mm and 0.008 m^3/s, respectively. Determine the depth and top width of the V-notch when it discharges 0.5 m^3/s.

Solution

Let $C_d = 0.62$, $H = 80$ mm $= 0.08$ m, $Q = 0.008\ \text{m}^3/\text{s}$ and $Q_1 = 0.5\ \text{m}^3/\text{s}$.

Let H be the depth and B the width of V-notch as shown in Figure 11.6.

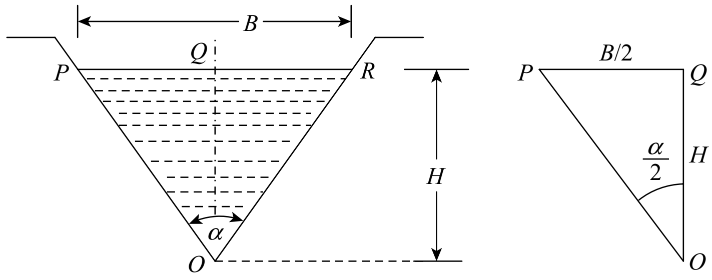

Figure 11.6

Since $$Q = \frac{8}{15} C_d \sqrt{2g} \tan\frac{\alpha}{2} H^{5/2} \qquad \text{[Equation (11.17)]}$$

$$0.008 = \frac{8}{15} \times 0.62 \times \sqrt{2 \times 9.81} \times \tan\frac{\alpha}{2} \times 0.08^{5/2}$$

$$\therefore \tan\frac{\alpha}{2} = \left[\frac{0.008 \times 15}{8 \times 0.62 \times \sqrt{19.62} \times 0.08^{5/2}}\right] = 3.017346$$

Again using Equation (11.17) for the discharge $Q_1 = 0.5\ \text{m}^3/\text{s}$, we get:

$$0.5 = \frac{8}{15} \times 0.62 \times \sqrt{2 \times 9.81} \times 3.017346 \times H^{5/2}$$

$$\therefore H = \left[\frac{0.5 \times 15}{8 \times 0.62 \times \sqrt{19.62} \times 3.017346}\right]^{2/5} = \mathbf{0.418256\ m}$$

Since $$\tan\frac{\alpha}{2} = \frac{(B/2)}{H} \qquad \text{[Figure (11.6)]}$$

$$\therefore B = 2H \tan\frac{\alpha}{2} = 2 \times 0.418256 \times 3.017346 = \mathbf{2.524\ m}$$

11.7 ❐ DISCHARGE OVER A TRAPEZOIDAL NOTCH OR WEIR

A trapezoidal notch or weir is a combination of a rectangular and a triangular notch or weir as shown in Figure 11.7. Thus, the total discharge over a trapezoidal weir or notch is the sum of the two discharges, namely discharge through a rectangular notch or weir and discharge through a triangular notch or weir.

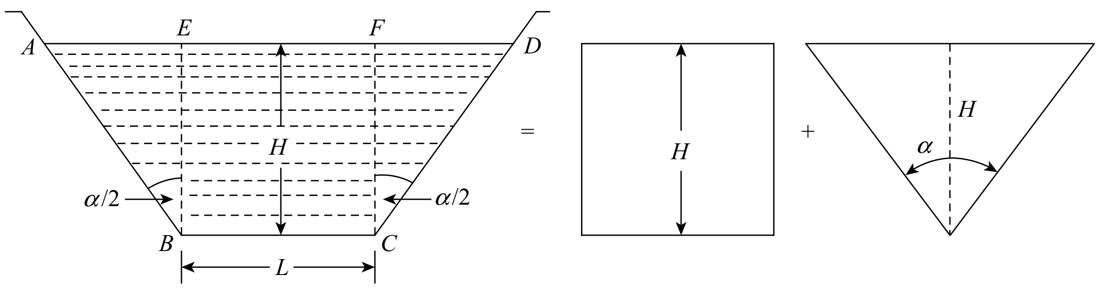

Figure 11.7 *Discharge from a trapezoidal notch or weir*

Let L be the length of the crest of the notch (i.e., rectangular portion), H be the height of water over the notch, $(\alpha/2)$ is the angle of inclination of its sides with the vertical, C_{d1} and C_{d2} be the coefficient of discharges, respectively for the rectangular portion (i.e., *EBCF*) and the triangular portion (i.e., *AEB* and *DFC*) of the trapezoidal notch or weir.

The discharge through the rectangular portion of the trapezoidal notch or weir is given by,

$$Q_1 = (2/3)C_{d1}L\sqrt{2g}H^{3/2} \quad \text{[From Equation (11.2)]}$$

The discharge through the triangular portion of the trapezoidal notch or weir is given by,

$$Q_2 = \frac{8}{15}C_{d2}\sqrt{2g}\tan\frac{\alpha}{2}H^{5/2} \quad \text{[From Equation (11.17)]}$$

Total discharge through the trapezoidal notch or weir is equal to the sum of Q_1 and Q_2.

Thus
$$Q = \frac{2}{3}C_{d1}L\sqrt{2g}H^{3/2} + \frac{8}{15}C_{d2}\sqrt{2g}\tan\frac{\alpha}{2}H^{5/2} \tag{11.22}$$

If the coefficient of discharge for the whole of the trapezoidal notch or weir is taken as C_d, then Equation (11.22) can be written as follows.

$$\boxed{\therefore Q = C_d\sqrt{2g}H^{3/2}\left[\frac{2}{3}L + \frac{8}{15}H\tan\frac{\alpha}{2}\right]} \tag{11.23}$$

Example 11.20 A trapezoidal notch is 0.8 m wide at the top and 0.32 m at the bottom and 0.24 m in height. If the head of water on the notch is 0.1 m, then determine the discharge through the notch in litres per second. Take coefficient of discharges for rectangular and triangular portions as 0.62 and 0.6, respectively.

Solution

Refer Figure 11.8. Let $AD = 0.8$ m, $BC = L = 0.32$ m, $EB = 0.24$ m, $H = 0.1$ m, $C_{d1} = 0.62$ and $C_{d2} = 0.6$.

From right-angled triangle *AEB*, we get:

$$\tan\frac{\alpha}{2} = \frac{AE}{EB} = \frac{(AD-EF)/2}{EB} = \frac{(0.8-0.32)/2}{0.24} = 1$$

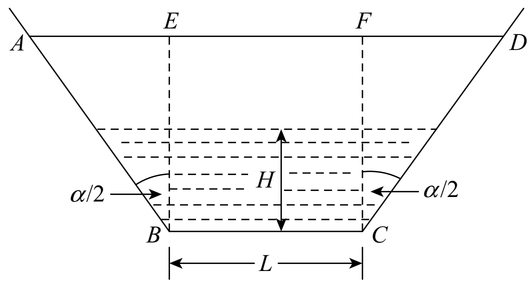

Figure 11.8

Since
$$Q = \frac{2}{3}C_{d1}L\sqrt{2g}H^{3/2} + \frac{8}{15}C_{d2}\sqrt{2g}\tan\frac{\alpha}{2}H^{5/2}$$

Thus
$$Q = \frac{2}{3}\times 0.62\times 0.32\times\sqrt{2\times 9.81}\times 0.1^{3/2} + \frac{8}{15}\times 0.6\times\sqrt{2\times 9.81}\times 1\times 0.1^{5/2}$$

$$\therefore Q = 0.01853 + 0.00448 = \mathbf{0.02301\ m^3/s\ or\ 23.01\ \mathit{l}/s}$$

11.8 ❐ CIPOLLETTI WEIR OR NOTCH

We know that the full discharge through a rectangular weir is reduced due to end contractions. To compensate this reduction in discharge, the sides of the rectangular weir are widened to form trapezoidal weir of increased area as shown in Figure 11.7. The sloping sides of which have an inclination of 1 horizontal to 4 vertical. Such weir is known as Cipolletti weir and it is named after the notable Italian engineer, Cipolletti. This weir is extensively used for measuring water supplied for irrigation purposes.

The discharge through a Cipolletti weir will be equal to the discharge through a rectangular weir of the same base L without any end contractions. It can be proved that in Cipolletti weir, $\tan(\alpha/2) = 1/4$, where $(\alpha/2)$ is the angle by which each side slopes outwardly from the vertical.

Discharge for a rectangular weir with two end contractions is given by Equation (11.10) as follows.

$$Q = \frac{2}{3} C_d (L - 0.2H)\sqrt{2g} H^{3/2}$$

By opening the brackets, we get:

$$Q = \frac{2}{3} C_d \sqrt{2g} L H^{3/2} - \frac{2}{15} C_d \sqrt{2g} H^{5/2}$$

Thus, reduction in discharge due to end contractions $(2/15)C_d \sqrt{2g} H^{5/2}$ is to be compensated by additional discharge due to the additional triangular area (Figure 11.7) which is given by Equation (11.17) as $Q = (8/15)C_d \sqrt{2g}\tan(\alpha/2)H^{5/2}$.

Therefore, equating these two expressions, we get:

$$\frac{8}{15} C_d \sqrt{2g}\tan\frac{\alpha}{2} H^{5/2} = \frac{2}{15} C_d \sqrt{2g} H^{5/2}$$

$$\boxed{\therefore \tan\frac{\alpha}{2} = \frac{1}{4} \Rightarrow \frac{\alpha}{2} = 14°2'}$$

Since the reduction in discharge over a rectangular weir due to end contractions is compensated by the additional discharge due to additional triangular area. Thus, the resultant discharge through a Cipolletti weir is equal to the discharge through its rectangular portion without end contractions. This is same as the discharge given by Equation (11.2) for the discharge over a suppressed rectangular weir as given below.

$$Q = (2/3) C_d L \sqrt{2g} H^{3/2} \qquad (11.24)$$

If the velocity of approach is also considered, then Equation (11.24) is written as follows.

$$Q = (2/3) C_d \sqrt{2g} \times [(H + h_a)^{3/2} - h_a^{3/2}] \qquad (11.25)$$

Here $$h_a = \frac{V_a^2}{2g}$$

Cipolletti proposed the following expression for discharge over a Cipolletti weir.

$$\boxed{Q = 1.866 L H^{3/2}} \qquad (11.26)$$

By comparing Equations (11.24) and (11.26), the value of coefficient of discharge C_d for a Cipolletti weir is found as 0.632.

Example 11.21 The water flows over a Cipolletti weir such that the head over the sill is 0.72 m. If the base of the weir is 2.5 m wide and its coefficient of discharge is 0.62, then determine the quantity of water flowing over the weir. Also determine the width of the weir at the top water surface.

Solution

Refer Figure 11.7. Let $H = 0.72$ m, $BC = L = 2.5$ m and $C_d = 0.62$.

Let L_{top} be the width of the weir at the top water surface.

Since $$Q = (2/3) C_d L \sqrt{2g} \times H^{3/2}$$

$$\therefore Q = \frac{2}{3} \times 0.62 \times 2.5 \times \sqrt{2 \times 9.81} \times 0.72^{3/2} = \mathbf{2.7963\ m^3/s}$$

From right-angled triangle AEB, we get:

$$AE = EB \tan\frac{\alpha}{2} = 0.72 \times \frac{1}{4} = 0.18 \text{ m} \qquad [\because \tan(\alpha/2) = 1/4]$$

$$L_{top} = L + 2AE = 2.5 + 2 \times 0.18 = \mathbf{2.86\ m} \qquad [\because AE = FD]$$

Example 11.22 Determine the discharge over a Cipolletti weir of crest length 0.75 m when the head over the weir is 0.45 m. Take the coefficient of discharge as 0.6. Also determine the discharge if the channel is 1 m wide and 0.625 m deep and the velocity of approach is taken into consideration.

Solution

Let $L = 0.75$ m, $H = 0.45$ m, $C_d = 0.6$, $b_c = 1$ m and $d_c = 0.625$ m.

(i) $Q = (2/3)C_d L\sqrt{2g} \times H^{3/2}$

$$\therefore Q = \frac{2}{3} \times 0.6 \times 0.75 \times \sqrt{2 \times 9.81} \times 0.45^{3/2} = \mathbf{0.40113\ m^3/s}$$

(ii) $A = b_c d_c = 1 \times 0.625 = 0.625\ \text{m}^2$

$$V_a = \frac{Q}{A} = \frac{0.40113}{0.625} = 0.642\ \text{m/s}$$

$$h_a = \frac{V_a^2}{2g} = \frac{0.642^2}{2 \times 9.81} = 0.021\ \text{m}$$

Since $$Q = (2/3)C_d L\sqrt{2g} \times [(H + h_a)^{3/2} - h_a^{3/2}]$$

Thus $$Q = (2/3) \times 0.6 \times 0.75 \times \sqrt{2 \times 9.81} \times [(0.45 + 0.021)^{3/2} - 0.021^{3/2}]$$

$$\therefore Q = \mathbf{0.4255\ m^3/s\ or\ 425.5\ \mathit{l}/s}$$

11.9 ❐ DISCHARGE OVER A STEPPED NOTCH

Figure 11.9 shows a stepped notch which is a combination of rectangular notches. The discharge Q through a stepped notch is the sum of the discharges passing through each rectangular notch.

Let L_1 be the length of the crest of the notch 1, H_1 be the height of water above the crest of notch 1 and Q_1 be the discharge over the notch 1. Similarly, L_2, H_2, Q_2 and L_3, H_3, Q_3 be the corresponding values for notch 2 and 3, respectively.

$$Q_1 = (2/3)C_d L_1 \sqrt{2g} H_1^{3/2}$$

$$Q_2 = (2/3)C_d L_2 \sqrt{2g}(H_2^{3/2} - H_1^{3/2})$$

$$Q_3 = (2/3)C_d L_3 \sqrt{2g}(H_3^{3/2} - H_2^{3/2})$$

The discharge Q through the stepped notch is given by,

$$Q = \frac{2}{3}C_d L_1 \sqrt{2g} H_1^{3/2} + \frac{2}{3}C_d L_2 \sqrt{2g}(H_2^{3/2} - H_1^{3/2}) + \frac{2}{3}C_d L_3 \sqrt{2g}(H_3^{3/2} - H_2^{3/2})$$

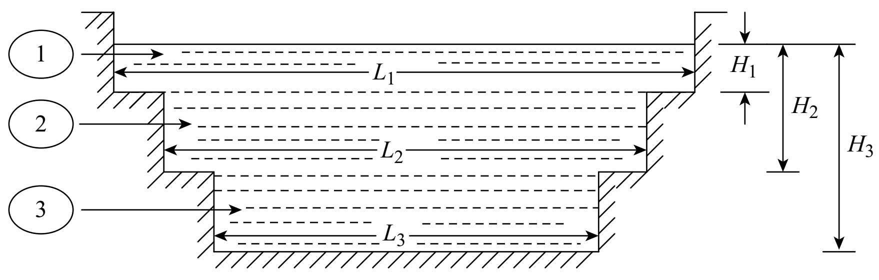

Figure 11.9 *Stepped notch*

Example 11.23 Determine the discharge over a stepped rectangular notch as shown in Figure 11.9, when $L_1 = 0.9$ m, $H_1 = 0.4$ m, $L_2 = 0.6$ m, $H_2 = 0.7$ m, $L_3 = 0.3$ m and $H_3 = 0.9$ m. Take the coefficient of discharge for the entire rectangular notch as 0.62.

Solution

Refer Figure 11.9. Let $L_1 = 0.9$ m, $H_1 = 0.4$ m, $L_2 = 0.6$ m, $H_2 = 0.7$ m, $L_3 = 0.3$ m, $H_3 = 0.9$ m and $C_d = 0.62$.

Since $$Q_1 = (2/3)C_d L_1 \sqrt{2g} H_1^{3/2}$$

$$\therefore Q_1 = (2/3) \times 0.62 \times 0.9 \times \sqrt{2 \times 9.81} \times 0.4^{3/2} = 0.41685 \text{ m}^3/\text{s}$$

Since $$Q_2 = (2/3)C_d L_2 \sqrt{2g}(H_2^{3/2} - H_1^{3/2})$$

$$\therefore Q_2 = (2/3) \times 0.62 \times 0.6 \times \sqrt{2 \times 9.81} \times (0.7^{3/2} - 0.4^{3/2}) = 0.36545 \text{ m}^3/\text{s}$$

Since $$Q_3 = (2/3)C_d L_3 \sqrt{2g}(H_3^{3/2} - H_2^{3/2})$$

$$\therefore Q_3 = (2/3) \times 0.62 \times 0.3 \times \sqrt{2 \times 9.81} \times (0.9^{3/2} - 0.7^{3/2}) = 0.14728 \text{ m}^3/\text{s}$$

$$Q = Q_1 + Q_2 + Q_3 = 0.41685 + 0.36545 + 0.14728 = \mathbf{0.92958 \text{ m}^3/\text{s}}$$

11.10 ❐ DISCHARGE OVER A BROAD-CRESTED WEIR

A broad-crested weir means a weir having a wide crest as shown in Figure 11.10. For a broad-crested weir, $2L > H$, here L is the length (thickness) of the weir and H is the head of water on the upstream side of the weir.

Let h be the head of water on the downstream side, V be the velocity of the water on the downstream side and C_d be the coefficient of discharge. Now applying Bernoulli's equation at point 1, where water is assumed to be still and point 2, where water velocity is V, we get the following expression.

$$0 + 0 + H = 0 + \frac{V^2}{2g} + h$$

$$\frac{V^2}{2g} = (H - h) \Rightarrow V = \sqrt{2g(H - h)}$$

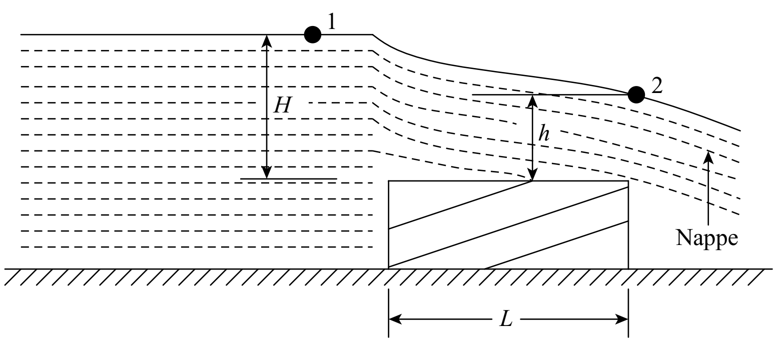

Figure 11.10 *Broad-crested weir*

The discharge (Q) over the weir is given by,

$$Q = C_d \times \text{Area of flow} \times \text{Velocity} = C_d Lh\sqrt{2g(H-h)}$$

Thus
$$Q = C_d L\sqrt{2g}\sqrt{(Hh^2 - h^3)} \tag{11.27}$$

The discharge will be maximum if $(Hh^2 - h^3)$ is maximum.

$$\frac{d}{dh}(Hh^2 - h^3) = 0$$

$$2Hh - 3h^2 = 0 \Rightarrow 2H - 3h = 0$$

$$\boxed{\therefore h = \frac{2}{3}H}$$

For obtaining maximum discharge (Q_{max}), substituting the value of h in Equation (11.27), we get the below expression.

$$Q_{max} = C_d L\sqrt{2g}\sqrt{H\left(\frac{2}{3}H\right)^2 - \left(\frac{2}{3}H\right)^3}$$

$$Q_{max} = C_d L\sqrt{2g}\sqrt{\frac{4}{9}H^3 - \frac{8}{27}H^3} = C_d L\sqrt{2\times 9.81}\sqrt{\frac{4}{27}H^3}$$

$$\boxed{\therefore Q_{max} = 1.705 C_d L H^{3/2}} \tag{11.28}$$

If the velocity of approach (V_a) is also considered, then Equation (11.28) is written as follows.

$$\boxed{Q_{max} = 1.705 C_d L(H + h_a)^{3/2}} \tag{11.29}$$

Here
$$h_a = \frac{V_a{}^2}{2g}$$

Example 11.24 Determine the maximum discharge over a broad-crested weir when the head of water above the crest is 0.4 m, the length of weir is 40 m and the channel has a cross-sectional area of 40 m^2 on the upstream side. Take coefficient of discharge as 0.62. Also determine the maximum discharge when the velocity of approach is considered.

Solution

Let $H = 0.4$ m, $L = 40$ m, $A = 40$ m^2 and $C_d = 0.62$.

$$Q_{max} = 1.705 C_d L H^{3/2} = 1.705\times 0.62\times 40\times 0.4^{3/2} = \mathbf{10.6971\ m^3/s}$$

$$V_a = \frac{Q}{A} = \frac{10.6971}{40} = 0.2674 \text{ m/s}$$

$$h_a = \frac{V_a{}^2}{2g} = \frac{0.2674^2}{2\times 9.81} = 0.00364 \text{ m}$$

Since
$$Q_{max} = 1.705 C_d L(H + h_a)^{3/2}$$

$$\therefore Q_{max} = 1.705\times 0.62\times 40\times (0.4 + 0.00364)^{3/2} = \mathbf{10.84345\ m^3/s}$$

11.11 ❒ DISCHARGE OVER A NARROW-CRESTED WEIR

A narrow-crested weir (sharp-crested weir) shown in Figure 11.11 is similar to a rectangular weir or notch. For a narrow crested weir, $2L < H$, here L is the length of the weir and H is the head of water on the upstream side of the weir.

The discharge (Q) over the narrow-crested weir is given by the same equation as a rectangular weir or notch as given below.

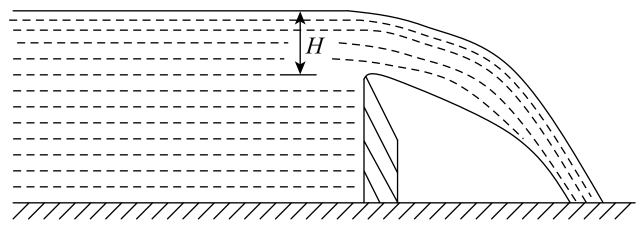

Figure 11.11 *Narrow or sharp-crested weir*

$$\boxed{Q = (2/3)C_d L\sqrt{2g}H^{3/2}} \tag{11.30}$$

11.12 ❒ DISCHARGE OVER AN OGEE WEIR

An Ogee weir (Figure 11.12) is generally employed as a spillway of a dam in which the crest rises to a maximum height of about $0.115H$ (here H is the head of water on the upstream side of the weir) and then falls down. This phenomenon is also known as Ogee spillway which is a portion of a dam over which the excess water from the dam flows to the downstream side.

The discharge over an Ogee weir is same as that of a rectangular weir and is given by,

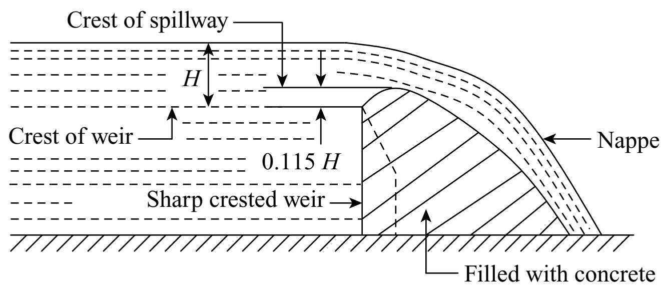

Figure 11.12 *Ogee weir*

$$Q = (2/3)C_d L\sqrt{2g}H^{3/2} \tag{11.31}$$

Example 11.25 Determine the discharge over an Ogee weir when the head of water above the crest is 0.5 m, the length of weir is 5 m and the coefficient of discharge is 0.6.

Solution

Let $H = 0.5$ m, $L = 5$ m and $C_d = 0.6$.

Since
$$Q = (2/3)C_d L\sqrt{2g}H^{3/2}$$

$$\therefore Q = (2/3)\times 0.6\times 5\times\sqrt{2\times 9.81}\times 0.5^{3/2} = \mathbf{3.1321\ m^3/s}$$

11.13 ❒ DISCHARGE OVER A SUBMERGED OR DROWNED WEIR

If a weir in which the water level on its downstream is above its crest, then the weir is known as submerged weir or drowned weir as shown in Figure 11.13. The submerged weir has large discharge capacity than a freely discharging weir. During floods, often a weir constructed across the river becomes a submerged weir.

Let Q be the total discharge over the submerged weir, H be the height of water on the upstream side of the weir, h be the height of water on the downstream side of the weir and L be length of the weir. The total discharge over a submerged weir may be evaluated by dividing it into two parts as, (i) Q_1 be the discharge through the portion between the upstream and downstream water surface having head $(H - h)$ that may be treated as a free weir and (ii) Q_2 be the discharge through the portion between the downstream water surface and the crest of the weir having head h that may be treated as a drowned weir.

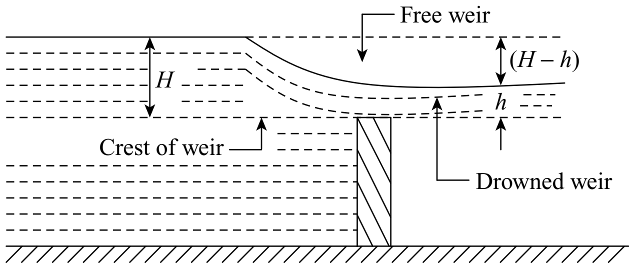

Figure 11.13 *Submerged weir*

Thus, if C_{d1} and C_{d2} be the coefficients of discharge for the free and drowned portions, respectively, then we get the following expression.

$$Q_1 = (2/3)C_{d1}L\sqrt{2g}(H-h)^{3/2}$$

$$Q_2 = C_{d2} \times \text{Area of flow} \times \text{Velocity of flow} = C_{d2}Lh\sqrt{2g(H-h)}$$

Thus $$Q = Q_1 + Q_2$$

$$\boxed{\therefore Q = (2/3)C_{d1}L\sqrt{2g}(H-h)^{3/2} + C_{d2}Lh\sqrt{2g(H-h)}} \quad (11.32)$$

Example 11.26 A submerged weir of length 2 m has heads of water on the upstream and downstream sides as 20 cm and 10 cm, respectively. Determine the discharge over the weir if the coefficients of discharge for the free and drowned portions are 0.6 and 0.8, respectively.

Solution

Let $L = 2$ m, $H = 20$ cm $= 0.2$ m, $h = 10$ cm $= 0.1$ m, $C_{d1} = 0.6$ and $C_{d2} = 0.8$.

Since $$Q = (2/3)C_{d1}L\sqrt{2g}\,(H-h)^{3/2} + C_{d2}Lh\sqrt{2g(H-h)}$$

Thus $$Q = \frac{2}{3}\times 0.6\times 2\times\sqrt{2\times 9.81}\times(0.2-0.1)^{3/2} + 0.8\times 2\times 0.1\times\sqrt{2\times 9.81\times(0.2-0.1)}$$

$$\therefore Q = 0.11206 + 0.22411 = \mathbf{0.33617\ m^3/s}$$

11.14 ❐ VENTILATION OF SUPPRESSED WEIR

The nappe emerging from the suppressed weir touches the side walls of the channel. The air traps in the space between the sidewalls of the channel, the falling nappe, the weir and the bottom of the channel. Gradually, the air is carried away with the flowing water and thereby, it reduces the pressure in the space below the nappe, which may fall below atmospheric pressure. This negative pressure (vacuum pressure) under the nappe results in drawing more water and it eventually increases the actual discharge over the weir. To overcome this difficulty, ventilation of the weir is done, i.e., holes are made in the side walls of the channel just below the lower nappe (Figure 11.14). This region connects with the atmospheric air outside and thus, a constant discharge can be maintained. These holes provided for the circulation of air are called ventilation holes and the weirs are called ventilated weirs.

Based on the extent of vacuum pressure and ventilation, some of the important types of nappe are given below.

1. **Free nappe (or fully aerated nappe):** When atmospheric pressure exists always in the space below the emerging nappe and it is not allowed to be reduced, then it is known as free nappe (Figure 11.14(a)). This is a standard case and the weir discharges freely.

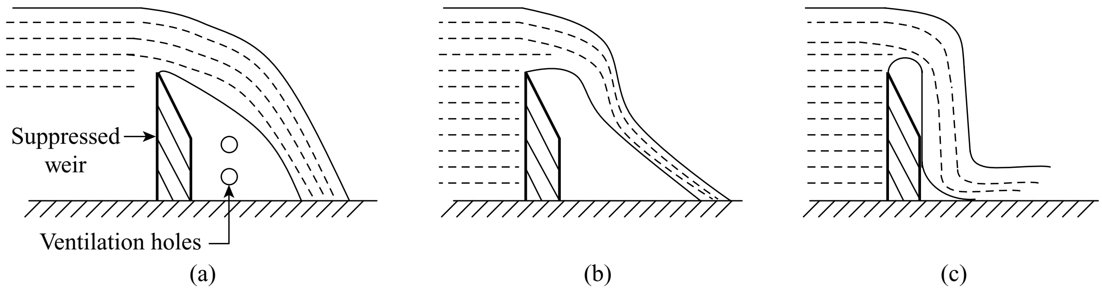

Figure 11.14 *Types of nappe over a suppressed weir*

2. **Depressed nappe:** When the space between the weir and the nappe is partially ventilated, then the emerging nappe is known as depressed nappe (Figure 11.14(b)). The discharge over a weir with depressed nappe will be about 5% to 7% more than that obtained with a free nappe.
3. **Clinging nappe:** When there is no air below the nappe, then the nappe may adhere to the downstream surface of the weir. Such a nappe is called clinging nappe (Figure 11.14(c)). The discharge of clinging nappe is about 25% to 30% more than that obtained with a free nappe.

11.15 ❒ TIME OF EMPTYING A RESERVOIR WITH RECTANGULAR WEIR OR NOTCH

Let L be the length of weir or notch, A be the cross-sectional area of reservoir or tank, H_1 be the initial height of liquid above the crest, H_2 be the final height of liquid above the crest, C_d be the coefficient of discharge and T be the time taken to lower the height of liquid from H_1 to H_2. Let h be the height of liquid surface above the crest of weir or notch at any instant, dh be the fall in the height of liquid in a small interval of time dT (Figure 11.15).

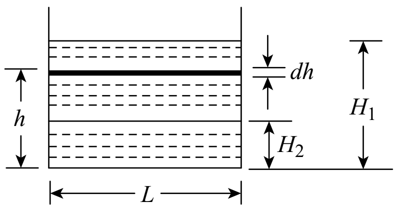

Figure 11.15 *Emptying a reservoir by a rectangular weir or notch*

Decrease in the volume of liquid in the tank = Discharge × Time

$$-Adh = QdT \quad \text{(Negative sign shows } h \text{ decreases with increase in } T\text{)}$$

$$dT = \frac{-Adh}{Q} = \frac{-Adh}{(2/3)C_d L\sqrt{2g}h^{3/2}}$$

Thus, total time can be obtained by integrating the above expressions as follows.

$$\int_0^T dT = \int_{H_1}^{H_2} \frac{-Adh}{(2/3)C_d L\sqrt{2g}h^{3/2}}$$

$$T = \frac{-A}{(2/3)C_d L\sqrt{2g}} \int_{H_1}^{H_2} h^{-3/2} dh = \frac{-3A}{2C_d L\sqrt{2g}} \left[\frac{h^{(-3/2)+1}}{(-3/2)+1} \right]_{H_1}^{H_2}$$

$$T = \frac{-3A}{2C_d L\sqrt{2g}} \left[\frac{1}{(-1/2)h^{1/2}} \right]_{H_1}^{H_2} = \frac{-3A \times (-2)}{2C_d L\sqrt{2g}} \left[\frac{1}{H_2^{1/2}} - \frac{1}{H_1^{1/2}} \right]$$

$$\boxed{\therefore T = \frac{3A}{C_d L\sqrt{2g}} \left[\frac{1}{\sqrt{H_2}} - \frac{1}{\sqrt{H_1}} \right]} \tag{11.33}$$

Example 11.27 Calculate the time required to lower the water level from 4 m to 3 m in a reservoir of dimensions 50 m × 50 m by a rectangular notch of length 1.3 m and coefficient of discharge as 0.623.

Solution

Let $H_1 = 4$ m, $H_2 = 3$ m, $A = 50 \times 50 = 2500$ m^2, $L = 1.3$ m and $C_d = 0.623$.

Since
$$T = \frac{3A}{C_d L\sqrt{2g}}\left[\frac{1}{\sqrt{H_2}} - \frac{1}{\sqrt{H_1}}\right]$$

$$\therefore T = \frac{3\times 2500}{0.623\times 1.3\times\sqrt{2\times 9.81}}\times\left[\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right] = \mathbf{161.712\ s}$$

11.16 ❐ TIME OF EMPTYING A RESERVOIR WITH TRIANGULAR WEIR OR NOTCH

Let A be the cross-sectional area of reservoir or tank, H_1 be the initial height of liquid above the apex of notch, H_2 be the final height of liquid above apex of notch, α be the angle of notch, C_d be the coefficient of discharge and T be the time taken to lower the height of liquid from H_1 to H_2. Let h be the height of liquid surface above the crest of weir or notch at any instant, dh be the fall in the height of liquid in a small interval of time dT (Figure 11.16).

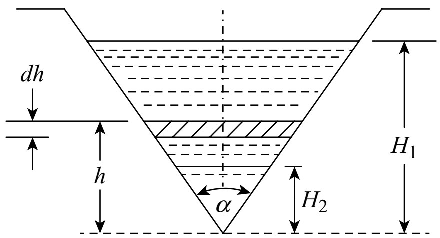

Figure 11.16 *Emptying a reservoir by a triangular weir or notch*

Decrease in the volume of liquid in the tank = Discharge × Time

$-Adh = QdT$ (Negative sign shows h decreases with increase in T)

$$dT = \frac{-Adh}{Q} = \frac{-Adh}{(8/15)C_d\sqrt{2g}\tan(\alpha/2)h^{5/2}}$$

Thus, total time can be obtained by integrating the above expressions as follows.

$$\int_0^T dT = \int_{H_1}^{H_2}\frac{-Adh}{(8/15)C_d\sqrt{2g}\tan(\alpha/2)h^{5/2}}$$

$$T = \frac{-A}{(8/15)C_d\sqrt{2g}\tan(\alpha/2)}\int_{H_1}^{H_2} h^{-5/2}dh$$

$$T = \frac{-15A}{8C_d\sqrt{2g}\tan(\alpha/2)}\left[\frac{h^{(-5/2)+1}}{(-5/2)+1}\right]_{H_1}^{H_2}$$

$$T = \frac{-15A}{8C_d\sqrt{2g}\tan(\alpha/2)}\left(-\frac{2}{3}\right)\left[\frac{1}{h^{3/2}}\right]_{H_1}^{H_2}$$

$$T = \frac{5A}{4C_d\sqrt{2g}\tan(\alpha/2)}\left[\frac{1}{H_2^{3/2}} - \frac{1}{H_1^{3/2}}\right]$$

$$\boxed{\therefore T = \frac{5A}{4C_d\sqrt{2g}\tan(\alpha/2)}\left[\frac{1}{H_2^{3/2}} - \frac{1}{H_1^{3/2}}\right]} \quad (11.34)$$

Example 11.28 Calculate the time required to lower the water level from 3 m to 2 m in a reservoir of dimensions 60 m × 60 m by a right angled V-notch having a coefficient of discharge as 0.62.

Solution

Let $H_1 = 3$ m, $H_2 = 2$ m, $A = 60 \times 60 = 3600$ m^2, $\alpha = 90°$ and $C_d = 0.62$.

Since $$T = \frac{5A}{4C_d\sqrt{2g}\tan(\alpha/2)}\left[\frac{1}{H_2^{3/2}} - \frac{1}{H_1^{3/2}}\right]$$

$$\therefore T = \frac{5\times 3600}{4\times 0.62\times\sqrt{2\times 9.81}\times\tan(90°/2)}\times\left[\frac{1}{2^{3/2}} - \frac{1}{3^{3/2}}\right] = \mathbf{263.983\ s}$$

Summary

1. A notch is an opening in the side of a tank such that the liquid surface in the tank is below the top edge of the opening.
2. A weir is a concrete or masonry structure built across a river (stream or open channel) over which the liquid flows.
3. The sheet of water flowing through a notch or over a weir is known as the nappe.
4. The bottom edge of a notch or the top of the weir over which water flows is known as sill or crest.
5. The height of the crest above the bottom of the tank is known as the crest height.
6. The discharge over a rectangular notch or weir is $Q = (2/3) C_d L\sqrt{2g}H^{3/2}$, here C_d is the coefficient of discharge, L is the length of notch or weir and H is the height of water over the crest.
7. The discharge over a rectangular notch or weir considering velocity of approach V_a is

   $$Q = (2/3)C_d L\sqrt{2g}\times[(H + h_a)^{3/2} - h_a^{3/2}],$$

   here $h_a = V_a^2/(2g)$.
8. Empirical formulae for discharge over rectangular weirs:

   (a) **Francis's formula:**

   (i) $Q = (2/3)C_d(L - 0.1nH)\sqrt{2g}H^{3/2}$

   (For n end contractions)

   (ii) $Q = (2/3)C_d(L - 0.2H)\sqrt{2g}H^{3/2}$

   (For 2 end contractions)

   (iii) $Q = 1.84LH^{3/2}$

   (For suppressed end contractions)

   (b) **Bazin's formula:** $Q = mL\sqrt{2g}H^{3/2}$, here $m = 0.405 + (0.003/H)$

   (c) **Rehbock's formula:** $Q = mL\sqrt{2g}(H + 0.0011)^{3/2}$,

   Here, $m = \left[0.403 + \frac{0.053(H + 0.0011)}{z}\right]$ and z is the height of weir.
9. The discharge over a triangular notch or weir is $Q = (8/15) C_d\sqrt{2g}\tan(\alpha/2)H^{5/2}$, here α is the vertex angle.
10. Effect on discharge due to error in measurement of head.

    (i) **A rectangular notch or weir:** $dQ/Q = 1.5(dH/H)$

    (ii) **A triangular notch or weir:** $dQ/Q = 2.5(dH/H)$

    Here, dQ is the error in discharge Q and dH is the error in the measurement of the head H.
11. **Discharge over a trapezoidal notch or weir:**

    $Q = (2/3)C_{d1}L\sqrt{2g}H^{3/2} + (8/15)C_{d2}\sqrt{2g}\tan(\alpha/2)H^{5/2}$,

    here C_{d1} and C_{d2} are the coefficient of discharges, respectively, for the rectangular and triangular portions, respectively.
12. Cipolletti weir is a trapezoidal weir with side slopes of 1 horizontal to 4 vertical.
13. **Maximum discharge over a broad-crested weir:** $Q_{\max} = 1.705C_d LH^{3/2}$
14. **Discharge over a narrow-crested weir or an Ogee weir or a Cipolletti weir:** $Q = (2/3)C_d L\sqrt{2g}H^{3/2}$
15. **Discharge over a submerged or drowned weir:**

    $$Q = (2/3)C_{d1}L\sqrt{2g}(H - h)^{3/2} + C_{d2}Lh\sqrt{2g(H - h)}$$

Here, C_{d1} and C_{d2} are the coefficients of discharge for the free and the drowned portions, respectively, H is the height of water on the upstream side of the weir and h is the height of water on the downstream side of the weir.

16. Time for emptying a reservoir with the following conditions.

(i) **Rectangular weir or notch:**

$$T = \frac{3A}{C_d L\sqrt{2g}}\left[\frac{1}{\sqrt{H_2}} - \frac{1}{\sqrt{H_1}}\right]$$

(ii) **Triangular weir or notch:**

$$T = \frac{5A}{4C_d\sqrt{2g}\tan(\alpha/2)}\left[\frac{1}{H_2^{3/2}} - \frac{1}{H_1^{3/2}}\right]$$

Here, H_1 and H_2 are the initial and final heights of liquid above the crest respectively and α is the angle of notch.

Multiple-choice Questions

1. A weir of length L and having head of water H is said to be broad-crested if
(a) $L = 2H$.
(b) $L = H$.
(c) $L = H/2$.
(d) None of the above.

2. If H is the height of water over the crest, then the discharge over a rectangular notch is directly proportional to
(a) H.
(b) $H^{1/2}$.
(c) $H^{3/2}$.
(d) $H^{5/2}$.

3. If H is the height of water over the crest, then the discharge over a triangular notch is directly proportional to
(a) H.
(b) $H^{1/2}$.
(c) $H^{3/2}$.
(d) $H^{5/2}$.

4. The ratio of percentage error in the discharge over rectangular notch and the percentage error in the measuring head is
(a) 1/2
(b) 1
(c) 1.5
(d) None of the above.

5. The ratio of percentage error in the discharge over triangular notch and the percentage error in the measuring head is
(a) 1
(b) 1.5
(c) 2
(d) 2.5

6. The horizontal to vertical side slope in the case of a Cipolletti weir is
(a) $1:2$.
(b) $1:3$.
(c) $1:4$.
(d) $1:5$.

Review Questions

1. Define and compare a notch and a weir. Also give their classifications.

2. Define nappe and crest. Also derive an expression for the discharge over a rectangular notch in terms of head of water over the crest of it.

3. Derive an expression for the error in discharge due to error in the measurement of head over (i) a rectangular weir or notch and (ii) a triangular weir or notch.

4. What do you mean by velocity of approach? Also find an expression for the discharge over a rectangular notch or weir with velocity of approach.

5. What do you mean by end contraction? What is the effect of end contraction on the discharge through a weir?

6. Obtain an expression for discharge over a triangular notch in terms of head of water over its crest. Also mention the advantages of a triangular notch over a rectangular notch.

7. Obtain the expressions for discharge over (i) a trapezoidal notch or weir and (ii) a submerged weir.

8. Obtain the expressions for discharge over (i) a Cipolletti notch or weir and (ii) a stepped notch.

9. Derive an expression for maximum discharge over a broad-crested weir.

10. Briefly explain the narrow-crested weir and an Ogee weir.

11. Write short notes on ventilation of suppressed weir and the important types of nappe.

12. Derive an expression for the time required to empty a tank with (i) a rectangular notch and (ii) a triangular notch

Problems

1. A rectangular notch has a discharge of 0.5 m^3/s when the height of water over the crest is half the breadth of the notch. If coefficient of discharge is 0.62, then find the breadth of the notch.

[**Ans.** 0.902 m]

2. A rectangular notch is required to measure a maximum discharge of 400 litres per second. If the maximum head of liquid over its still is not to exceed beyond 0.5 m, then what should be the width of the notch? Take coefficient of discharge as 0.62.

[**Ans.** 0.618 m]

3. The head of water passing through a rectangular notch 0.2 m broad is measured as 0.1 m. Calculate the discharge, if coefficient of discharge is 0.6. Also find the percentage error in the discharge if the permissible error in the measurement of head is 1%.

[**Ans.** 0.0112 m^3/s, 1.5%]

4. The head of water over a 100 m long rectangular weir is 1.5 m. If the velocity of approach is 0.5 m/s and coefficient of discharge is 0.62, then determine discharge over the weir.

[**Ans.** 340.38 m^3/s]

5. A rectangular weir of crest length 0.5 m is provided in a rectangular channel of width 1 m wide and depth 0.8 m. Determine the discharge over the weir if the head of water over its crest is 0.2 m and the coefficient of discharge is 0.623. Consider the velocity of approach and neglect end contractions.

[**Ans.** 0.0826 m^3/s]

6. A rectangular channel of width 3 m is provided with a sharp-crested rectangular weir of 100 cm height across it. Determine the discharge over the weir if the head of water over its crest is 0.4 m, coefficient of discharge is 0.6 and velocity of approach is considered.

[**Ans.** 1.369 m^3/s]

7. The water flows over a rectangular weir of width 1 m at a depth of 15 cm and afterwards, it passes through a triangular right-angled weir. If the coefficients of discharge for the rectangular and triangular weir are 0.62 and 0.59, respectively, then determine the depth of water over the triangular weir.

[**Ans.** 0.357 m]

8. Determine the depth and top width of a V-notch capable of discharging a maximum of 500 litres per second and such that the head shall be 6 cm for a discharge of 5 litres per second. Its coefficient of discharge is same as that of a similar right-angled V-notch for which $Q = 1.407H^{5/2}$.

[**Ans.** 0.3786 m, 3.0515 m]

9. Determine the discharge over a triangular notch of angle 60° when the head over the V-notch is 0.5 m and the coefficient of discharge is 0.6.

[**Ans.**0.1447 m^3/s]

10. Determine the coefficient of discharge of a right angled V-notch if the quantity of water collected in one minute is 217 litres with a constant head of 9 cm. Assume that there is no lateral contraction.

[**Ans.** 0.63]

11. Determine the coefficient of discharge of a right angled V-notch if the quantity of water collected in one minute is 0.1 m^3/s and it works under a constant head of 6.5 cm.

[**Ans.** 0.656]

12. The water flows through a triangular right-angled weir first and then over a rectangular weir of 1 m width. If the coefficients of discharge of the triangular and rectangular weirs are 0.6 and 0.62, respectively and the depth of the triangular weir is 30 cm, then determine the depth of water over the rectangular weir.

[**Ans.** 0.1134 m]

13. A right angled V-notch having coefficient of discharge as 0.62 is used to measure the discharge. Determine the flow rate if the head $(H + dH)$ measured above the crest is given as (0.2 ± 0.01) m.

[**Ans.** $(0.0262 \pm 0.003275)\ m^3/s$]

14. A sharp-edged rectangular notch 0.5 m broad has been used to measure the discharge estimated to be about 30 litres per second. Determine the percentage error in computing the discharge that would be introduced by an error of 2 mm in observing the head over the notch. Take coefficient of discharge as 0.63.

[**Ans.** 2.96%]

15. Determine the discharge over a rectangular weir 3 m long working under a constant head of 0.25 m using Francis's and Bazin's formulae when (i) end contractions are suppressed and (ii) end contractions are considered.

[**Ans.** 0.69 m^3/s, 0.693 m^3/s, 0.678 m^3/s]

16. A weir 36 m long is divided into 12 equal bays by vertical posts, each 0.6 m wide. Determine the discharge over the weir if the head over the crest is 1.2 m and the velocity of approach is 2 m/s.

[**Ans.** 75.267 m^3/s]

17. The head of water over a triangular notch of angle 60° is 0.4 m and the coefficient of discharge is 0.6. Determine the limiting values of head if the flow measured by it is to be within an accuracy of ±1.5%.

[**Ans.** 0.4024 m or 0.3976 m]

18. Determine the discharge over a Cipolletti weir of length 1.5 m when the head over the weir is 1 m and the coefficient of discharge is 0.62.

[**Ans.** 2.746 m^3/s]

19. Determine the discharge over a Cipolletti weir of crest length 0.6 m when the head over the weir is 0.4 m. Take the coefficient of discharge as 0.62. Also determine the discharge if the channel is 0.8 m wide and 0.6 m deep and the velocity of approach is taken into consideration.

[**Ans.** 0.2779 m^3/s, 0.2934 m^3/s]

20. The horizontal base of a sharp-edged trapezoidal weir is 0.1 m wide and its top and depths are 0.5 m and 0.3 m, respectively. Determine the discharge over the weir in litres per second when the upstream water surface is 0.25 m above the weir crest. Neglect the velocity of approach and take coefficient of discharge as 0.62 for both the rectangular and triangular portions.

[**Ans.** 53.39 litres/s]

21. Determine the discharge over a Cipolletti weir of crest length 0.6 m when the head over the weir is 0.225 m. Take the coefficient of discharge as 0.62. Also determine the discharge if the channel is 1 m wide and 0.5 m deep and the velocity of approach is taken into consideration.

[**Ans.** 0.11724 m^3/s, 0.1193 m^3/s]

22. Find the maximum discharge over a broad-crested weir of length 60 m and height 60 cm of water above the crest of the weir. Take coefficient of discharge as 0.62. Also determine the maximum discharge when the velocity of approach is considered and the cross-sectional area of the channel on the upstream side of the weir is 45 m^2.

[**Ans.** 29.478 m^3/s, 47.01 m^3/s]

23. Using Francis and Bazin's formulae determine the discharge over an Ogee weir which is 4 m long with suppressed end contractions and discharges water under a head of 0.45 m.

[**Ans.** 2.2217 m^3/s, 2.202 m^3/s]

24. A submerged weir of length 2 m has heads of water on the upstream and downstream sides as 15 cm and 7.5 cm, respectively. Determine the discharge over the weir if the coefficients of discharge for free and drowned portions are 0.58 and 0.8, respectively.

[**Ans.** 0.21595 m^3/s]

25. A submerged weir spans the entire width of a rectangular channel 5 m wide. The crest of weir is 1 m above the bed of the channel. If the depths of water on the upstream and downstream sides are 1.6 m and 1.2 m, respectively and the coefficients of discharge for free and drowned portions are 0.6 and 0.8, respectively, then determine the discharge.

[**Ans.** 4.482 m^3/s]

26. Water flows over a rectangular sharp-crested weir of length 1 m, the head over the sill of the weir is 66 cm. The approach channel is 1.4 m wide and the depth of the flow in the channel is 1.2 m. Taking coefficient of discharge as 0.6 and neglecting the velocity of approach, determine the discharge over the weir. Also determine the discharge if the velocity of approach and the effect of end contractions are considered.

[**Ans.** 0.824 m^3/s, 0.8431 m^3/s]

27. Calculate the time required to lower the water level from 4 m to 3 m in a reservoir of dimensions 60 m × 60 m by (i) a rectangular notch of length 1.5 m and (ii) a right-angled V-notch. Take coefficient of discharge as 0.62 in each case.

[**Ans.** 202.8 s, 110.5 s]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (c) **2.** (c) **3.** (d) **4.** (c) **5.** (d)
6. (c)

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Chapter **12**

Laminar Flow (Viscous Flow)

12.1 ❒ INTRODUCTION

All real fluids are viscous in nature. Viscosity is the property of a fluid that resists the movement of one layer of fluid over an adjacent layer. It produces tangential or shear stresses in a moving fluid. Depending upon the domination of viscosity, viscous flow can be classified as laminar flow and turbulent flow. In laminar flow, the fluid particles move along straight parallel paths in layers (laminae or sheets) and there is no mixing of fluid particles between two adjacent layers. This occurs at low velocity so that forces due to viscosity predominate over the inertial forces. The viscosity induces relative motion within the fluid when fluid layers slide over each other. The gradient of velocity between the layers gives rise to shear stresses. The magnitude of viscous shear stress varies from point to point, being maximum at the boundary and gradually, decreases with increase in the distance from the boundary. In order to overcome the shear resistance to flow, the pressure drops from section to section in the direction of flow, so that a pressure gradient exists. In a turbulent flow, the motion of the fluid particles is irregular, where it moves in heterogeneous fashion and the pathlines are erratic curves.

In previous chapters, the effect of viscosity has not been considered when the fluid flows. This idealization of non-viscous flow greatly simplifies the mathematics involved in the analysis. However, such an idealization fails to explain many phenomena where viscous force is very large and in such cases it is not desirable to neglect these forces. In this chapter, the effect of viscosity has been introduced for formulating the mathematical solutions of different practical problems. Expressions relating to shear stress and pressure gradients in laminar flow is developed and applied to analyse many cases of laminar flow, such as Couette flow, Poiseuille flow and flow through annulus. Relationships among the velocity field, flow rate, shear stress, pressure gradient and geometry of the laminar flow field have also been established in this chapter.

12.2 ❒ REYNOLDS EXPERIMENTS

Osborne Reynolds (1883), an English scientist performed experiments on the setup shown schematically in Figure 12.1. It consists of a long horizontal glass tube of bell-mouthed entrance fitted to a constant head tank containing water. To control the water flow through the glass tube, a regulating valve is fitted at its outlet. A jet of dye (aniline) of same specific weight as water is introduced from a small tank containing dye.

The observations made by Reynolds are given below.

1. When the velocity of flow was low, the dye filament in the glass tube was in the form of a stable straight line and moved so steadily that it hardly appeared to be in motion. This was the case of laminar flow (Figure 12.2(a)) in which the loss of pressure head (h_f) was observed to be proportional to the velocity (V), i.e., $h_f \propto V$.

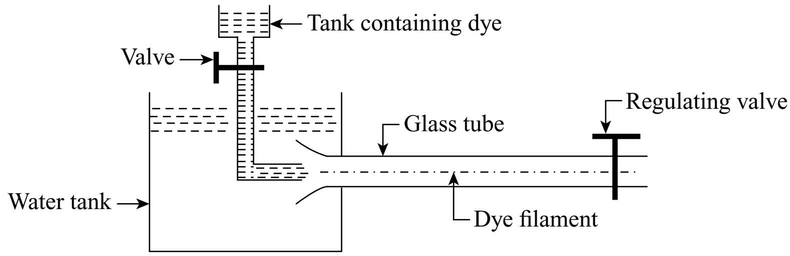

Figure 12.1 *Schematics of Reynolds experimental set-up*

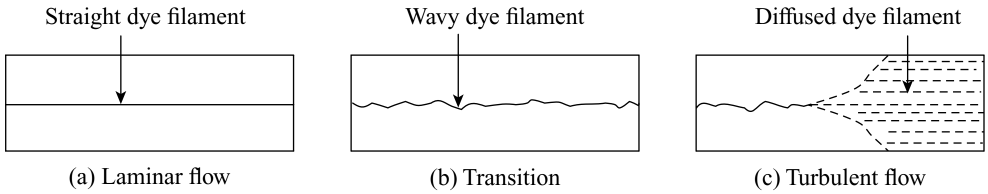

Figure 12.2 *Stages of dye filament*

2. When the velocity of flow was increased, a stage reached at which the dye filament was no longer a straight line but it became irregular and wavy as shown in Figure 12.2(b). This showed that the flow was no longer laminar, i.e., it was a transitional state.
3. When the velocity of flow was further increased, the irregularity and the waviness of the dye filament increases and finally, the dye filament disappeared or diffused in water as shown in Figure 12.2(c). Such a flow wherein the motions are randomized and irregular is called turbulent flow and the velocity at which it starts is called lower critical velocity. The flow velocity at the instant when the dye filament breaks completely and gets diffused throughout the flow is called higher critical velocity. In case of turbulent flow, the loss of pressure head was observed to be proportional to V^n, i.e., $h_f \propto V^n$, where n varies from 1.75 to 2.

Reynolds found that the fluid flow is governed by the parameters, namely density of fluid (ρ), mean flow velocity (V), dynamic viscosity of the fluid (μ), kinematic viscosity of the fluid ($\nu = \mu/\rho$) and the characteristic dimension of the stream cross section, for example, diameter of the pipe (D). By grouping these variables, a dimensionless quantity was formed called Reynolds number as given below.

$$\boxed{\mathrm{Re} = \frac{\rho VD}{\mu} = \frac{VD}{(\mu/\rho)} = \frac{VD}{\nu}} \tag{12.1}$$

Reynolds number represents the ratio of inertia force to viscous force. At low Reynolds number, the viscous force predominates and the flow is laminar, whereas at higher Reynolds number, the inertia force predominates and consequently, the fluid layers break up into a turbulent flow.

The values of Reynolds number for the type of flow in circular pipes and tubes are (i) laminar flow when Re < 2000, (ii) turbulent flow when Re > 4000 and (iii) unpredictable flow or transitional flow (i.e., transition from laminar to turbulent) when 2000 < Re < 4000.

The lower critical Reynolds number $(\mathrm{Re})_{cr}$ is the value of Reynolds number below which disturbances of any magnitude is eventually damped by viscous action. The approximate values of $(\mathrm{Re})_{cr}$ for some cases are (i) $(\mathrm{Re})_{cr} = 1000$ for parallel walls, (ii) $(\mathrm{Re})_{cr} = 50$ for open channel flows and (iii) $(\mathrm{Re})_{cr} = 1$ for flow around a sphere.

12.3 ❐ NAVIER-STOKES EQUATIONS OF MOTION

Consider a three-dimensional fluid element of dimensions dx, dy and dz having an infinitesimal volume $dx\,dy\,dz$ in a flow field which has three velocity components u, v and w along x, y and z directions, respectively, as shown in Figure 12.3. Assuming viscosity (μ) and density (ρ) of the fluid is throughout constant. The motion of the fluid element is influenced by normal forces due to pressure p, i.e., F_p, body force (gravity force) F_b, shear forces F_τ and inertia forces F_i.

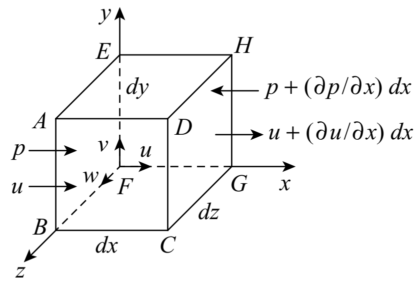

Figure 12.3 *Three-dimensional infinitesimal fluid element*

(i) **Normal forces due to pressure:** The net pressure force in x-direction is given by,

$$F_{px} = p \times dydz - \left(p + \frac{\partial p}{\partial x}dx\right) \times dydz = -\frac{\partial p}{\partial x}dxdydz$$

(ii) **Body or gravity force:** Let F_{bx}, F_{by} and F_{bz} be the components of the body force per unit mass in x, y and z directions, respectively. The body force acting on the fluid element in the direction of x-coordinate is given below.

$$= F_{bx} \times \rho \times dxdydz$$

(iii) **Shear forces:** Let $F_{\tau x}$, $F_{\tau y}$ and $F_{\tau z}$ be the components of shear force per unit mass in x, y and z directions, respectively. The shear force acting on the fluid element in the direction of x-coordinate is given below.

$$= F_{\tau x} \times \rho \times dxdydz$$

(iv) **Inertia forces:** The inertia force acting on the fluid element in the direction of x-coordinate is given by the product of mass and acceleration as follows.

$$(F_i)_x = \rho \times dxdydz \times a_x = \rho \times dxdydz \times \frac{du}{dt}$$

Here $$a_x = \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \qquad \text{[Equation (6.13a)]}$$

According to Newton's second law of motion, the summation of forces acting on the fluid element in any direction equals the inertia forces in that direction. Thus, along x-direction, we get the following expression.

$$-\frac{\partial p}{\partial x}dxdydz + F_{bx}\rho dxdydz + F_{\tau x}\rho dxdydz = \rho dxdydz\frac{du}{dt}$$

$$F_{bx} - \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{du}{dt} - F_{\tau x} \qquad (12.2a)$$

Similarly, along y and z directions, respectively, we get the below expressions.

$$F_{by} - \frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{dv}{dt} - F_{\tau y} \qquad (12.2b)$$

$$F_{bz} - \frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{dw}{dt} - F_{\tau z} \qquad (12.2c)$$

Since shear stress due to viscosity on a particular surface equals the rate of change of velocity in a direction normal to the surface. Thus, shear force in terms of shear stress τ acting on face $ABFE$ is given below.

$$= -\tau \times (dydz) = -\mu\frac{\partial u}{\partial x} \times dydz$$

Shear force acting on face *DCGH* is given by,

$$= \mu \frac{\partial}{\partial x}\left(u + \frac{\partial u}{\partial x} dx\right) \times dydz = \mu\left(\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} dx\right) dydz$$

It should be noted that the shear force (resistance force) acting on the opposite faces have opposite sign and both of these forces are directed opposite to the respective pressure forces.

The resultant shear force along x-direction can be obtained by the algebraic sum of forces acting on the faces *ABFE* and *DCGH* as given below.

$$= -\mu \frac{\partial u}{\partial x} dydz + \mu\left(\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} dx\right) dydz = \mu \frac{\partial^2 u}{\partial x^2} dxdydz \qquad \text{(i)}$$

Similarly, the resultant shear force along x-axis acting on the faces *BCGF* and *ADHE* is given below.

$$= \mu \frac{\partial^2 u}{\partial y^2} dxdydz \qquad \text{(ii)}$$

The resultant shear force along x-axis acting on the faces *EFGH* and *ABCD* is given by,

$$= \mu \frac{\partial^2 u}{\partial z^2} dxdydz \qquad \text{(iii)}$$

The total viscous resistance parallel to x-axis on all six faces can be obtained by adding expressions (i), (ii) and (iii) as given below.

$$= \mu \frac{\partial^2 u}{\partial x^2} dxdydz + \mu \frac{\partial^2 u}{\partial y^2} dxdydz + \mu \frac{\partial^2 u}{\partial z^2} dxdydz$$

$$= \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) dxdydz \qquad \text{(iv)}$$

The shear force per unit mass $F_{\tau x}$ can be obtained by dividing expression (iv) by $\rho dx\, dy\, dz$, we get the following expression.

$$F_{\tau x} = \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

Similarly, we can obtain the below expression.

$$F_{\tau y} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \text{ and } F_{\tau z} = \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Substituting the values of $F_{\tau x}$, $F_{\tau y}$ and $F_{\tau z}$ in Equations (12.2a), (12.2b) and (12.2c), we get:

$$F_{bx} - \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{du}{dt} - \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad \text{(12.3a)}$$

$$F_{by} - \frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{dv}{dt} - \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \qquad \text{(12.3b)}$$

$$F_{bz} - \frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{dw}{dt} - \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \qquad \text{(12.3c)}$$

The Equations (12.3a), (12.3b) and (12.3c) are called Navier-Stokes equations for constant viscosity and density. These equations are fundamental to general analysis of a viscous flow. Since these equations are non-linear second order differential equations, finding the general solution to these equations is not possible. However, the solution can be obtained for different flow situations by making suitable assumptions and thus, by neglecting some of the terms in the equation. Some simple solutions, for simple flow situations are described in this chapter.

12.4 ❐ RELATION BETWEEN SHEAR STRESS AND PRESSURE GRADIENT

Consider a three-dimensional fluid element of dimensions dx, dy and dz in a flow field which has three velocity components u, v and w along x, y and z directions, respectively, as shown in Figure 12.4(a). Due to viscous effects, there is relative motion between different layers of fluid, i.e., the velocity distribution is non-uniform as shown in Figure 12.4(b). The velocity gradient across the two layers setup shear stresses.

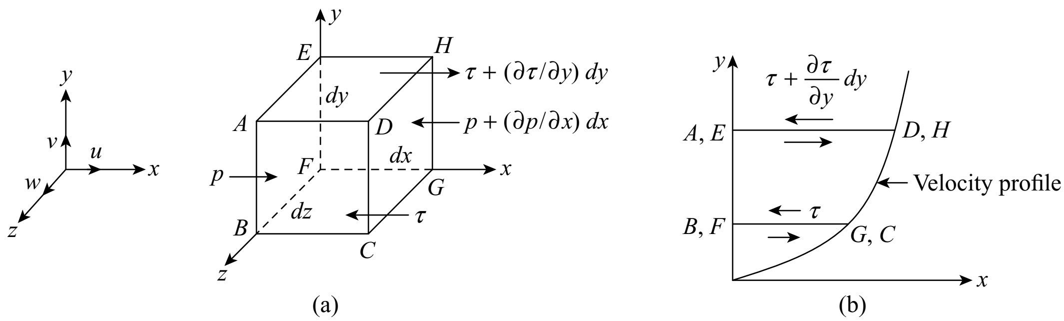

Figure 12.4 *Forces on a fluid element in laminar flow*

For two-dimensional steady flows, there will be no shear stresses on the vertical faces $ABCD$ and $EFGH$. Thus, the only forces acting on the fluid element in the direction of flow x will be the pressure and shear forces.

Let τ be the shear stress on the lower face $BFGC$ and $\tau + (\partial\tau/\partial y)dy$ be the shear stress on the upper face $ADHE$, then the net shear force on the element is given below.

$$= \left[\left(\tau + \frac{\partial \tau}{\partial y}dy\right)dxdz - \tau dxdz\right] = \frac{\partial \tau}{\partial y}dxdydz$$

Net pressure force becomes,

$$= \left[pdydz - \left(p + \frac{\partial p}{\partial x}dx\right)dydz\right] = -\frac{\partial p}{\partial x}dxdydz$$

For steady and uniform flow, there is no acceleration in the direction of motion. Thus, the sum of these forces in the x-direction must be equal to zero as given below.

$$\frac{\partial \tau}{\partial y}dxdydz - \frac{\partial p}{\partial x}dxdydz = 0$$

$$\boxed{\therefore \frac{\partial \tau}{\partial y} = \frac{\partial p}{\partial x}} \tag{12.4}$$

The Equation (12.4) shows that in a steady uniform laminar flow, the pressure gradient in the direction of flow is equal to the shear stress gradient in the normal direction. This equation is applicable to both the laminar and turbulent flows and holds good for all types of boundary geometry. For a Newtonian fluid, $\tau = \mu(du/dy)$ and thus, Equation (12.4) is written as follows.

$$\boxed{\frac{\partial p}{\partial x} = \mu \frac{d^2u}{dy^2}} \tag{12.5}$$

12.5 ❒ LAMINAR FLOW IN CIRCULAR PIPES (HAGEN-POISEUILLE THEORY)

Shear stress distribution The Figure 12.5 shows a horizontal circular pipe of radius R through which steady laminar flow of an incompressible fluid occurs from the left to right. Consider a small cylindrical fluid element of radius r and length dx located at a distance y from the bottom internal surface of the pipe.

The forces acting on the small element in the flow direction are the net pressure force and the shear force on the surface. If p be the intensity of pressure on the face PQ (left end of the element), then the intensity of pressure on the face RS (right end of the element) will be $p+(\partial p/\partial x)dx$. If τ be the shear stress, then shear force on the surface of fluid element will be $\tau \times 2\pi r dx$. Thus, the forces acting on the fluid element are as follows.

(i) Pressure force: $p \times \pi r^2$ on the face PQ

(ii) pressure force, $\left(p + \frac{\partial p}{\partial x} dx\right) \times \pi r^2$ on the face RS

(iii) Shear force: $\tau \times 2\pi r dx$ on the surface of fluid element.

Since the flow is steady and the pipe is of uniform size, the acceleration is zero. Therefore, the sum of net pressure force and viscous force in the x-direction must be zero as given below.

$$p \times \pi r^2 - \left(p + \frac{\partial p}{\partial x} dx\right)\pi r^2 - \tau \times 2\pi r dx = 0$$

$$-\frac{\partial p}{\partial x} dx \pi r^2 - \tau \times 2\pi r dx = 0 \Rightarrow -\frac{\partial p}{\partial x} r - 2\tau = 0$$

$$\therefore \tau = -\frac{\partial p}{\partial x}\frac{r}{2} \tag{12.6}$$

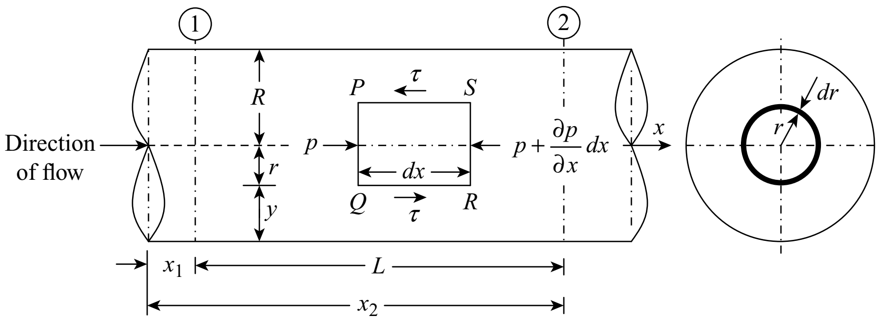

Figure 12.5 *Laminar flow through a circular pipe*

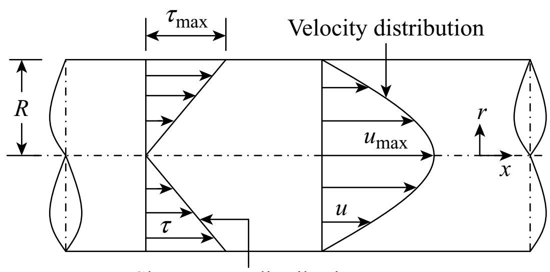

Figure 12.6 *Shear stress and velocity distribution for laminar flow in a pipe*

In the above equation, the negative sign indicates that the pressure decreases in the direction of flow. The Equation (12.6) shows that the shear stress (τ) varies linearly along the radius of the pipe as shown in Figure 12.6. At the centre of the pipe $r = 0$, $\tau = 0$ and at the pipe wall $r = R$ and thus, τ is maximum as given below.

$$\tau_{\max} = -\frac{\partial p}{\partial x}\frac{R}{2} \tag{12.7}$$

Since $$-\frac{\partial p}{\partial x} = \frac{-(p_2 - p_1)}{x_2 - x_1} = \frac{p_1 - p_2}{L} = \frac{\Delta p}{L}$$

Thus $$\tau_{\max} = -\frac{\partial p}{\partial x}\frac{R}{2} = \frac{\Delta p}{L}\frac{R}{2}$$

Velocity distribution From Newton's law of viscosity, we get:

$$\tau = \mu \frac{du}{dy} \tag{12.8}$$

Since $$y = R - r$$

Thus $$dy = -dr$$

$$\therefore \tau = -\mu \frac{du}{dr} \tag{12.9}$$

Comparing Equations (12.6) and (12.9), we get:

$$-\frac{\partial p}{\partial x}\frac{r}{2} = -\mu \frac{du}{dr} \Rightarrow du = \frac{1}{2\mu}\frac{\partial p}{\partial x} r\,dr$$

Upon integration, we get:

$$u = \frac{1}{4\mu}\frac{\partial p}{\partial x} r^2 + k \tag{12.10}$$

Here, k is the constant of integration whose value can be obtained from the boundary condition that at $r = R$, $u = 0$.

Thus $$0 = \frac{1}{4\mu}\frac{\partial p}{\partial x} R^2 + k \Rightarrow k = -\frac{1}{4\mu}\frac{\partial p}{\partial x} R^2$$

Substituting the value of k in Equation (12.10), we get:

$$u = \frac{1}{4\mu}\frac{\partial p}{\partial x}r^2 - \frac{1}{4\mu}\frac{\partial p}{\partial x}R^2 = -\frac{1}{4\mu}\frac{\partial p}{\partial x}[R^2 - r^2] \tag{12.11}$$

or

$$u = -\frac{1}{4\mu}\frac{\partial p}{\partial x}R^2\left[1-\left(\frac{r}{R}\right)^2\right] \tag{12.11a}$$

The Equation (12.11) shows that the velocity distribution is parabolic and the surface of velocity distribution is a paraboloid of revolution as shown in Figure 12.6.

Ratio of maximum velocity to average velocity The maximum velocity occurs at the centre, i.e., when $r = 0$. Thus, maximum velocity (u_{max}) can be obtained from Equation (12.11) as follows.

$$u_{max} = -\frac{1}{4\mu}\frac{\partial p}{\partial x}R^2 \tag{12.12}$$

From Equations (12.11(a)) and (12.12), we get:

$$u = u_{max}\left[1-\left(\frac{r}{R}\right)^2\right] \tag{12.13}$$

The discharge (dQ) through an elementary ring of thickness dr at a radial distance r (Figure 12.5) is given by,

$$dQ = u \times 2\pi r dr = -\frac{1}{4\mu}\frac{\partial p}{\partial x}[R^2 - r^2]\times 2\pi r dr \qquad \text{[Substitute Equation (12.11)]}$$

The total discharge can be obtained by integrating the above equation and it is given below.

$$\int dQ = \frac{\pi}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\int_0^R (rR^2 - r^3)dr$$

Thus

$$Q = \frac{\pi}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left[\frac{r^2R^2}{2} - \frac{r^4}{4}\right]_0^R = \frac{\pi}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left[\frac{R^4}{2} - \frac{R^4}{4}\right] = \frac{\pi}{8\mu}\left(-\frac{\partial p}{\partial x}\right)R^4 \tag{12.14}$$

Average velocity of flow (V or u_{av}) is given by,

$$V = \frac{Q}{A} = \frac{\pi}{8\mu}\left(\frac{\partial p}{\partial x}\right)R^4 \times \frac{1}{\pi R^2} = \frac{1}{8\mu}\left(-\frac{\partial p}{\partial x}\right)R^2 \tag{12.15}$$

$$V = \frac{1}{2}\times\frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right)R^2 = \frac{u_{max}}{2} \qquad \text{[Substitute Equation (12.12)]} \tag{12.16}$$

Thus

$$\boxed{\frac{u_{max}}{V} = 2}$$

The point where the local velocity (u) is equal to the average velocity (V) can be located by combining Equations (12.13) and (12.16) as follows.

$$u_{max}\left[1-\left(\frac{r}{R}\right)^2\right] = \frac{u_{max}}{2}$$

or $$1-\left(\frac{r}{R}\right)^2=\frac{1}{2}$$

Thus $$\boxed{r=\frac{R}{\sqrt{2}}=0.707R} \tag{12.17}$$

Loss in pressure head over a length of pipe From Equation (12.15), we get:

$$-\frac{\partial p}{\partial x}=\frac{8\mu V}{R^2}$$

The pressure difference (Δp) between sections 1 and 2 shown in Figure 12.5 at distances x_1 and x_2 can be obtained by integrating the above equation as given below.

$$-\int_1^2 \partial p=\frac{8\mu V}{R^2}\int_1^2 \partial x$$

$$-(p_2-p_1)=\frac{8\mu V}{R^2}(x_2-x_1)$$

or $$(p_1-p_2)=\frac{8\mu V}{R^2}(x_2-x_1)$$

Since $\Delta p=(p_1-p_2)$, $R=D/2$, and $L=(x_2-x_1)$

$$\boxed{\therefore\ \Delta p=\frac{8\mu VL}{(D/2)^2}=\frac{32\mu VL}{D^2}} \tag{12.18}$$

Here, D is the diameter of pipe and L is the length.

The Equation (12.18) is known as Hagen-Poiseuille equation (pronounced as Har'-gen Pwah-zoy'-yuh equation) which can also be written as follows.

$$\frac{\Delta p}{\rho g}=\frac{32\mu VL}{\rho g D^2} \tag{12.19}$$

Here, $[\Delta p/(\rho g)]=h_f$ represents the drop in pressure head between two sections. From Equation (12.19), we get:

$$\boxed{h_f=\frac{\Delta p}{\rho g}=\frac{32\mu VL}{\rho g D^2}} \tag{12.20}$$

From Equation (12.20), it can be observed that the frictional head loss over length of the pipe varies directly proportional to the velocity of flow of fluid and inversely as the square of the diameter of the pipe.

The loss of head due to frictional resistance in a pipe of length L and diameter D may also be expressed by Darcy-Weisbach equation as given below.

$$h_f=\frac{4fLV^2}{2gD} \tag{12.21}$$

Here, f is the coefficient of friction (Darcy coefficient of friction) and V is the average velocity of flow. Sometimes Equation (12.21) may also be used in the following form.

$h_f=\dfrac{f_f LV^2}{2gD}$, where $f_f=4f$ is called the friction factor or Darcy friction factor.

By simplifying Equations (12.20) and (12.21), we get:

$$\frac{4fLV^2}{2gD} = \frac{32\mu VL}{\rho g D^2}$$

Thus
$$\boxed{f = \frac{16}{(\rho VD)/\mu} = \frac{16}{\text{Re}}} \tag{12.22}$$

And
$$\boxed{f_f = \frac{64}{\text{Re}}} \tag{12.22a}$$

The Equation (12.21) is also known as Fanning equation and f is known as Fanning friction factor.

Example 12.1 An oil of viscosity 0.9 poise and specific gravity 0.89 flows through a horizontal pipe of diameter 90 mm and length 9 m. If 800 N of the oil is collected in a tank in 20 seconds, then determine the pressure difference between the two ends of the pipe.

Solution

Let $\mu = 0.9$ poise $= 0.09$ Ns/m^2, $S_{oil} = 0.89$, $D = 90$ mm $= 0.09$ m, $L = 9$ m and weight of oil collected in 20 s = 800 N.

Let W be the weight of oil collected per second, ρ be its density, V be its average velocity and Δp is the pressure difference between the two ends of the pipe.

$$\rho = S_{oil}\rho_w = 0.89 \times 1000 = 890 \text{ kg/m}^3$$

$$A = \frac{\pi}{4}D^2 = \frac{\pi}{4} \times 0.09^2 = 0.006362 \text{ m}^2$$

$$W = \frac{800}{20} = 40 \text{ N/s}$$

Thus
$$Q = \frac{W}{\rho g} = \frac{40}{890 \times 9.81} = 0.00458 \text{ m}^3/\text{s}$$

$$V = \frac{Q}{A} = \frac{0.00458}{0.006362} = 0.72 \text{ m/s}$$

$$\text{Re} = \frac{\rho VD}{\mu} = \frac{890 \times 0.72 \times 0.09}{0.09} = 640.8$$

Since Re < 2000, the flow is laminar.

$$\Delta p = \frac{32\mu VL}{D^2} = \frac{32 \times 0.09 \times 0.72 \times 9}{0.09^2} = \mathbf{2304 \text{ N/m}^2}$$

Example 12.2 The maximum velocity of flow in a pipe of diameter 250 mm is measured to be 2.4 m/s. If the flow through the pipe is laminar, then determine the average velocity and the radius at which it occurs. Also determine the velocity at 40 mm from the wall of the pipe.

Solution

Let $D = 250$ mm $= 0.25$ m, $u_{max} = 2.4$ m/s and $y = 40$ mm $= 0.04$ m. Let r be the radius at which average velocity occurs and r_1 is the radius at 40 mm from the wall of the pipe.

$$R = \frac{D}{2} = \frac{0.25}{2} = 0.125 \text{ m}$$

$$V = \frac{u_{max}}{2} = \frac{2.4}{2} = \mathbf{1.2\ m/s}$$

$$r = 0.707R = 0.707 \times 0.125 = \mathbf{0.0884\ m\ or\ 88.4\ mm}$$

$$r_1 = R - y = 0.125 - 0.04 = 0.085 \text{ m}$$

The velocity at a radius of r_1 is given by,

$$u = u_{max}\left[1 - \left(\frac{r_1}{R}\right)^2\right] = 2.4 \times \left[1 - \left(\frac{0.085}{0.125}\right)^2\right] = \mathbf{1.29\ m/s}$$

Example 12.3 An oil of absolute viscosity 0.1 Ns/m^2 and relative density 0.9 is pumped through a 32 mm diameter pipe. If the pressure drop per metre length of the pipe is 20 kN/m^2, then find (i) the mass flow rate, (ii) maximum shear stress, (iii) type of flow and (iv) power required per metre length of the pipe to maintain the flow.

Solution

Let $\mu = 0.1$ Ns/m^2, $S_{oil} = 0.9$, $D = 32$ mm $= 0.032$ m, $\Delta p = 20$ kN/m^2 and $L = 1$ m.

$$\rho = S_{oil}\,\rho_w = 0.9 \times 1000 = 900 \text{ kg/m}^3$$

$$A = \frac{\pi}{4}D^2 = \frac{\pi}{4} \times 0.032^2 = 0.000804 \text{ m}^2$$

$$R = \frac{D}{2} = \frac{0.032}{2} = 0.016 \text{ m}$$

(i) $\because \Delta p = \dfrac{32\mu VL}{D^2}$

$$\therefore V = \frac{\Delta p D^2}{32\mu L} = \frac{20 \times 10^3 \times 0.032^2}{32 \times 0.1 \times 1} = 6.4 \text{ m/s}$$

$$Q = AV = 0.000804 \times 6.4 = 0.005146 \text{ m}^3/\text{s}$$

The mass flow rate (m) is given by,

$$m = \rho Q = 900 \times 0.005146 = \mathbf{4.6314\ kg/s}$$

(ii) Maximum shear stress will be at the pipe wall and is given by,

$$\tau_{max} = -\frac{\partial p}{\partial x}\frac{R}{2} = \frac{\Delta p}{L}\frac{R}{2} = \left(\frac{20 \times 10^3}{1}\right) \times \frac{0.016}{2} = \mathbf{160\ N/m^2}$$

(iii) $\text{Re} = \dfrac{\rho VD}{\mu} = \dfrac{900 \times 6.4 \times 0.032}{0.1} = 1843.2$

Since Re < 2000, the flow is laminar.

(iv) $h_f = \dfrac{\Delta p}{\rho g} = \dfrac{20\times 10^3}{900\times 9.81} = 2.2653$ m of oil

The power required for maintaining the flow per metre length of the pipe is given below.

$$P = \rho g Q h_f = 900\times 9.81\times 0.005146\times 2.2653 = \mathbf{102.922\ W}$$

Example 12.4 Lubricating oil of absolute viscosity 0.12 Ns/m^2 and relative density 0.86 flows through a 40 mm diameter pipe. If the rate of flow of oil through the pipe is 3 litres per second, then find (i) the pressure drop in a length of 280 m and (ii) shear stress at the pipe wall.

Solution

Let $\mu = 0.12$ Ns/m^2, $S_{\text{oil}} = 0.86$, $D = 40$ mm $= 0.04$ m, $Q = 3\ l/\text{s} = 0.003\ \text{m}^3/\text{s}$ and $L = 280$ m.

$$\rho = S_{\text{oil}}\rho_w = 0.86\times 1000 = 860\ \text{kg/m}^3$$

$$A = \frac{\pi}{4}D^2 = \frac{\pi}{4}\times 0.04^2 = 0.001257\ \text{m}^2$$

$$R = \frac{D}{2} = \frac{0.04}{2} = 0.02\ \text{m}$$

(i) $V = \dfrac{Q}{A} = \dfrac{0.003}{0.001257} = 2.387$ m/s

$$\text{Re} = \frac{\rho V D}{\mu} = \frac{860\times 2.387\times 0.04}{0.12} = 684.27$$

Since Re < 2000, the flow is laminar.

$$\Delta p = \frac{32\mu V L}{D^2} = \frac{32\times 0.12\times 2.387\times 280}{0.04^2} = \mathbf{1604064\ N/m^2}$$

(ii) Shear stress at the pipe wall is given by,

$$\tau_{\max} = -\frac{\partial p}{\partial x}\frac{R}{2} = \frac{\Delta p}{L}\frac{R}{2} = \left(\frac{1604064}{280}\right)\times\frac{0.02}{2} = \mathbf{57.288\ N/m^2}$$

Example 12.5 An oil of absolute viscosity 9 poise and specific gravity 0.88 is flowing through a horizontal pipe of diameter 50 mm. If the pressure drop in 80 m length of the pipe is 1620 kPa, then find (i) the rate of flow of oil, (ii) centre line velocity, (iii) total frictional drag over 80 m length, (iv) power required to maintain the flow, (v) velocity gradient at the wall of the pipe and (vi) velocity and shear stress at 6 mm from the wall.

Solution

Let $\mu = 9$ poise $= 0.9$ Ns/m^2, $S_{\text{oil}} = 0.88$, $D = 50$ mm $= 0.05$ m, $L = 80$ m, $\Delta p = 1620$ kPa and $y = 6$ mm $= 0.006$ m.

$$\rho = S_{\text{oil}}\rho_w = 0.88\times 1000 = 880\ \text{kg/m}^3$$

$$A = \frac{\pi}{4}D^2 = \frac{\pi}{4}\times 0.05^2 = 0.0019635\ \text{m}^2$$

$$R = \frac{D}{2} = \frac{0.05}{2} = 0.025\ \text{m}$$

(i) $\because \Delta p = \frac{32\mu VL}{D^2}$

$$\therefore V = \frac{\Delta p D^2}{32\mu L} = \frac{1620\times10^3\times0.05^2}{32\times0.9\times80} = 1.76 \text{ m/s}$$

$$\text{Re} = \frac{\rho VD}{\mu} = \frac{880\times1.76\times0.05}{0.9} = 86.04$$

Since Re < 2000, the flow is laminar.

$$Q = AV = 0.0019635\times1.76 = \mathbf{0.003456\ m^3/s}$$

(ii) $u_{max} = 2V = 2\times1.76 = \mathbf{3.52\ m/s}$

(iii) $\tau_{max} = -\frac{\partial p}{\partial x}\frac{R}{2} = \frac{\Delta p}{L}\frac{R}{2} = \left(\frac{1620\times10^3}{80}\right)\times\frac{0.025}{2} = 253.125 \text{ N/m}^2$

The frictional drag (F_D) is given by,

$$F_D = \tau_{max}\times\pi DL = 253.125\times\pi\times0.05\times80 = \mathbf{3180.862\ N}$$

(iv) Power required for maintaining the flow is given by,

$$P = \frac{F_D V}{1000} = \frac{3180.862\times1.76}{1000} = \mathbf{5.5983\ kW}$$

(v) Velocity gradient at the wall of the pipe is given by,

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\tau_{max}}{\mu} = \frac{253.125}{0.9} = \mathbf{281.25\ s^{-1}}$$

(vi) $r = R - y = 0.025 - 0.006 = 0.019$ m

Since $$u = \frac{1}{4\mu}\frac{\partial p}{\partial x}[R^2 - r^2] = \frac{1}{4\mu}\frac{\Delta p}{L}[R^2 - r^2]$$

$$\therefore u = \frac{1}{4\times0.9}\times\frac{1620\times10^3}{80}\times[0.025^2 - 0.019^2] = \mathbf{1.485\ m/s}$$

Since $$\frac{\tau}{r} = \frac{\tau_{max}}{R} \quad \text{[Linear variation]}$$

$$\therefore \tau = \frac{\tau_{max} r}{R} = \frac{253.125\times0.019}{0.025} = \mathbf{192.375\ N/m^2}$$

Example 12.6 A fluid of viscosity 0.9 Ns/m^2 and relative density 1.2 flows through a horizontal pipe of diameter 120 mm. If the flow is laminar and the maximum shear stress at the pipe wall is 200 N/m^2, then find (i) the pressure gradient, (ii) average velocity and (iii) Reynolds number of the flow.

Solution

Let $\mu = 0.9$ Ns/m^2, $S_{fluid} = 1.2$, $D = 120$ mm $= 0.12$ m and $\tau_{max} = 200$ N/m^2.

$$\rho = S_{\text{fluid}} \rho_w = 1.2 \times 1000 = 1200 \text{ kg/m}^3$$

$$R = \frac{D}{2} = \frac{0.12}{2} = 0.06 \text{ m}$$

$$A = \frac{\pi}{4} D^2 = \frac{\pi}{4} \times 0.12^2 = 0.01131 \text{ m}^2$$

(i) $\because \tau_{\max} = -\frac{\partial p}{\partial x}\frac{R}{2}$

Thus $$\frac{\partial p}{\partial x} = -\frac{2\tau_{\max}}{R} = -\frac{2 \times 200}{0.06} = \mathbf{-6666.67 \ N/m^2 \ per \ m}$$

The negative sign indicates that pressure decreases in the direction of flow.

(ii) $V = \frac{1}{8\mu}\left(-\frac{\partial p}{\partial x}\right)R^2 = \frac{1}{8 \times 0.9} \times 6666.67 \times 0.06^2 = \mathbf{3.333 \ m/s}$

(iii) $\text{Re} = \frac{\rho VD}{\mu} = \frac{1200 \times 3.333 \times 0.12}{0.9} = \mathbf{533.28}$

Example 12.7 A lubricating oil of absolute viscosity 0.14 Ns/m^2 and relative density 0.9 flows through a pipe of diameter 24 mm and length 3 m at one-tenth of critical velocity for which Reynolds number is 2400. Find (i) the velocity of flow through the pipe, (ii) loss of head required for maintaining the flow and (ii) power required to overcome viscous resistance to the flow.

Solution

Let $\mu = 0.14$ Ns/m^2, $S_{\text{oil}} = 0.9$, $D = 24$ mm $= 0.024$ m, $L = 3$ m, $V = (V_{cr}/10)$ and Re = 2400.

$$\rho = S_{\text{oil}} \rho_w = 0.9 \times 1000 = 900 \text{ kg/m}^3$$

$$A = \frac{\pi}{4} D^2 = \frac{\pi}{4} \times 0.024^2 = 0.0004524 \text{ m}^2$$

(i) $\text{Re} = \frac{\rho V_{cr} D}{\mu}$

Thus $$V_{cr} = \frac{\text{Re}\,\mu}{\rho D} = \frac{2400 \times 0.14}{900 \times 0.024} = 15.55 \text{ m/s}$$

$$V = \frac{V_{cr}}{10} = \frac{15.55}{10} = \mathbf{1.555 \ m/s}$$

(ii) $h_f = \frac{32\mu VL}{\rho g D^2} = \frac{32 \times 0.14 \times 1.555 \times 3}{900 \times 9.81 \times 0.024^2} = \mathbf{4.11 \ m}$

(iii) $Q = AV = \frac{\pi}{4} D^2 \times V = \frac{\pi}{4} \times 0.024^2 \times 1.555 = 0.0007035 \text{ m}^3/\text{s}$

Power required to overcome viscous resistance of flow is given by,

$$P = \rho g Q h_f = 900 \times 9.81 \times 0.0007035 \times 4.11 = \mathbf{25.528 \ W}$$

Example 12.8 Oil of absolute viscosity 0.15 Ns/m^2 and specific gravity 0.85 flows through a pipe of diameter 0.3 m. If the head loss in 3000 m length of pipe is 20 m and the flow is laminar, then find (i) the velocity of flow through the pipe, (ii) Reynolds number, (iii) coefficient of friction (Fanning) and (iv) Darcy friction factor.

Solution

Let $\mu = 0.15$ Ns/m^2, $S_{\text{oil}} = 0.85$, $D = 0.3$ m, $L = 3000$ m and $h_f = 20$ m.

$$\rho = S_{\text{oil}}\rho_w = 0.85 \times 1000 = 850 \text{ kg/m}^3$$

(i) $\because h_f = \dfrac{32\mu VL}{\rho g D^2}$

$$\therefore V = \frac{\rho g D^2 h_f}{32\mu L} = \frac{850 \times 9.81 \times 0.3^2 \times 20}{32 \times 0.15 \times 3000} = \mathbf{1.042 \text{ m/s}}$$

(ii) $\text{Re} = \dfrac{\rho VD}{\mu} = \dfrac{850 \times 1.042 \times 0.3}{0.15} = \mathbf{1771.4}$

(iii) $f = \dfrac{16}{\text{Re}} = \dfrac{16}{1771.4} = \mathbf{0.009}$

(iv) $f_f = 4f = 4 \times 0.009 = \mathbf{0.036}$

Example 12.9 An oil of relative density 0.82 is pumped through a 0.15 m diameter and 3000 m long horizontal pipe at the rate of 15 litres per second. If the pump has an efficiency of 68% and requires 7.35 kW power to pump the oil, then find the dynamic viscosity of the oil and the type of flow.

Solution

Let $S_{\text{oil}} = 0.82$, $D = 0.15$ m, $L = 3000$ m, $Q = 15$ l/s $= 0.015$ m^3/s, $\eta_p = 0.68$ and $P = 7.35$ kW.

$$\rho = S_{\text{oil}}\rho_w = 0.82 \times 1000 = 820 \text{ kg/m}^3$$

$$A = \frac{\pi}{4}D^2 = \frac{\pi}{4} \times 0.15^2 = 0.017671 \text{ m}^2$$

$$V = \frac{Q}{A} = \frac{0.015}{0.017671} = 0.849 \text{ m/s}$$

Since
$$P = \frac{\rho g Q h_f}{\eta_p}$$

Thus
$$h_f = \frac{\eta_p P}{\rho g Q} = \frac{0.68 \times 7.35 \times 10^3}{820 \times 9.81 \times 0.015} = 41.421\, m$$

Also
$$h_f = \frac{32\mu VL}{\rho g D^2}$$

Thus
$$41.421 = \frac{32\mu \times 0.849 \times 3000}{820 \times 9.81 \times 0.15^2}$$

$$\therefore \mu = \frac{41.421 \times 820 \times 9.81 \times 0.15^2}{32 \times 0.849 \times 3000} = \mathbf{0.092 \text{ Ns/m}^2}$$

$$\text{Re} = \frac{\rho VD}{\mu} = \frac{820 \times 0.849 \times 0.15}{0.092} = 1135.08$$

Since Re < 2000, the flow is laminar.

Example 12.10 The laminar flow of glycerine through a horizontal pipe of diameter 0.1 m is 0.012 m^3/s. If the absolute viscosity of glycerine is 0.9 Ns/m^2 and kinematic viscosity is 9 stokes, then what power is required per kilometre of pipeline to overcome the viscous resistance to the flow of glycerine?

Solution

Let $D = 0.1$ m, $Q = 0.012$ m^3/s, $\mu = 0.9$ Ns/m^2, $\nu = 9$ stokes $= 9 \times 10^{-4}$ m^2/s and $L = 1$ km $= 1000$ m. Let P be the power required to overcome viscous resistance.

$$\rho = \frac{\mu}{\nu} = \frac{0.9}{9 \times 10^{-4}} = 1000 \text{ kg/m}^3$$

$$A = \frac{\pi}{4} D^2 = \frac{\pi}{4} \times 0.1^2 = 0.007854 \text{ m}^2$$

$$V = \frac{Q}{A} = \frac{0.012}{0.007854} = 1.528 \text{ m/s}$$

$$h_f = \frac{32 \mu VL}{\rho g D^2} = \frac{32 \times 0.9 \times 1.528 \times 1000}{1000 \times 9.81 \times 0.1^2} = 448.59 \text{ m}$$

$$P = \frac{\rho g Q h_f}{1000} = \frac{1000 \times 9.81 \times 0.012 \times 448.59}{1000} = \mathbf{52.808 \text{ kW}}$$

Example 12.11 A crude oil of viscosity 0.14 Ns/m^2 and relative density 0.92 flows through a 25 mm diameter vertical pipe. If the pressure gauges fixed at 15 m apart measure 540 kN/m^2 and 180 kN/m^2, the lower value of the gauge is at the higher level, then determine the direction and the rate of flow through the pipe.

Solution

Refer Figure 12.7. Let $\mu = 0.14$ Ns/m^2, $S_{\text{oil}} = 0.92$, $D = 25$ mm $= 0.025$ m, $L = 15$ m, $p_1 = 540$ kN/m^2 and $p_2 = 180$ kN/m^2.

$$\rho = S_{\text{oil}} \rho_w = 0.92 \times 1000 = 920 \text{ kg/m}^3$$

$$A = \frac{\pi}{4} D^2 = \frac{\pi}{4} \times 0.025^2 = 0.000491 \text{ m}^2$$

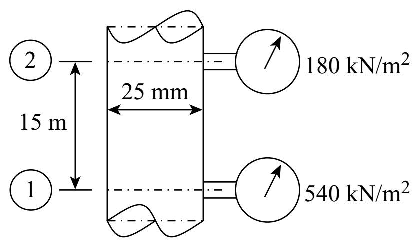

Figure 12.7

Let the level at point 1 be the datum and h_1 and h_2 be the piezometric heads at points 1 and 2, respectively. As the pipe is of uniform cross section, the velocity head is same at the points 1 and 2.

$$h_1 = \frac{p_1}{\rho g} + z_1 = \frac{540 \times 10^3}{920 \times 9.81} + 0 = 59.8325 \text{ m}$$

$$h_2 = \frac{p_2}{\rho g} + z_2 = \frac{180 \times 10^3}{920 \times 9.81} + 15 = 34.9441 \text{ m}$$

$\because h_1 > h_2$, flow takes place from point 1 to 2, i.e., in upward direction.

$$h_f = h_1 - h_2 = 59.8325 - 34.9441 = 24.8884 \text{ m}$$

Also $$h_f = \frac{32\mu VL}{\rho g D^2}$$

Thus $$24.8884 = \frac{32 \times 0.14 \times V \times 15}{920 \times 9.81 \times 0.025^2}$$

$$\therefore V = \frac{24.8884 \times 920 \times 9.81 \times 0.025^2}{32 \times 0.14 \times 15} = \mathbf{2.09\ m/s}$$

$$\text{Re} = \frac{\rho VD}{\mu} = \frac{920 \times 2.09 \times 0.025}{0.14} = 343.36$$

Since Re < 2000, the flow is laminar.

$$Q = AV = 0.000491 \times 2.09 = \mathbf{0.00103\ m^3/s}$$

Example 12.12 A pipe of diameter 0.04 m and length 300 m slopes upwards at the rate of 1 in 30. If an oil of viscosity 0.88 Ns/m^2 and specific gravity 0.89 is pumped at the rate of 0.003 m^3/s, then determine (i) the type of flow, (ii) pressure difference between the two ends, (iii) power of the pump assuming the overall efficiency of 0.68, (iv) centre line velocity and (v) velocity gradient at the pipe wall.

Solution

Refer Figure 12.8. Let $D = 0.04$ m, $L = 300$ m, slope = 1 in 30, $\mu = 0.88$ Ns/m^2, $S_{\text{oil}} = 0.89$, $Q = 0.003$ m^3/s and $\eta_o = 0.68$.

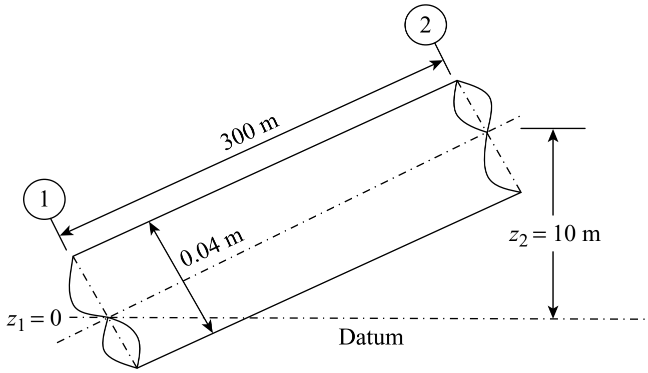

Figure 12.8

$$R = \frac{D}{2} = \frac{0.04}{2} = 0.02 \text{ m}$$

$$\rho = S_{\text{oil}} \rho_w = 0.89 \times 1000 = 890 \text{ kg/m}^3$$

$$A = \frac{\pi}{4} D^2 = \frac{\pi}{4} \times 0.04^2 = 0.001257 \text{ m}^2$$

(i) $V = \dfrac{Q}{A} = \dfrac{0.003}{0.001257} = 2.387$ m/s

$$\text{Re} = \frac{\rho VD}{\mu} = \frac{890 \times 2.387 \times 0.04}{0.88} = 96.565$$

Since Re < 2000, the flow is laminar.

(ii) $\dfrac{p_1}{\rho g}+\dfrac{V_1^2}{2g}+z_1=\dfrac{p_2}{\rho g}+\dfrac{V_2^2}{2g}+z_2+h_f$ [Bernoulli's equation]

But $V_1=V_2,\ z_1=0,\ z_2=(1/30)\times 300=10$ m, and $h_f=(32\mu VL)/(\rho g D^2)$

Thus $$\frac{p_1-p_2}{\rho g}=10+\frac{32\mu VL}{\rho g D^2}$$

or $$\Delta p=10\rho g+\frac{32\mu VL}{D^2} \quad [\Delta p=p_1-p_2]$$

$$\therefore \Delta p=10\times 890\times 9.81+\frac{32\times 0.88\times 2.387\times 300}{0.04^2}=\mathbf{12690669\ N/m^2}$$

(iii) $P=\dfrac{\rho g Q h_f}{\eta_p}=\dfrac{\rho g Q}{\eta_p}\times\dfrac{\Delta p}{\rho g}=\dfrac{Q\Delta p}{\eta_p}=\dfrac{0.003\times 12690669}{0.68\times 10^3}=\mathbf{55.988\ kW}$

(iv) $u_{max}=2V=2\times 2.387=\mathbf{4.774\ m/s}$

(v) $\tau_{max}=\dfrac{\Delta p}{L}\dfrac{R}{2}=\left(\dfrac{12690669}{300}\right)\times\dfrac{0.02}{2}=423.0223\ \mathrm{N/m^2}$

$$\left(\frac{\partial u}{\partial y}\right)_{y=0}=\frac{\tau_{max}}{\mu}=\frac{423.0223}{0.88}=\mathbf{480.71\ s^{-1}}$$

Example 12.13 Water flows in a pipe of diameter 0.25 m. The shear stress at a point 25 mm from the pipe axis is 0.1 kPa. If coefficient of friction is 0.04, then determine (i) Reynolds number and type of flow and (ii) shear stress at the pipe wall.

Solution

Let $D=0.25$ m, $r=25$ mm $=0.025$ m, $\tau=0.1$ kPa $=0.1$ kN/m^2 and $f=0.04$.

(i) $\mathrm{Re}=\dfrac{16}{f}=\dfrac{16}{0.04}=\mathbf{400}$

Since Re < 2000, the flow is laminar.

(ii) $R=\dfrac{D}{2}=\dfrac{0.25}{2}=0.125$ m

Since $$\frac{\tau}{r}=\frac{\tau_{max}}{R} \quad \text{[Linear variation]}$$

$$\therefore \tau_{max}=\frac{\tau R}{r}=\frac{0.1\times 0.125}{0.025}=\mathbf{0.5\ kN/m^2}$$

Example 12.14 A pipe of diameter 0.25 m and length 10000 m slopes upwards at a slope of 1 in 250 m of pipe length traversed. An oil of viscosity 0.15 Ns/m^2 and specific gravity 0.86 is required to be discharged through it at the rate of 0.025 m^3/s. Determine (i) the head lost due to friction and (ii) power required to drive the pump.

Solution

Let $D=0.25$ m, $L=10000$ m, $i=1$ in 250, $\mu=0.15$ Ns/m^2, $S_{oil}=0.86$ and $Q=0.025$ m^3/s.

$$\rho = S_{\text{oil}} \rho_w = 0.86 \times 1000 = 860 \text{ kg/m}^3$$

$$A = \frac{\pi}{4} D^2 = \frac{\pi}{4} \times 0.25^2 = 0.0491 \text{ m}^2$$

(i) $V = \dfrac{Q}{A} = \dfrac{0.025}{0.0491} = 0.5092 \text{ m/s}$

$$\text{Re} = \frac{\rho V D}{\mu} = \frac{860 \times 0.5092 \times 0.25}{0.15} = 729.85$$

Since $\text{Re} < 2000$, the flow is laminar.

$$h_f = \frac{32 \mu V L}{\rho g D^2} = \frac{32 \times 0.15 \times 0.5092 \times 10000}{860 \times 9.81 \times 0.25^2} = \mathbf{46.353 \text{ m}}$$

(ii) Let h be the height through which oil is to be lifted and it is given by,

$$h = i \times L = \frac{1}{250} \times 10000 = 40 \text{ m}$$

Thus, the total head (H) against which the pump is to work is given by,

$$H = h + h_f = 40 + 46.353 = 86.353 \text{ m}$$

Power required to pump the oil is given by,

$$P = \frac{\rho g Q H}{1000} = \frac{860 \times 9.81 \times 0.025 \times 86.353}{1000} = \mathbf{18.213 \text{ kW}}$$

12.6 ❐ LAMINAR FLOW THROUGH ANNULUS

Velocity distribution Consider a steady laminar flow of an incompressible fluid through horizontal annular space between two concentric circular pipes as shown in Figure 12.9. Let R_1 be the outer radius and R_2 be the inner radius of the annulus. A small fluid element of the shape of a small concentric cylinder of length dx and thickness dr at a radial distance r considered as free body. The forces acting on the small cylindrical fluid element in the flow direction are normal pressure forces and shear forces as shown in Figure 12.9.

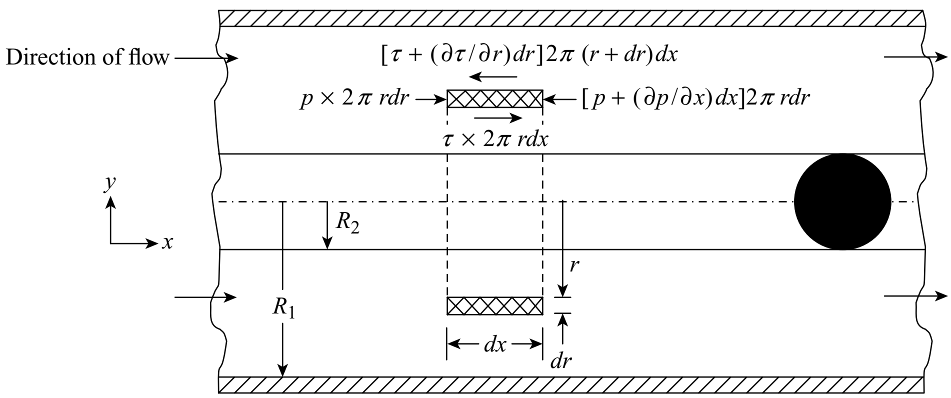

Figure 12.9 *Laminar flow through an annulus*

Thus, the forces acting on the fluid element are as follows.

(i) **Pressure forces:** $p \times 2\pi r dr$ and $\left[p + \dfrac{\partial p}{\partial x} dx\right] 2\pi r dr$

(ii) **Shear forces:** $\tau \times 2\pi r dx$ and $\left[\tau + \dfrac{\partial \tau}{\partial r} dr\right] 2\pi (r + dr) dx$

Since the flow is steady and uniform, the summation of the forces on the fluid element in the direction of flow is zero as given below.

$$p \times 2\pi r dr - \left[\left(p + \frac{\partial p}{\partial x} dx\right) 2\pi r dr\right] + \tau \times 2\pi r dx - \left[\left(\tau + \frac{\partial \tau}{\partial r} dr\right) 2\pi (r + dr) dx\right] = 0$$

$$\left(-\frac{\partial p}{\partial x} dx\right) 2\pi r dr - 2\pi \tau dr dx - 2\pi \frac{\partial \tau}{\partial r} dr r dx - 2\pi \frac{\partial \tau}{\partial r} dr dr dx = 0$$

$$-\frac{\partial p}{\partial x} - \frac{\tau}{r} - \frac{\partial \tau}{\partial r} = 0 \quad \text{[By neglecting } (dr)^2 \text{ term and dividing by } 2\pi r\, dr\, dx]$$

$$r \frac{\partial p}{\partial x} + r \frac{\partial \tau}{\partial r} + \tau = 0 \Rightarrow r \frac{\partial p}{\partial x} + \frac{d}{dr}(r\tau) = 0$$

$$r \frac{\partial p}{\partial x} + \frac{d}{dr}\left(-\mu r \frac{du}{dr}\right) = 0 \Rightarrow \frac{d}{dr}\left(r \frac{du}{dr}\right) = \frac{r}{\mu} \frac{\partial p}{\partial x} \qquad \text{(i)}$$

Integrating expression (i) with respect to r, we get:

$$\left(r \frac{du}{dr}\right) = \frac{r^2}{2\mu} \frac{\partial p}{\partial x} + k_1$$

$$\frac{du}{dr} = \frac{r}{2\mu} \frac{\partial p}{\partial x} + \frac{k_1}{r} \qquad \text{(ii)}$$

Integrating expression (ii) with respect to r, we get:

$$u = \frac{r^2}{4\mu} \frac{\partial p}{\partial x} + k_1 \ln r + k_2 \qquad \text{(iii)}$$

The constants of integration k_1 and k_2 can be determined from the boundary conditions, (i) at $r = R_1$, $u = 0$ and (ii) at $r = R_2$, $u = 0$

Applying first boundary condition in expression (iii), we get:

$$0 = \frac{R_1^2}{4\mu} \frac{\partial p}{\partial x} + k_1 \ln R_1 + k_2 \qquad \text{(iv)}$$

Applying second boundary condition in expression (iii), we get:

$$0 = \frac{R_2^2}{4\mu} \frac{\partial p}{\partial x} + k_1 \ln R_2 + k_2 \qquad \text{(v)}$$

Equating expressions (iv) and (v), we get:

$$\frac{R_1^2}{4\mu} \frac{\partial p}{\partial x} + k_1 \ln R_1 + k_2 = \frac{R_2^2}{4\mu} \frac{\partial p}{\partial x} + k_1 \ln R_2 + k_2$$

$$k_1(\ln R_1 - \ln R_2) = \frac{R_2^2}{4\mu}\frac{\partial p}{\partial x} - \frac{R_1^2}{4\mu}\frac{\partial p}{\partial x}$$

$$k_1 \ln\frac{R_1}{R_2} = -\frac{(R_1^2 - R_2^2)}{4\mu}\frac{\partial p}{\partial x}$$

$$k_1 = -\frac{(R_1^2 - R_2^2)}{4\mu \ln(R_1/R_2)}\frac{\partial p}{\partial x} \qquad \text{(vi)}$$

Substituting the value of k_1 in expression (iv), we get:

$$0 = \frac{R_1^2}{4\mu}\frac{\partial p}{\partial x} - \frac{(R_1^2 - R_2^2)}{4\mu \ln(R_1/R_2)}\frac{\partial p}{\partial x}\ln R_1 + k_2$$

Thus
$$k_2 = -\frac{1}{4\mu}\frac{\partial p}{\partial x}\left[R_1^2 - \frac{(R_1^2 - R_2^2)}{\ln(R_1/R_2)}\ln R_1\right] \qquad \text{(vii)}$$

Substituting the value of k_1 and k_2 in expression (iii), we get:

$$u = \frac{r^2}{4\mu}\frac{\partial p}{\partial x} - \frac{(R_1^2 - R_2^2)}{4\mu \ln(R_1/R_2)}\frac{\partial p}{\partial x}\ln r - \frac{1}{4\mu}\frac{\partial p}{\partial x}\left[R_1^2 - \frac{(R_1^2 - R_2^2)}{\ln(R_1/R_2)}\ln R_1\right]$$

$$u = -\frac{1}{4\mu}\frac{\partial p}{\partial x}\left[R_1^2 - r^2 - \frac{(R_1^2 - R_2^2)}{\ln(R_1/R_2)}(\ln R_1 - \ln r)\right]$$

Thus
$$u = -\frac{1}{4\mu}\frac{\partial p}{\partial x}\left[R_1^2 - r^2 - \frac{(R_1^2 - R_2^2)}{\ln(R_1/R_2)}\ln\left(\frac{R_1}{r}\right)\right] \qquad (12.23)$$

In order to locate the point where maximum velocity occurs, differentiating Equation (12.23) with respect to r and equating it to zero, we get the below expression.

$$\frac{\partial u}{\partial r} = -\frac{1}{4\mu}\frac{\partial p}{\partial x}\left[-2r + \frac{1}{r}\frac{(R_1^2 - R_2^2)}{\ln(R_1/R_2)}\right] = 0$$

$$\therefore r = \left[\frac{R_1^2 - R_2^2}{2\ln(R_1/R_2)}\right]^{1/2} \qquad (12.24)$$

By substituting this value of r in Equation (12.23), the value of maximum velocity can be obtained.

Shear stress The shear stress is given by,

$$\tau = -\mu\frac{\partial u}{\partial r} = -\mu\frac{\partial}{\partial r}\left[-\frac{1}{4\mu}\frac{\partial p}{\partial x}\left\{R_1^2 - r^2 - \frac{(R_1^2 - R_2^2)}{\ln(R_1/R_2)}\ln\left(\frac{R_1}{r}\right)\right\}\right]$$

Thus
$$\tau = \frac{1}{4}\frac{\partial p}{\partial x}\left[-2r - \frac{1}{r}\frac{(R_1^2 - R_2^2)}{\ln(R_1/R_2)}\left(-\frac{1}{r}\right)\right] = -\frac{1}{4}\frac{\partial p}{\partial x}\left[2r - \frac{1}{r}\frac{(R_1^2 - R_2^2)}{\ln(R_1/R_2)}\right] \qquad (12.25)$$

Discharge The discharge (Q) through an annulus is given by,

$$Q = \int_{R_1}^{R_2} u \times 2\pi r dr = -\int_{R_1}^{R_2} \frac{1}{4\mu}\frac{\partial p}{\partial x}\left[R_1^2 - r^2 - \frac{(R_1^2 - R_2^2)}{\ln(R_1/R_2)}\ln\left(\frac{R_1}{r}\right)\right] \times 2\pi r dr$$

Thus
$$Q = -\frac{\pi}{8\mu}\frac{\partial p}{\partial x}\left[R_1^4 - R_2^4 - \frac{(R_1^2 - R_2^2)^2}{\ln(R_1/R_2)}\right] \tag{12.26}$$

Average velocity The average velocity (V) through the annulus is given by,

$$V = \frac{Q}{A} = -\frac{\pi}{8\mu}\frac{\partial p}{\partial x} - \left[R_1^4 - R_2^4 - \frac{(R_1^2 - R_2^2)^2}{\ln(R_1/R_2)}\right] \times \frac{1}{\pi(R_1^2 - R_2^2)}$$

$$\therefore V = -\frac{1}{8\mu}\frac{\partial p}{\partial x}\left[R_1^2 + R_2^2 - \frac{(R_1^2 - R_2^2)}{\ln(R_1/R_2)}\right] \tag{12.27}$$

12.7 ❒ LAMINAR FLOW BETWEEN TWO PARALLEL PLATES WHEN BOTH PLATES ARE AT REST

Velocity distribution Consider a laminar flow of fluid between two parallel fixed plates located at a distance b apart as shown in Figure 12.10. Take a small rectangular fluid element of length dx, thickness dy and width unity at a distance y from the bottom plate.

The forces acting on the small element in the flow direction are the net pressure force and the shear forces on the surface. If p be the intensity of pressure on the face PQ (left end of the element), then the intensity of pressure on the face RS (right end of the element) will be $p + (\partial p / \partial x)dx$. If τ be the shear stress on the face QR (bottom surface of the element), then the shear force on the face PS (top surface of the element) will be $\tau + (\partial \tau/\partial y)dy$. Thus, the forces acting on the fluid element are as follows.

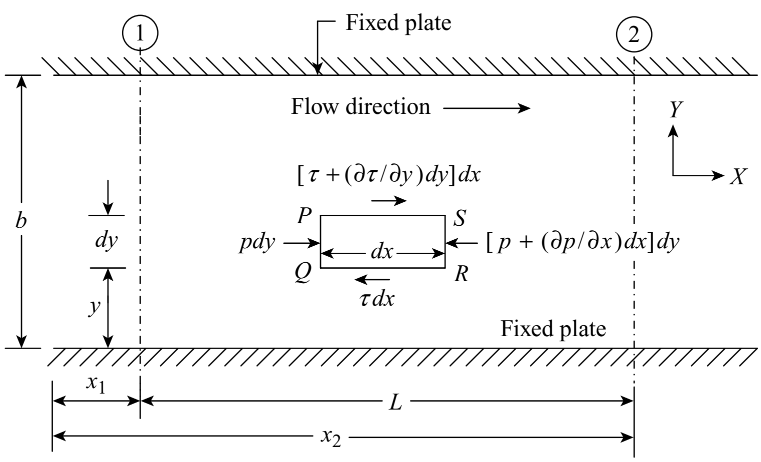

Figure 12.10 *Laminar flow between two parallel fixed plates*

(i) Pressure force on the face: $PQ = p \times dy \times 1 = pdy$

(ii) Pressure force on the face: $RS = \left[p + \frac{\partial p}{\partial x} dx \right] \times dy \times 1 = \left[p + \frac{\partial p}{\partial x} dx \right] dy$

(iii) Shear force on the face: $QR = \tau \times dx \times 1 = \tau dx$

(iv) Shear force on the face: $PS = \left[\tau + \frac{\partial \tau}{\partial y} dy \right] \times dx \times 1 = \left[\tau + \frac{\partial \tau}{\partial y} dy \right] dx$

For steady and uniform flow, there is no acceleration and hence, the resultant force in the direction of flow is zero. Therefore, the sum of net pressure force and viscous force in the x-direction is zero as given below.

$$pdy - \left(p + \frac{\partial p}{\partial x} dx \right) dy - \tau dx + \left(\tau + \frac{\partial \tau}{\partial y} dy \right) dx = 0$$

$$-\frac{\partial p}{\partial x} dxdy + \frac{\partial \tau}{\partial y} dydx = 0$$

Thus

$$\frac{\partial \tau}{\partial y} = \frac{\partial p}{\partial x} \tag{12.28}$$

$$\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y} \right) = \frac{\partial p}{\partial x} \Rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu} \frac{\partial p}{\partial x}$$

Integrating the above equation with respect to y, we get:

$$\frac{\partial u}{\partial y} = \frac{1}{\mu} \frac{\partial p}{\partial x} y + k_1$$

Integrating again, we get:

$$u = \frac{1}{\mu} \frac{\partial p}{\partial x} \frac{y^2}{2} + k_1 y + k_2 \tag{12.29}$$

The constants of integration k_1 and k_2 can be determined from the boundary conditions, (i) at $y = 0$, $u = 0$ and (ii) at $y = b$, $u = 0$

Applying boundary conditions (i) in Equation (12.29), we get:

$$0 = 0 + k_1 \times 0 + k_2 \Rightarrow k_2 = 0$$

Now substitute $k_2 = 0$ and applying boundary conditions (ii) in Equation (12.29), we get:

$$0 = \frac{1}{\mu} \frac{\partial p}{\partial x} \frac{b^2}{2} + k_1 b + 0 \Rightarrow k_1 = -\frac{1}{2\mu} \frac{\partial p}{\partial x} b$$

Substituting the value of k_1 and k_2 in Equation (12.29), we get:

$$u = \frac{1}{\mu} \frac{\partial p}{\partial x} \frac{y^2}{2} - \frac{1}{2\mu} \frac{\partial p}{\partial x} by = -\frac{1}{2\mu} \frac{\partial p}{\partial x} (by - y^2) \tag{12.30}$$

Since μ, $(\partial p / \partial x)$ and b are constant, velocity varies with the square of y. Therefore, Equation (12.30) indicates that the velocity distribution across the section of parallel plates is parabolic as shown in Figure (12.11).

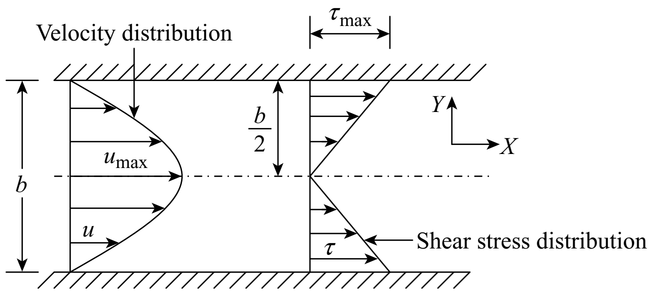

Figure 12.11 *Velocity distribution and shear stress distribution*

For obtaining maximum velocity (u_{max}), substitute $(\partial u/\partial y) = 0$ as given below.

$$\frac{\partial}{\partial y}\left[-\frac{1}{2\mu}\frac{\partial p}{\partial x}(by - y^2)\right] = 0 \Rightarrow b - 2y = 0$$

$$\boxed{\therefore y = \frac{b}{2}}$$

The maximum velocity can be obtained by substituting $y = (b/2)$ in Equation (12.30) as follows.

$$u_{max} = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left[b\left(\frac{b}{2}\right) - \left(\frac{b}{2}\right)^2\right] = -\frac{1}{8\mu}\frac{\partial p}{\partial x}b^2 \quad (12.31)$$

Shear stress distribution From Newton's law of viscosity, we get:

$$\tau = \mu\frac{\partial u}{\partial y} = \mu\frac{\partial}{\partial y}\left[-\frac{1}{2\mu}\frac{\partial p}{\partial x}(by - y^2)\right] \quad \text{[Substitute Equation (12.30)]}$$

Thus

$$\tau = -\frac{1}{2}\frac{\partial p}{\partial x}(b - 2y) \quad (12.32)$$

The Equation (12.32) indicates that shear stress varies linearly with y and its minimum and maximum values are given as follows.

(i) At $y = \frac{b}{2}$: $\tau = 0$ (minimum value)

(ii) At $y = 0$: $\tau_{max} = \left(-\frac{\partial p}{\partial x}\right)\frac{b}{2}$ (maximum value) (12.33)

The shear stress distribution is shown in Figure 12.11.

Discharge and average velocity The discharge (dQ) through an elementary strip of thickness dy is given by,

$$dQ = \text{Velocity} \times \text{Area} = u \times dy \times 1 = -\frac{1}{2\mu}\frac{\partial p}{\partial x}[by - y^2]dy$$

The total discharge can be obtained by integrating the above equation as given below.

$$\int dQ = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\int_0^b (by - y^2)dy$$

$$Q = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left[\frac{by^2}{2} - \frac{y^3}{3}\right]_0^b = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left[\frac{b^3}{2} - \frac{b^3}{3}\right] = -\frac{1}{12\mu}\frac{\partial p}{\partial x}b^3 \quad (12.34)$$

The average velocity of flow V (or u_{av}) is given by,

$$V = \frac{Q}{A} = -\frac{1}{12\mu}\frac{\partial p}{\partial x}b^3 \times \frac{1}{(b\times 1)} = -\frac{1}{12\mu}\frac{\partial p}{\partial x}b^2 \quad (12.35)$$

Ratio of maximum velocity to average velocity The ratio of maximum velocity to average velocity is given by,

$$\frac{u_{max}}{V} = \frac{-\frac{1}{8\mu}\frac{\partial p}{\partial x}b^2}{-\frac{1}{12\mu}\frac{\partial p}{\partial x}b^2} = \frac{3}{2} \quad (12.36)$$

Loss in pressure head over a length of plates From Equation (12.35), we get:

$$\partial p = -\frac{12\mu V}{b^2}\partial x$$

The pressure difference between sections 1 and 2 as shown in Figure 12.10 at distances x_1 and x_2 can be obtained by integrating the above expression as follows.

$$\int_1^2 \partial p = -\frac{12\mu V}{b^2}\int_1^2 \partial x \Rightarrow (p_2 - p_1) = -\frac{12\mu V}{b^2}(x_2 - x_1)$$

Since $\Delta p = (p_1 - p_2)$ and $(x_2 - x_1) = L$

$$\boxed{\therefore \Delta p = \frac{12\mu VL}{b^2}} \quad (12.37)$$

From Equation (12.37), we get:

$$\boxed{\frac{\Delta p}{\rho g} = h_f = \frac{12\mu VL}{\rho g b^2}} \quad (12.38)$$

Here, $[\Delta p/(\rho g)] = h_f$ represents the drop in pressure head between two sections. From Equation (12.38), it can be observed that the frictional head loss over the length of plates varies directly proportional to the velocity of flow of fluid and inversely as the square of the distance between the plates.

Example 12.15 Two parallel plates kept 10 cm apart have laminar flow of oil between them with a maximum velocity of 1.5 m/s. If the viscosity of oil is 25 poise, then compute (i) the discharge per metre width, (ii) maximum shear stress at the plates, (iii) difference in pressure between two points 20 m apart, (iv) velocity gradient at the plates and (v) velocity at 2 cm from the plate.

Solution

Let $b = 10$ cm $= 0.1$ m, $u_{max} = 1.5$ m/s, $\mu = 25$ poise $= 2.5$ Ns/m^2, $L = 20$ m and $y = 2$ cm $= 0.02$ m. Let q be the discharge per metre width of the plate.

(i) From Equation (12.36), we get:

$$V = \frac{2}{3}u_{max} = \frac{2}{3}\times 1.5 = 1 \text{ m/s}$$

$$q = Vb = 1\times 0.1 = \mathbf{0.1\ m^3/s\ per\ m}$$

(ii) From Equation (12.35), we get:

$$\left(-\frac{\partial p}{\partial x}\right) = \frac{12\mu V}{b^2} = \frac{12\times 2.5\times 1}{0.1^2} = 3000 \text{ N/m}^2\text{/m}$$

$$\tau_{max} = \left(-\frac{\partial p}{\partial x}\right)\frac{b}{2} = 3000\times\frac{0.1}{2} = \mathbf{150\ N/m^2}$$

(iii) $\Delta p = \dfrac{12\mu VL}{b^2} = \dfrac{12\times 2.5\times 1\times 20}{0.1^2} = \mathbf{60000\ N/m^2}$

(iv) $\left(\dfrac{\partial u}{\partial y}\right)_{y=0} = \dfrac{\tau_{max}}{\mu} = \dfrac{150}{2.5} = \mathbf{60\ s^{-1}}$

(v) $u = \dfrac{1}{2\mu}\left(-\dfrac{\partial p}{\partial x}\right)(by - y^2) = \dfrac{3000\times(0.1\times 0.02 - 0.02^2)}{2\times 2.5} = \mathbf{0.96\ m/s}$

Example 12.16 Two parallel plates are 5 mm apart and a steady laminar flow of oil is occurring between them. If the pressure drop is 10 kPa per metre length of the plates and viscosity of oil is 0.06 Ns/m^2, then calculate (i) the discharge per metre width, (ii) maximum shear stress and (iii) maximum velocity of flow.

Solution

Let $b = 5$ mm $= 0.005$ m, $\Delta p/L = 10$ kPa/m, $L = 1$ m and $\mu = 0.06$ Ns/m^2. Let q be the discharge per metre of width of the plate.

(i) $\because \Delta p = \dfrac{12\mu VL}{b^2}$

$$\therefore V = \frac{\Delta p b^2}{12\mu L} = \frac{10\times 10^3\times 0.005^2}{12\times 0.06\times 1} = 0.3472 \text{ m/s}$$

$$q = Vb = 0.3472\times 0.005 = \mathbf{1.736\times 10^{-3}\ m^3/s\ per\ m}$$

(ii) $\tau_{max} = \left(-\dfrac{\partial p}{\partial x}\right)\dfrac{b}{2} = 10\times 10^3\times\dfrac{0.005}{2} = \mathbf{25\ N/m^2}$

(iii) From Equation (12.36), we get:

$$u_{max} = \frac{3}{2}V = \frac{3}{2}\times 0.3472 = \mathbf{0.5208\ m/s}$$

Example 12.17 A container is filled completely by an oil of viscosity 0.2 Ns/m^2. The container has a horizontal crack in its end wall which is 0.4 m wide and 0.04 m thick in the direction of flow. If the pressure difference between two faces of the crack is 12 kN/m^2 and the crack forms a gap of 0.5 mm between the parallel surfaces, then calculate (i) the rate of leakage of the oil through the crack, (ii) maximum leakage velocity, (iii) shear stress at the wall and (iv) velocity gradient at the wall.

Solution

Let $\mu = 0.2$ Ns/m^2, $w = 0.4$ m, $L = 0.04$ m, $\Delta p = 12$ kN/m^2 and $b = 0.5$ mm $= 0.0005$ m.

The oil leakage through the crack corresponds to the case of flow between two parallel fixed plates.

(i) $\because \Delta p = \dfrac{12\mu VL}{b^2}$

$$\therefore V = \frac{\Delta p b^2}{12\mu L} = \frac{12 \times 10^3 \times 0.0005^2}{12 \times 0.2 \times 0.04} = 0.03125 \text{ m/s}$$

$$Q = Vwb = 0.03125 \times 0.4 \times 0.0005 = \mathbf{6.25 \times 10^{-6}\ m^3/s}$$

(ii) From Equation (12.36), we get:

$$u_{max} = \frac{3}{2}V = \frac{3}{2} \times 0.03125 = \mathbf{0.0469\ m/s}$$

(iii) $\tau_{max} = \left(-\dfrac{\partial p}{\partial x}\right)\dfrac{b}{2} = \dfrac{12 \times 10^3}{0.04} \times \dfrac{0.0005}{2} = \mathbf{75\ N/m^2}$

(iv) $\left(\dfrac{\partial u}{\partial y}\right)_{y=0} = \dfrac{\tau_{max}}{\mu} = \dfrac{75}{0.2} = \mathbf{375\ s^{-1}}$

12.8 ❐ LAMINAR FLOW BETWEEN TWO PARALLEL PLATES WHEN ONE PLATE MOVES AND OTHER AT REST (COUETTE FLOW)

Velocity distribution Consider a laminar flow of fluid between two parallel flat plates located at a distance b apart as shown in Figure 12.12. The lower plate is fixed and the upper plate moves with a uniform velocity V. This type of flow is known as Couette flow and it is named after M.F.A. Couette.

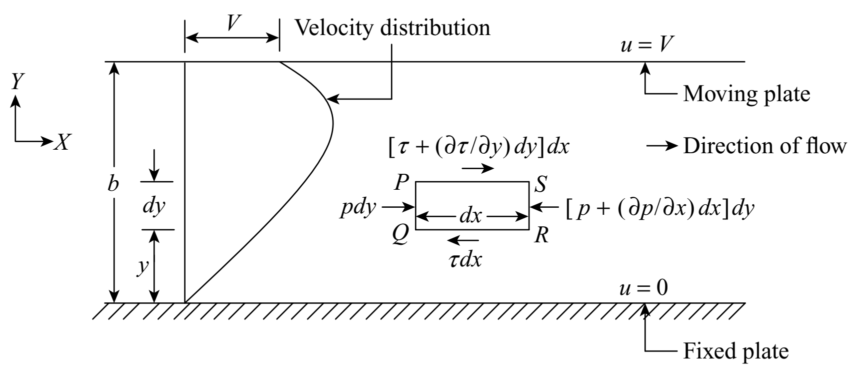

Figure 12.12 *Couette flow*

The Equation (12.29) derived earlier for laminar flow between two fixed parallel plates is also applicable for Couette flow.

Thus
$$u = \frac{1}{\mu}\frac{\partial p}{\partial x}\frac{y^2}{2} + k_1 y + k_2 \tag{12.29}$$

The boundary conditions for Couette flow are

(i) at $y = 0$, $u = 0$ and

(ii) at $y = b$, $u = V$.

Applying boundary conditions (i) in Equation (12.29), we get:

$$0 = 0 + k_1 \times 0 + k_2 \Rightarrow k_2 = 0$$

Now substitute $k_2 = 0$ and applying boundary conditions (ii) in Equation (12.29), we get:

$$V = \frac{1}{\mu}\frac{\partial p}{\partial x}\frac{b^2}{2} + k_1 b + 0 \Rightarrow k_1 = \frac{V}{b} - \frac{1}{2\mu}\frac{\partial p}{\partial x} b$$

Substituting the value of k_1 and k_2 in Equation (12.29), we get:

$$u = \frac{1}{\mu}\frac{\partial p}{\partial x}\frac{y^2}{2} - \left(\frac{V}{b} - \frac{1}{2\mu}\frac{\partial p}{\partial x} b\right) y = \frac{V}{b} y - \frac{1}{2\mu}\frac{\partial p}{\partial x}(by - y^2) \tag{12.39}$$

The Equation (12.39) shows that the velocity distribution in Couette flow depends upon the velocity of the moving plate V and the pressure gradient $(\partial p/\partial x)$. The velocity profiles for different pressure gradients may be either positive or negative.

If $V = 0$, i.e., the upper plate is fixed, then Equation (12.39) reduces to Equation (12.30). If there is no pressure gradient [i.e., $(\partial p/\partial x) = 0$], then Equation (12.39) is written as follows.

$$u = \frac{V}{b} y \tag{12.40}$$

The Equation (12.40) shows that the velocity distribution is linear. This type of flow is known as plain Couette flow or simple shear flow.

The Equation (12.39) can be transformed into non-dimensional form by dividing both sides by V and by rearranging as,

$$\frac{u}{V} = \frac{y}{b} + \frac{1}{2\mu V}\left(-\frac{\partial p}{\partial x}\right) b^2 \left(\frac{y}{b} - \frac{y^2}{b^2}\right) = \frac{y}{b} + \left[\frac{b^2}{2\mu V}\left(-\frac{\partial p}{\partial x}\right)\right]\frac{y}{b}\left(1 - \frac{y}{b}\right) \tag{12.41}$$

In Equation (12.41), the term $[\{b^2/(2\mu V)\}(-\partial p/\partial x)]$ is constant and it is termed as non-dimensional pressure gradient ϕ. Thus, Equation (12.41) is written as follows.

$$\frac{u}{V} = \frac{y}{b} + \phi \frac{y}{b}\left(1 - \frac{y}{b}\right) \tag{12.42}$$

A family of velocity distribution curves can be plotted in terms of (y/b) and (u/V) for different values of ϕ as shown in Figure 12.13. The positive values of ϕ (i.e., $\phi > 0$) indicates pressure drop in the direction of flow and thus, the velocity distribution is positive over the entire width of the plate. The negative values of ϕ (i.e., $\phi < 0$) indicate an increase in fluid pressure in the direction of flow and thus, for some layers of fluid, there exists a backward flow due to an adverse pressure gradient in the fluid.

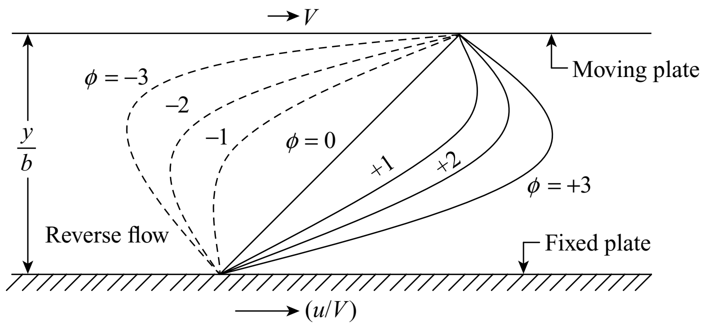

Figure 12.13 *Non-dimensional velocity curves for Couette flow*

Shear stress distribution From Newton's law of viscosity, we get:

$$\tau = \mu \frac{\partial u}{\partial y} = \mu \frac{\partial}{\partial y}\left[\frac{V}{b} y - \frac{1}{2\mu}\frac{\partial p}{\partial x}(by - y^2)\right] \quad \text{[Substitute Equation (12.39)]}$$

Thus

$$\tau = \mu\left[\frac{V}{b} - \frac{1}{2\mu}\frac{\partial p}{\partial x}(b - 2y)\right] = \frac{\mu V}{b} - \frac{1}{2}\frac{\partial p}{\partial x}(b - 2y) \tag{12.43}$$

The Equation (12.43) indicates that shear stress varies linearly with distance y from the boundary as shown in Figure 12.14. The shear stress distribution is asymmetrical with the values of y at different locations as given below.

(i) At $y = 0 : \tau = \frac{\mu V}{b} - \frac{\partial p}{\partial x}\frac{b}{2}$

(ii) At $y = \frac{b}{2} : \tau = \frac{\mu V}{b}$

(iii) At $y = b : \tau = \frac{\mu V}{b} + \frac{\partial p}{\partial x}\frac{b}{2}$

Discharge The discharge per unit width (Q) is given by,

$$Q = \int_0^b u\,dy = \int_0^b \left[\frac{V}{b} y - \frac{1}{2\mu}\frac{\partial p}{\partial x}(by - y^2)\right] dy \quad \text{[Substitute Equation (12.39)]}$$

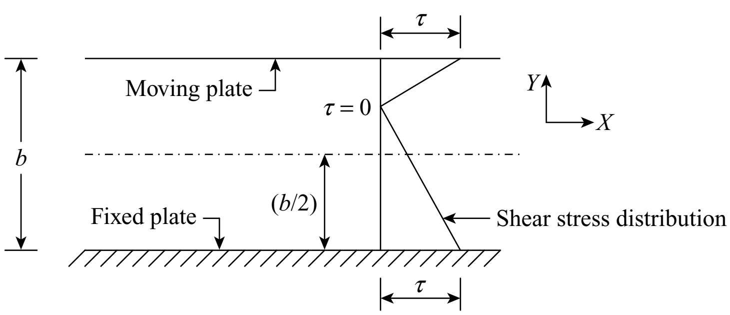

Figure 12.14 *Shear stress distribution for Couette flow*

$$Q = \left[\frac{V}{b}\frac{y^2}{2} - \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(\frac{by^2}{2} - \frac{y^3}{3}\right)\right]_0^b = \frac{V}{b}\frac{b^2}{2} - \frac{1}{2\mu}\frac{\partial p}{\partial x}\left[\frac{bb^2}{2} - \frac{b^3}{3}\right]$$

Thus $$Q = \frac{Vb}{2} - \frac{1}{12\mu}\frac{\partial p}{\partial x}b^3 \qquad (12.44)$$

Average velocity Average velocity of flow (u_{av}) is given by,

$$u_{av} = \frac{Q}{A} = \left(\frac{Vb}{2} - \frac{1}{12\mu}\frac{\partial p}{\partial x}b^3\right) \times \frac{1}{(b \times 1)} = \frac{V}{2} - \frac{1}{12\mu}\frac{\partial p}{\partial x}b^2 \qquad (12.45)$$

Maximum velocity The location of maximum velocity ($u_{\max}$) can be obtained by differentiating Equation (12.42) with respect to dy and equating with zero as given below.

$$\frac{du}{dy} = \frac{d}{dy}\left[V\left\{\frac{y}{b} + \phi\frac{y}{b}\left(1 - \frac{y}{b}\right)\right\}\right] = 0 \Rightarrow \frac{V}{b} + \frac{V\phi}{b}\left(1 - \frac{2y}{b}\right) = 0$$

Thus $$\frac{y}{b} = \frac{1}{2} + \frac{1}{2\phi} \qquad (12.46)$$

The value of maximum velocity can be obtained by substituting Equation (12.46) in Equation (12.42) as given below.

$$u_{\max} = \frac{(1+\phi)^2}{4\phi}V \qquad (12.47)$$

Example 12.18 Two horizontal plates 10 mm apart contain oil of viscosity 0.86 poise. If the upper plate is moving with 1.2 m/s with respect to the lower plate which is stationary and the pressure difference between two sections 80 m apart is 80 kPa, then calculate the velocity distribution, shear stress on the upper plate and discharge per unit width.

Solution

Let $b = 10$ mm $= 0.01$ m, $\mu = 0.86$ poise $= 0.086$ Ns/m^2, $V = 1.2$ m/s, $(x_2 - x_1) = 80$ m and $(p_2 - p_1) = \Delta p = 80$ kPa.

$$\frac{\partial p}{\partial x} = -\frac{\Delta p}{(x_2 - x_1)} = -\frac{80 \times 10^3}{80} = -10^3 \text{ N/m}^2 \text{ per m}$$

Since $$u = \frac{V}{b}y - \frac{1}{2\mu}\frac{\partial p}{\partial x}(by - y^2)$$

Thus $$u = \frac{1.2}{0.01}y - \frac{(-10^3) \times (0.01y - y^2)}{2 \times 0.086} = y(120 + 58.14 - 5813.95y)$$

$$\therefore u = \mathbf{y(178.14 - 5813.95y)}$$

Since $$\tau = \frac{\mu V}{b} - \frac{1}{2}\frac{\partial p}{\partial x}(b - 2y)$$

$$\therefore \tau = \frac{0.086 \times 1.2}{0.01} - \frac{(-10^3) \times (0.01 - 2 \times 0.01)}{2} = \mathbf{5.32\ N/m^2}$$

Since $$Q = \frac{Vb}{2} - \frac{1}{12\mu}\frac{\partial p}{\partial x} b^3$$

$$\therefore Q = \frac{1.2 \times 0.01}{2} - \frac{(-10^3) \times 0.01^3}{12 \times 0.086} = \mathbf{0.00697\ m^3/s}$$

12.9 ❐ POWER ABSORBED IN BEARINGS

Lubrication of bearings is an example of laminar flow of viscous fluids. In a bearing, a very thin film of lubricating oil is maintained between its stationary surface and that of the rotating shaft. Highly viscous oil used for lubrication of bearings offers great resistance which causes great power loss, whereas light oil may not be able to maintain a required oil film between the rotating part and stationary metal surface. Therefore, an oil of suitable viscosity should be used for the lubrication of bearings. A thin film of lubricating oil formed between the stationary and rotating surfaces may acquire high pressure, which is capable of supporting load and friction. The power lost or absorbed due to viscous resistance is described below in three types of bearings, namely journal bearing, foot step bearing and collar bearing.

12.9.1 Journal Bearing

A journal bearing consists of a sleeve which may be partially or completely wrapped around a rotating shaft or journal as shown in Figure 12.15. Journal bearing is designed to support a radial load. A thin film of lubricating oil separates the shaft and the bearing and it offers viscous resistance to the rotating shaft.

Let D be the diameter of the shaft, R be the radius of the shaft, N be the speed of the shaft in rpm, $\omega = (2\pi N)/60$ be the angular speed of the shaft, V be the tangential speed of the shaft, L be the length of the bearing and t be the thickness of the oil film.

Now $$V = \omega R = \frac{2\pi N}{60} \times \frac{D}{2} = \frac{\pi DN}{60}$$

Since the thickness of oil film is very small, a linear velocity distribution can be assumed.

$$\frac{du}{dy} = \frac{V-0}{t} = \frac{V}{t} = \frac{\pi DN}{60} \times \frac{1}{t}$$

Shear stress (τ) in the oil is given by,

$$\tau = \mu \frac{du}{dy} = \mu \times \frac{\pi DN}{60t}$$

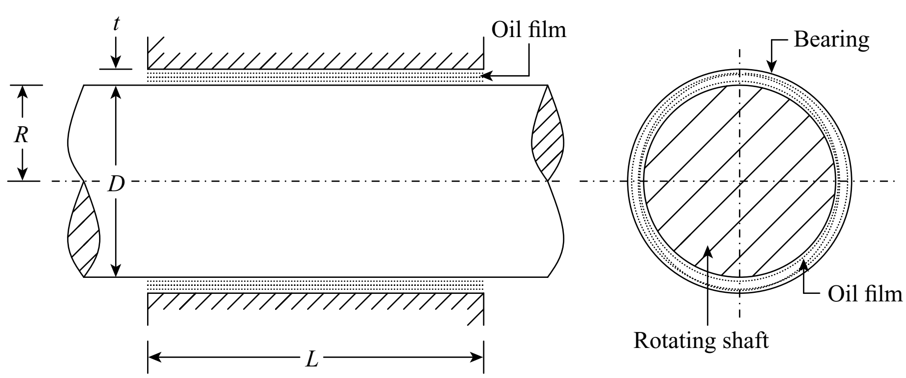

Figure 12.15 *Journal bearing*

Shear force or viscous resistance (F) is given by,

$$F = \tau \times \text{Surface area} = \tau \times \pi DL = \frac{\mu\pi DN}{60t} \times \pi DL = \frac{\mu\pi^2 D^2 NL}{60t}$$

Torque (T) required to overcome the viscous resistance is given by,

$$T = F \times \frac{D}{2} = \frac{\mu\pi^2 D^2 NL}{60t} \times \frac{D}{2} = \frac{\mu\pi^2 D^3 NL}{120t} \tag{12.48}$$

Power (P) absorbed in overcoming the viscous resistance is given by,

$$P = T\omega = \frac{\mu\pi^2 D^3 NL}{120t} \times \frac{2\pi N}{60} = \left(\frac{\mu\pi^3 D^3 N^2 L}{3600t}\right) \text{watts} \tag{12.49}$$

Example 12.19 A shaft of diameter 0.1 m rotates at 80 rpm in a 0.1 m long journal bearing. If the shaft and bearing are separated by a distance of 1 mm of oil with dynamic viscosity of 0.008 Ns/m^2, then find the power absorbed by the bearing.

Solution

Let $D = 0.1$ m, $N = 80$ rpm, $L = 0.1$ m, $t = 1$ mm $= 0.001$ m and $\mu = 0.008$ Ns/m^2.

$$P = \frac{\mu\pi^3 D^3 N^2 L}{3600t} = \frac{0.008 \times \pi^3 \times 0.1^3 \times 80^2 \times 0.1}{3600 \times 0.001} = \mathbf{0.0441\ Watts}$$

12.9.2 Foot Step Bearing

The Figure 12.16 illustrates a foot step bearing in which one end of a vertical shaft rests in the bearing and a thin film of lubricating oil separates the surface of the shaft and the bearing.

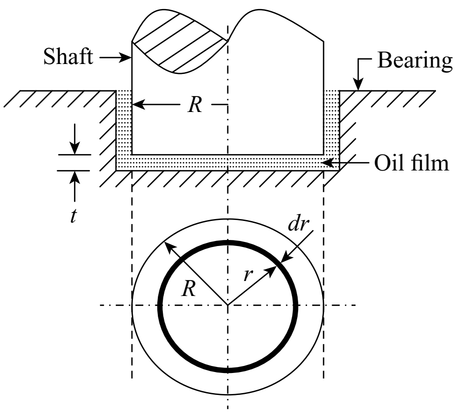

Figure 12.16 *Foot step bearing*

Let R be the radius of the shaft, N be the speed of the shaft in rpm, $\omega = (2\pi N)/60$ be the angular speed of the shaft, V be the tangential speed of the shaft and t be the thickness of the oil film. Consider an elementary ring of thickness dr at a radial distance r from the axis of the shaft. The area of the elementary ring is equal to $2\pi r dr$. Since the thickness of oil film is very small, a linear velocity distribution can be assumed.

Thus $$\frac{du}{dy} = \frac{V - 0}{t} = \frac{V}{t}$$

$$V = \omega r = \frac{2\pi N}{60} \times r$$

The shear stress (τ) on the elementary oil ring is given by,

$$\tau = \mu\frac{du}{dy} = \mu \times \frac{V}{t} = \mu \times \frac{2\pi Nr}{60} \times \frac{1}{t} = \frac{2\pi Nr\mu}{60t}$$

Shear force or viscous resistance (dF) on the elementary ring is given by,

$$dF = \tau \times \text{area of ring} = \tau \times 2\pi r dr = \frac{2\pi Nr\mu}{60t} \times 2\pi r dr = \frac{\mu\pi^2 N}{15t} r^2 dr$$

Thus, the torque (dT) required to overcome the viscous resistance on the ring is given by,

$$dT = dF \times r = \frac{\mu\pi^2 N}{15t} r^2 dr \times r = \frac{\mu\pi^2 N}{15t} r^3 dr$$

Total torque (T) required to overcome the viscous resistance is given by,

$$T = \int_0^R \frac{\mu\pi^2 N}{15t} r^3 dr = \frac{\mu\pi^2 N}{15t}\left[\frac{r^4}{4}\right]_0^R = \frac{\mu\pi^2 N R^4}{60t} \tag{12.50}$$

Power (P) absorbed in overcoming the viscous resistance is given by,

$$P = T\omega = \frac{\mu\pi^2 N R^4}{60t} \times \frac{2\pi N}{60} = \frac{\mu\pi^3 N^2 R^4}{1800t} \text{ Watts} \tag{12.51}$$

Example 12.20 The lower end of a vertical shaft of diameter 0.12 m rotates in a foot step bearing at a speed of 800 rpm. Both surfaces at the end of the vertical shaft and that of the bearing are separated by an oil film thickness of 0.6 mm. If the dynamic viscosity of the oil is 0.12 Ns/m^2, then determine the torque and the power absorbed in the bearing.

Solution

Let $D = 0.12$ m, $N = 800$ rpm, $t = 0.6$ mm $= 0.0006$ m and $\mu = 0.12$ Ns/m^2.

$$T = \frac{\mu\pi^2 N R^4}{60t} = \frac{0.12 \times \pi^2 \times 800 \times (0.12/2)^4}{60 \times 0.0006} = \mathbf{0.3411\ Nm}$$

$$P = T\omega = \frac{2\pi NT}{60} = \frac{2 \times \pi \times 800 \times 0.3411}{60} = \mathbf{28.576\ Watts}$$

12.9.3 Collar Bearing

The collar bearing shown in Figure 12.17 supports the axial thrust of the rotating shaft. The face of the collar is separated from the surface of the bearing by an oil film of uniform thickness which is maintained by a forced lubrication system.

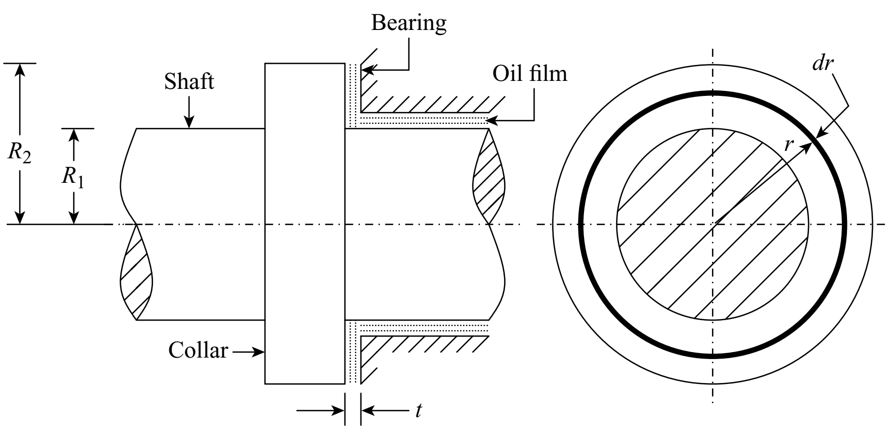

Figure 12.17 *Collar bearing*

Let R_1 be the internal radius of the collar, R_2 be the external radius of the collar, N be the speed of the shaft in rpm, $\omega = (2\pi N)/60$ be the angular speed of the shaft, V be the tangential speed of the shaft and t be the thickness of the oil film.

Consider an elementary ring of thickness dr at a radial distance r from the axis of the shaft. Area of the elementary ring is equal to $2\pi r dr$. Since the thickness of oil film is very small, a linear velocity distribution can be assumed.

$$\frac{du}{dy} = \frac{V-0}{t} = \frac{V}{t}$$

$$V = \omega r = \frac{2\pi N r}{60}$$

Shear stress (τ) on the elementary oil ring is given by,

$$\tau = \mu \frac{du}{dy} = \mu \times \frac{V}{t} = \mu \times \frac{2\pi N r}{60} \times \frac{1}{t} = \frac{2\pi N r \mu}{60t}$$

Shear force or viscous resistance (dF) on the elementary ring is given by,

$$dF = \tau \times \text{area of ring} = \tau \times 2\pi r dr = \frac{2\pi N r \mu}{60t} \times 2\pi r dr = \frac{\mu \pi^2 N}{15t} r^2 dr$$

Torque (dT) required to overcome the viscous resistance on the ring is given by,

$$dT = dF \times r = \frac{\mu \pi^2 N}{15t} r^2 dr \times r = \frac{\mu \pi^2 N}{15t} r^3 dr$$

Total torque (T) required to overcome the viscous resistance is given by,

$$T = \int_{R_1}^{R_2} \frac{\mu \pi^2 N}{15t} r^3 dr = \frac{\mu \pi^2 N}{15t} \left[\frac{r^4}{4} \right]_{R_1}^{R_2} = \frac{\mu \pi^2 N (R_2^4 - R_1^4)}{60t} \tag{12.52}$$

Power (P) absorbed in overcoming the viscous resistance is given by,

$$P = T\omega = \frac{\mu \pi^2 N (R_2^4 - R_1^4)}{60t} \times \frac{2\pi N}{60} = \frac{\mu \pi^3 N^2 (R_2^4 - R_1^4)}{1800t} \text{ Watts} \tag{12.53}$$

Example 12.21 A collar bearing used to take the thrust of a shaft has internal and external diameters as 0.16 m and 0.22 m, respectively. An oil film of thickness 0.2 mm and viscosity 0.088 Ns/m^2 is maintained between the collar surface and the bearing. If the shaft rotates at 250 rpm, then determine the torque and the power lost in overcoming the viscous resistance.

Solution

Let $D_1 = 0.16$ m, $D_2 = 0.22$ m, $t = 0.2$ mm $= 0.0002$ m, $\mu = 0.088$ Ns/m^2 and $N = 250$ rpm.

$$R_1 = \frac{D_1}{2} = \frac{0.16}{2} = 0.08 \text{ m} \text{ and } R_2 = \frac{D_2}{2} = \frac{0.22}{2} = 0.11 \text{ m}$$

$$T = \frac{\mu \pi^2 N (R_2^4 - R_1^4)}{60t} = \frac{0.088\pi^2 \times 250 \times (0.11^4 - 0.08^4)}{60 \times 0.0002} = \mathbf{1.908 \text{ Nm}}$$

$$P = T\omega = \frac{2\pi N T}{60} = \frac{2 \times \pi \times 250 \times 1.908}{60} = \mathbf{49.951 \text{ Watts}}$$

12.10 ❐ MOVEMENT OF PISTON IN DASHPOT

A dashpot is a hydraulic device used for damping vibrations of machines. A simple dashpot mechanism consisting of a piston moving in a concentric cylinder containing highly viscous oil is shown in Figure 12.18. The piston is connected with the machine element whose motion is to be restrained. The downwards movement of the piston increases the pressure of oil below it, as a result of which the viscous oil moves upwards through the annular space between the piston and the cylinder. Conversely, when the piston moves upwards, the oil is displaced downwards and thus, it moves to the space below the piston. The flow of oil offers resistance to the movement of piston which damps the mechanical vibrations of the machine.

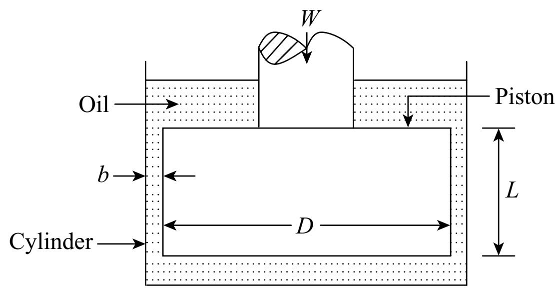

Figure 12.18 *A dashpot mechanism*

Let D be the diameter of the piston, L be the length of the piston, W be the weight of the piston, u be the velocity of the piston, μ be the viscosity of oil, V be the average velocity of the oil in the clearance, b be the clearance between the piston and the dashpot, and Δp be the difference of pressure intensities between the two ends of the piston. The flow of oil through the clearance of the dashpot behaves as laminar flow between two parallel plates. Therefore the difference of pressure between the two ends of the piston is given below.

$$\Delta p = \frac{12\mu VL}{b^2} \tag{i}$$

The difference of pressure at the two ends of the piston is also given by,

$$\Delta p = \frac{W}{A} = \frac{W}{(\pi/4)D^2} = \frac{4W}{\pi D^2} \tag{ii}$$

Simplifying expressions (i) and (ii), we get:

$$\frac{12\mu VL}{b^2} = \frac{4W}{\pi D^2} \Rightarrow V = \frac{Wb^2}{3\pi\mu LD^2} \tag{iii}$$

Using continuity equation, we get:

$$\text{Rate of oil flow in the clearance} = \text{Rate of oil flow in the dashpot}$$

$$\text{Average velocity} \times \text{Clearance area} = \text{Velocity of piston} \times \text{Area of piston}$$

$$V \times \pi Db = u \times \frac{\pi}{4} D^2 \Rightarrow V = \frac{uD}{4b} \tag{iv}$$

Simplifying expressions (iii) and (iv), we get:

$$\frac{Wb^2}{3\pi\mu LD^2} = \frac{uD}{4b}$$

Thus

$$\boxed{\mu = \frac{4Wb^3}{3\pi uLD^3}} \tag{12.54}$$

Example 12.22 The piston of an oil dashpot used for damping vibrations falls with a uniform speed and covers 60 mm in 120 seconds. If an additional weight of 1.5 N is placed on the top of the piston, then it falls through 60 mm in 96 seconds with uniform speed. The diameter of the piston is 60 mm and its length is 96 mm. If the clearance between the piston and the cylinder is 1.24 mm which is uniform throughout, then determine the viscosity of the oil.

Solution

Let $y = 60$ mm = 0.06 m, $T = 120$ s, $w = 1.5$ N, $T' = 96$ s, $D = 60$ mm = 0.06 m, $L = 96$ mm = 0.096 m and $b = 1.24$ mm = 0.00124 m.

Let W be the weight of the piston, u be the velocity of piston without additional weight and u' be the velocity of piston with additional weight.

$$u = \frac{y}{T} = \frac{60}{120} = 0.5 \text{ mm/s} \text{ and } u' = \frac{y}{T'} = \frac{60}{96} = 0.625 \text{ mm/s}$$

Since
$$\mu = \frac{4Wb^3}{3\pi u LD^3} = \frac{4(W+1.5)b^3}{3\pi u' LD^3}$$

Thus
$$\frac{W}{W+1.5} = \frac{u}{u'} = \frac{0.5}{0.625} = 0.8$$

$$W = 0.8W + 1.2 \Rightarrow 0.2W = 1.2$$

$$\therefore W = \frac{1.2}{0.2} = 6\ N$$

$$\mu = \frac{4Wb^3}{3\pi u LD^3} = \frac{4\times 6\times 0.00124^3}{3\times\pi\times 0.5\times 10^{-3}\times 0.096\times 0.06^3} = \mathbf{0.4683\ Ns/m^2}$$

12.11 ❐ MEASUREMENT OF VISCOSITY (VISCOMETERS)

The apparatus used to determine the viscosity of fluid is called a viscometer. In this section, some of the viscometers, namely (i) capillary tube viscometer, (ii) rotating cylinder viscometer, (iii) falling sphere viscometer and (iv) efflux (orifice type) viscometer are discussed. The viscometers are based on three principles, namely Newton's law of viscosity, Hagen-Poiseuille law and Stoke's law.

12.11.1 Capillary Tube Viscometer

This viscometer uses the Hagen-Poiseuille law for determining the viscosity of the liquid. A capillary tube viscometer shown in Figure 12.19 consists of a capillary tube attached horizontally very close to the bottom of a tank filled with the liquid whose viscosity is to be measured. The level of the liquid in the tank is maintained constant so as to ensure steady flow through the tube. The liquid passing through the capillary tube is collected in a measuring tank for a given period and thus, the rate of flow of the liquid is measured. The pressure head is measured by a piezometer at a point far away from the tank.

Let h be the difference of pressure head for the length L of the capillary tube, D be the diameter of the tube, Q be the rate of discharge through the tube, ρ be the density of the liquid and μ be the viscosity of the liquid. Recalling Hagen-Poiseuille equation, we get the following expression.

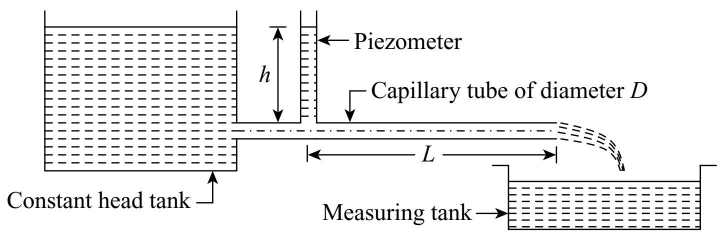

Figure 12.19 *Capillary tube viscometer*

$$h = \frac{32\mu VL}{\rho g D^2}$$

But
$$V = \frac{Q}{A} = \frac{Q}{(\pi/4)D^2}$$

Thus
$$h = \frac{32\mu L}{\rho g D^2} \times \frac{Q}{(\pi/4)D^2} = \frac{128\mu QL}{\pi \rho g D^4}$$

$$\boxed{\therefore \mu = \frac{\pi \rho g h D^4}{128QL}} \tag{12.55}$$

It can be seen from Equation (12.55) that the diameter D is raised to the fourth power, whereas all other variables occur as first power. Thus, any error in the measurement of the diameter D would significantly change the result and hence, the diameter of the capillary tube should be measured accurately.

Example 12.23 A capillary tube viscometer of diameter 40 mm is used to measure the viscosity of an oil of specific gravity 0.86. The difference of pressure head between two points 1.4 m apart is 0.4 m of water. If the mass of oil collected in a measuring tank is 50 kg in 100 seconds, then determine the viscosity of the oil.

Solution

Let $D = 40 \text{ mm} = 0.04 \text{ m}$, $S_{\text{oil}} = 0.86$, $L = 1.4 \text{ m}$, $h = 0.4 \text{ m}$, $\text{mass} = 50 \text{ kg}$ and $T = 100 \text{ s}$. Let m be the mass of oil per second, ρ be its density and Q be its discharge.

$$m = \frac{\text{mass}}{T} = \frac{50}{100} = 0.5 \text{ kg/s}$$

$$\rho = S_{\text{oil}}\,\rho_w = 0.86 \times 1000 = 860 \text{ kg/m}^3$$

$$Q = \frac{m}{\rho} = \frac{0.5}{860} = 0.0005814 \text{ m}^3/\text{s}$$

$$\mu = \frac{\pi \rho g h D^4}{128QL} = \frac{\pi \times 860 \times 9.81 \times 0.4 \times 0.04^4}{128 \times 0.0005814 \times 1.4} = \mathbf{0.2605 \text{ Ns/m}^2}$$

12.11.2 Rotating Cylinder Viscometer

The rotating cylinder viscometer uses Newton's law of viscosity for determining the viscosity of a fluid. It consists of two coaxial cylinders of radii R_1 and R_2 in such a way that the annular space $(R_2 - R_1) = b$ is left between the cylinders on the side and t at the bottom as shown in Figure 12.20. The annular space is filled with the liquid whose viscosity is to be determined. The outer cylinder is rotated at a fixed speed N while the inner cylinder is kept stationary and is suspended by a torsion wire.

The torque generated due to the rotation of the outer cylinder is transmitted to the inner cylinder through the oil film in the annular space. This causes the rotation of the torsion wire which can be measured by attaching a dial and pointer to it.

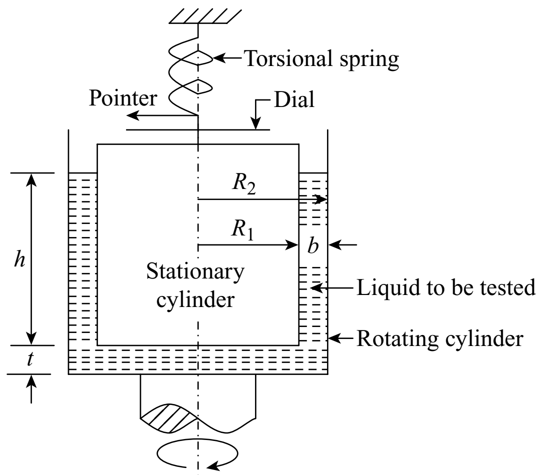

Figure 12.20 *Rotating cylinder viscometer*

Let T be the total torque exerted on the inner cylinder, T_b be the torque due to liquid in annular space, T_t be the torque due to liquid at the bottom, h be depth of liquid in the cylinder and u be the tangential velocity.

$$T_b = (\text{Shear stress} \times \text{Area}) \times \text{Radius} = (\tau \times 2\pi R_1 h) \times R_1 = \mu \frac{du}{dy} \times 2\pi R_1 h R_1$$

Since the thickness of oil film is very small, a linear velocity distribution can be assumed.

$$\frac{du}{dy} = \frac{u}{b} = \frac{\pi D_2 N}{60b} = \frac{2\pi R_2 N}{60b}$$

Thus $$T_b = \mu \frac{2\pi R_2 N}{60b} \times 2\pi R_1 h R_1 = \frac{\mu \pi^2 N R_1^2 R_2 h}{15b}$$

$$T_t = \frac{\mu \pi^2 N R_1^4}{60t} \quad \text{[Using Equation (12.50)]}$$

Thus $$T = T_b + T_t = \frac{\mu \pi^2 N R_1^2 R_2 h}{15b} + \frac{\mu \pi^2 N R_1^4}{60t} = \frac{\mu \pi^2 N}{15}\left[\frac{R_1^2 R_2 h}{b} + \frac{R_1^4}{4t}\right]$$

Thus $$\mu = \frac{T}{\frac{\pi^2 N}{15}\left[\frac{R_1^2 R_2 h}{b} + \frac{R_1^4}{4t}\right]} \tag{12.56}$$

Example 12.24 The radii of the outer and inner cylinders in a rotating cylinder viscometer are 0.04 m and 0.038 m, respectively. The outer cylinder rotates at 300 rpm. The annular space between the cylinders is filled with a liquid up to a height of 0.125 m and the clearance at the bottom of the two cylinders is 0.003 m. If the torque produced on the inner cylinder is 0.0015 Nm, then determine the viscosity of the liquid.

Solution

Let $R_2 = 0.04$ m, $R_1 = 0.038$ m, $N = 300$ rpm, $h = 0.125$ m, $t = 0.003$ m and $T = 0.0015$ Nm.

$$b = R_2 - R_1 = 0.04 - 0.038 = 0.002 \text{ m}$$

$$\mu = \frac{T}{\frac{\pi^2 N}{15}\left[\frac{R_1^2 R_2 h}{b} + \frac{R_1^4}{4t}\right]} = \frac{0.0015}{\frac{\pi^2 \times 300}{15} \times \left[\frac{0.038^2 \times 0.04 \times 0.125}{0.002} + \frac{0.038^4}{4 \times 0.003}\right]}$$

$$\therefore \mu = \mathbf{2.01 \times 10^{-3}\ Ns/m^2}$$

12.11.3 Falling Sphere Viscometer

This method of measuring the viscosity of a liquid is based on the Stoke's law. A falling sphere viscometer consists of a tall vertical transparent tube which is filled with the liquid whose viscosity is to be determined. This tube is surrounded by a constant temperature bath tub as shown in Figure 12.21.

A small spherical steel ball is released to fall vertically into the test liquid. The time (T) to traverse a known vertical distance (L) between two fixed marks on the tube is noted to calculate the terminal velocity, $V = (L/T)$ of the ball. For the equilibrium of the ball under steady state condition, the drag force (F_D) and the buoyancy force (F_B) must be added to balance the weight of the ball (W).

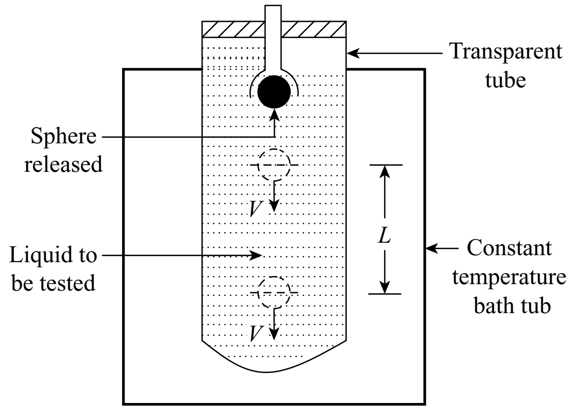

Figure 12.21 *Falling sphere viscometer*

According to Stoke's law, the expression for drag force (F_D) on a ball of diameter d moving at a velocity V in a liquid of viscosity μ is given below.

$$F_D = 3\pi\mu Vd \tag{12.57}$$

It may be noted that the equation for drag used above is valid only for Re < 1.

Let ρ_l be the density of the liquid and ρ_s be the density of the steel ball.

$$F_B = \text{Volume of liquid displaced} \times \text{Density of liquid} \times g = \frac{\pi}{6}d^3 \times \rho_l \times g$$

$$W = \text{Volume of sphere} \times \text{Density of sphere} \times g = \frac{\pi}{6}d^3 \times \rho_s \times g$$

$$F_D + F_B = W \qquad \text{[For equilibrium]}$$

Thus

$$3\pi\mu Vd + \frac{\pi}{6}d^3\rho_l g = \frac{\pi}{6}d^3\rho_s g$$

$$3\pi\mu Vd = \frac{\pi}{6}d^3(\rho_s - \rho_l)g$$

$$\boxed{\therefore \mu = \frac{gd^2(\rho_s - \rho_l)}{18V}} \tag{12.58}$$

The Equation (12.58) is true only for the motion of small sphere in viscous fluids. The falling sphere viscometer measures viscosity accurately only if Stoke's law is applicable.

Example 12.25 In a falling sphere viscometer, a lubricating oil of density 860 kg/m^3 is filled in 0.1 m inside the diameter tube. A steel sphere of diameter 5 mm is made to fall 1 m in 25 seconds. If the density of the sphere is 8500 kg/m^3, then find the viscosity of the oil.

Solution

Let $\rho_l = 860$ kg/m^3, $D = 0.1$ m, $d = 5$ mm $= 0.005$ m, $L = 1$ m, $T = 25$ s and $\rho_s = 8500$ kg/m^3.

$$V = \frac{L}{T} = \frac{1}{25} = 0.04 \text{ m/s}$$

$$\mu = \frac{gd^2(\rho_s - \rho_l)}{18V} = \frac{9.81 \times 0.005^2 \times (8500 - 860)}{18 \times 0.04} = \mathbf{2.6024\ Ns/m^2}$$

12.11.4 Efflux Viscometer

An efflux viscometer mainly consists of (i) a container maintained at a required temperature and (ii) an aperture (an orifice, a nozzle or a capillary) through which a fixed volume of liquid whose viscosity is to be determined is drained out. The time taken to discharge the fixed volume of liquid under standard condition is taken as a measure of the viscosity of the liquid. These viscometers are calibrated with some standard liquid and the viscosity is expressed in terms of ratios of time taken with respect to the standard liquid. The Saybolt viscometer, Redwood viscometer and Eagler viscometer works based on this principle.

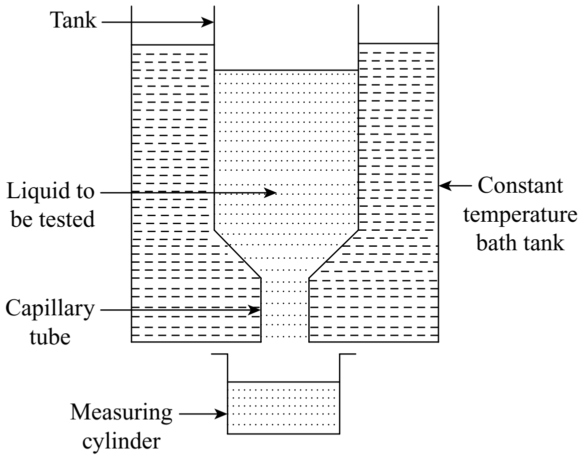

Figure 12.22 *Saybolt viscometer*

Figure 12.22 shows a schematic view of the Saybolt viscometer which consists of a tank having a short capillary tube at its bottom. This tank contains the liquid whose viscosity is to be determined and it is surrounded by another tank called constant temperature bath. The time taken by 60 cm^3 of liquid to pass through the capillary tube at a standard temperature is noted down. The initial height of the liquid in the tank is fixed at a predetermined height. From time measurement, the kinematic viscosity of the liquid can be known by the use of empirical formula or the calibration curve. The relation between kinematic viscosity (ν) in stokes and the time (T) in seconds for a Saybolt viscometer is given below.

$$\boxed{\nu = 0.24T - \frac{190}{T}} \tag{12.59}$$

Summary

1. **Laminar flow:** The fluid particles move along straight parallel paths in layers.
2. **Reynolds number (Re):** The ratio of inertia force to viscous force.
3. For laminar flow: $\text{Re} < 2000$, (ii) for turbulent flow: $\text{Re} > 4000$ and (iii) for transitional flow: $2000 < \text{Re} < 4000$.
4. In a steady uniform laminar flow, the pressure gradient ($\partial p/\partial x$) in the direction of flow is equal to the shear stress gradient ($\partial \tau/\partial y$) in the normal direction.
5. **For laminar flow in circular pipes:**
   (i) **Shear stress:** $\tau = -\frac{\partial p}{\partial x}\frac{r}{2}$
   (ii) **Velocity:** $u = -\frac{1}{4\mu}\frac{\partial p}{\partial x}R^2\left[1 - \left(\frac{r}{R}\right)^2\right]$
   (iii) **Ratio of maximum velocity (u_{max}) to average velocity (V):** $\frac{u_{max}}{V} = 2$
   (iv) **Drop in pressure head:** $h_f = \frac{p_1 - p_2}{\rho g} = \frac{32\mu VL}{\rho g D^2}$
   (v) **Discharge:** $Q = \frac{\pi}{8\mu}\left(-\frac{\partial p}{\partial x}\right)R^4$

   Here, r is the radius at any point, R is the radius of the pipe, D is the diameter of the pipe, L is the length of the pipe, ρ is the density of the liquid and μ is the viscosity of the liquid.
6. **For laminar flow through annulus:**
   (i) **Velocity:** $u = -\frac{1}{4\mu}\frac{\partial p}{\partial x}\left[R_1^2 - r^2 - \frac{(R_1^2 - R_2^2)}{\ln(R_1/R_2)}\ln\left(\frac{R_1}{r}\right)\right]$

(ii) **Shear stress:** $\tau = -\frac{1}{4}\frac{\partial p}{\partial x}\left[2r - \frac{1}{r}\frac{(R_1^2 - R_2^2)}{\ln(R_1/R_2)}\right]$

(iii) **Discharge:** $Q = -\frac{\pi}{8\mu}\frac{\partial p}{\partial x}\left[R_1^4 - R_2^4 - \frac{(R_1^2 - R_2^2)^2}{\ln(R_1/R_2)}\right]$

(iv) **Averagevelocity:** $V = -\frac{1}{8\mu}\frac{\partial p}{\partial x}\left[R_1^2 + R_2^2 - \frac{(R_1^2 - R_2^2)}{\ln(R_1/R_2)}\right]$

Here, r is the radial distance at any point, R_1 is the outer radius and R_2 is the inner radius of the annulus.

7. **For laminar flow between two stationary parallel plates:**

(i) **Velocity:** $u = -\frac{1}{2\mu}\frac{\partial p}{\partial x}(by - y^2)$

(ii) **Maximum velocity:** $u_{\max} = -\frac{1}{8\mu}\frac{\partial p}{\partial x}b^2$

(iii) **Shear stress:** $\tau = -\frac{1}{2}\frac{\partial p}{\partial x}(b - 2y)$

(iv) **Discharge:** $Q = -\frac{1}{12\mu}\frac{\partial p}{\partial x}b^3$

(v) **Average velocity:** $V = -\frac{1}{12\mu}\frac{\partial p}{\partial x}b^2$

(vi) **Ratio of maximum velocity to average velocity:** $\frac{u_{\max}}{V} = \frac{3}{2}$

(vii) **Drop in pressure head:** $\frac{p_1 - p_2}{\rho g} = h_f = \frac{12\mu VL}{\rho g b^2}$

Here, y is the distance at any point and b is the distance between the plates.

8. **For Couette flow:**

(i) **Velocity:** $u = \frac{V}{b}y - \frac{1}{2\mu}\frac{\partial p}{\partial x}(by - y^2)$

(ii) **Shear stress:** $\tau = \frac{\mu V}{b} - \frac{1}{2}\frac{\partial p}{\partial x}(b - 2y)$

(iii) **Discharge:** $Q = \frac{Vb}{2} - \frac{1}{12\mu}\frac{\partial p}{\partial x}b^3$

(iv) **Average velocity:** $u_{av} = \frac{V}{2} - \frac{1}{12\mu}\frac{\partial p}{\partial x}b^2$

(v) **Maximum velocity:** $u_{\max} = \frac{(1+\phi)^2}{4\phi}V$

Here, y is the distance at any point, b is the distance between the plates, V is the velocity of the upper plate and $\phi = \frac{b^2}{2\mu V}\left(-\frac{\partial p}{\partial x}\right)$ is a constant.

9. **Power (P) absorbed in journal bearing:** $P = \frac{\mu\pi^3 D^3 N^2 L}{3600t}$ Watts, here D is the diameter of the shaft, N is the speed of the shaft in rpm, L is the length of the bearing and t is the thickness of the oil film.

10. Power absorbed in foot step bearing is $P = \frac{\mu\pi^3 N^2 R^4}{1800t}$ Watts, here R is the radius of the shaft, N is the speed of the shaft in rpm, μ is the viscosity of the oil and t is the thickness of the oil film.

11. Power absorbed in collar bearing is $P = \frac{\mu\pi^3 N^2 (R_2^4 - R_1^4)}{1800t}$ Watts, here R_1 is the internal radius of the collar, R_2 is the external radius of the collar, N be the speed of the shaft in rpm and t is the thickness of the oil film.

12. The viscosity determined by dash pot is $\mu = \frac{4Wb^3}{3\pi u L D^3}$, here D is the diameter of the piston, L is the length of the piston, W is the weight of the piston, u is the velocity of the piston and b is the clearance between the piston and the dashpot.

13. The viscosity determined by capillary tube viscometer is $\mu = \frac{\pi\rho g h D^4}{128QL}$, here h is the difference of pressure head for length L of the capillary tube, D is the diameter of the tube and Q is the rate of discharge through the tube.

14. Viscosity by rotating cylinder viscometer is $\mu = T/\left[\frac{\pi^2 N}{15}\left\{\frac{R_1^2 R_2 h}{b} + \frac{R_1^4}{4t}\right\}\right]$, here R_1 and R_2 are the radii such that $(R_2 - R_1) = b$ is the space left between the cylinders on the side and t at the bottom, N is the speed of outer cylinder, T is the total torque exerted on the inner cylinder and h is the depth of liquid in the cylinder.

15. Viscosity by falling sphere viscometer is $\mu = \frac{gd^2(\rho_s - \rho_l)}{18V}$, here d is the diameter of the ball, V is the velocity of the ball in the liquid, ρ_s is the density of the steel ball and ρ_l is the density of the liquid.

16. The relation between kinematic viscosity (ν) in stokes and the time (T) in seconds for a Saybolt viscometer is $\nu = 0.24T - (190/T)$.

Multiple-choice Questions

1. In laminar flow through a pipe
   (a) Velocity varies.
   (b) Reynolds number < 2000.
   (c) Reynolds number > 2000.
   (d) Fluid particles move in a zigzag manner.
2. The velocity distribution for steady laminar flow in a pipe is
   (a) Exponential.
   (b) Parabolic.
   (c) Linear.
   (d) None of the above.
3. The ratio of maximum velocity to average velocity in a laminar flow through a pipe is
   (a) 0.6
   (b) 1.3
   (c) 1.67
   (d) 2.
4. At what radius (r) from the centre of a pipe of radius (R), the average velocity occurs?
   (a) $r = R/\sqrt{2}$.
   (b) $r = \sqrt{3}R$.
   (c) $r = \sqrt{2/3}R$.
   (d) None of the above.
5. For a laminar flow through a pipe, the loss of head
   (a) Is directly proportional to viscosity.
   (b) Varies directly as the length of the pipe.
   (c) Is directly proportional to the velocity.
   (d) Varies as the square of the diameter of the pipe.
6. For a dashpot system, the piston velocity varies directly as
   (a) Viscosity of liquid.
   (b) Square of piston diameter.
   (c) Clearance between the piston and cylinder.
   (d) Load on the piston.
7. For viscous flow, the coefficient of friction (f) in terms of Reynolds number (Re) is equal to
   (a) $64/\text{Re}$.
   (b) $32/\text{Re}$.
   (c) $16/\text{Re}$.
   (d) $8/\text{Re}$.
8. In a laminar flow between two stationary parallel plates, the ratio of maximum velocity to average velocity is equal to
   (a) 1
   (b) 1.5
   (c) 2
   (d) 2.5.

Review Questions

1. Define viscous flow and explain the Reynolds experiment with a neat sketch?
2. Derive Navier-Stokes equations.
3. Derive the relationship between shear stress and pressure gradient.
4. Derive expressions for the velocity and shear stress distributions for laminar flow through a circular pipe.
5. Prove that the maximum velocity in a circular pipe for laminar flow is twice the average velocity of the flow.
6. Obtain an expression for the Hagen-Poiseuille equation.
7. Derive expressions for (i) velocity distribution, (ii) shear stress, (iii) discharge, and (iv) average velocity of flow through an annulus.
8. Derive expressions for velocity and shear stress distributions for laminar flow between two stationary parallel plates.
9. Obtain expression for the difference of pressure head for a given length of the laminar flow between two parallel fixed plates.
10. Define the Couette flow and derive expressions for the velocity and shear stress distributions in it.
11. Derive expressions for the discharge, average velocity and maximum velocity for the Couette flow.
12. Derive expressions for the power absorbed in overcoming viscous resistance in (i) journal bearing and (ii) foot step bearing.
13. Derive an expression for the power absorbed in overcoming viscous resistance in a collar bearing.
14. Derive an expression for the viscosity measured in a dashpot mechanism.
15. Explain capillary tube viscometer with a neat sketch. Also derive an expression for the coefficient of viscosity measured by it.
16. Derive an expression for the coefficient of viscosity measured by rotating cylinder viscometer.
17. Briefly explain the falling sphere viscometer. Also derive an expression for the coefficient of viscosity measured by it.
18. Write short note on efflux viscometers.

Problems

1. An oil of viscosity 0.097 Ns/m^2 and specific gravity 0.9 flows through a horizontal circular pipe of diameter and length 10 cm and 10 m, respectively. If 50 kg of the oil is collected in 15 seconds, then find the difference of pressure at the two ends of the pipe.

[**Ans.** 1461.98 N/m^2]

2. A laminar flow occurs in a pipe of diameter 400 mm. If the maximum velocity is 2 m/s, then determine the average velocity and the radius at which it occurs. Also determine the velocity at 10 mm from the wall of the pipe.

[**Ans.** 1 m/s, 0.1414 m, 0.195 m/s]

3. A fluid of specific gravity 1.2 and viscosity 5 poise flows through a pipe of diameter 0.1 m. If the maximum shear stress at the pipe wall is 147.15 N/m^2, then determine the pressure gradient, average velocity and Reynolds number of flow.

[**Ans.** –5.89 kN/m^3, 3.68 m/s, 883.2]

4. A crude oil of viscosity 1 Ns/m^2 and relative density 0.6 flows through a horizontal pipe of diameter 3 cm. If the pressure drop in 50 m length of the pipe is 3200 kPa, then determine (i) the discharge of oil, (ii) centre line velocity, (iii) wall shear stress and the total drag over 50 m length of pipe, (iv) power required to maintain the flow and (v) velocity gradient at the pipe wall.

[**Ans.** 0.001272 m^3/s, 3.6 m/s, 480 N/m^2, 2261.95 N, 4.0715 kW, 480 s^{-1}]

5. An oil of viscosity 1.5 poise and relative density 0.9 flows through a horizontal pipe of diameter 5.5 cm and length 325 m. If the oil flow rate through the pipe is 0.0037 m^3/s, then determine (i) the pressure drop in a length of 325 m and (ii) shear stress at the wall.

[**Ans.** 804.496 kPa, 34.04 N/m^2]

6. An oil of viscosity 1.2 poise and relative density 0.8 flows through a horizontal pipe of diameter 60 cm and length 500 m. If the oil flow rate through the pipe is 5 litres per second, then determine the shear stress at the wall.

[**Ans.** 0.02832 N/m^2]

7. The glycerine having viscosity as 8 poise flows through a horizontal pipe of diameter 10 cm at a rate of 10 litres per second. Determine the power required per kilometre of pipeline to overcome the viscous resistance.

[**Ans.** 32.594 kW]

8. An oil of viscosity 0.1 Ns/m^2 and specific gravity 0.85 is pumped through a 3 cm diameter pipe. If the pressure drop per metre length of pipe is 20 kPa, then determine (i) the mass flow rate, (ii) shear stress at the pipe wall, (iii) Reynolds number and (iv) power required per 50 m length of the pipe to maintain the flow.

[**Ans.** 3.38 kg/s, 150 Pa, 1434.375, 3.974 kW]

9. A pipe of diameter 0.2 m and length 10000 m slopes upwards at a slope of 1 in 200 m of pipe length traversed. An oil of viscosity 0.15 Ns/m^2 and specific gravity 0.85 is required to be discharged through it at the rate of 0.025 m^3/s. Determine (i) the head lost due to friction and (ii) power required to drive the pump.

[**Ans.** 114.55 m, 34.3 kW]

10. An oil of viscosity 0.09 Ns/m^2 and relative density 0.85 is flowing through an inclined pipe of 20 mm diameter. Determine the inclination of the pipe when a discharge of 0.014 m^3/s is maintained through the pipe in such a way that pressure along the length remains constant.

[**Ans.** 39.9°]

11. Water flows in a pipe of diameter 0.2 m. The shear stress at a point 3 cm from the pipe axis is 0.12 kPa. If the coefficient of friction is 0.04, then determine the Reynolds number and the shear stress at the pipe wall.

[**Ans.** 400, 0.4 kPa]

12. Water flows between two large parallel plates at a distance of 0.2 cm apart. If the dynamic viscosity of water is 0.002 Ns/m^2, then determine the maximum velocity, the pressure drop per unit length and the shear stress at the walls of the plates if the average velocity is 0.25 m/s.

[**Ans.** 0.375 m/s, 1500 Pa/m, 1.5 Pa]

13. Oil flows between two parallel fixed plates of width 17.5 cm kept at a distance of 7.5 cm apart. If the drop of pressure in a length of 1.25 m is 4 kPa and the dynamic viscosity of oil is 1.5 Ns/m^2, then determine the discharge of oil in litres per second between the plates.

[**Ans.** 13.125 *l*/s]

14. A laminar flow of an oil of dynamic viscosity 25 poise is maintained between two horizontal parallel fixed plates kept at a distance of 15 cm apart. If maximum velocity of the oil is 2.5 m/s, then determine the pressure gradient, the shear stress at the plates and the discharge per metre width for the flow of oil.

[**Ans.** –2222.22 Pa/m, 166.67 Pa, 0.25 m^3/s]

15. An oil of viscosity 0.5 poise flows between two stationary parallel plates 100 cm wide maintained 2 cm apart. If the velocity midway between the plates is 2.5 m/s, then determine the pressure gradient along the flow, the average velocity and the discharge of the oil.

[**Ans.** 2.5 kPa/m, 1.67 m/s, 0.0334 m^3/s]

16. A 10 cm thick wall has 5 cm wide and 0.2 cm deep horizontal crack. If the difference of pressure between the two ends of the crack is 2950 Pa and viscosity of water is 0.001 Ns/m^2, then determine the rate of leakage of water in litres per second through the crack.

[**Ans.** 0.983 litres/s]

17. Determine the direction and amount of flow per metre width between two parallel plates when one is moving relative to the other with a velocity of 3 m/s in the negative direction, if $(\partial p/\partial x) = -100$ MN/m^2 per m, viscosity is 0.4 poise, and the distance between the plates is 1 mm.

[**Ans.** 0.2068 m^3/s]

18. A shaft of 0.1 m diameter rotates at 60 rpm in a 0.2 m long bearing. Find the power absorbed in the bearing when the two surfaces are uniformly separated by a distance of 0.5 mm and assuming linear velocity distribution in the lubricating oil having dynamic viscosity of 0.06 poise.

[**Ans.** 0.0744 watts]

19. A shaft of diameter 0.06 m rotates centrally at a speed of 750 rpm in a journal bearing of 0.0602 m diameter and length 0.1 m. If the annular space between the shaft and the bearing is filled with oil of 0.85 poise viscosity, then determine the power absorbed in the bearing.

[**Ans.** 88.95 watts]

20. A shaft of diameter 0.1 m rotates at a speed of 35 rpm centrally in a journal bearing of length 0.2 m. If the annular space between the shaft and the bearing is 0.02 mm and the power absorbed in the bearing is 200 Watts, then find the viscosity of the oil.

[**Ans.** 1.896 Ns/m^2]

21. Determine the power required to rotate a vertical shaft of diameter 100 mm at 750 rpm. The lower end of the shaft rests in a foot step bearing. The end of the shaft and surface of the bearing are both flat and are separated by an oil film of thickness 0.5 mm. The viscosity of the oil is 1.5 poise.

[**Ans.** 18.166 W]

22. A circular disc of diameter 0.24 m rotates at 750 rpm. If the disc has a clearance of 0.5 mm from the bottom flat plate and the clearance is filled with an oil of viscosity 1.5 poise, then determine the power required to rotate the disc.

[**Ans.** 602.762 W]

23. The internal and external diameters of a collar bearing are 0.2 m and 0.3 m, respectively. An oil film thickness of 0.25 mm and of viscosity 0.8 poise is maintained between the collar surface and the bearing. If the shaft rotates at 150 rpm, then find the torque and power lost in overcoming the viscous resistance.

[**Ans.** 3.2076 Nm, 50.385 W]

24. An oil dashpot consists of a piston moving in a cylinder having oil is used for damping vibrations. The piston falls with a uniform speed and covers 50 mm in 100 seconds. If an additional weight of 1.34 N is placed on the top of the piston, then it falls through 50 mm in 86 seconds with uniform speed. The diameter of the piston is 75 mm and its length is 100 mm. The clearance between the piston and the cylinder is 1.2 mm which is uniform throughout. Find the viscosity of the oil in poise.

[**Ans.** 2.861 poise]

25. A capillary tube viscometer of diameter 2 mm and length 0.1 m measures viscosity of a liquid as 0.3 poise. If the difference of pressure between the two ends of the tube is 7 kPa, then determine the discharge through the capillary tube.

[**Ans.** 9.163×10^{-7} m^3/s]

26. A rotating viscometer is used for determining the viscosity of a liquid. The radii of inner and outer cylinders are 0.1 m and 0.1025 m, respectively. The liquid is filled in the annular space up to a height of 0.3 m and the clearance at the bottom of the cylinder is 0.005 m. If the outer cylinder rotates at 300 rpm and the torque was registered as 8 Nm, then determine the viscosity of the liquid.

[**Ans.** 0.3166 Ns/m^2]

27. A sphere of diameter 3 mm is made to fall 160 mm in 20 seconds in a viscous liquid of density 950 kg/m^3. If the density of the sphere is 7650 kg/m^3, then find the viscosity of the viscous liquid.

[**Ans.** 4.108 Ns/m^2]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (b)	**2.** (b)	**3.** (d)	**4.** (a)	**5.** (b)
6. (d)	**7.** (c)	**8.** (b)		

Chapter 13

Turbulent Flow in Pipes

13.1 ❐ INTRODUCTION

In pipes, turbulent flow occurs when Reynolds number (Re) is greater than 4000. Mostly, the flow in pipes is turbulent in which the fluid motion is irregular and chaotic. In turbulent flow, the fluid particles move haphazardly, remains in a state of disorder and develop large scale eddies (lump of particles) which causes complete mixing of the fluid. Therefore, in turbulent flow, the mass, momentum and heat transfer get enhanced.

In turbulent flow, the velocity distribution is relatively uniform and it tends to follow power law and logarithmic law. The velocity profile of turbulent flow is more flat than the corresponding laminar flow (Figure 13.1) and it becomes even flatter with increasing Reynolds number. In turbulent flow, the velocity gradient near the pipe wall is very large and therefore, shear stress at the wall of the pipe is very high. There are irregular velocity and pressure fluctuations due to which the analytical treatment of turbulent flow is extremely complicated. The velocity fluctuations cause an additional shear stress (or frictional resistance) to flow which is in addition to the viscous shear stress and it is known as turbulent shear stress.

The laminar flow is an idealization to get some analytical or numerical solution. In practical life, most of the engineering flows are turbulent, for example, flow in rivers, channels, flow past an obstruction, the rising smoke, etc. Simple mathematical relationship between frictional head loss for turbulent flow does not exist as flow is random. In this chapter, some semi-empirical theories developed for turbulent flow are discussed and some information on turbulence and turbulent flow in pipes has been provided.

13.2 ❐ LOSS OF HEAD IN PIPES (DARCY-WEISBACH EQUATION)

Consider a control volume between sections 1 and 2 of a horizontal pipe having a steady flow of any fluid as shown in Figure 13.1.

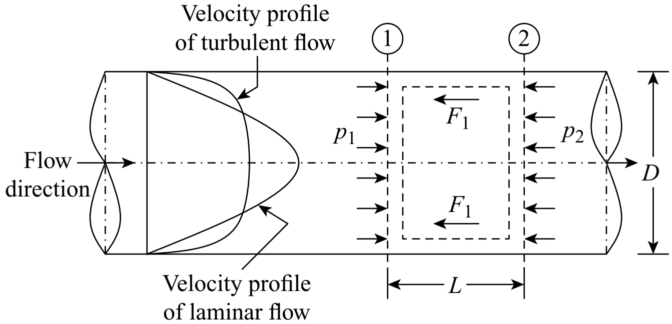

Figure 13.1 *Velocity distribution and forces on control volume in a pipe flow*

Let L be the length of the pipe between sections 1 and 2, D be the diameter of the pipe, A be the area of cross section of the pipe, V be the average flow velocity, f' be the dimensional parameter whose value depends upon the material and nature of the pipe surface, p_1 and p_2 be the intensities of pressure at sections 1 and 2, respectively, P be the wetted perimeter, τ_o be the shear stress on the pipe walls, $F_1 = \tau_o \times \pi DL$ be the frictional resistance force, h_f be the loss of head due to friction, ρ be the density and $w = \rho g$ be the weight density of flowing fluid.

There is a frictional resistance to the flow and thus, pressure continuously decreases along the length of the pipe as given in the below expression.

$$p_1 A = p_2 A + F_1$$

$$F_1 = (p_1 - p_2)A \qquad \text{(i)}$$

Experimentally, the frictional resistance force is found proportional to L, P and V^n, where n varies from 1.5 to 2 and for turbulent flow $n = 2$.

Thus $$F_1 = f' PLV^2 \qquad \text{(ii)}$$

Under equilibrium conditions, from expressions (i) and (ii), we get:

$$(p_1 - p_2)A = f' PLV^2$$

$$\frac{p_1 - p_2}{w} = \frac{f' PLV^2}{wA}$$

Since $$h_f = \frac{p_1 - p_2}{w}$$

Thus $$h_f = \frac{f'}{w}\left(\frac{P}{A}\right)LV^2 \qquad (13.1)$$

The ratio (A/P) is known as the hydraulic mean depth (or hydraulic radius) and it is denoted by m. Thus, by multiplying and dividing by $2g$, we get the below expression.

$$h_f = \frac{2gf'}{w}\left(\frac{LV^2}{2gm}\right)$$

As the term $[(LV^2)/(2gm)]$ has the same dimensions as that of h_f, the term $(2gf')/w$ is a non-dimensional quantity and it can be replaced by f.

Thus $$h_f = f\frac{LV^2}{2gm} \qquad (13.1a)$$

The hydraulic radius for a circular pipe is given by,

$$m = \frac{A}{P} = \frac{(\pi/4)D^2}{\pi D} = \frac{D}{4}$$

By substituting the above value of m in Equation (13.1a), we get:

$$\boxed{h_f = \frac{fLV^2}{2g(D/4)} = \frac{4fLV^2}{2gD}} \qquad (13.2)$$

The Equation (13.2) is known as Darcy-Weisbach equation. It is valid for all types of flows provided a proper value of f is chosen. The factor f is known as Darcy coefficient of friction (or coefficient of friction) and it can be expressed in terms of shear stress as given below.

$$(p_1 - p_2)A = F_1 = \tau_o \pi DL$$

$$p_1 - p_2 = \frac{\tau_o \pi DL}{A} = \frac{\tau_o \pi DL}{(\pi/4)D^2} = \frac{4\tau_o L}{D}$$

or

$$\frac{p_1 - p_2}{w} = \frac{4\tau_o L}{wD}$$

Thus

$$h_f = \frac{4\tau_o L}{wD} = \frac{4\tau_o L}{\rho g D} \tag{13.2a}$$

$$\frac{4fLV^2}{2gD} = \frac{4\tau_o L}{\rho g D} \quad \text{[Equating (13.2) and (13.2a)]}$$

Thus

$$\tau_o = \frac{f\rho V^2}{2} \tag{13.3}$$

$$f = \frac{2\tau_o}{\rho V^2}$$

The above equation can also be rearranged by multiplying both sides by 4 as given below.

$$4f = \frac{8\tau_o}{\rho V^2} \Rightarrow \sqrt{\frac{\tau_o}{\rho}} = V\sqrt{\frac{4f}{8}} = V\sqrt{\frac{f_f}{8}}$$

The quantity $\sqrt{\tau_o/\rho}$ has the dimensions of velocity. It is known as shear velocity (or friction velocity or friction shearing velocity) and it is denoted by u_s. In some books, it is denoted by u_* or u^*. In the above equation, f_f is friction factor which is equal to $4f$.

Thus

$$\boxed{u_s = \sqrt{\frac{\tau_o}{\rho}} = V\sqrt{\frac{4f}{8}} = V\sqrt{\frac{f_f}{8}}} \tag{13.4}$$

The Equation (13.4) holds good for both the smooth and rough pipes.

Example 13.1 The petrol of specific gravity 0.74 flows at a rate of 0.06 m^3/s through a pipe of length 1250 m and diameter 0.25 m. If the coefficient of friction is $f = 0.002$ in the Darcy-Weisbach equation, then determine (i) the loss of head due to friction, (ii) shear stress on the pipe wall, (iii) shear velocity and (iv) power required to maintain the flow.

Solution

Let $S_{\text{petrol}} = 0.74$, $Q = 0.06$ m^3/s, $L = 1250$ m, $D = 0.25$ m and $f = 0.002$.

$$V = \frac{Q}{A} = \frac{Q}{(\pi/4)D^2} = \frac{0.06}{(\pi/4)\times 0.25^2} = 1.222 \text{ m/s}$$

(i) $h_f = \dfrac{4fLV^2}{2gD} = \dfrac{4\times 0.002\times 1250\times 1.222^2}{2\times 9.81\times 0.25} = \mathbf{3.044\ m}$

(ii) $\rho = S_{\text{petrol}}\,\rho_w = 0.74\times 1000 = 740 \text{ kg/m}^3$

$$\tau_o = \frac{f\rho V^2}{2} = \frac{0.002\times 740\times 1.222^2}{2} = \mathbf{1.105\ N/m^2}$$

(iii) $u_s = V\sqrt{\frac{4f}{8}} = 1.222 \times \sqrt{\frac{4 \times 0.002}{8}} = \mathbf{0.0386\ m/s}$

(iv) $P = \frac{\rho g Q h_f}{1000} = \frac{740 \times 9.81 \times 0.06 \times 3.044}{1000} = \mathbf{1.326\ kW}$

13.3 ❐ CHARACTERISTICS OF TURBULENT FLOW (TURBULENCE)

13.3.1 Classification of Turbulence

Turbulence is the disturbance that causes transfer of fluid particles from one region to the other and it is three-dimensional in character. It can be generated by the flow of fluid layers with different velocities over one another or by frictional forces at the solid walls. Turbulence can be classified into the following main groups.

1. **Wall turbulence:** The turbulence generated and affected by stationary wall is called wall turbulence. It occurs in the boundary layer flows and in the vicinity of solid surface.
2. **Free turbulence:** The turbulence generated by two adjacent layers of fluid away from the solid boundary is called free turbulence. It occurs in wakes, jets, mixing layers and in the outer part of boundary layer flows.
3. **Homogeneous turbulence:** Turbulence having the same structure quantitatively in all parts of the flow field is called homogeneous turbulence.
4. **Isotropic and anisotropic turbulence:** Turbulence in which the statistical features have no directional preference and the gradient of the mean velocity does not exist is called isotropic turbulence. However, if the mean velocity has a gradient, then the turbulence is known as anisotropic. Isotropic turbulence will always be homogeneous.
5. **Convective turbulence:** Turbulence due to conversion of potential energy into kinetic energy by the process of mixing is called convective turbulence. It occurs in the turbulent flow in the annular space between the concentric rotating cylinders.
6. **Fully developed turbulence:** When the flow as a whole approaches to a certain invariant state in terms of appropriate variables, then the turbulence is called fully developed.
7. **Fine and large scale turbulence:** When the size of eddies are small, then the turbulence is called fine scale turbulence and when the size of eddies are large, then the turbulence is called large scale turbulence.

13.3.2 Mean and Fluctuating Velocities

The instantaneous velocity (u_i) at any point in the turbulent flow in x-direction is made up of average velocity component (time mean component) (u) and a randomly varying velocity component (u'), i.e., $u_i = u + u'$. Similarly, $v_i = v + v'$ and $w_i = w + w'$ are the velocity components in y-direction and z-direction, respectively. The average velocity components u, v and w are the functions of position only, whereas the fluctuating velocity components u', v' and w' vary with time (t) also. The average velocity is also known as temporal velocity. The fluctuating velocity components may be positive or negative as shown in Figure 13.2.

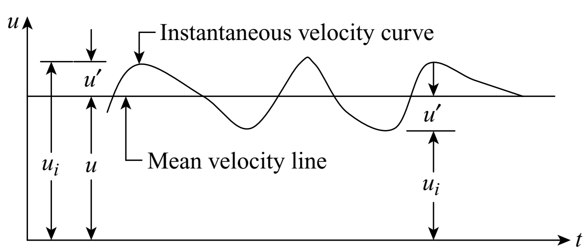

Figure 13.2 *Mean and fluctuating velocities*

The expression for average velocity u and the time average of the random component $\overline{u'}$ over a long interval of time t is respectively given below.

$$u = \frac{1}{t}\int_0^t u_i dt \tag{13.5}$$

$$\overline{u'} = \frac{1}{t}\int_0^t u' dt = \frac{1}{t}\int_0^t (u_i - u) dt = \frac{1}{t}\int_0^t u_i dt - \frac{1}{t}\int_0^t u dt = u - u = 0 \tag{13.6}$$

Similarly,

$$v = \frac{1}{t}\int_0^t v_i dt,\ w = \frac{1}{t}\int_0^t w_i dt,\ \overline{v'} = 0 \text{ and } \overline{w'} = 0$$

Thus, it can be seen that the time average of the fluctuating velocities are zero. Like velocity, the other properties, such as pressure, temperature and density also fluctuate and their instantaneous values can be respectively given as $p_i = p + p'$, $T_i = T + T'$ and $\rho_i = \rho + \rho'$.

13.3.3 Degree and Intensity of Turbulence

The degree of turbulence (or magnitude of turbulence) can be expressed in terms of root mean square of velocity fluctuations $\overline{u'^2}$, $\overline{v'^2}$ and $\overline{w'^2}$ as given below.

$$\text{Degree of turbulence} = \sqrt{\frac{1}{3}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)} \tag{13.7}$$

Here, the mean square of the velocity fluctuations is given below.

$$\overline{u'^2} = \frac{1}{t}\int_0^t u'^2 dt,\ \overline{v'^2} = \frac{1}{t}\int_0^t v'^2 dt \text{ and } \overline{w'^2} = \frac{1}{t}\int_0^t w'^2 dt \tag{13.8}$$

Intensity of turbulence is a measure of the level of turbulence. It can be obtained as the ratio of the degree of turbulence to the average flow velocity, $V = \sqrt{u^2 + v^2 + w^2}$ at a point in the turbulent flow field as given below.

$$\text{Intensity of turbulence} = \frac{1}{V}\sqrt{\frac{1}{3}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)} \tag{13.9}$$

For a unidirectional flow, $v = w = 0$ and $V = u$ and thus, Equation (13.9) is written as follows.

$$\text{Intensity of turbulence} = \frac{1}{u}\sqrt{\frac{1}{3}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)} \tag{13.10}$$

The intensity of turbulence is observed to increase with the increase in velocity fluctuations.

13.3.4 Scale of Turbulence

In addition to the intensity of turbulence, the scale of turbulence, i.e., the size of eddy is also required for the analysis of turbulent flow. The mean time interval between reversals in the sign of u', v' and w' gives a measure of the size of eddies. The average size of the eddy which is equal to the product of mean velocity (u) and the average time interval (dt) is known as the scale of turbulence in the x-direction. The relation between these factors is mathematically expressed as follows.

$$\text{Scale of turbulence} = u dt$$

The scale of turbulence depends upon the boundary conditions. The energy dissipation in turbulent flow depends upon the intensity of turbulence and the size of the eddy. Energy dissipation will be higher with high intensity of turbulence and smaller eddy size.

13.3.5 Kinetic Energy of Turbulence

The kinetic energy of turbulence per unit mass is given by,

$$\text{K.E. of turbulence/Mass} = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right) \tag{13.11}$$

The kinetic energy of turbulence in a boundary layer is maximum near the wall and it decreases towards the free stream.

13.3.6 Reynolds Equations of Turbulence

The continuity equation for a steady incompressible flow is given by,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

When we incorporate the mean and fluctuating components in the above equation, it becomes the continuity equation for the turbulent flow as given below.

$$\frac{\partial (u+u')}{\partial x} + \frac{\partial (v+v')}{\partial y} + \frac{\partial (w+w')}{\partial z} = 0 \tag{i}$$

The following two rules are used for applying the time average process.

(i) Integration with respect to time is independent of differentiation with respect to space coordinates.

(ii) According to Equation (13.5), $\frac{1}{t}\int_0^t u_i dt = \frac{1}{t}\int_0^t (u+u')\, dt = u.$

Time average of first term $\frac{\partial (u+u')}{\partial x}$ in expression (i) is given by,

$$\overline{\frac{\partial (u+u')}{\partial x}} = \frac{1}{t}\int_0^t \left[\frac{\partial}{\partial x}(u+u')\right] dt = \frac{\partial}{\partial x}\left[\frac{1}{t}\int_0^t (u+u')dt\right] = \frac{\partial u}{\partial x}$$

Similarly, the time average of 2nd and 3rd terms is respectively given below.

$$\overline{\frac{\partial (v+v')}{\partial y}} = \frac{\partial v}{\partial y} \text{ and } \overline{\frac{\partial (w+w')}{\partial z}} = \frac{\partial w}{\partial z}$$

Therefore, the continuity equation for turbulent flow is given below.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{ii}$$

Now subtracting (ii) from (i), we get:

$$\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right) = 0 \tag{iii}$$

From expressions (ii) and (iii), it can be seen that the continuity equation holds good both for the mean and fluctuating components of velocity.

Now considering the steady, incompressible, Newtonian, isotropic, turbulent flow at constant temperature and without body force state, the Navier-Stokes equation is given below.

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

or

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u$$

Replacing u, v, w and p by $(u+u')$, $(v+v')$, $(w+w')$ and $(p+p')$, respectively, we get:

$$(u+u')\frac{\partial(u+u')}{\partial x}+(v+v')\frac{\partial(u+u')}{\partial y}+(w+w')\frac{\partial(u+u')}{\partial z}=-\frac{1}{\rho}\frac{\partial(p+p')}{\partial x}+\nu\nabla^2(u+u')$$

Expanding the first term of the above equation by taking time average and using the facts,

$$\overline{u'}=0,\ \overline{\frac{\partial u'}{\partial x}}=0 \text{ and } u'\frac{\partial u'}{\partial x}=\frac{\overline{u'\partial u'}}{\partial x}$$

we get: $$(u+u')\frac{\partial(u+u')}{\partial x}=u\frac{\partial u}{\partial x}+\frac{\overline{u'\partial u'}}{\partial x}$$

Similarly, the remaining terms can be expanded and thus, we obtain the time average of x-direction equation for the fluid motion as follows.

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u-\left[\frac{\overline{u'\partial u'}}{\partial x}+\frac{\overline{v'\partial u'}}{\partial y}+\frac{\overline{w'\partial u'}}{\partial z}\right]$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u-\left[\frac{\partial}{\partial x}\left(\overline{u'^2}\right)+\frac{\partial}{\partial y}\left(\overline{u'v'}\right)+\frac{\partial}{\partial z}\left(\overline{u'w'}\right)\right] \quad (13.12a)$$

Similarly, the time average of y and z-directions equations for the fluid motion can be respectively obtained as follows.

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v-\left[\frac{\partial}{\partial x}\left(\overline{u'v'}\right)+\frac{\partial}{\partial y}\left(\overline{v'^2}\right)+\frac{\partial}{\partial z}\left(\overline{u'w'}\right)\right] \quad (13.12b)$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 w-\left[\frac{\partial}{\partial x}\left(\overline{u'w'}\right)+\frac{\partial}{\partial y}\left(\overline{v'w'}\right)+\frac{\partial}{\partial z}\left(\overline{w'^2}\right)\right] \quad (13.12c)$$

The Equations (13.12a), (13.12b) and (13.12c) are called the Reynolds equations for the motion of turbulent flow. When these equations are compared with Navier-Stokes equations, then we find the following additional terms.

$$-\left[\frac{\partial}{\partial x}\left(\overline{u'^2}\right)+\frac{\partial}{\partial y}\left(\overline{u'v'}\right)+\frac{\partial}{\partial z}\left(\overline{u'w'}\right)\right]$$

$$-\left[\frac{\partial}{\partial x}\left(\overline{u'v'}\right)+\frac{\partial}{\partial y}\left(\overline{v'^2}\right)+\frac{\partial}{\partial z}\left(\overline{v'w'}\right)\right]$$

$$-\left[\frac{\partial}{\partial x}\left(\overline{u'w'}\right)+\frac{\partial}{\partial y}\left(\overline{v'w'}\right)+\frac{\partial}{\partial z}\left(\overline{w'^2}\right)\right]$$

These above terms represent the stress components and they are known as Reynolds stresses (or turbulent stresses or eddy stresses).

13.4 ❒ SHEAR STRESSES IN TURBULENT FLOW

The velocity fluctuations in turbulent flow cause a continuous interchange of fluid masses between the adjacent layers which is accompanied by a momentum transfer. The momentum transport between adjacent layers results in additional shear stresses of high magnitude. A number of semi-empirical theories also known as turbulence models have been developed to determine the magnitude of the turbulent shear stress. A few of these theories are discussed below.

13.4.1 Boussinesq's Theory

The viscous shear stress (τ_v) obtained from Newton's law of viscosity is given below.

$$\tau_v = \mu \frac{du}{dy} \tag{13.13}$$

Similar to the above expression, J. Boussinesq, a French mathematician in 1877 proposed the following expression for turbulent shear stress (τ_t) as given below.

$$\tau_t = \eta \frac{du}{dy} \tag{13.14}$$

Here, η is called eddy viscosity (apparent or virtual viscosity) which is analogous to the absolute viscosity (μ) and u is the average velocity at a distance y from boundary.

Similar to kinematic viscosity ($\nu = \mu/\rho$), the kinematic eddy viscosity (ε) can also be obtained by dividing the eddy viscosity by mass density of the fluid as given below.

$$\varepsilon = \frac{\eta}{\rho} \tag{13.15}$$

Both the eddy viscosity and kinematic eddy viscosity are mainly the functions of the characteristics of flow and vary from point to point in the flow.

The total shear stress (τ) in turbulent flow can be given as the sum of the shear stress by Newton's law of viscosity and the shear stress given by Boussinesq as stated below.

$$\tau = \tau_v + \tau_t$$

$$\tau = \mu \frac{du}{dy} + \eta \frac{du}{dy} = (\mu + \eta) \frac{du}{dy} \tag{13.16}$$

For laminar flow, $\eta = 0$, whereas for turbulent flow, η is several thousand times larger than μ. Therefore, for evaluating the shear stress in turbulent flow, the shear stress due to fluid viscosity can be neglected. As the value of η cannot be predicted, the Boussinesq's equation has only limited practical use.

13.4.2 Reynolds Theory

The expression for turbulent shear stress (τ_t) between the two layers of a fluid at a small distance apart developed by Reynolds (1886) is given below.

$$\tau_t = \rho u' v' \tag{13.17}$$

Here, u' and v' are the fluctuating components of velocities in the x and y-directions, respectively.

Since both u' and v' are fluctuating, the magnitude of τ_t will also vary. Generally, for the analysis of turbulent flow problems, the time average value of turbulent shear stress is considered which is also known as Reynolds stress and the expression is as follows.

$$\boxed{\overline{\tau_t} = \overline{\rho u' v'} = \rho\, \overline{u' v'}} \tag{13.18}$$

Here, $\overline{u'v'}$ is the time average of the product of the fluctuating components u' and v' and it is usually negative.

13.4.3 Prandtl's Mixing Length Theory

Ludwig Prandtl, a German engineer in 1925 gave the mixing length hypothesis by means of which the turbulent shear stress can be expressed in terms of measurable quantities related to the average flow characteristics. According to Prandtl, mixing length (l) is the distance between two layers in transverse direction such that a lump of fluid particles travel from one layer to the other and mixes with the fluid particles of the adjacent layer in such a way that their momentum in x-direction remains same (Figure 13.3).

It was also assumed that the velocity fluctuation components u' and v' are of the same order and are related to the mixing length as given below.

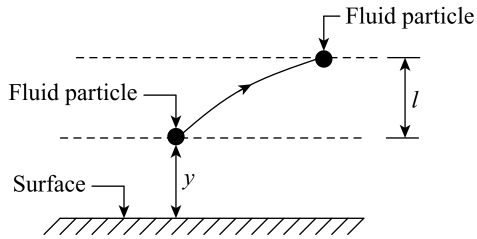

Figure 13.3 *Mixing length*

$$\overline{u'} = \overline{v'} = l\frac{du}{dy} \tag{13.19}$$

Thus

$$\overline{u'v'} = l^2\left(\frac{du}{dy}\right)^2 \tag{13.20}$$

According to Prandtl, shear stress in turbulent flow can be obtained by substituting Equation (13.20) in Equation (13.18) as given below. For the sake of convenience the bar sign on τ denoting the time average quantity has been omitted.

Thus

$$\boxed{\tau_t = \rho l^2\left(\frac{du}{dy}\right)^2} \tag{13.21}$$

The expression for total shear stress (τ) at any point is the sum of the viscous shear stress and turbulent shear stress as given below.

$$\tau = \mu\frac{du}{dy} + \rho l^2\left(\frac{du}{dy}\right)^2 \tag{13.22}$$

However, the viscous shear stress is negligible when compared with the turbulent shear stress and therefore, the shear stress can be assumed due to turbulence only. Since it is possible to establish a relationship between mixing length (l) and the characteristic length of the flow, the Equation (13.22) is used for determining the shear stress in various turbulent flow problems.

13.4.4 Von Karman Similarity Concept

Theodore von Karman extended the Prandtl's momentum theory by considering the dependence of mixing length on the distribution of average flow velocity. Karman proposed that the mixing length (l) is the ratio of the first derivative of mean velocity to the second derivative of mean velocity. The mathematical expression for mixing length is given below.

$$l = k_t\frac{(du/dy)}{(d^2u/dy^2)} \tag{13.23}$$

Here, k_t is the turbulence constant.

The turbulent shear stress can be obtained by substituting Equation (13.23) in Equation (13.21) as given below.

$$\boxed{\tau_t = \rho {k_t}^2\frac{(du/dy)^4}{(d^2u/dy^2)^2}} \tag{13.24}$$

It is to be noted that neither Prandtl's hypothesis nor Karman's concept is valid for determining shear stress at the pipe centreline and at the wall surface.

Example 13.2 The velocity profile for turbulent flow of water in a pipe of diameter 0.7 m is given by $u = 5 + (1/5)\ln y$, where velocity u is in metre per second and the distance y from the wall is measured in metres. If shear stress at a point 0.1 m from the wall is measured as 8 N/m^2, then find the turbulence viscosity, mixing length and turbulence constant.

Solution

Let $D = 0.7$ m, $u = 5 + (1/5)\ln y$, $y = 0.1$ m and $\tau_t = 8$ N/m^2.

$$\frac{du}{dy} = \frac{1}{5y} = \frac{1}{5 \times 0.1} = 2 \text{ s}^{-1}$$

$$\frac{d^2u}{dy^2} = -\frac{1}{5y^2} = -\frac{1}{5 \times 0.1^2} = -20 \text{ s}^{-1}$$

$$\eta = \frac{\tau_t}{(du/dy)} = \frac{8}{2} = \mathbf{4 \text{ Ns/m}^2}$$

From Equation (13.21), we get:

$$l = \sqrt{\frac{\tau_t}{\rho_w (du/dy)^2}} = \sqrt{\frac{8}{1000 \times 2^2}} = \mathbf{0.04472 \text{ m}}$$

Turbulence constant can be given from Equation (13.24) as follows.

$$k_t = \sqrt{\frac{\tau_t}{\rho_w} \frac{(d^2u/dy^2)^2}{(du/dy)^4}} = \sqrt{\frac{8}{1000} \times \frac{(-20)^2}{2^4}} = \mathbf{0.4472}$$

13.5 ❐ UNIVERSAL VELOCITY DISTRIBUTION EQUATION

In turbulent flow through circular pipes, Prandtl assumed that the mixing length (l) is a linear function of the distance y from the pipe wall and its mathematical expression is given below.

$$\boxed{l = \kappa y} \tag{13.25}$$

Here, κ (Greek kappa) is a proportionality constant and is known as Karman universal constant. As per Nikuradse's experimental results, the value of κ is found to be 0.4.

By neglecting viscous shear stress, the turbulent shear stress (τ or τ_t) can be obtained by substituting Equation (13.25) in Equation (13.21) as given below.

$$\tau = \rho \kappa^2 y^2 \left(\frac{du}{dy}\right)^2$$

$$\frac{du}{dy} = \frac{1}{\kappa y}\sqrt{\frac{\tau}{\rho}} \tag{13.26}$$

For regions very close to the boundary of the pipe, the shear stress (τ) is constant and it is approximately equal to τ_o, i.e., turbulent shear stress at the boundary of the pipe. Thus, Equation (13.26) is written as follows.

$$\frac{du}{dy} = \frac{1}{\kappa y}\sqrt{\frac{\tau_o}{\rho}} \tag{13.27}$$

$$\frac{du}{dy} = \frac{u_s}{\kappa y} \qquad [\because u_s = \sqrt{\tau_o/\rho}]$$

As for the given turbulent flow, u_s is constant and thus, the above equation can be integrated to obtain the equation for velocity as given below.

$$u = \frac{u_s}{\kappa} \ln y + C \tag{13.28}$$

Here, C is a constant of integration and it can be obtained by applying boundary conditions. From Equation (13.28), it can be seen that velocity distribution in turbulent flow is logarithmic in nature. Now applying boundary condition, the velocity is maximum at the centre of the pipe, i.e., $u = u_{max}$ at $y = R$ and we get the following expression.

$$u_{max} = \frac{u_s}{\kappa} \ln R + C \tag{13.29}$$

Thus $$C = u_{max} - \frac{u_s}{\kappa} \ln R$$

Substituting the above value of C in Equation (13.28), we get:

$$u = \frac{u_s}{\kappa} \ln y + u_{max} - \frac{u_s}{\kappa} \ln R = u_{max} + \frac{u_s}{\kappa} \ln \frac{y}{R}$$

$$u = u_{max} + \frac{u_s}{0.4} \ln \frac{y}{R} = u_{max} + 2.5 u_s \ln \frac{y}{R} \qquad [\because \kappa = 0.4] \tag{13.30}$$

The Equation (13.30) is called Prandtl's universal velocity distribution equation which is applicable to both smooth and rough pipes. This equation may also be written in non-dimensional form as given below.

$$\boxed{\frac{u_{max} - u}{u_s} = 2.5 \ln \frac{R}{y}} \tag{13.31}$$

Since $$\ln \frac{R}{y} = 2.3 \log_{10} \frac{R}{y}$$

From Equation (13.31), we get:

$$\boxed{\frac{u_{max} - u}{u_s} = 2.5 \times 2.3 \log_{10} \frac{R}{y} = 5.75 \log_{10} \frac{R}{y}} \tag{13.31a}$$

The Equation (13.31) is called velocity defect law in which the difference $(u_{max} - u)$ is known as velocity defect. This equation shows that the ratio of velocity defect to shear velocity is a function of (R/y) alone and it appears to be independent of the nature of the boundary. However, it is experimentally found that the nature of boundary (i.e., rough or smooth) in pipe affects the velocity near the boundary. Thus, it may be stated that Prandtl's equation is applicable only to the turbulent flow in the central region of the pipe. For hydrodynamically smooth and rough boundaries, different velocity distribution equations are to be derived.

Example 13.3 A pipe of diameter 124 mm carries water. The velocities at the pipe centre and 40 mm from the pipe centre are 3 m/s and 2 m/s, respectively. Determine (i) the shear velocity and (ii) wall shearing stress.

Solution

Let $D = 124$ mm $= 0.124$ m, $r = 40$ mm $= 0.04$ m, $u_{max} = 3$ m/s and $u = 2$ m/s.

$$R = \frac{D}{2} = \frac{0.124}{2} = 0.062 \text{ m}$$

$$y = R - r = 0.062 - 0.04 = 0.022 \text{ m}$$

(i) From Equation (13.31a), we get:

$$u_s = \frac{u_{\max} - u}{5.75\log_{10}(R/y)} = \frac{3-2}{5.75\log_{10}(0.062/0.022)} = \mathbf{0.3865\ m/s}$$

(ii) From Equation (13.4), we get:

$$\tau_o = \rho_w {u_s}^2 = 1000 \times 0.3865^2 = \mathbf{149.38225\ N/m^2}$$

13.6 ❐ HYDRODYNAMICALLY SMOOTH AND ROUGH BOUNDARIES

The pipe wall surface over which fluid flows contains irregularities that vary in shape, size and spacing. The absolute roughness is called the average height of the irregularities projecting from the pipe surface. In general, all boundaries are rough. Let k be the average height of the irregularities projecting from the surface of a boundary as illustrated in Figure 13.4.

Generally, if the average height of the irregularities (k) of the boundary on its surface is large, then the boundary is called rough boundary and if the value of k is small, then the boundary is known as smooth boundary. However, for proper classification of smooth and rough boundaries, the flow and fluid characteristics have to be considered. In circular pipes for turbulent flow, there is a laminar sublayer of height δ' in the immediate neighbourhood of the boundary where viscous shear stress predominates while the shear stress due to turbulence is negligible. However, the flow outside the laminar sublayer is turbulent and thus, shear stress due to turbulence is large when compared to viscous stress. Depending upon the values of k and δ', the boundary is known as rough or smooth.

If the value of k is much less than δ', then the boundary is called smooth boundary (Figure13.4a). Various sizes of eddy present in the flow outside the laminar sublayer try to penetrate through the laminar sublayer. However, due to greater thickness of the laminar sublayer, these cannot reach the surface irregularities and as a result, the boundary acts as a smooth boundary. Such type of boundary is called hydrodynamically smooth boundary.

If the value of k is much larger than δ', then the boundary is called rough boundary (Figure13.4b). With the increase in Reynolds number, δ' becomes much smaller than k. Thus, the irregularities of the surface project through the laminar sublayer and eddies come in contact with them and consequently, large amount of energy loss occurs. Such type of boundary is called hydrodynamically rough boundary.

According to Nikuradse's experimental results, the boundary behaves as follows.

1. If $\frac{k}{\delta'} < 0.25$, then the boundary is hydrodynamically smooth.
2. If $\frac{k}{\delta'} > 6$, then the boundary is hydrodynamically rough.
3. If $0.25 < \frac{k}{\delta'} < 6$, then the boundary is in transition.

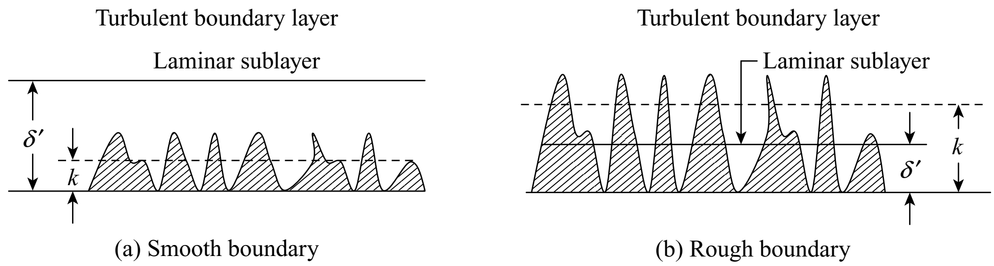

Figure 13.4 *Smooth and rough boundaries*

In terms of roughness Reynolds number, $(\text{Re})_r = (u_s k)/\nu$, the boundary behaves as follows.

1. If $(\text{Re})_r < 4$, then the boundary is smooth.
2. If $(\text{Re})_r > 100$, then the boundary is rough.
3. If $4 < (\text{Re})_r < 100$, then the boundary is in transition.

Example 13.4 A pipe carrying water has average irregularities projecting from the surface of the pipe as 0.01 mm. If the shear stress developed is 5.5 N/m^2, then what type of boundary is it? For water, take density as 1000 kg/m^3 and dynamic viscosity as 0.001 Ns/m^2.

Solution

Let $k = 0.01\text{ mm} = 0.01\times10^{-3}\text{ m}$, $\tau_o = 5.5\text{ N/m}^2$, $\rho_w = 1000\text{ kg/m}^3$ and $\mu = 0.001\text{ Ns/m}^2$.

$$u_s = \sqrt{\frac{\tau_o}{\rho_w}} = \sqrt{\frac{5.5}{1000}} = 0.0742\text{ m/s}$$

$$(\text{Re})_r = \frac{\rho_w u_s k}{\mu} = \frac{1000\times0.0742\times0.01\times10^{-3}}{0.001} = 0.742$$

Since $(\text{Re})_r < 4$, the boundary is smooth.

Example 13.5 A pipe carrying oil of specific gravity 0.85 has average irregularities projecting from the surface of the pipe as 0.25 mm. If the shear stress developed is 6 N/m^2, then what type of boundary is it? For oil, take kinematic viscosity as 0.01 stoke.

Solution

Let $S_{\text{oil}} = 0.85$, $k = 0.25\text{ mm} = 0.25\times10^{-3}\text{ m}$, $\tau_o = 6\text{ N/m}^2$ and $\nu = 0.01\text{ stoke} = 0.01\times10^{-4}\text{ m}^2\text{/s}$.

$$\rho = S_{\text{oil}}\,\rho_w = 0.85\times1000 = 850\text{ kg/m}^3$$

$$u_s = \sqrt{\frac{\tau_o}{\rho}} = \sqrt{\frac{6}{850}} = 0.084\text{ m/s}$$

$$(\text{Re})_r = \frac{u_s k}{\nu} = \frac{0.084\times0.25\times10^{-3}}{0.01\times10^{-4}} = 21$$

Since $(\text{Re})_r$ lies between 4 and 100, the boundary is in transition.

13.7 ❐ VELOCITY DISTRIBUTION FOR TURBULENT FLOW IN SMOOTH PIPES

The velocity distribution for turbulent flow in pipes is given by Equation (13.28) as follows.

$$u = \frac{u_s}{\kappa}\ln y + C$$

The velocity distribution for turbulent flow has a peculiarity that the velocity will be zero at a certain finite distance above the boundary say $y = y'$. Thus, the above equation is written as follows.

$$0 = \frac{u_s}{\kappa}\ln y' + C \Rightarrow C = -\frac{u_s}{\kappa}\ln y'$$

Substituting this value of C in the equation of velocity distribution, we get:

$$u = \frac{u_s}{\kappa}\ln y - \frac{u_s}{\kappa}\ln y' = \frac{u_s}{\kappa}\ln\frac{y}{y'}$$

Substituting the value of $\kappa = 0.4$ in the above equation, we get:

$$u = 2.5u_s \ln\frac{y}{y'}$$

$$\frac{u}{u_s} = 2.5\times 2.3\log_{10}\frac{y}{y'} = 5.75\log_{10}\frac{y}{y'} \tag{13.32}$$

For turbulent flow in smooth pipes according to Nikuradse, we get:

$$\boxed{\delta' = \frac{11.6\nu}{u_s}} \text{ and}$$

$$y' = \frac{\delta'}{107} = \frac{11.6\nu}{107u_s} = \frac{0.108\nu}{u_s}$$

Substituting the value of y' in Equation (13.32), we get:

$$\frac{u}{u_s} = 5.75\ \log_{10}\left(y\times\frac{u_s}{0.108\nu}\right) = 5.75\ \log_{10}\left(\frac{u_s y}{\nu}\times 9.259\right)$$

Thus
$$\boxed{\frac{u}{u_s} = 5.75\ \log_{10}\frac{u_s y}{\nu} + 5.75\log_{10} 9.259 = 5.75\ \log_{10}\frac{u_s y}{\nu} + 5.5} \tag{13.33}$$

The Equation (13.33) is known as Karman-Prandtl equation for the velocity distribution near hydrodynamically smooth boundaries.

Example 13.6 A smooth pipe of diameter 100 mm and length 1000 m carries water at the rate of 0.009 m^3/s. If coefficient of friction f is given as $f = 0.0791/(\text{Re}^{1/4})$ and kinematic viscosity of water is 0.015 stokes, then find (i) the loss of head, (ii) wall shearing stress, (iii) centreline velocity, (iv) shear stress and velocity at 25 mm from the pipe wall and (v) the thickness of laminar sublayer.

Solution

Let $D = 100\text{ mm} = 0.1\text{ m}$, $L = 1000\text{ m}$, $Q = 0.009\text{ m}^3/\text{s}$, $f = 0.0791/(\text{Re}^{1/4})$, $\nu = 0.015\text{ stoke} = 0.015\times 10^{-4}\text{ m}^2/\text{s}$ and $y = 25\text{ mm} = 0.025\text{ m}$.

$$R = \frac{D}{2} = \frac{0.1}{2} = 0.05\text{ m}$$

$$V = \frac{Q}{A} = \frac{Q}{(\pi/4)D^2} = \frac{0.009}{(\pi/4)\times 0.1^2} = 1.146\text{ m/s}$$

Thus
$$\text{Re} = \frac{VD}{\nu} = \frac{1.146\times 0.1}{0.015\times 10^{-4}} = 76400$$

Since Re > 4000, the flow is turbulent.

Thus
$$f = \frac{0.0791}{\text{Re}^{0.25}} = \frac{0.0791}{76400^{0.25}} = 0.00476$$

(i) $$h_f = \frac{4fLV^2}{2gD} = \frac{4 \times 0.00476 \times 1000 \times 1.146^2}{2 \times 9.81 \times 0.1} = \mathbf{12.745\ m}$$

(ii) $$\tau_o = \frac{f\rho_w V^2}{2} = \frac{0.00476 \times 1000 \times 1.146^2}{2} = \mathbf{3.1257\ N/m^2}$$

(iii) $$u_s = \sqrt{\frac{\tau_o}{\rho_w}} = \sqrt{\frac{3.1257}{1000}} = 0.05591 \text{ m/s}$$

The centreline velocity (u_{max}) will be at $y = R$.

From Equation (13.33), we get:
$$\frac{u_{max}}{u_s} = 5.75 \log_{10} \frac{u_s R}{\nu} + 5.5$$

Thus
$$\frac{u_{max}}{0.05591} = 5.75 \log_{10}\left(\frac{0.05591 \times 0.05}{0.015 \times 10^{-4}}\right) + 5.5$$
$$\therefore u_{max} = 0.05591 \times 24.30462 = \mathbf{1.36\ m/s}$$

(iv) Shear stress (τ) at any distance (r) from the centre is given by,
$$\tau = -\frac{\partial p}{\partial x}\frac{r}{2} \qquad \text{(i)}$$

Shear stress at pipe wall τ_o, i.e., at $r = R$ is given by,
$$\tau_o = -\frac{\partial p}{\partial x}\frac{R}{2} \qquad \text{(ii)}$$
$$\frac{\tau}{\tau_o} = \frac{r}{R} \qquad \text{[Dividing (i) by (ii)]}$$

Thus
$$\tau = \frac{\tau_o r}{R}$$

At $y = 25$ mm, $r = R - y = 50 - 25 = 25$ mm $= 0.025$ m
$$\therefore \tau = \frac{3.1257 \times 0.025}{0.05} = \mathbf{1.56285\ N/m^2}$$

Since
$$\frac{u}{u_s} = 5.75 \log_{10} \frac{u_s y}{\nu} + 5.5$$

Thus, velocity (u) at $y = 25$ mm $= 0.025$ m is given by,
$$\frac{u}{0.05591} = 5.75 \log_{10} \frac{0.05591 \times 0.025}{0.015 \times 10^{-4}} + 5.5$$
$$\therefore u = 0.05591 \times 22.574 = \mathbf{1.262\ m/s}$$

(v) $$\delta' = \frac{11.6\nu}{u_s} = \frac{11.6 \times 0.015 \times 10^{-4}}{0.05591} = \mathbf{3.112 \times 10^{-4}\ m \ \ or\ \ 0.3112\ mm}$$

13.8 ❒ VELOCITY DISTRIBUTION FOR TURBULENT FLOW IN ROUGH PIPES

In rough pipes, the laminar sublayer is very small. The surface irregularities protrude beyond the laminar sublayer and hence, the sublayer is totally destroyed. For rough pipes (cement coated), Nikuradse and others found that y' is directly proportional to k and $y' = (k/30)$. Substituting this value of y' in Equation (13.32), we get the below expression.

$$\frac{u}{u_s} = 5.75 \log_{10}\left[\frac{y}{(k/30)}\right] = 5.75 \log_{10}\frac{y}{k} + 5.75 \log_{10} 30$$

Thus
$$\boxed{\frac{u}{u_s} = 5.75 \log_{10}\frac{y}{k} + 8.5} \qquad (13.34)$$

Example 13.7 A rough pipe of diameter 80 mm carries water. If the velocity at a point 3 cm from the wall is 24% more than the velocity at a point 1 cm from the pipe wall, then determine the average height of the roughness.

Solution

Let $D = 80$ mm $= 0.08$ m, u be the velocity at a point 1 cm from the pipe wall, then $1.24u$ be the velocity at a point 3 cm from the pipe wall.

Since
$$\frac{u}{u_s} = 5.75 \log_{10}\frac{y}{k} + 8.5 \qquad \text{[Equation (13.34)]}$$

At $y = 1\,\text{cm}$, the velocity is u and thus, Equation (13.34) is written as follows.

$$\frac{u}{u_s} = 5.75 \log_{10}\frac{1}{k} + 8.5 \qquad \text{(i)}$$

At $y = 3$ cm, the velocity is $1.24u$ and thus, Equation (13.34) is written as follows.

$$\frac{1.24u}{u_s} = 5.75 \log_{10}\frac{3}{k} + 8.5 \qquad \text{(ii)}$$

Thus
$$1.24 = \frac{5.75 \log_{10}(3/k) + 8.5}{5.75 \log_{10}(1/k) + 8.5} \qquad \text{[Dividing (ii) by (i)]}$$

$$1.24 \times [5.75 \log_{10}(1/k) + 8.5] = 5.75 \log_{10}(3/k) + 8.5$$

$$7.13 \log_{10} 1 - 7.13 \log_{10} k + 10.54 = 5.75 \log_{10} 3 - 5.75 \log_{10} k + 8.5$$

$$0 - 7.13 \log_{10} k + 10.54 = 2.74345 - 5.75 \log_{10} k + 8.5$$

$$-1.38 \log_{10} k = 0.70345$$

$$\log_{10} k = -\frac{0.70345}{1.38}$$

$$\therefore k = 10^{-\left(\frac{0.70345}{1.38}\right)} = \mathbf{0.3092\ cm\ or\ 3.092\ mm}$$

13.9 ❒ VELOCITY DISTRIBUTION IN TERMS OF AVERAGE VELOCITY

Let a fluid flow through a pipe of radius R as shown in Figure 13.5. Consider an elementary circular ring of thickness dr and radius r at a distance y from the pipe wall, i.e., $y = (R - r)$.

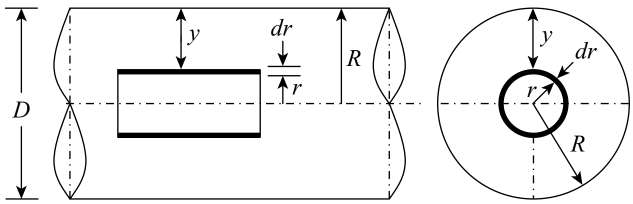

Figure 13.5 *Velocity distribution and forces on control volume in a pipe flow*

The discharge (dQ) through the ring is given by the product of area and velocity as follows.

$$dQ = 2\pi r dr \times u$$

The total discharge (Q) through the pipe is given by integrating the above expression.

$$Q = \int_0^R 2\pi r dr \times u \tag{13.35}$$

1. **Smooth pipes:** The velocity distribution for smooth pipe is given from Equation (13.33) as follows.

$$u = \left[5.75 \log_{10} \frac{u_s y}{\nu} + 5.5\right] u_s = \left[5.75 \log_{10} \frac{u_s (R-r)}{\nu} + 5.5\right] u_s$$

Substituting the above value of u in Equation (13.35), we get:

$$Q = \int_0^R \left[5.75 \log_{10} \frac{u_s (R-r)}{\nu} + 5.5\right] u_s \times 2\pi r dr$$

The average velocity ($\bar{u}$) is given by,

$$\bar{u} = \frac{Q}{A} = \frac{Q}{\pi R^2} = \frac{1}{\pi R^2} \int_0^R \left[5.75 \log_{10} \frac{u_s (R-r)}{\nu} + 5.5\right] u_s \times 2\pi r dr$$

Integrating and simplifying the above expression, we get:

$$\boxed{\frac{\bar{u}}{u_s} = 5.75 \log_{10} \frac{u_s R}{\nu} + 1.75} \tag{13.36}$$

2. **Rough pipes:** The velocity distribution for rough pipe is given from Equation (13.34) as follows.

$$u = \left[5.75 \log_{10} \frac{y}{k} + 8.5\right] u_s = \left[5.75 \log_{10} \frac{(R-r)}{k} + 8.5\right] u_s$$

Substituting the above value of u in Equation (13.35), we get:

$$Q = \int_0^R \left[5.75 \log_{10} \frac{(R-r)}{k} + 8.5\right] u_s \times 2\pi r dr$$

The average velocity ($\bar{u}$) is given by,

$$\bar{u} = \frac{Q}{A} = \frac{Q}{\pi R^2} = \frac{1}{\pi R^2}\int_0^R \left[5.75 \log_{10}\frac{(R-r)}{k} + 8.5\right] u_s \times 2\pi r dr$$

Integrating and simplifying the above expression, we get:

$$\boxed{\frac{\bar{u}}{u_s} = 5.75 \log_{10}\frac{R}{k} + 4.75} \quad (13.37)$$

3. **Relationship between velocity at any point and average velocity for smooth and rough pipes:**
For smooth pipes, subtracting Equation (13.36) from Equation (13.33), we get:

$$\frac{u}{u_s} - \frac{\bar{u}}{u_s} = \left[5.75 \log_{10}\frac{u_s y}{\nu} + 5.5\right] - \left[5.75 \log_{10}\frac{u_s R}{\nu} + 1.75\right]$$

$$\frac{u - \bar{u}}{u_s} = 5.75 \log_{10}\frac{u_s y}{\nu} \times \frac{\nu}{u_s R} + 3.75$$

Thus
$$\boxed{\frac{u - \bar{u}}{u_s} = 5.75 \log_{10}\frac{y}{R} + 3.75} \quad (13.38)$$

At the centreline of the pipe Equation (13.38) becomes,

$$\frac{u - \bar{u}}{u_s} = 5.75 \log_{10}\frac{R}{R} + 3.75 = 3.75 \quad (13.38a)$$

Similarly, for rough pipes, subtracting Equation (13.37) from Equation (13.34), we get:

$$\frac{u}{u_s} - \frac{\bar{u}}{u_s} = \left[5.75 \log_{10}\frac{y}{k} + 8.5\right] - \left[5.75 \log_{10}\frac{R}{k} + 4.75\right]$$

$$\frac{u - \bar{u}}{u_s} = 5.75 \log_{10}\frac{y}{R} + 3.75 \quad (13.39)$$

Since Equations (13.38) and (13.39) are identical, velocity distribution in smooth as well as rough pipes is the same.

Example 13.8 Find the distance y from the pipe wall at which the local velocity u is equal to the average velocity $\bar{u}$ for turbulent flow in a pipe of radius R.

Solution

Since
$$\frac{u - \bar{u}}{u_s} = 5.75 \log_{10}\frac{y}{R} + 3.75$$

Thus
$$\frac{\bar{u} - \bar{u}}{u_s} = 5.75 \log_{10}\frac{y}{R} + 3.75 \quad [\because u = \bar{u}]$$

$$5.75 \log_{10}\frac{y}{R} + 3.75 = 0$$

$$\log_{10}\frac{y}{R} = -\frac{3.75}{5.75}$$

$$\frac{y}{R} = 10^{-\left(\frac{3.75}{5.75}\right)} = 0.223$$

$$\therefore \mathbf{y = 0.223\,R}$$

Example 13.9 If turbulent flow of water is maintained in a pipe of diameter 0.3 m , then determine the distance from the wall surface of the pipe when local velocity is half of the average velocity and shear velocity is (1/20$^{\text{th}}$) of the average velocity.

Solution

Let $D = 0.3$ m, $u = (\bar{u}/2)$ and $u_s = (\bar{u}/20)$.

$$R = \frac{D}{2} = \frac{0.3}{2} = 0.15 \text{ m}$$

Since
$$\frac{u - \bar{u}}{u_s} = 5.75 \log_{10} \frac{y}{R} + 3.75$$

$$\frac{(\bar{u}/2) - \bar{u}}{(\bar{u}/20)} = 5.75 \log_{10} \frac{y}{R} + 3.75$$

$$-10 = 5.75 \log_{10} \frac{y}{0.15} + 3.75$$

$$\log_{10} \frac{y}{0.15} = \frac{-10 - 3.75}{5.75} = -2.3913$$

$$\frac{y}{0.15} = 10^{-2.3913} = 0.004062$$

$$\therefore y = 0.004062 \times 0.15 = \mathbf{0.00061 \text{ m or } 0.61 \text{ mm}}$$

13.10 ❒ POWER LAW FOR VELOCITY DISTRIBUTION IN SMOOTH PIPES

According to Nikuradse, the velocity distribution for turbulent flow in smooth pipe can be expressed by an exponential equation instead of logarithmic equation. The following exponential form (i.e., velocity distribution equation) for turbulent flow in smooth pipe was suggested as follows.

$$\frac{u}{u_{\max}} = \left(\frac{y}{R}\right)^{1/n} \tag{13.40}$$

Here, exponent ($1/n$) depends on Reynolds number (Re) and it decreases with increasing Re, but it tends to approach a constant value asymptotically for very large Re.

$$\frac{1}{n} = \frac{1}{6} \text{ for Re} = 4000$$

$$\frac{1}{n} = \frac{1}{7} \text{ for Re} = 1.1 \times 10^5$$

$$\frac{1}{n} = \frac{1}{10} \text{ for Re} \geq 2 \times 10^6$$

Thus, for (1/n) = (1/7), the velocity distribution Equation (13.40) is written as follows.

$$\boxed{\frac{u}{u_{max}} = \left(\frac{y}{R}\right)^{1/7}} \tag{13.41}$$

The Equation (13.41) is known as $1/7^{th}$ power law of velocity distribution for smooth pipes.

13.11 ❐ RESISTANCE TO FLOW OF FLUID IN SMOOTH AND ROUGH PIPES

The loss of head due to friction in pipes may be calculated correctly by Darcy-Weisbach equation (Equation 13.2) only if coefficient of friction (f) is evaluated accurately. The coefficient of friction depends upon the Reynolds number (Re) and relative roughness (k/D).

Thus $$f = \phi\left[\text{Re}, (k/D)\right] \tag{13.42}$$

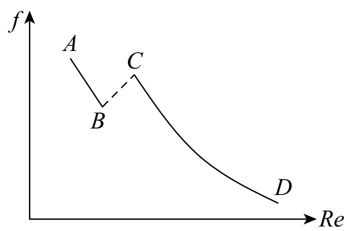

Figure 13.6 *Variation of friction coefficient with Reynolds number*

Sometimes (k/D) is replaced by relative smoothness (R/k), here k is the average height of pipe wall roughness protrusions, D is the diameter of the pipe and R is the radius of the pipe. The Equation (13.42) is the general equation which is applicable to laminar as well as turbulent flow in pipes. The general graphical relationship between f and Re is shown in Figure 13.6 in which *AB* represents the laminar flow, *BC* represents the transition zone and *CD* represents the turbulent flow.

The relationship between friction factor (f_f), Reynolds number (Re) and relative roughness (k/D) over a wide range can be shown by the Moody diagram (or Moody chart) as graphically shown in Figure 13.7. The friction factor is given as $f_f = 4f$ and it can be known from the Moody diagram that was prepared by L. F. Moody (American engineer) for ordinary commercial pipes. The logarithmic plot of f_f versus Re for various values of k/D is known as Stanton diagram.

The Moody diagram is the best means for predicting the values of f_f for circular pipes when the value of relative roughness is known. It can also be used for non-circular pipes by replacing the diameter by the hydraulic diameter. The following observations can be made from the given diagram (Figure 13.7).

(i) In laminar flow, the friction factor (f_f) is independent of pipe roughness (k/D) and it decreases with increase in Reynolds number (Re).

(ii) For Re > 2000, there are two regions, namely transition region and turbulent region.

(iii) In transition region, f_f depends on Re and k/D.

(iv) In turbulent region, f_f solely depends on k/D and it is independent of Re, since f_f curves corresponding to relative roughness values are nearly horizontal.

1. **Variation of coefficient of friction '*f*' for laminar flow.** In previous chapter, the coefficient of friction (f) for laminar flow in pipes is derived as follows.

$$\boxed{f = \frac{16}{\text{Re}}} \tag{13.43}$$

From Equation (13.43), it can be observed that coefficient of friction f for laminar flow varies inversely with Re and it is independent of (k/D). The friction factor can be given as $f_f = 4f = 64/\text{Re}$.

2. **Variation of coefficient of friction '*f*' for turbulent flow.** Depending on whether the boundary is smooth or rough, the coefficient of friction f may be given as follows.

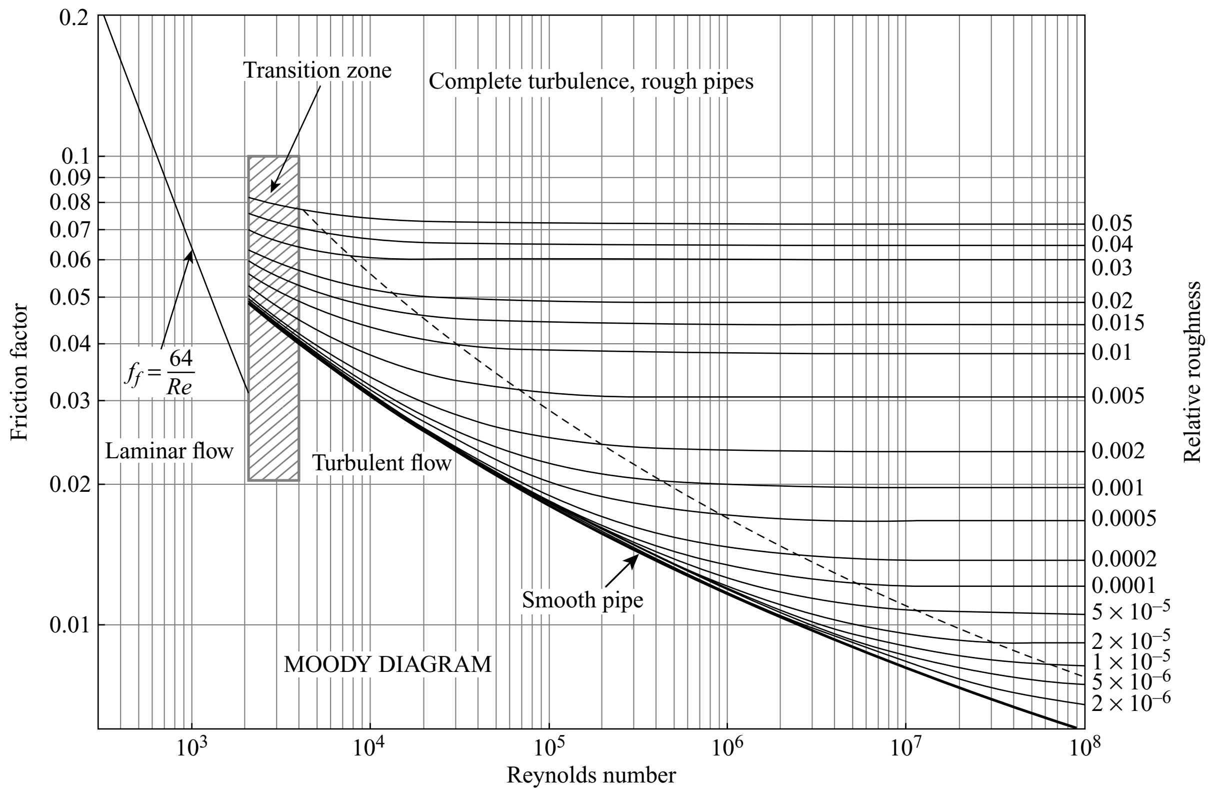

Figure 13.7 *Moody diagram for friction factor*

(a) **Variation of coefficient of friction '*f*' for turbulent flow in smooth pipes:** The value of f for smooth pipes for $4000 < \text{Re} < 10^5$ given by Blasius is as follows.

$$\boxed{f = \frac{0.0791}{\text{Re}^{1/4}}} \tag{13.44}$$

Since $V = \bar{u}$ and $\sqrt{\tau_o/\rho} = u_s$

Thus $f = \dfrac{2\tau_o}{\rho V^2} = \dfrac{2}{\bar{u}^2}\left(\sqrt{\dfrac{\tau_o}{\rho}}\right)^2 = \dfrac{2}{\bar{u}^2} \times u_s^2$ [From Equation (13.3)]

Thus $u_s = \bar{u}\sqrt{\dfrac{f}{2}}$ (i)

Now $\dfrac{\bar{u}}{u_s} = 5.75 \log_{10} \dfrac{u_s R}{\nu} + 1.75$ [Equation (13.36)]

The value of f for $\text{Re} > 10^5$ can be obtained by substituting expression (i) in Equation (13.36) as given below.

$$\frac{\bar{u}}{\bar{u}\sqrt{f/2}} = 5.75 \log_{10} \frac{\bar{u}\sqrt{f/2}\, R}{\nu} + 1.75$$

After substituting $R = (D/2)$ and simplifying, we get:

$$\frac{1}{\sqrt{4f}} = 2.03 \log_{10}\left(\frac{\bar{u}D}{\nu}\sqrt{4f}\right) - 0.91$$

$$\frac{1}{\sqrt{4f}} = 2.03 \log_{10}\left(\text{Re}\sqrt{4f}\right) - 0.91 \quad (13.45)$$

Equation (13.45) is valid up to $\text{Re} = 4\times10^6$ and in terms of friction factor, it is written as follows.

$$\frac{1}{\sqrt{f_f}} = 2.03 \log_{10}\left(\text{Re}\sqrt{f_f}\right) - 0.91 \quad (13.45a)$$

The Karman-Prandtl resistance equation for turbulent flow in smooth pipe which is applicable in the range of $\text{Re} = 5\times10^4$ to 4×10^7 is given below.

$$\frac{1}{\sqrt{4f}} = 2.0 \log_{10}\left(\text{Re}\sqrt{4f}\right) - 0.8 \quad (13.46)$$

or $$\frac{1}{\sqrt{f_f}} = 2.0 \log_{10}\left(\text{Re}\sqrt{f_f}\right) - 0.8 \quad (13.46a)$$

The equation valid for $\text{Re} = 4\times10^3$ to 3.2×10^6 given by Nikuradse is as follows.

$$f = 0.0008 + \frac{0.05525}{\text{Re}^{0.237}} \quad (13.47)$$

(b) **Variation of coefficient of friction '*f*' for turbulent flow in rough pipes:**

$$\frac{\bar{u}}{u_s} = 5.75 \log_{10}\frac{R}{k} + 4.75 \qquad \text{[Equation (13.37)]}$$

The value of f for turbulent flow in rough pipe can be obtained by substituting $u_s = \bar{u}\sqrt{f/2}$ in Equation (13.37) as given below.

$$\frac{\bar{u}}{\bar{u}\sqrt{f/2}} = 5.75 \log_{10}\frac{R}{k} + 4.75$$

After simplifying, we get:

$$\frac{1}{\sqrt{4f}} = 2.03 \log_{10}\frac{R}{k} + 1.68 \quad (13.48)$$

or $$\frac{1}{\sqrt{f_f}} = 2.03 \log_{10}\frac{R}{k} + 1.68 \quad (13.48a)$$

However, Nikuradse suggested the following expressions.

$$\boxed{\frac{1}{\sqrt{4f}} = 2\log_{10}\frac{R}{k} + 1.74} \quad (13.49)$$

or $$\frac{1}{\sqrt{f_f}} = 2\log_{10}\frac{R}{k} + 1.74 \quad (13.49a)$$

(c) **Value of coefficient of friction 'f' for commercial pipes:** The following empirical relation was developed by Colebrook and White for determining the value of f for smooth as well as rough pipes.

$$\frac{1}{\sqrt{4f}} - 2\log_{10}\frac{R}{k} = 1.74 - 2\log_{10}\left[1 + 18.7\frac{(R/k)}{\text{Re}\sqrt{4f}}\right] \quad (13.50)$$

or

$$\frac{1}{\sqrt{f_f}} - 2\log_{10}\frac{R}{k} = 1.74 - 2\log_{10}\left[1 + 18.7\frac{(R/k)}{\text{Re}\sqrt{f_f}}\right] \quad (13.50a)$$

Example 13.10 The velocity at the centre of the pipe and 10 cm away from the centre are 2.2 m/s and 1.8 m/s, respectively. If the diameter of pipe is 30 cm and flow is turbulent, then determine (i) the discharge through the pipe, (ii) coefficient of friction and (iii) height of roughness projections.

Solution

Let $r = 10$ cm, $u_{max} = 2.2$ m/s, $u = 1.8$ m/s and $D = 30$ cm.

$$R = \frac{D}{2} = \frac{30}{2} = 15 \text{ cm}$$

(i) $y = R - r = 15 - 10 = 5$ cm

Since $$\frac{u_{max} - u}{u_s} = 5.75\log_{10}\frac{R}{y}$$

Thus $$\frac{2.2 - 1.8}{u_s} = 5.75\log_{10}\frac{15}{5} = 2.74345$$

$$\therefore u_s = \frac{2.2 - 1.8}{2.74345} = 0.1458 \text{ m/s}$$

Since $$\frac{u - \bar{u}}{u_s} = 5.75\log_{10}\frac{y}{R} + 3.75$$

At $y = R$, velocity $u = u_{max}$ and thus, from the above equation, we get the following result.

$$\frac{u_{max} - \bar{u}}{u_s} = 5.75\log_{10}\frac{R}{R} + 3.75 = 3.75$$

$$\frac{2.2 - \bar{u}}{0.1458} = 3.75$$

$$\therefore \bar{u} = 2.2 - 3.75 \times 0.1458 = 1.653 \text{ m/s}$$

$$Q = \frac{\pi}{4}D^2\bar{u} = \frac{\pi}{4} \times 0.3^2 \times 1.653 = \mathbf{0.11684\ m^3/s}$$

(ii) $u_s = \bar{u}\sqrt{\frac{f}{2}}$

$$0.1458 = 1.653 \times \sqrt{\frac{f}{2}}$$

$$\therefore f = 2 \times \left(\frac{0.1458}{1.653}\right)^2 = \mathbf{0.01556}$$

(iii) $\frac{1}{\sqrt{4f}} = 2\log_{10}\frac{R}{k} + 1.74$

$$\frac{1}{\sqrt{4\times 0.01556}} = 2\log_{10}\frac{0.15}{k} + 1.74$$

$$4.01 = 2\log_{10}\frac{0.15}{k} + 1.74$$

$$\log_{10}\frac{0.15}{k} = \frac{4.01-1.74}{2} = 1.135$$

$$\frac{0.15}{k} = 10^{1.135} = 13.646$$

$$\therefore k = \frac{0.15}{13.646} = \mathbf{0.01099\ m\ or\ 1.099\ cm}$$

Example 13.11 A pipe of diameter 0.25 m and length 450 m carries oil with mass density 860 kg/m^3 and dynamic viscosity 0.006 Ns/m^2. Determine the head lost due to friction and the power required maintaining the oil flow at a rate of 0.08 m^3/s.

Solution

Let $D = 0.25$ m, $L = 450$ m, $\rho = 860$ kg/m^3, $\mu = 0.006$ Ns/m^2 and $Q = 0.08$ m^3/s.

Let h_f and P be the head lost due to friction and power, respectively.

$$V = \frac{Q}{A} = \frac{Q}{(\pi/4)D^2} = \frac{0.08}{(\pi/4)\times 0.25^2} = 1.63 \text{ m/s}$$

$$\text{Re} = \frac{\rho VD}{\mu} = \frac{860\times 1.63\times 0.25}{0.006} = 58408.33$$

Since Re > 4000, the flow is turbulent.

Thus
$$f = \frac{0.0791}{\text{Re}^{0.25}} = \frac{0.0791}{58408.33^{0.25}} = 0.0051$$

$$h_f = \frac{4fLV^2}{2gD} = \frac{4\times 0.0051\times 450\times 1.63^2}{2\times 9.81\times 0.25} = \mathbf{4.9725\ m}$$

$$P = \frac{\rho g Q h_f}{1000} = \frac{860\times 9.81\times 0.08\times 4.9725}{1000} = \mathbf{3.324\ kW}$$

Example 13.12 A pipe of diameter 25 cm and length 750 m carries oil with specific gravity 0.96 and dynamic viscosity 0.96 Ns/m^2 at a rate of 0.15 m^3/s. If the viscosity of oil decreases by a factor of 10, then compare the pumping cost when the same quantity of oil is conveyed.

Solution

Let $D = 25$ cm $= 0.25$ m, $L = 750$ m, $S_{\text{oil}} = 0.96$, $\mu_1 = 0.96$ Ns/m^2, $Q = 0.15$ m^3/s and $\mu_2 = (0.96/10) = 0.096$ Ns/m^2.

$$\rho = S_{\text{oil}}\rho_w = 0.96\times 1000 = 960 \text{ kg/m}^3$$

$$V = \frac{Q}{A} = \frac{Q}{(\pi/4)D^2} = \frac{0.15}{(\pi/4)\times 0.25^2} = 3.056 \text{ m/s}$$

$$\text{Re}_1 = \frac{\rho VD}{\mu_1} = \frac{960\times 3.056\times 0.25}{0.96} = 764$$

Since $\text{Re}_1 < 2000$, the flow is laminar. Thus, frictional loss can be given as follows.

$$h_{f1} = \frac{32\mu_1 VL}{\rho g D^2} = \frac{32\times 0.96\times 3.056\times 750}{960\times 9.81\times 0.25^2} = 119.623 \text{ m}$$

Power required for pumping the oil is given by,

$$P_1 = \frac{\rho g Q h_{f1}}{1000} = \frac{960\times 9.81\times 0.15\times 119.623}{1000} = 168.984 \text{ kW}$$

Now
$$\text{Re}_2 = \frac{\rho VD}{\mu_2} = \frac{960\times 3.056\times 0.25}{0.096} = 7640$$

Since $\text{Re}_2 > 4000$, the flow is turbulent. Thus, the coefficient of friction is given as follows.

$$f = \frac{0.0791}{\text{Re}^{0.25}} = \frac{0.0791}{7640^{0.25}} = 0.00846$$

$$h_{f2} = \frac{4fLV^2}{2gD} = \frac{4\times 0.00846\times 750\times 3.056^2}{2\times 9.81\times 0.25} = 48.324 \text{ m}$$

Power required for pumping the oil is given by,

$$P_2 = \frac{\rho g Q h_{f2}}{1000} = \frac{960\times 9.81\times 0.15\times 48.324}{1000} = 68.264 \text{ kW}$$

Ratio of pumping cost is given by,

$$\frac{P_2}{P_1} = \frac{68.264}{168.984} = \mathbf{0.404}$$

Example 13.13 A smooth pipe of diameter 0.46 m and length 900 m carries water with kinematic viscosity 0.019 stokes. Determine the head lost due to friction, wall shear stress, centreline velocity and the thickness of laminar sublayer for maintaining water flow at a rate of 0.06 m^3/s.

Solution

Let $D = 0.46$ m, $L = 900$ m, $\nu = 0.019$ stokes $= 0.019\times 10^{-4}$ m^2/s and $Q = 0.06$ m^3/s.

$$V = \frac{Q}{A} = \frac{Q}{(\pi/4)D^2} = \frac{0.06}{(\pi/4)\times 0.46^2} = 0.361 \text{ m/s}$$

$$\text{Re} = \frac{VD}{\nu} = \frac{0.361\times 0.46}{0.019\times 10^{-4}} = 87400$$

Since $\text{Re} > 4000$, the flow is turbulent. Thus, the coefficient of friction is given below.

$$f = \frac{0.0791}{\text{Re}^{0.25}} = \frac{0.0791}{87400^{0.25}} = 0.0046$$

$$h_f = \frac{4fLV^2}{2gD} = \frac{4\times 0.0046\times 900\times 0.361^2}{2\times 9.81\times 0.46} = \mathbf{0.2391\ m}$$

$$\tau_o = \frac{f\rho_w V^2}{2} = \frac{0.0046\times 1000\times 0.361^2}{2} = \mathbf{0.29974\ N/m^2}$$

$$u_s = \sqrt{\frac{\tau_o}{\rho_w}} = \sqrt{\frac{0.29974}{1000}} = 0.0173 \text{ m/s}$$

Centreline velocity (u_{max}) will be at $y = R = 0.23$ m.

From Equation (13.33), we get:

$$\frac{u_{max}}{u_s} = 5.75 \log_{10}\frac{u_s y}{\nu} + 5.5$$

Thus $$\frac{u_{max}}{0.0173} = 5.75 \log_{10}\left(\frac{0.0173\times 0.23}{0.019\times 10^{-4}}\right) + 5.5 = 24.596$$

$$\therefore u_{max} = 0.0173\times 24.596 = \mathbf{0.4255\ m/s}$$

$$\delta' = \frac{11.6\nu}{u_s} = \frac{11.6\times 0.019\times 10^{-4}}{0.0173} = \mathbf{0.001274\ m\ or\ 1.274\ mm}$$

Example 13.14 A rough pipe of diameter 0.3 m and length 3000 m carrying water has average height of roughness of 0.3 mm. Determine (i) the head lost due to friction and (ii) power required for maintaining water flow rate of 0.1 m^3/s.

Solution

Let $D = 0.3$ m, $L = 3000$ m, $k = 0.3$ mm $= 0.3\times 10^{-3}$ m and $Q = 0.1$ m^3/s.

$$R = \frac{D}{2} = \frac{0.3}{2} = 0.15 \text{ m}$$

(i) $$V = \frac{Q}{A} = \frac{Q}{(\pi/4)D^2} = \frac{0.1}{(\pi/4)\times 0.3^2} = 1.415 \text{ m/s}$$

Since $$\frac{1}{\sqrt{4f}} = 2\log_{10}\frac{R}{k} + 1.74$$

Thus $$\frac{1}{\sqrt{4f}} = 2\log_{10}\frac{0.15}{0.3\times 10^{-3}} + 1.74 = 7.138$$

$$\therefore f = \frac{1}{4}\times\left(\frac{1}{7.138}\right)^2 = 0.00491$$

$$h_f = \frac{4fLV^2}{2gD} = \frac{4\times 0.00491\times 3000\times 1.415^2}{2\times 9.81\times 0.3} = \mathbf{20.043\ m}$$

(ii) $$P = \frac{\rho_w g Q h_f}{1000} = \frac{1000\times 9.81\times 0.1\times 20.043}{1000} = \mathbf{19.6622\ kW}$$

Example 13.15 A turbulent flow of water is maintained in a rough pipe of diameter 0.3 m and length 500 m. The velocity and velocity gradient at a distance of 3 cm from the wall are observed to be 2.5 m/s and 12 s^{-1}. Using Prandtl's mixing length hypothesis and assuming linear variation of shear stress, determine (i) the wall shear stress, (ii) protrusion height, (iii) coefficient of friction, (iv) mean velocity, (v) head lost due to friction and (vi) power required for maintaining the water flow rate of 0.15 m^3/s.

Solution

Let $D = 0.3$ m, $L = 500$ m, $y = 3$ cm $= 0.03$ m, $u = 2.5$ m/s, $(du/dy) = 12\ s^{-1}$ and $Q = 0.15\ m^3/s$.

$$R = \frac{D}{2} = \frac{0.3}{2} = 0.15 \text{ m}$$

(i) $\tau = \rho_w \kappa^2 y^2 \left(\frac{du}{dy}\right)^2$ [From Prandtl's hypothesis]

$$\therefore \tau = 1000 \times 0.4^2 \times 0.03^2 \times 12^2 = 20.736 \text{ N/m}^2 \qquad [\because \kappa = 0.4]$$

Since $\tau = \tau_o\left[1 - \frac{y}{R}\right]$ [Linear variation]

Thus $20.736 = \tau_o\left(1 - \frac{0.03}{0.15}\right) = 0.8\tau_o$

$$\therefore \tau_o = \frac{20.736}{0.8} = \mathbf{25.92\ N/m^2}$$

(ii) $u_s = \sqrt{\frac{\tau_o}{\rho_w}} = \sqrt{\frac{25.92}{1000}} = 0.161$ m/s

Since $\frac{u}{u_s} = 5.75 \log_{10}\frac{y}{k} + 8.5$

Thus $\frac{2.5}{0.161} = 5.75 \log_{10}\frac{0.03}{k} + 8.5$

$$5.75 \log_{10}\frac{0.03}{k} = 15.528 - 8.5 = 7.018$$

$$\log_{10}\frac{0.03}{k} = \frac{7.018}{5.75} = 1.2223$$

$$\frac{0.03}{k} = 10^{1.2223} = 16.684$$

$$\therefore k = \frac{0.03}{16.684} = \mathbf{0.001798\ m}$$

(iii) $\frac{1}{\sqrt{4f}} = 2\log_{10}\frac{R}{k} + 1.74$

$$\frac{1}{\sqrt{4f}} = 2\log_{10}\frac{0.15}{0.001798} + 1.74 = 5.5826$$

$$\therefore f = \frac{1}{4} \times \left(\frac{1}{5.5826}\right)^2 = \mathbf{0.008}$$

(iv) $\frac{\bar{u}}{u_s} = 5.75 \log_{10} \frac{R}{k} + 4.75$

$$\frac{\bar{u}}{0.161} = 5.75 \log_{10} \frac{0.15}{0.001798} + 4.75 = 15.7975$$

$$\therefore \bar{u} = 15.7975 \times 0.161 = \mathbf{2.5434\ m/s}$$

(v) $\bar{u} = V = 2.5434$ m/s

$$h_f = \frac{4fLV^2}{2gD} = \frac{4 \times 0.008 \times 500 \times 2.5434^2}{2 \times 9.81 \times 0.3} = \mathbf{17.5845\ m}$$

(vi) $P = \frac{\rho_w g Q h_f}{1000} = \frac{1000 \times 9.81 \times 0.15 \times 17.5845}{1000} = \mathbf{25.8756\ kW}$

Summary

1. Turbulent flow occurs when Reynolds number (Re) is greater than 4000.
2. **Darcy-Weisbach Equation:** $h_f = (4fLV^2)/(2gD)$, here h_f is the loss of head due to friction, L is the length of the pipe, D is the diameter of the pipe, V is the average flow velocity and f is coefficient of friction.
3. **Coefficient of friction in terms of shear stress (τ_o):** $f = (2\tau_o)/(\rho V^2)$.
4. Turbulence is the disturbance that causes transfer of fluid particles from one region to the other.
5. The instantaneous velocity (u_i) at any point in the turbulent flow is made up of average velocity component (u) and a randomly varying velocity component (u'), i.e., $u_i = u + u'$. Similarly $v_i = v + v'$ and $w_i = w + w'$.
6. Degree of turbulence $= \sqrt{(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})/3}$
7. Intensity of turbulence $= (1/V)\sqrt{(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})/3}$
8. Scale of turbulence $= udt$
9. K.E. of turbulence/Mass $= (\overline{u'^2} + \overline{v'^2} + \overline{w'^2})/2$
10. **The total shear stress (τ) in turbulent flow:** $\tau = \mu \frac{du}{dy} + \eta \frac{du}{dy} = (\mu + \eta)\frac{du}{dy}$
11. According to Reynolds theory, the Reynolds stress is $\bar{\tau}_t = \overline{\rho u'v'} = \rho\, \overline{u'v'}$.
12. According to Prandtl's mixing length theory, the total shear stress (τ) is
$$\tau = \mu(du/dy) + \rho l^2 \left(du/dy\right)^2.$$
13. **Prandtl's universal velocity distribution equation:** $u = u_{max} + 2.5\, u_s \ln(y/R)$.
14. **Shear velocity (or friction velocity):** $u_s = \sqrt{\tau_o/\rho}$
15. (i) If $(k/\delta') < 0.25$, then smooth boundary, (ii) if $(k/\delta') > 6$, then rough boundary and (iii) if $0.25 < (k/\delta') < 6$, then boundary in transition, here k is the height of the irregularities and δ' is the thickness of laminar sublayer.
16. **Karman-Prandtl equation for smooth pipes:**
$$u/u_s = 5.75 \log_{10}(u_s y/\nu) + 5.5$$
17. **Velocity distribution for turbulent flow in rough pipes:**
$$u/u_s = 5.75 \log_{10}(y/k) + 8.5$$
18. **Velocity distribution for smooth pipe:**
$$\bar{u}/u_s = 5.75 \log_{10}(u_s R/\nu) + 1.75$$
19. **Velocity distribution for rough pipe:**
$$\bar{u}/u_s = 5.75 \log_{10}(R/k) + 4.75$$
20. **Relationship between velocity at any point and average velocity for smooth and rough pipes:**
$$(u - \bar{u})/u_s = 5.75 \log_{10}(y/R) + 3.75$$

21. 1/7th power law for velocity distribution in smooth pipes: $u/u_{\max} = (y/R)^{1/7}$.

22. Coefficient of friction (f) for laminar flow: $f = 16/\text{Re}$.

23. The value of f for turbulent flow in smooth pipes when $4000 < \text{Re} < 10^5$ (Blasius equation): $f = 0.0791/\text{Re}^{0.25}$.

24. The value of f for turbulent flow in smooth pipes when $\text{Re} = 4 \times 10^3$ to 3.2×10^6: $f = 0.0008 + 0.05525/\text{Re}^{0.237}$.

25. The value of f for turbulent flow in rough pipes given by Nikuradse: $1/\sqrt{4f} = 2\log_{10}(R/k) + 1.74$.

Multiple-choice Questions

1. The relation for coefficient of friction (f) in terms of shear stress (τ_o) is
(a) $f = (2\tau_o)/(\rho V^2)$.
(b) $f = \tau_o/(\rho V^2)$.
(c) $f = \tau_o/(2\rho V^2)$.
(d) None of the above.
Here, ρ is the fluid density and V is fluid velocity.

2. The flow in water supply pipes is generally
(a) Turbulent.
(b) Laminar.
(c) Transition.
(d) None of the above.

3. Shear stress according to Prandtl's mixing length theory is given by
(a) $\rho l(du/dy)^2$.
(b) $l^2(du/dy)^2$.
(c) ρl^2.
(d) $\rho l^2(du/dy)^2$.
Here, ρ is the fluid density, l is the mixing length and (du/dy) is the velocity gradient.

4. In a turbulent flow through a pipe, the shear stress is
(a) Maximum at the centre and deceases logarithmically towards the pipe wall.
(b) Maximum at the centre and decreases linearly towards the wall.
(c) Maximum at the wall and decreases linearly to a zero value at the centre.
(d) None of the above.

5. The velocity distribution in laminar sublayer in a fully turbulent pipe flow is
(a) Linear.
(b) Logarithmic.
(c) Parabolic.
(d) Exponential.

6. For a hydraulically smooth pipe, the relation between height of the irregularities (k) and thickness of laminar sublayer (δ') is given by
(a) $(k/\delta') < 0.25$.
(b) $(\delta'/k) < 0.25$.
(c) $(k/\delta') > 6$.
(d) None of the above.

7. For a hydraulically rough pipe, the roughness Reynolds number $[(\text{Re})_r]$ is
(a) $(\text{Re})_r < 4$.
(b) $(\text{Re})_r > 100$.
(c) $(\text{Re})_r > 1000$.
(d) None of the above.

8. Friction velocity (u_s) in a pipe in terms of shear stress (τ_o) and density (ρ) is equal to
(a) $\sqrt{\tau_o/\rho}$.
(b) $\sqrt{\rho/\tau_o}$.
(c) τ_o/ρ.
(d) ρ/τ_o.

Review Questions

1. Define turbulent flow and mention the factors which decide the type of flow in a pipe. Also give a comparison for velocity distribution in laminar and turbulent flow.

2. Derive an expression for the Darcy-Weisbach equation.

3. Define turbulence and also give its classification.

4. Derive expressions for the Reynolds equations and Reynolds stresses for the motion of turbulent flow.

5. Define the terms (i) mean and fluctuating velocities, (ii) degree and intensity of turbulence, (iii) scale of turbulence and (iv) kinetic energy of turbulence.

6. Obtain an expression for shear stress on the basis of Prandtl mixing length theory.
7. What is von Karman similarity concept? What are its limitations?
8. Briefly discuss (i) Boussinesq's theory and (ii) Reynolds theory.
9. Obtain an expression for the Prandtl's universal velocity distribution equation.
10. What do you mean by hydrodynamically smooth and rough boundaries? Also give its criteria.
11. Derive expressions for the velocity distribution for turbulent flow in smooth and rough pipes.
12. Derive expressions for the velocity distribution in terms of average velocity for smooth and rough pipes.
13. Derive an expression for the difference of local velocity and average velocity for turbulent flow through smooth or rough pipe.

Problems

1. A turbulent flow of water in a pipe of diameter 800 mm has velocity profile as $u = 4 + (1/4)\ln y$, where velocity u is in m/s and the distance y from the wall is measured in m. If shear stress at a point 0.1 m from the wall is measured as 1.25 N/m^2, then find the turbulence viscosity, mixing length and turbulence constant.
[**Ans.** 0.5 Ns/m^2, 0.01414 m, 0.1414]
2. If in a pipe of diameter 300 mm having turbulent flow, the centre velocity is 7.5 m/s and that at 50 mm from the pipe wall is 6 m/s, then determine the shear friction velocity.
[**Ans.** 0.546 m/s]
3. In a pipe of diameter 0.1 m carrying water has the velocities at pipe centre and 30 mm from the pipe centre are measured as 2 m/s and 1.5 m/s, respectively. Determine the wall shearing stress.
[**Ans.** 47.742 N/m^2]
4. In a rough pipe of diameter 90 mm, the velocity of flow increases by 20% as the pitot tube is moved from a point 2.5 cm to a point 1 cm from the pipe wall. Determine the height of roughness.
[**Ans.** 3.08 mm]
5. A smooth pipe of diameter 76 mm and length 1000 m carries water at the rate of 0.48 m^3/min. If coefficient of friction is given as $f = 0.0791/(\mathrm{Re}^{1/4})$ and kinematic viscosity of water is 0.015 stokes, then find (i) the loss of head, (ii) wall shearing stress, (iii) centreline velocity, (iv) shear stress and velocity at 25 mm from the pipe wall and (v) thickness of laminar sublayer.
[**Ans.** 38.17 m, 7.114 N/m^2, 2.08 m/s, 2.434 N/m^2, 1.99 m/s, 0.2064 mm]
6. A pipe carrying water has turbulent flow with shear stress 20 N/m^2 and surface irregularity $k = 0.3$ mm. Determine whether the pipe behaves as hydrodynamically rough or smooth, if the kinematic viscosity of fluid is one stoke.
[**Ans.** smooth]
7. A rough pipe of diameter 6 cm carries water. If the velocity at a point 2 cm from the wall is 24% more than the velocity at a point 1 cm from the pipe wall, then determine the average height of the roughness.
[**Ans.** 1.675 cm]
8. For turbulent flow in a pipe of diameter 300 mm, determine the distance from the pipe wall at which the local velocity is equal to the average velocity.
[**Ans.** 33.4 mm]
9. A rough pipe of diameter 0.5 m and length 300 m carrying water with a velocity of 3 m/s has an absolute roughness of 0.25 mm. If the kinematic viscosity of water is 0.9 centistoke, find (i) the type of flow and (ii) head loss in friction.
[**Ans.** 0.0042, 4.624 m]
10. A turbulent flow of water is maintained in a pipe of 5 cm diameter. If the velocity at the centre of the pipe is 2.4 m/s and 1.5 cm away from the centre is 1.4 m/s, then determine the shear stress at the pipe wall.
[**Ans.** 190.97 N/m^2]
11. The velocity at the centre of the pipe and 10 cm away from the centre are 2.4 m/s and 2 m/s. If the pipe diameter is 30 cm and the flow is turbulent, then determine the average velocity and discharge through the pipe.
[**Ans.** 1.853 m/s, 0.13098 m^3/s]
12. A rough pipe of diameter 0.1 m carries water at a rate of 0.05 m^3/s. If the average height of the protrusions on the pipe surface is 0.15 mm and kinematic viscosity of water is one centistoke, then determine (i) the coefficient of friction, (ii) maximum velocity, (iii) stress at the pipe surface, (iv) shear velocity and (v) thickness of laminar sublayer.
[**Ans.** 0.00543, 6.37 m/s, 110.17 N/m^2, 0.332 m/s, 0.0035 cm]
13. A pipe of length 50 m and diameter 300 mm carries water at a velocity of 3.2 m/s. If the kinematic viscosity of water is 0.01 stoke, then determine the head lost due to friction by using Darcy relation.
[**Ans.** 0.88 m]
14. Air flows through a 0.25 m diameter rough pipe with average velocity of 8 m/s. Determine the coefficient of friction and wall shear stress when the average roughness of pipe

surface is 0.5 mm and the flow is turbulent. The values of density and kinematic viscosity for air are 1.22 kg/m^3 and 1.5×10^{-5} m^2/s, respectively.

[**Ans.** 0.00585, 0.2284 N/m^2]

15. A smooth pipe of length 750 m and diameter 0.4 m carries water at a rate of 40 litres per second. If the kinematic viscosity of water is 0.018 stokes, then find (i) the coefficient of friction, (ii) head lost due to friction, (iii) wall shear stress, (iv) centreline velocity and (v) thickness of laminar sublayer.

[**Ans.** 0.00485, 0.188 m, 0.2457 N/m^2, 0.38 m/s, 1.33 mm]

16. A fluid with a kinematic viscosity of one centistoke and specific gravity of 1.1 flows through a pipe of diameter 0.3 m and length 1000 m. If the flow rate is 66.67 litres per second, then find the maximum pressure and power rating of the pump.

[**Ans.** 23.61 kPa, 1.574 kW]

17. A rough pipe of diameter 0.48 m and length 850 m carries water at a rate of 500 litres per second. If the average height of roughness is 0.15 mm, then determine the coefficient of friction, wall shear stress, centreline velocity and velocity at a distance of 0.2 m from the pipe wall.

[**Ans.** 0.00376, 14.352 N/m^2, 3.225 m/s, 3.171 m/s]

18. For turbulent flow in a rough pipe of radius R, the coefficient of friction $f = 0.01$, determine the local velocity at a radial distance of $R/4$ from the axis of pipe. Also determine the velocity at the centre of the pipe if the average velocity is 0.4 m/s.

[**Ans.** 0.485 m/s, 0.505 m/s]

19. A turbulent flow of water is maintained in a rough pipe of diameter 0.24 m. The velocity and velocity gradient at a distance of 3 cm from the wall are 2 m/s and 10 s^{-1}, respectively. Using Prandtl mixing length theory, determine (i) the wall shear stress, (ii) protrusion height, (iii) coefficient of friction and (iv) average velocity.

[**Ans.** 19.2 N/m^2, 0.0028 m, 0.00998, 1.96 m/s]

20. The velocity in a rough pipe of diameter 10 cm is observed by a pitot tube. When the pitot tube is moved from 1.5 cm to 3 cm from the wall, the increase in velocity is observed to be 10%. Determine the roughness of the pipe and coefficient of friction for the pipe.

[**Ans.** 0.44 mm, 0.0073]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (a)	**2.** (a)	**3.** (d)	**4.** (c)	**5.** (a)
6. (a)	**7.** (b)	**8.** (a)		

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Chapter **14**

Flow Through Pipes

14.1 ❒ INTRODUCTION

A pipe (commonly circular in section) is a closed conduit through which fluids flow under pressure. Since the fluids flow under pressure, the pipes always run completely full and there is no free surface. Atmospheric pressure exists in a pipe running partially full and thus, it behaves like an open channel such as sewer pipes. A fluid flowing through the pipe is subjected to resistance due to shear forces between fluid particles and boundary walls of the pipe and between the fluid particles themselves due to viscosity of fluid. This resistance to the flow is known as frictional resistance and it causes loss of a certain amount of fluid energy. Pipes are commonly used for distribution of water, drainage collection and for supplying oil and gases. The flow in a pipe is characterized as internal flow. This chapter deals with the analysis of internal flow of a liquid with constant viscosity and density. In this chapter, various problems of pipe flow based on major and minor energy losses, pipes in series, parallel and branches, flow through syphon and nozzles, water hammer, and power transmission through pipes are discussed and solved with the help of empirical and theoretical formulae.

14.2 ❒ ENERGY LOSS (HEAD LOSS) IN PIPES

Resistance offered to the flowing fluid through a pipe results in loss of energy. These losses may be major losses and minor losses.

14.2.1 Major Losses

When fluid flows through a pipe, the major loss of energy is caused by friction. In long pipelines, this loss forms the major portion of the total loss and thus, the other losses may be neglected. Therefore, this loss of energy due to friction is known as major loss.

14.2.2 Minor Losses

The minor energy losses occur due to change in the velocity of flowing fluid in the magnitude or direction. In long pipes, these losses are small when compared to the friction losses and they are known as minor losses. However, in short pipes, these losses may sometimes supersede the friction loss. The minor head losses may be (i) due to sudden enlargement, (ii) due to sudden contraction, (iii) at the entrance of a pipe, (iv) at the exit of a pipe, (v) due to obstruction in a pipe, (vi) due to bend in a pipe (vii) in various pipe fittings.

14.3 ❐ FORMULAE FOR MAJOR ENERGY LOSS IN PIPES

The following formulae are used to determine the major energy loss in the analysis of the pipe flow problems.

14.3.1 Darcy-Weisbach Formula

This formula is commonly used to determine the loss of head due to friction in pipes and the expression is as follows.

$$\boxed{h_f = \frac{4fLV^2}{2gD}} \tag{14.1}$$

$$\boxed{h_f = \frac{4fL}{2gD}\left[\frac{Q}{A}\right]^2 = \frac{4fL}{2gD}\left[\frac{Q}{(\pi/4)D^2}\right]^2 = \frac{32fLQ^2}{\pi^2 gD^5}} \tag{14.1a}$$

Here, h_f is the loss of head due to friction, L is the length of the pipe, D is the diameter of the pipe, A is the area of cross-section of the pipe, V is the average flow velocity, Q is the discharge and f is the coefficient of friction which can be calculated on the basis of Reynolds number (Re) as given below.

$$\boxed{f = \frac{16}{\text{Re}} \text{ (When Re} < 2000)} \tag{14.2}$$

$$\boxed{f = \frac{0.0791}{\text{Re}^{0.25}} \text{ (When Re varies from 4000 to } 10^6)} \tag{14.3}$$

14.3.2 Chezy's Formula

In Equation (13.1) given in Chapter 13, h_f is the loss of head due to friction, w is the weight density of fluid, f' is the dimensional parameter, P is the wetted perimeter, A is the area of cross section of the pipe, L is the length of the pipe and V is the mean (average) flow velocity.

$$h_f = \frac{f'}{w}\left(\frac{P}{A}\right)LV^2 \qquad \text{[Equation (13.1)]}$$

$$V = \sqrt{\frac{w}{f'}}\sqrt{\frac{A}{P} \times \frac{h_f}{L}}$$

Here, $C = \sqrt{w/f'}$ is the Chezy's constant and the dimensions are $[L^{1/2}T^{-1}]$, $m = (A/P)$ is the hydraulic mean depth (or hydraulic radius) and $i = (h_f/L)$ is the loss of head per unit length of pipe.

Thus $$\boxed{V = C\sqrt{m\,i}} \tag{14.4}$$

The Equation (14.4) is called Chezy's formula invented by the French engineer Chezy (1775). This formula may be used for computing head loss due to friction in pipes. However, it is not commonly used. The hydraulic engineers use C without bothering about the dimensions even though it is very important.

14.3.3 Manning's Formula

This formula is often used for the analysis of the pipe flow problems and the expression is given below.

$$\boxed{V = \frac{1}{n} m^{2/3} i^{1/2}} \tag{14.5}$$

Here, V is the average flow velocity, n is the Manning's roughness whose value depends on the nature of the boundary, m is the hydraulic mean depth and $i = (h_f/L)$ is the loss of head per unit length of pipe. The hydraulic engineers use n without bothering about the dimensions even though it is very important.

14.3.4 Hazen William's Formula

Earlier, this formula was widely used for designing water supply systems but nowadays it is not used. Due to its empirical nature, the range of applicability of this formula is limited and if used outside its data, the level of error is significant. This formula is applicable to pipes having diameter more than 5 cm and flow velocity less than 3 m/s and it is given by the following expression.

$$\boxed{V = 0.848\, k\, m^{0.63}\, i^{0.54}} \tag{14.6}$$

Here, V is the average flow velocity, k is the coefficient whose value depends on the nature of the boundary, m is the hydraulic mean depth, $i = (h_f/L)$ is the loss of head per unit length of pipe. The value of k for very smooth pipes is taken as 130 and for extremely smooth and straight pipes it is taken as 140.

Example 14.1 Find the head lost due to friction in a pipe of diameter 0.15 m and length 60 m carrying water at a velocity of 2.5 m/s, using (i) Darcy-Weisbach formula and (ii) Chezy's formula for which $C = 58$. Take kinematic viscosity of water as 0.012 stoke.

Solution

Let $D = 0.15$ m, $L = 60$ m, $V = 2.5$ m/s, $C = 58$ and $\nu = 0.012 \text{ stoke} = 0.012\times10^{-4}\ \text{m}^2/\text{s}$.

(i) $$\text{Re} = \frac{VD}{\nu} = \frac{2.5\times0.15}{0.012\times10^{-4}} = 312500$$

Since Re > 4000, the flow is turbulent.

Thus $$f = \frac{0.0791}{\text{Re}^{0.25}} = \frac{0.0791}{312500^{0.25}} = 0.003345$$

$$h_f = \frac{4fLV^2}{2gD} = \frac{4\times0.003345\times60\times2.5^2}{2\times9.8\times0.15} = \mathbf{1.705\ m}$$

(ii) $$m = \frac{A}{P} = \frac{(\pi/4)D^2}{\pi D} = \frac{D}{4} = \frac{0.15}{4} = 0.0375\ \text{m}$$

Since $$V = C\sqrt{m\,i} = C\sqrt{m\times\frac{h_f}{L}}$$

Thus $$2.5 = 58\times\sqrt{0.0375\times\frac{h_f}{60}}$$

$$\therefore h_f = \left(\frac{2.5}{58}\right)^2\times\frac{60}{0.0375} = \mathbf{2.973\ m}$$

Example 14.2 If an oil with specific gravity 0.75 and kinematic viscosity 0.25 stoke flows at a rate of 0.3 m^3/s through a pipe of diameter 0.3 m and length 500 m, then find the head lost due to friction and the power required to maintain the flow.

Solution

Let $S_{oil} = 0.75$, $\nu = 0.25$ stoke $= 0.25\times10^{-4}$ m^2/s, $Q = 0.3$ m^3/s, $D = 0.3$ m and $L = 500$ m.

$$\rho = S_{oil}\rho_w = 0.75\times1000 = 750 \text{ kg/m}^3$$

$$V = \frac{Q}{A} = \frac{Q}{(\pi/4)D^2} = \frac{0.3}{(\pi/4)\times0.3^2} = 4.244 \text{ m/s}$$

$$\text{Re} = \frac{VD}{\nu} = \frac{4.244\times0.3}{0.25\times10^{-4}} = 50928$$

Since Re > 4000, the flow is turbulent.

$$f = \frac{0.0791}{\text{Re}^{0.25}} = \frac{0.0791}{50928^{0.25}} = 0.00526$$

$$h_f = \frac{4fLV^2}{2gD} = \frac{4\times0.00526\times500\times4.244^2}{2\times9.81\times0.3} = \mathbf{32.192\ m}$$

$$P = \frac{\rho g Q h_f}{1000} = \frac{750\times9.81\times0.3\times32.192}{1000} = \mathbf{71.056\ kW}$$

Example 14.3 Using Manning's formula, find the head lost due to friction in a pipe of diameter 0.2 m and length 1000 m carrying water at rate of 0.05 m^3/s when Manning's roughness is 0.016.

Solution

Let $D = 0.2$ m, $L = 1000$ m, $Q = 0.05$ m^3/s and $n = 0.016$.

$$V = \frac{Q}{A} = \frac{Q}{(\pi/4)D^2} = \frac{0.05}{(\pi/4)\times0.2^2} = 1.59 \text{ m/s}$$

$$m = \frac{A}{P} = \frac{(\pi/4)D^2}{\pi D} = \frac{D}{4} = \frac{0.2}{4} = 0.05 \text{ m}$$

Since
$$V = \frac{1}{n}m^{2/3}i^{1/2} = \frac{1}{n}m^{2/3}\left(\frac{h_f}{L}\right)^{1/2}$$

Thus
$$1.59 = \frac{1}{0.016}\times0.05^{2/3}\times\left(\frac{h_f}{1000}\right)^{1/2}$$

$$\therefore h_f = 1000\times\left[\frac{1.59\times0.016}{0.05^{2/3}}\right]^2 = \mathbf{35.135\ m}$$

Example 14.4 Using Hazen-William's formula find the head lost due to friction in a pipe of diameter 0.1 m and length 1000 m carrying water at rate of 0.02 m^3/s. Take $k = 100$.

Solution

Let $D = 0.1$ m, $L = 1000$ m, $Q = 0.02$ m^3/s and $k = 100$.

$$V = \frac{Q}{A} = \frac{Q}{(\pi/4)D^2} = \frac{0.02}{(\pi/4)\times 0.1^2} = 2.5465 \text{ m/s}$$

$$m = \frac{A}{P} = \frac{(\pi/4)D^2}{\pi D} = \frac{D}{4} = \frac{0.1}{4} = 0.025 \text{ m}$$

Since $$V = 0.848km^{0.63}\, i^{0.54} = 0.848\, k\, m^{0.63}\left(\frac{h_f}{L}\right)^{0.54}$$

Thus $$2.5465 = 0.848\times 100\times 0.025^{0.63}\times\left(\frac{h_f}{1000}\right)^{0.54}$$

$$\therefore h_f = 1000\times\left[\frac{2.5465}{0.848\times 100\times 0.025^{0.63}}\right]^{1/0.54} = \mathbf{112.128\ m}$$

Example 14.5 A pipe of diameter 0.3 m and length 2000 m carries water at the rate of 0.05 m^3/s. Find the head loss in pipe if coefficient of friction is given by $f = [0.002 + (0.09/\text{Re}^{0.5})]$ and kinematic viscosity of water is 0.012 stoke.

Solution

Let $D = 0.3$ m, $L = 2000$ m, $Q = 0.02\ \text{m}^3/\text{s}$, $f = [0.002 + (0.09/\text{Re}^{0.5})]$ and $\nu = 0.012 \text{ stoke} = 0.012\times 10^{-4}\ \text{m}^2/\text{s}$.

$$V = \frac{Q}{A} = \frac{Q}{(\pi/4)D^2} = \frac{0.05}{(\pi/4)\times 0.3^2} = 0.707 \text{ m/s}$$

$$\text{Re} = \frac{VD}{\nu} = \frac{0.707\times 0.3}{0.012\times 10^{-4}} = 176750$$

$$f = 0.002 + \frac{0.09}{\text{Re}^{0.5}} = 0.002 + \frac{0.09}{176750^{0.5}} = 0.002214$$

$$h_f = \frac{4fLV^2}{2gD} = \frac{4\times 0.002214\times 2000\times 0.707^2}{2\times 9.81\times 0.3} = \mathbf{1.5041\ m}$$

Example 14.6 A hostel consisting of 1000 students is supplied water by a pumping station situated at 2500 m from the hostel. The total amount of water required is to be supplied within 8 hours and each student requires 200 litres of water per day. If the friction loss through pipe is 40 m of water, then find the diameter of the pipe required. Use Darcy's formula and take $f = 0.001$.

Solution

Let $n = 1000$, $L = 2500$ m, $t = 8$ hours, $q = 200$ l/day $= 0.2\ \text{m}^3$/day, $h_f = 40$ m and $f = 0.001$.

Let Q be the maximum flow per second and D be the required diameter of the pipe.

$$Q = \frac{qn}{t} = \frac{0.2\times 1000}{8\times 3600} = 0.006944\ \text{m}^3/\text{s}$$

$$V = \frac{Q}{A} = \frac{0.006944}{(\pi/4)D^2} = \frac{0.00884}{D^2}$$

Since $$h_f = \frac{4fLV^2}{D \times 2g}$$

Thus $$40 = \frac{4 \times 0.001 \times 2500}{2 \times 9.81 \times D} \times \left(\frac{0.00884}{D^2}\right)^2$$

$$D^5 = \frac{4 \times 0.001 \times 2500 \times 0.00884^2}{2 \times 9.81 \times 40} = 9.9574 \times 10^{-7}$$

$$\therefore D = (9.9574 \times 10^{-7})^{1/5} = \mathbf{0.06304\ m}$$

14.4 ❐ MINOR ENERGY LOSSES IN PIPES

The minor energy losses (or head losses) are caused due to change in magnitude or direction of velocity of flowing fluid. The minor head losses are also known as secondary losses. The minor head losses include (i) loss of head due to sudden enlargement, (ii) loss of head due to sudden contraction, (iii) loss of head at the entrance of a pipe, (iv) loss of head at the exit of a pipe, (v) loss of head due to obstruction in a pipe, (vi) loss of head due to bend in a pipe and (vii) loss of head in various pipe fittings.

The expressions for the above mentioned minor head losses are derived in the following sections.

14.4.1 Loss of Head Due to Sudden Enlargement

Consider that a fluid flows through a pipe in which there is a sudden enlargement of cross section from A_1 to A_2 as shown in Figure 14.1. Due to sudden change in the cross-sectional area of the flow passage, the flow separates from the boundary. It forms regions of separation in which turbulent eddies are formed which results in the loss of energy.

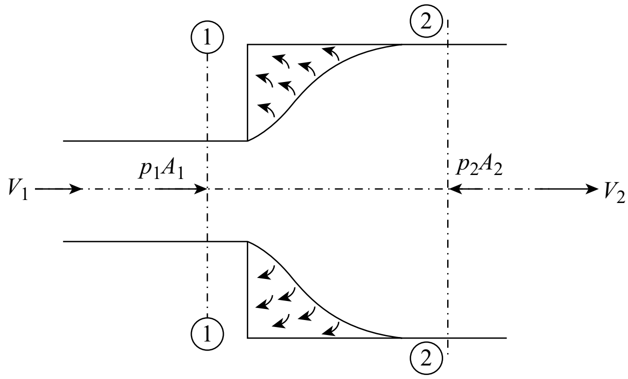


Figure 14.1 *Sudden enlargement in a pipe*

Consider section 1–1 and section 2–2 before and after the enlargement respectively. Let p_1, V_1 and A_1 be the pressure, velocity and area, respectively, at section 1–1 and p_2, V_2 and A_2 be the corresponding values at section 2–2 and $(h_L)_e$ be the loss of head due to sudden enlargement. Experimentally, the intensity of pressure of the eddies (p_o) in area $(A_2 - A_1)$ is found equal to pressure p_1.

Apply Bernoulli's equation to sections 1–1 and 2–2, we get:

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + (h_L)_e \qquad [\because z_1 = z_2]$$

$$(h_L)_e = \left(\frac{p_1 - p_2}{\rho g}\right) + \left(\frac{V_1^2 - V_2^2}{2g}\right) \qquad \text{(i)}$$

For the control volume between sections 1–1 and 2–2, the force applied in the direction of flow is equal to the rate of change of momentum, i.e., $F = \rho Q(V_2 - V_1)$.

Thus $$p_1A_1 - p_2A_2 + p_o(A_2 - A_1) = \rho Q(V_2 - V_1)$$

$$p_1A_1 - p_2A_2 + p_1(A_2 - A_1) = \rho Q(V_2 - V_1) \qquad [\because p_o = p_1]$$

$$p_1A_1 - p_2A_2 + p_1A_2 - p_1A_1 = \rho A_2V_2(V_2 - V_1) \qquad [\because Q = A_2V_2]$$

Rearranging and dividing both sides by g, we get:

$$\frac{p_1 - p_2}{\rho g} = \frac{V_2(V_2 - V_1)}{g} \tag{ii}$$

Substituting expression (ii) in expression (i), we get:

$$(h_L)_e = \frac{V_2(V_2 - V_1)}{g} + \left(\frac{V_1^2 - V_2^2}{2g}\right) = \frac{2V_2^2 - 2V_1V_2 + V_1^2 - V_2^2}{2g}$$

Thus

$$\boxed{(h_L)_e = \frac{V_1^2 + V_2^2 - 2V_1V_2}{2g} = \frac{(V_1 - V_2)^2}{2g}} \tag{14.7}$$

The Equation (14.7) is also sometimes known as Borda–Carnot equation for head loss and the expression is given below.

$$(h_L)_e = \frac{V_1^2}{2g}\left(1 - \frac{A_1}{A_2}\right)^2 = \frac{V_2^2}{2g}\left(\frac{A_2}{A_1} - 1\right)^2 \qquad [\because A_1V_1 = A_2V_2] \tag{14.8}$$

Example 14.7 A horizontal pipe of diameter 0.2 m is suddenly enlarged to 0.3 m. If the rate of flow of water through the pipe is 0.2 m^3/s and the intensity of pressure in the smaller pipe is 100 kPa, then find (i) the head loss due to sudden enlargement, (ii) power loss due to enlargement and (iii) intensity of pressure in the larger diameter pipe.

Solution

Let $D_1 = 0.2$ m, $D_2 = 0.3$ m, $Q = 0.2\ \text{m}^3/\text{s}$ and $p_1 = 100$ kPa.

(i) $V_1 = \dfrac{Q}{A_1} = \dfrac{Q}{(\pi/4)D_1^2} = \dfrac{0.2}{(\pi/4)\times 0.2^2} = 6.3662$ m/s

$$V_2 = \frac{Q}{A_2} = \frac{Q}{(\pi/4)D_2^2} = \frac{0.2}{(\pi/4)\times 0.3^2} = 2.83 \text{ m/s}$$

$$(h_L)_e = \frac{(V_1 - V_2)^2}{2g} = \frac{(6.3662 - 2.83)^2}{2\times 9.81} = \mathbf{0.6373\ m}$$

(ii) $P = \dfrac{\rho_w g Q (h_L)_e}{1000} = \dfrac{1000\times 9.81\times 0.2\times 0.6373}{1000} = \mathbf{1.2504\ kW}$

(iii) $\dfrac{p_1}{\rho_w g} + \dfrac{V_1^2}{2g} + z_1 = \dfrac{p_2}{\rho_w g} + \dfrac{V_2^2}{2g} + z_2 + (h_L)_e$ [Bernoulli's equation]

$$p_2 = \rho_w g\left[\frac{p_1}{\rho_w g} + \frac{V_1^2}{2g} - \frac{V_2^2}{2g} - (h_L)_e\right] \qquad [\because z_1 = z_2]$$

Thus

$$p_2 = 1000\times 9.81\times\left[\frac{100\times 10^3}{1000\times 9.81} + \frac{6.3662^2}{2\times 9.81} - \frac{2.83^2}{2\times 9.81} - 0.6373\right]$$

$$\therefore p_2 = \frac{110007.9}{10^3} \approx \mathbf{110.01\ kPa}$$

Example 14.8 For sudden expansion in a pipe flow, workout the optimum ratio between the diameter of the pipe before expansion and the diameter of pipe after expansion so that the pressure rise is maximum.

Solution

From Bernoulli's theorem, we get:

$$\frac{p_2}{\rho g} - \frac{p_1}{\rho g} = \frac{V_1^2}{2g} - \frac{V_2^2}{2g} - (h_L)_e \qquad [\because z_1 = z_2]$$

$$p_2 - p_1 = \rho g\left[\frac{V_1^2}{2g} - \frac{V_2^2}{2g} - \frac{(V_1 - V_2)^2}{2g}\right] \qquad \text{[Substitute value of } (h_L)_e]$$

$$V_2 = \frac{A_1}{A_2}V_1 = \frac{(\pi/4)D_1^2}{(\pi/4)D_2^2}V_1 = \left(\frac{D_1}{D_2}\right)^2 V_1 = k^2 V_1 \qquad \text{[From Continuity equation]}$$

Thus

$$p_2 - p_1 = \Delta p = \rho g\left[\frac{V_1^2}{2g} - \frac{(k^2V_1)^2}{2g} - \frac{(V_1 - k^2V_1)^2}{2g}\right]$$

$$\Delta p = \rho g \frac{V_1^2}{2g}[1 - k^4 - (1-k^2)^2]$$

For maximum pressure rise: $d(\Delta p)/dk = 0$

$$\frac{d}{dk}\left[\rho g \frac{V_1^2}{2g}\{1 - k^4 - (1-k^2)^2\}\right] = 0$$

Thus

$$-4k^3 - 2(1-k^2)(-2k) = 0$$

$$4k - 8k^3 = 0$$

$$4k(1 - 2k^2) = 0$$

$$(1 - 2k^2) = 0 \qquad [\because k \neq 0]$$

$$k = \frac{1}{\sqrt{2}} \qquad \text{[Neglect –ve root]}$$

$$\therefore \boxed{\left(\frac{D_1}{D_2}\right) = \frac{1}{\sqrt{2}}}$$

Example 14.9 An oil of specific gravity 0.8 flows in 80 mm diameter pipeline. A sudden enlargement occurs into a second pipeline of such a diameter that maximum pressure rise is obtained. If the oil flow rate through the pipeline is 0.0125 m^3/s, then find (i) the loss of energy in the sudden enlargement and (ii) differential gauge length showed by an oil-mercury manometer connected between the two pipes.

Solution

Let $S_{oil} = 0.8$, $D_1 = 80 \text{ mm} = 0.08 \text{ m}$ and $Q = 0.0125 \text{ m}^3/\text{s}$.

(i) $V_1 = \dfrac{Q}{A_1} = \dfrac{Q}{(\pi/4)D_1^2} = \dfrac{0.0125}{(\pi/4)\times 0.08^2} = 2.487 \text{ m/s}$

Diameter of the second pipe for maximum pressure rise is given from previous example.

$$D_2 = \sqrt{2}\times D_1 = \sqrt{2}\times 0.08 = 0.11314 \text{ m}$$

$$V_2 = \frac{Q}{A_2} = \frac{Q}{(\pi/4)D_2^2} = \frac{0.0125}{(\pi/4)\times 0.11314^2} = 1.243 \text{ m/s}$$

$$(h_L)_e = \frac{(V_1 - V_2)^2}{2g} = \frac{(2.487 - 1.243)^2}{2\times 9.81} = \mathbf{0.0789\ m}$$

(ii) From Bernoulli's theorem, we get:

$$\frac{p_2 - p_1}{\rho g} = \frac{V_1^2 - V_2^2}{2g} - (h_L)_e = \frac{2.487^2 - 1.243^2}{2\times 9.81} - 0.0789 = 0.1576 \text{ m}$$

Let y be the reading of the U-tube oil-mercury manometer.

$$\frac{p_2 - p_1}{\rho g} = y\left(\frac{S_m}{S_{\text{oil}}} - 1\right)$$

$$0.1576 = y\left(\frac{13.6}{0.8} - 1\right)$$

$$\therefore y = \frac{0.1576}{16} = \mathbf{0.00985\ m}$$

14.4.2 Loss of Head Due to Sudden Contraction

Consider a fluid flowing through a pipe in which there is a sudden contraction of cross section from A_1 to A_2 as shown in Figure 14.2. Due to sudden reduction in the cross-sectional area of the flow passage, the streamlines converge to a minimum cross section known as vena-contracta (section C–C) and then expand to fill the downstream pipe flow. Consider section 1–1 and section 2–2 before and after the contraction, respectively. Let p_1, V_1 and A_1 be the pressure, velocity and area, respectively, at section 1–1 and p_2, V_2 and A_2 be the corresponding values at the section 2–2 and V_c and A_c be the velocity and area of the flow at section C–C and $(h_L)_c$ be the loss of head due to sudden contraction.

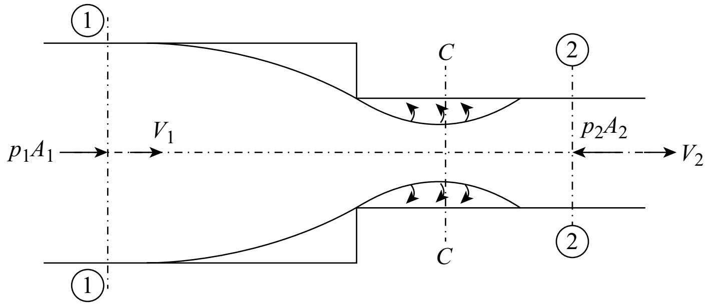


Figure 14.2 *Sudden contraction in a pipe*

The loss of head is negligible during contraction of the flow (i.e., from section 1-1 to section C-C) and thus, the main loss of head is due to the sudden enlargement beyond the vena-contracta (i.e., from section C-C to section 2-2). In fact, $(h_L)_c$ is the loss of head due to sudden enlargement from section C-C to the section 2-2 and it is given from Equation (14.7) as follows.

$$(h_L)_c = \frac{(V_c - V_2)^2}{2g} = \frac{V_2^2}{2g}\left(\frac{V_c}{V_2} - 1\right)^2 \qquad \text{(i)}$$

$$A_c V_c = A_2 V_2 \qquad \text{[Continuity equation]}$$

$$\frac{V_c}{V_2} = \frac{A_2}{A_c} = \frac{1}{(A_c / A_2)} = \frac{1}{C_c}$$

Thus $$V_c = \frac{V_2}{C_c}$$

Substituting this value of $V_c = (V_2/C_c)$ in expression (i), we get:

$$(h_L)_c = \frac{V_2^2}{2g}\left(\frac{V_2}{C_c V_2} - 1\right)^2 = \frac{V_2^2}{2g}\left(\frac{1}{C_c} - 1\right)^2 \qquad (14.9)$$

$$(h_L)_c = \frac{kV_2^2}{2g} \qquad (14.9a)$$

Here, $k = \left(\frac{1}{C_c} - 1\right)^2$

For $C_c = 0.62$, $k = 0.376$

$$(h_L)_c = 0.376\frac{V_2^2}{2g} \qquad (14.10)$$

However, if value of C_c is not given, then assume $k = 0.5$, then Equation (14.9a) is written as follows.

$$\boxed{(h_L)_c = 0.5\frac{V_2^2}{2g}} \qquad (14.11)$$

Example 14.10 A pipe carrying 0.12 m^3/s of water suddenly reduces from 0.4 m to 0.2 m diameter. If the loss of head is 0.3 m, then find the coefficient of contraction.

Solution

Let $Q = 0.12 \text{ m}^3/\text{s}$, $D_1 = 0.4$ m, $D_2 = 0.2$ m and $(h_L)_c = 0.3$ m.

$$V_2 = \frac{Q}{A_2} = \frac{Q}{(\pi/4)D_2^2} = \frac{0.12}{(\pi/4)\times 0.2^2} = 3.82 \text{ m/s}$$

Since $$(h_L)_c = \frac{V_2^2}{2g}\left(\frac{1}{C_c} - 1\right)^2$$

Thus $$0.3 = \frac{3.82^2}{2\times 9.81}\times\left(\frac{1}{C_c} - 1\right)^2$$

$$\frac{1}{C_c} = \sqrt{\frac{0.3\times 2\times 9.81}{3.82^2}} + 1 = 1.63511$$

$$\therefore C_c = \frac{1}{1.63511} = \mathbf{0.6116}$$

Example 14.11 The diameter of a horizontal pipe suddenly reduces from 0.4 m to 0.2 m due to which pressure changes from 125 kN/m^2 to 105 kN/m^2. If the coefficient of contraction is 0.62, then find the flow rate of water.

Solution

Let $D_1 = 0.4$ m, $D_2 = 0.2$ m, $p_1 = 125\ \text{kN/m}^2$, $p_2 = 105\ \text{kN/m}^2$ and $C_c = 0.62$.

$$(h_L)_c = \frac{V_2^2}{2g}\left(\frac{1}{C_c}-1\right)^2 = \frac{V_2^2}{2g}\left(\frac{1}{0.62}-1\right)^2 = 0.376\frac{V_2^2}{2g}$$

From continuity equation, we get:

$$V_1 = \frac{A_2V_2}{A_1} = \frac{(\pi/4)D_2^2}{(\pi/4)D_1^2}V_2 = \left(\frac{D_2}{D_1}\right)^2 V_2 = \left(\frac{0.2}{0.4}\right)^2 V_2 = \frac{V_2}{4}$$

Applying Bernoulli's theorem, we get:

$$\frac{p_1}{\rho_w g}+\frac{V_1^2}{2g} = \frac{p_2}{\rho_w g}+\frac{V_2^2}{2g}+(h_L)_c \qquad [\because z_1 = z_2]$$

Thus

$$\frac{125\times10^3}{1000\times9.81}+\frac{(V_2/4)^2}{2g} = \frac{105\times10^3}{1000\times9.81}+\frac{V_2^2}{2g}+0.376\frac{V_2^2}{2g}$$

$$12.7421+\frac{V_2^2}{16\times2g} = 10.7034+1.376\frac{V_2^2}{2g}$$

$$1.3135\frac{V_2^2}{2g} = 2.0387$$

$$\therefore V_2 = \sqrt{\frac{2.0387\times2g}{1.3135}} = \sqrt{\frac{2.0387\times2\times9.81}{1.3135}} = 5.5184\ \text{m/s}$$

$$Q = A_2V_2 = \frac{\pi}{4}D_2^2\times V_2 = \frac{\pi}{4}\times0.2^2\times5.5184 = \mathbf{0.1734\ m^3/s}$$

Example 14.12 The diameter of a horizontal pipe suddenly reduces from 0.5 m to 0.25 m due to which pressure changes from 135 kN/m^2 to 110 kN/m^2. If the flow rate of water is 0.325 m^3/s, then find the coefficient of contraction.

Solution

Let $D_1 = 0.5$ m, $D_2 = 0.25$ m, $p_1 = 135\ \text{kN/m}^2$, $p_2 = 110\ \text{kN/m}^2$ and $Q = 0.325\ \text{m}^3/\text{s}$.

$$V_1 = \frac{Q}{A_1} = \frac{Q}{(\pi/4)D_1^2} = \frac{0.325}{(\pi/4)\times0.5^2} = 1.6552\ \text{m/s}$$

From continuity equation, we get:

$$V_1 = \frac{A_2V_2}{A_1} = \frac{(\pi/4)D_2^2}{(\pi/4)D_1^2}V_2 = \left(\frac{D_2}{D_1}\right)^2 V_2 = \left(\frac{0.25}{0.5}\right)^2 V_2 = \frac{V_2}{4}$$

Thus

$$V_2 = 4V_1 = 4\times1.6552 = 6.6208\ \text{m/s}$$

Applying Bernoulli's theorem, we get:

$$\frac{p_1}{\rho_w g}+\frac{V_1^2}{2g}=\frac{p_2}{\rho_w g}+\frac{V_2^2}{2g}+(h_L)_c \qquad [\because z_1 = z_2]$$

$$\frac{p_1}{\rho_w g}+\frac{V_1^2}{2g}=\frac{p_2}{\rho_w g}+\frac{V_2^2}{2g}+\frac{V_2^2}{2g}\left(\frac{1}{C_c}-1\right)^2 \qquad \text{[Substituting the value of } (h_L)_c]$$

Thus $$\frac{135\times10^3}{1000\times9.81}+\frac{1.6552^2}{2\times9.81}=\frac{110\times10^3}{1000\times9.81}+\frac{6.6208^2}{2\times9.81}+\frac{6.6208^2}{2\times9.81}\times\left(\frac{1}{C_c}-1\right)^2$$

$$13.7615+0.13964=11.21305+2.2342+2.2342\times\left(\frac{1}{C_c}-1\right)^2$$

$$\frac{1}{C_c}=\sqrt{\frac{0.45389}{2.2342}}+1=1.45073$$

$$\therefore C_c=\frac{1}{1.45073}=\mathbf{0.6893}$$

Example 14.13 A horizontal pipeline carrying water suddenly contracts from a diameter of 0.65 m to 0.4 m. After a short distance, it suddenly enlarges to its original diameter. The pressure values at sections before contraction and at the middle of the contracted region were measured as 120 kPa and 75 kPa, respectively. If the coefficient of contraction is 0.64, then determine the flow rate through the pipe and the pressure of water after enlargement.

Solution

Refer Figure 14.3. Let $D_1 = D_3 = 0.65$ m, $D_2 = 0.4$ m, $p_1 = 120$ kPa, $p_2 = 75$ kPa and $C_c = 0.64$.

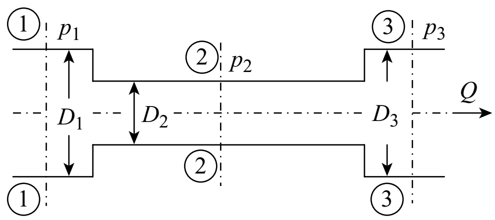


Figure 14.3

From continuity equation, we get:

$$V_1=\frac{A_2V_2}{A_1}=\frac{(\pi/4)D_2^2}{(\pi/4)D_1^2}V_2=\left(\frac{D_2}{D_1}\right)^2V_2=\left(\frac{0.4}{0.65}\right)^2V_2=0.3787V_2$$

$$(h_L)_c=\frac{V_2^2}{2g}\left(\frac{1}{C_c}-1\right)^2=\frac{V_2^2}{2\times9.81}\times\left(\frac{1}{0.64}-1\right)^2=0.01613V_2^2$$

Applying Bernoulli's theorem between sections 1 and 2, we get:

$$\frac{p_1}{\rho_w g}+\frac{V_1^2}{2g}=\frac{p_2}{\rho_w g}+\frac{V_2^2}{2g}+(h_L)_c \qquad [\because z_1 = z_2]$$

$$\frac{120\times10^3}{10^3\times9.81}+\frac{(0.3787V_2)^2}{2\times9.81}=\frac{75\times10^3}{10^3\times9.81}+\frac{V_2^2}{2\times9.81}+0.01613V_2^2$$

$$12.23241+0.0073V_2^2=7.64526+0.05097V_2^2+0.01613V_2^2$$

$$0.0598V_2^2=4.58715$$

$$\therefore V_2 = \sqrt{\frac{4.58715}{0.0598}} = 8.7583 \text{ m/s}$$

$$Q = A_2 V_2 = \frac{\pi}{4} D_2^2 \times V_2 = \frac{\pi}{4} \times 0.4^2 \times 8.7583 = \mathbf{1.101\ m^3/s}$$

Now $V_1 = 0.3787 V_2 = 0.3787 \times 8.7583 = 3.317$ m/s

Thus $V_1 = V_3 = 3.317$ m/s $\quad [\because D_1 = D_3]$

Applying Bernoulli's equation between section 2 and 3, we get:

$$\frac{p_2}{\rho_w g} + \frac{V_2^2}{2g} = \frac{p_3}{\rho_w g} + \frac{V_3^2}{2g} + (h_L)_e \qquad [\because z_2 = z_3]$$

$$\frac{p_2}{\rho_w g} + \frac{V_2^2}{2g} = \frac{p_3}{\rho_w g} + \frac{V_3^2}{2g} + \frac{(V_2 - V_3)^2}{2g} \qquad \text{[Substituting the value of } (h_L)_e]$$

$$\frac{p_3}{\rho_w g} = \left[\frac{p_2}{\rho_w g} + \frac{V_2^2}{2g} - \frac{V_3^2}{2g} - \frac{(V_2 - V_3)^2}{2g}\right]$$

Thus
$$\frac{p_3}{10^3 \times 9.81} = \left[\frac{75 \times 10^3}{10^3 \times 9.81} + \frac{8.7583^2}{2 \times 9.81} - \frac{3.317^2}{2 \times 9.81} - \frac{(8.7583 - 3.317)^2}{2 \times 9.81}\right]$$

$$\therefore p_3 = 9810 \times [7.6452 + 3.9097 - 0.5608 - 1.509] = \mathbf{93.05\ kN/m^2}$$

14.4.3 Loss of Head at the Inlet (Entrance) of a Pipe

When fluids from a large reservoir enter into a pipe, the streamlines first contract and then expand and thereby, it results in the loss of head. It can be considered as a loss of head due to sudden contraction which is denoted by $(h_L)_i$ and its value is given below.

$$(h_L)_i = k\frac{V^2}{2g} \tag{14.12}$$

Here, V is the mean velocity of liquid in the pipe and k is a constant whose value depends on shape at the entry. For a bell mouthed entry $k = 0.04$, for conical entry (with included angle 30° to 60°) $k = 0.18$ and for a sharp edged mouthpiece $k = 0.5$. However, in general, the expression for loss of head at the inlet of a pipe is given below.

$$\boxed{(h_L)_i = 0.5\frac{V^2}{2g}} \tag{14.13}$$

14.4.4 Loss of Head at the Outlet (Exit) of a Pipe

This loss occurs due to sudden expansion when the liquid comes out of a pipe and enters into a tank or discharges into the form of a free jet. It is denoted by $(h_L)_o$. If V is the mean velocity of liquid at the outlet of the pipe, then the expression for this loss is given below.

$$\boxed{(h_L)_o = \frac{V^2}{2g}} \tag{14.14}$$

14.4.5 Loss of Head Due to Obstruction in a Pipe

When a liquid in a pipe flows past any obstruction, there will be a loss of head due to sudden enlargement of the area of flow beyond the obstruction. A vena-contracta is formed beyond section 1–1 (Figure 14.4) after which the velocity of flow at section 2–2 becomes uniform and equals the velocity in the pipe.

Let V be the mean velocity of flow in the main pipe at section 2–2, V_c be the velocity of flow at vena-contracta at section C–C, A be the area of cross section of the main pipe at section 2–2, a be the maximum area of obstruction at section 1–1, $(A-a)$ be the area of pipe at section 1–1, A_c be the area of cross section at the vena-contracta, $C_c = [A_c/(A-a)]$ be the coefficient of contraction and $(h_L)_{obs}$ be the loss of head due to obstruction.

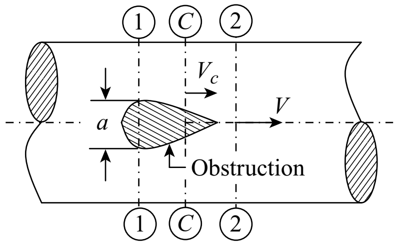


Figure 14.4 *Flow through a pipe having an obstruction*

$$A_c = C_c(A-a) \qquad [\because C_c = A_c/(A-a)]$$

$$A_c V_c = AV \qquad \text{[Continuity equation]}$$

$$C_c(A-a)V_c = AV$$

Thus $$V_c = \frac{AV}{C_c(A-a)}$$

The loss of head due to obstruction is equal to the loss of head due to sudden enlargement from vena-contracta at section C–C to section 2–2 and the expression is given below.

$$(h_L)_{\text{obs}} = \frac{(V_c - V)^2}{2g} \tag{i}$$

Substituting the value of V_c in expression (i), we get:

$$\boxed{(h_L)_{\text{obs}} = \frac{1}{2g}\left[\frac{AV}{C_c(A-a)} - V\right]^2 = \frac{V^2}{2g}\left[\frac{A}{C_c(A-a)} - 1\right]^2} \tag{14.15}$$

The value of C_c depends on the type of obstruction and generally, it varies from 0.60 to 0.66.

14.4.6 Loss of Head Due to Bend in a Pipe

The loss of head due to bend in a pipe $(h_L)_b$ is due to the separation of flow from the boundary and the formation of eddies resulting in the dissipation of energy in turbulence. The expression for loss due to bend in a pipe is given below.

$$\boxed{(h_L)_b = k\frac{V^2}{2g}} \tag{14.16}$$

Here, V is the mean velocity of liquid in the pipe and k is a constant whose value depends on the angle of the bend, radius of curvature of the bend and the diameter of the pipe. The value of k ranges from 0.19 to 0.42 and it is equal to 0.5 for right-angled bend.

14.4.7 Loss of Head in Various Pipe Fittings

All pipe fittings such as valves and couplings inserted into a pipe cause obstruction to flow and thus, it results in loss of head. The loss of head due to various pipe fittings is denoted by $(h_L)_f$ and the expression is given below.

$$\boxed{(h_L)_f = k\frac{V^2}{2g}} \tag{14.17}$$

Here, V is the mean velocity of liquid in the pipe and k is a constant whose value mainly depends on the type of fitting. The value of k for various valves ranges from 0.1 to 10.

Example 14.14 A pipe of length 100 m and diameter 200 mm carries water with a velocity of 4 m/s. For the pipe, if coefficient of friction is $f = 0.002$, then determine the total head loss in it.

Solution

Let $L = 100$ m, $D = 200$ mm $= 0.2$ m, $V = 4$ m/s and $f = 0.002$.

Since $$h_L = (h_L)_i + h_f + (h_L)_o = 0.5\frac{V^2}{2g} + \frac{4fLV^2}{2gD} + \frac{V^2}{2g} = \frac{V^2}{2g}\left[0.5 + \frac{4fL}{D} + 1\right]$$

$$\therefore h_L = \frac{4^2}{2\times 9.81}\times\left[0.5 + \frac{4\times 0.002\times 100}{0.2} + 1\right] = \mathbf{4.4852\ m}$$

Example 14.15 Water contained in a tank to a depth of 4 m above the entrance of a 0.3 m diameter pipe discharged through the end of the pipe is 75 m long. Determine the discharge if the pipe is laid with a slope of 1 in 100 and the friction coefficient is $f = 0.01$.

Solution

Refer Figure 14.5. Let $h_1 = 4$ m, $D = 0.3$ m, $L = 75$ m, $i = (1/100)$ and $f = 0.01$.

$$z_1 = i\times L = \frac{1}{100}\times 75 = 0.75 \text{ m}$$

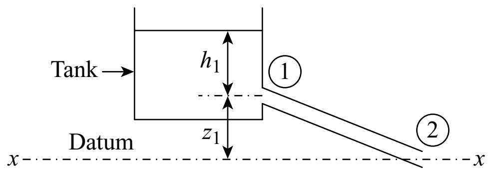


Figure 14.5

Applying Bernoulli's equation to the points 1 and 2 taking x–x as datum, we get:

$$z_1 + h_1 = z_2 + h_L = z_2 + [(h_L)_i + h_f + (h_L)_o] = z_2 + \left[0.5\frac{V^2}{2g} + \frac{4fLV^2}{2gD} + \frac{V^2}{2g}\right]$$

$$z_1 + h_1 = z_2 + \frac{V^2}{2g}\left[0.5 + \frac{4fL}{D} + 1\right]$$

Thus $$0.75 + 4 = 0 + \frac{V^2}{2\times 9.81}\times\left[0.5 + \frac{4\times 0.01\times 75}{0.3} + 1\right]$$

$$4.75 = \frac{11.5V^2}{19.62}$$

$$\therefore V = \sqrt{\frac{4.75\times 19.62}{11.5}} = 2.847 \text{ m/s}$$

$$Q = AV = \frac{\pi}{4}D^2\times V = \frac{\pi}{4}\times 0.3^2\times 2.847 = \mathbf{0.20124\ m^3/s}$$

Example 14.16 A pipe of diameter 30 mm from a large reservoir runs 5 m, then it suddenly enlarges to 60 mm and it runs 3 m, and then discharges as free jet with a velocity of 1.5 m/s. Determine the required height of water surface above the point of discharge if the friction coefficient $f = 0.008$.

Solution

Refer Figure 14.6. Let $D_1 = 30$ mm $= 0.03$ m, $L_1 = 5$ m, $D_2 = 60$ mm $= 0.06$ m, $L_2 = 3$ m, $V_2 = 1.5$ m/s and $f = 0.008$.

$$V_1 = \frac{A_2 V_2}{A_1} = \frac{(\pi/4)D_2^2}{(\pi/4)D_1^2} V_2 = \frac{0.06^2}{0.03^2} \times 1.5 = 6 \text{ m/s} \quad \text{[From Continuity equation]}$$

Total head loss (h_L) will be equal to the difference in reservoir level and the point of discharge as given in the below expression.

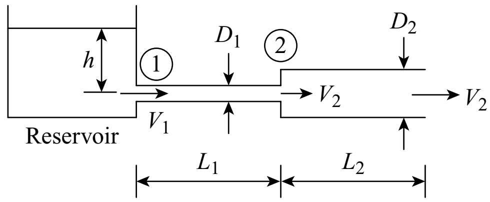


Figure 14.6

$$h_L = (h_L)_i + h_{f1} + (h_L)_e + h_{f2} + (h_L)_o$$

or
$$h_L = 0.5\frac{V_1^2}{2g} + \frac{4fL_1V_1^2}{2gD_1} + \frac{(V_1 - V_2)^2}{2g} + \frac{4fL_2V_2^2}{2gD_2} + \frac{V_2^2}{2g}$$

Thus
$$h_L = \frac{0.5\times 6^2}{2\times 9.81} + \frac{4\times 0.008\times 5\times 6^2}{2\times 9.81\times 0.03} + \frac{(6-1.5)^2}{2\times 9.81} + \frac{4\times 0.008\times 3\times 1.5^2}{2\times 9.81\times 0.06} + \frac{1.5^2}{2\times 9.81}$$

$$\therefore h_L = 0.9174 + 9.786 + 1.0321 + 0.1835 + 0.1147 = \mathbf{12.0337\ m}$$

Example 14.17 Two water tanks are connected by a pipeline of diameter 0.4 m and length 200 m. If the flow rate through the pipe is 0.35 m^3/s and the friction coefficient is $f = 0.009$, then determine the difference in head between the two tanks.

Solution

Refer Figure 14.7. Let $D = 0.4$ m, $L = 200$ m, $Q = 0.35 \text{ m}^3/\text{s}$ and $f = 0.009$.

Let h_1 and h_2 be the heights of water in the first and second tanks, respectively, above the centre of the pipe.

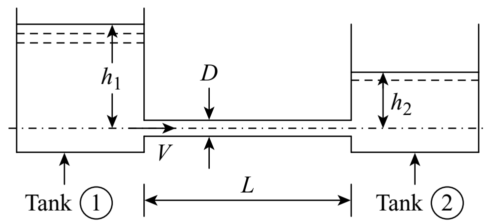


Figure 14.7

$$V = \frac{Q}{A} = \frac{Q}{(\pi/4)D^2} = \frac{0.35}{(\pi/4)\times 0.4^2} = 2.7852 \text{ m/s}$$

Difference between two reservoir level $(h_1 - h_2)$ will be equal to the total head loss (h_L).

$$h_L = (h_L)_i + h_f + (h_L)_o = \left[0.5\frac{V^2}{2g} + \frac{4fLV^2}{2gD} + \frac{V^2}{2g}\right] = \frac{V^2}{2g}\left[0.5 + \frac{4fL}{D} + 1\right]$$

$$\therefore (h_1 - h_2) = \frac{2.7852^2}{2\times 9.81} \times \left[0.5 + \frac{4\times 0.009\times 200}{0.4} + 1\right] = \mathbf{7.71\ m}$$

Example 14.18 A water tank is maintained at constant head of 5 m. The water is discharged through a horizontal pipe of diameter 0.1 m and length 50 m fitted with a valve at the end of the pipe. If the flow rate through the valve when half open is 0.016 m^3/s and the friction coefficient f is 0.008, then determine the value of loss coefficient of the valve.

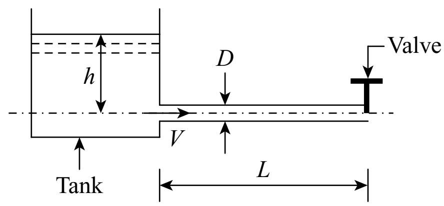


Figure 14.8

Solution

Refer Figure 14.8. Let $h = 5$ m, $D = 0.1$ m, $L = 50$ m, $Q = 0.016 \text{ m}^3/\text{s}$ and $f = 0.008$.

Let k be the loss coefficient of the valve fitted to the pipe.

$$V = \frac{Q}{A} = \frac{Q}{(\pi/4)D^2} = \frac{0.016}{(\pi/4)\times 0.1^2} = 2.037 \text{ m/s}$$

Total head (h) available at the tank will be equal to the total head loss (h_L) in the pipe.

$$h = h_L = (h_L)_i + h_f + (h_L)_o = \left[0.5\frac{V^2}{2g} + \frac{4fLV^2}{2gD} + k\frac{V^2}{2g}\right] = \frac{V^2}{2g}\left[0.5 + \frac{4fL}{D} + k\right]$$

Thus $$5 = \frac{2.037^2}{2\times 9.81}\times\left[0.5 + \frac{4\times 0.008\times 50}{0.1} + k\right]$$

$$5 = 0.2115\times[16.5 + k]$$

$$\therefore k = \frac{5}{0.2115} - 16.5 = \mathbf{7.1407}$$

14.5 ❐ HYDRAULIC GRADIENT LINE AND TOTAL ENERGY LINE

The concept of hydraulic gradient line (H.G.L.) and total energy line (T.E.L.) is a graphical representation of the sum of different heads along the length of the pipe and is quite useful in the study of flow through the pipes.

Hydraulic gradient line It is the line joining the piezometric heads (i.e., the sum of pressure head and datum head) at various points in a flow along the length of the pipe with respect to some reference line. Sometimes, the hydraulic gradient line is also known as piezometric head line. The mathematical expression for piezometric head is $[p/(\rho g) + z]$.

Total energy line It is the line joining the total heads (i.e., the sum of pressure head, kinetic head and datum head) at various points in a flow along the length of the pipe with respect to some reference line. The total energy line is also known as energy gradient line. The mathematical expression for total head is $[p/(\rho g) + V^2/(2g) + z]$.

The hydraulic and energy gradient lines can be obtained as described in the following problems.

Example 14.19 A water tank is maintained at a constant head of 4 m. Water is discharged to the atmosphere through a horizontal pipe of diameter 0.25 m and length 60 m connected to the tank. If the friction coefficient $f = 0.008$, then draw the total energy line and hydraulic gradient line.

Solution

Refer Figure 14.9. Let $h = 4$ m, $D = 0.25$ m, $L = 60$ m and $f = 0.008$.

Total head (h) available at the tank will be equal to the total head loss (h_L) in the pipe.

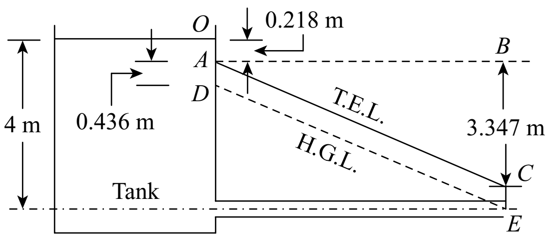


Figure 14.9 *(Not to the scale)*

$$h = h_L = (h_L)_i + h_f + (h_L)_o = \left[0.5\frac{V^2}{2g} + \frac{4fLV^2}{2gD} + \frac{V^2}{2g}\right]$$

$$= \frac{V^2}{2g}\left[0.5 + \frac{4fL}{D} + 1\right]$$

Thus $$4 = \frac{V^2}{2\times 9.81}\times\left[0.5 + \frac{4\times 0.008\times 60}{0.25} + 1\right]$$

$$4 = 0.4679V^2$$

$$\therefore V = \sqrt{\frac{4}{0.4679}} = 2.924 \text{ m/s}$$

$$(h_L)_i = \frac{0.5V^2}{2g} = \frac{0.5 \times 2.924^2}{2 \times 9.81} = 0.218 \text{ m}$$

$$h_f = \frac{4fLV^2}{2gD} = \frac{4 \times 0.008 \times 60 \times 2.924^2}{2 \times 9.81 \times 0.25} = 3.347 \text{ m}$$

Procedure for drawing total energy line:

(i) Take $OA = 0.218$ m as the head loss at the inlet.

(ii) Draw a dotted horizontal line AB through point 'A' equal to the length of the pipe. Locate a point C vertically below B taking BC equal to the loss of head due to friction in the pipe, i.e., 3.347 m.

(iii) Join AC and thus, OAC is the required total energy line.

Procedure for drawing hydraulic gradient line (H.G.L.) Hydraulic gradient line can be obtained by subtracting the velocity head from total energy line.

(i) Take $AD = \text{Velocity head} = \dfrac{V^2}{2g} = \dfrac{2.924^2}{2 \times 9.81} = 0.436 \text{ m}$.

(ii) Draw DE parallel to AC which represents the hydraulic gradient line.

Example 14.20 A horizontal pipeline of length 40 m is connected to a water tank at one end and discharges freely into the atmosphere at the other end. For the first 25 m of its length from the tank, the pipe is 150 mm in diameter and its diameter is suddenly enlarged to 300 mm. The height of water level in the tank is 8 m above the centre of the pipe. Take the friction coefficient $f = 0.01$ for both the sections and draw the total energy line and hydraulic gradient line.

Solution

Refer Figure 14.10. Let $L = 40$ m, $L_1 = 25$ m, $L_2 = 40 - 25 = 15$ m, $D_1 = 150$ mm $= 0.15$ m, $D_2 = 300$ mm $= 0.3$ m, $h = 8$ m and $f = 0.01$.

$$V_1 = \frac{A_2V_2}{A_1} = \frac{(\pi/4)D_2^2}{(\pi/4)D_1^2}V_2 = \frac{0.3^2}{0.15^2}V_2 = 4V_2 \text{ m/s} \qquad \text{[From Continuity equation]}$$

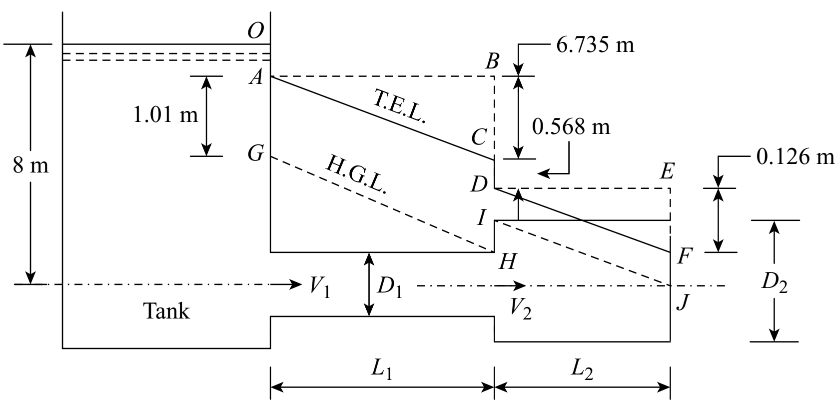


Figure 14.10 *(Not to the scale)*

Total head loss (h_L) will be equal to the difference in reservoir level and the point of discharge as given in the below expression.

$$h_L = h = (h_L)_i + h_{f1} + (h_L)_e + h_{f2} + (h_L)_o$$

or

$$h = \frac{0.5V_1^2}{2g} + \frac{4fL_1V_1^2}{2gD_1} + \frac{(V_1 - V_2)^2}{2g} + \frac{4fL_2V_2^2}{2gD_2} + \frac{V_2^2}{2g}$$

Thus

$$8 = \frac{0.5 \times (4V_2)^2}{2g} + \frac{4 \times 0.01 \times 25 \times (4V_2)^2}{2g \times 0.15} + \frac{(4V_2 - V_2)^2}{2g} + \frac{4 \times 0.01 \times 15 \times V_2^2}{2g \times 0.3} + \frac{V_2^2}{2g}$$

$$8 = \frac{8V_2^2}{2g} + \frac{106.67V_2^2}{2g} + \frac{9V_2^2}{2g} + \frac{2V_2^2}{2g} + \frac{V_2^2}{2g}$$

$$8 \times 2g = 126.67V_2^2$$

$$\therefore V_2 = \sqrt{\frac{8 \times 2 \times 9.81}{126.67}} = 1.113 \text{ m/s}$$

Thus

$$V_1 = 4V_2 = 4 \times 1.113 = 4.452 \text{ m/s}$$

$$(h_L)_i = \frac{0.5V_1^2}{2g} = \frac{0.5 \times 4.452^2}{2 \times 9.81} = 0.505 \text{ m}$$

$$h_{f1} = \frac{4fL_1V_1^2}{2gD_1} = \frac{4 \times 0.01 \times 25 \times 4.452^2}{2 \times 9.81 \times 0.15} = 6.735 \text{ m}$$

$$(h_L)_e = \frac{(V_1 - V_2)^2}{2g} = \frac{(4.452 - 1.113)^2}{2 \times 9.81} = 0.568 \text{ m}$$

$$h_{f2} = \frac{4fL_2V_2^2}{2gD_2} = \frac{4 \times 0.01 \times 15 \times 1.113^2}{2 \times 9.81 \times 0.3} = 0.126 \text{ m}$$

$$(h_L)_o = \frac{V_2^2}{2g} = \frac{1.113^2}{2 \times 9.81} = 0.063 \text{ m}$$

$$\text{Velocity head} = \frac{V_1^2}{2g} = \frac{4.452^2}{2 \times 9.81} = 1.01 \text{ m}$$

Procedure for drawing total energy line (T.E.L.):

(i) From point 'O' lying on the free surface of water, take $OA = 0.505$ m as the head loss at the inlet.

(ii) Draw a dotted horizontal line AB through point 'A' equal to the length of the pipe (i.e., 25 m). Locate a point C vertically below B taking BC equal to the loss of head due to friction in the pipe, i.e., 6.735 m.

(iii) Join AC. From C, draw a line vertically downward $CD = 0.568$ m as the head loss due to enlargement.

(iv) Draw a dotted horizontal line DE equal to the length of the pipe (i.e., 15 m). From E, draw a line vertically downward $EF = 0.126$ m which is equal to the loss of head due to friction in the pipe, i.e., 0.126 m.

(v) Thus $OACDF$ is the required total energy line.

Procedure for drawing hydraulic gradient line (H.G.L.) Hydraulic gradient line can be obtained by subtracting the velocity head from total energy line.

(i) Take AG = velocity head = 1.01 m.

(ii) Draw GH parallel to AC.

(iii) From point J which is the midpoint of the exit pipe (where potential head is zero), draw JI parallel to FD. Join IH.

(iv) Thus, $GHIJ$ represents the required hydraulic gradient line.

Example 14.21 The water tanks are connected by a pipeline consisting of two pipes, one of 0.14 m diameter and length 5 m and the other of diameter 0.21 m and 15 m length. Draw the energy gradient line and hydraulic gradient line, if the difference of water levels in the two tanks is 6 m and the coefficient of friction $f = 0.04$.

Solution

Refer Figure 14.11. Let $D_1 = 0.14$ m, $L_1 = 5$ m, $D_2 = 0.21$ m, $L_2 = 15$ m, $h = 6$ m and $f = 0.04$.

$$V_1 = \frac{A_2V_2}{A_1} = \frac{(\pi/4)D_2^2}{(\pi/4)D_1^2}V_2 = \frac{0.21^2}{0.14^2}V_2 = 2.25V_2 \text{ m/s} \qquad \text{[From Continuity equation]}$$

Total head loss (h_L) will be equal to the difference in water levels of the tanks.

$$h_L = h = (h_L)_i + h_{f1} + (h_L)_e + h_{f2} + (h_L)_o$$

or

$$h = \frac{0.5V_1^2}{2g} + \frac{4fL_1V_1^2}{2gD_1} + \frac{(V_1 - V_2)^2}{2g} + \frac{4fL_2V_2^2}{2gD_2} + \frac{V_2^2}{2g}$$

$$6 = \frac{0.5\times(2.25V_2)^2}{2g} + \frac{4\times 0.04\times 5\times(2.25V_2)^2}{2g\times 0.14} + \frac{(2.25V_2 - V_2)^2}{2g} + \frac{4\times 0.04\times 15V_2^2}{2g\times 0.21} + \frac{V_2^2}{2g}$$

$$6 = \frac{2.53125V_2^2}{2g} + \frac{28.9286V_2^2}{2g} + \frac{1.5625V_2^2}{2g} + \frac{11.4286V_2^2}{2g} + \frac{V_2^2}{2g}$$

$$6\times 2g = 45.451V_2^2$$

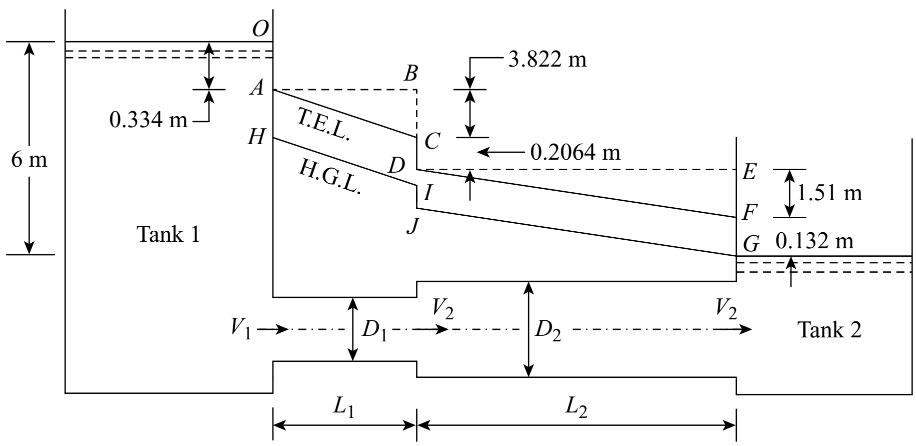


Figure 14.11 *(Not to the scale)*

$$\therefore V_2 = \sqrt{\frac{6\times 2\times 9.81}{45.451}} = 1.61 \text{ m/s}$$

$$V_1 = 2.25V_2 = 2.25\times 1.61 = 3.6225 \text{ m/s}$$

$$(h_L)_i = \frac{0.5V_1^2}{2g} = \frac{0.5\times 3.6225^2}{2\times 9.81} = 0.334 \text{ m}$$

$$h_{f1} = \frac{4fL_1V_1^2}{2gD_1} = \frac{4\times 0.04\times 5\times 3.6225^2}{2\times 9.81\times 0.14} = 3.822 \text{ m}$$

$$(h_L)_e = \frac{(V_1 - V_2)^2}{2g} = \frac{(3.6225-1.61)^2}{2\times 9.81} = 0.2064 \text{ m}$$

$$h_{f2} = \frac{4fL_2V_2^2}{2gD_2} = \frac{4\times 0.04\times 15\times 1.61^2}{2\times 9.81\times 0.21} = 1.51 \text{ m}$$

$$(h_L)_o = \frac{V_2^2}{2g} = \frac{1.61^2}{2\times 9.81} = 0.132 \text{ m}$$

Procedure for drawing total energy line (T.E.L.):

(i) From point 'O' lying on the free surface of water, take OA = 0.334 m as the head loss at the inlet.

(ii) Draw a dotted horizontal line AB through point 'A' equal to the length of the pipe (i.e., 5 m). Locate a point C vertically below B taking BC equal to the loss of head due to friction in the pipe, i.e., 3.822 m.

(iii) Join AC. From C, draw a line vertically downward CD = 0.2064 m as the head loss due to enlargement.

(iv) Draw a dotted horizontal line DE equal to the length of the pipe (i.e., 15 m). From E, draw a line vertically downward EF = 1.51 m which is equal to the loss of head due to friction in the pipe, i.e., 1.51 m. Join DF.

(v) From point F, draw a line vertically downward FG = 0.132 m which is equal to the loss at the exit of the pipe.

(v) Thus, $OACDFG$ is the required total energy line.

Procedure for drawing hydraulic gradient line (H.G.L.) Hydraulic gradient line can be drawn by subtracting $V_1^2/(2g)$ and $V_2^2/(2g)$ from total energy line.

(i) Vertical depth of H.G.L. from AC $= \dfrac{V_1^2}{2g} = \dfrac{3.6225^2}{2\times 9.81} = 0.669$ m

Thus, draw HI parallel to AC.

(ii) Vertical depth of H.G.L. from DF $= \dfrac{V_2^2}{2g} = \dfrac{1.61^2}{2\times 9.81} = 0.132$ m

Thus, draw JG parallel to DF and join IJ.

(iii) The line $HIJG$ represents the required hydraulic gradient line.

Example 14.22 The water is pumped at a flow rate of 20 litres per second into a pipeline ABC of length 200 m which is laid on an upward slope of 1 in 40. The length AB = 100 m and its diameter is 0.1 m and length BC = 100 m and its diameter is 0.2 m. The change in diameter at B is sudden. If the pressure of water at the entry point A is 196.2 kPa, and the coefficient of friction f = 0.008, then determine the pressure of water at point C. Also draw the total gradient and hydraulic gradient lines.

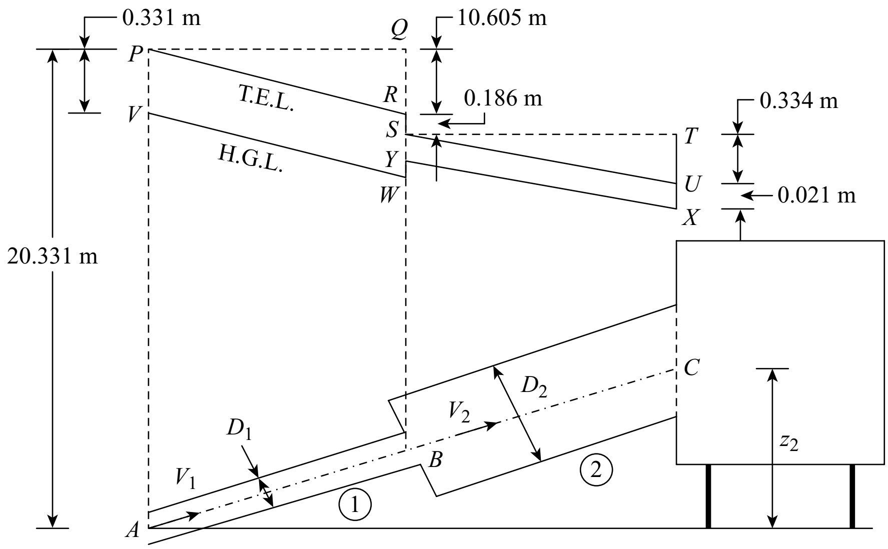


Figure 14.12 *(Not to the scale)*

Solution

Refer Figure 14.12. Let $Q = 20\ l/\text{s} = 0.02\ \text{m}^3/\text{s}$, $L = 200$ m, $i = 1$ in 40, $L_1 = 100$ m, $D_1 = 0.1$ m, $L_2 = 100$ m, $D_2 = 0.2$ m, $p_1 = 196.2$ kPa and $f = 0.008$.

$$V_1 = \frac{Q}{A_1} = \frac{Q}{(\pi/4)D_1^2} = \frac{0.02}{(\pi/4)\times 0.1^2} = 2.55\ \text{m/s}$$

$$V_2 = \frac{Q}{A_2} = \frac{Q}{(\pi/4)D_2^2} = \frac{0.02}{(\pi/4)\times 0.2^2} = 0.64\ \text{m/s}$$

Since $h_L = (h_L)_i + h_{f1} + (h_L)_e + h_{f2} + (h_L)_o$

Thus
$$h_L = \frac{0.5V_1^2}{2g} + \frac{4fL_1V_1^2}{2gD_1} + \frac{(V_1 - V_2)^2}{2g} + \frac{4fL_2V_2^2}{2gD_2} + \frac{V_2^2}{2g}$$

$$= \frac{0.5\times 2.55^2}{2\times 9.81} + \frac{4\times 0.008\times 100\times 2.55^2}{2\times 9.81\times 0.1} + \frac{(2.55-0.64)^2}{2\times 9.81} + \frac{4\times 0.008\times 100\times 0.64^2}{2\times 9.81\times 0.2} + \frac{0.64^2}{2\times 9.81}$$

$$\therefore h_L = 0.166 + 10.605 + 0.186 + 0.334 + 0.021 = 11.312\ \text{m}$$

Applying Bernoulli's equation to the points A and C, we get:

$$\frac{p_1}{\rho_w g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho_w g} + \frac{V_2^2}{2g} + z_2 + h_L$$

$$\frac{196.2\times 10^3}{10^3\times 9.81} + \frac{2.55^2}{2\times 9.81} + 0 = \frac{p_2}{10^3\times 9.81} + \frac{0.64^2}{2\times 9.81} + \left(\frac{1}{40}\times 200\right) + 11.312$$

$$20 + 0.331 = \frac{p_2}{9810} + 0.021 + 5 + 11.312$$

$$\frac{p_2}{9810} = 20.331 - 16.333 = 3.998$$

$$\therefore p_2 = \frac{3.998 \times 9810}{10^3} = \mathbf{39.2204\ kPa}$$

$$(h_L)_i = \frac{0.5V_1^2}{2g} = \frac{0.5 \times 2.55^2}{2 \times 9.81} = 0.166 \text{ m}$$

$$h_{f1} = \frac{4fL_1V_1^2}{2gD_1} = \frac{4 \times 0.008 \times 100 \times 2.55^2}{0.1 \times 2 \times 9.81} = 10.605 \text{ m}$$

$$(h_L)_e = \frac{(V_1 - V_2)^2}{2g} = \frac{(2.55 - 0.64)^2}{2 \times 9.81} = 0.186 \text{ m}$$

$$h_{f2} = \frac{4fL_2V_2^2}{2gD_2} = \frac{4 \times 0.008 \times 100 \times 0.64^2}{0.2 \times 2 \times 9.81} = 0.334 \text{ m}$$

Let datum line passes through point '*A*', then total energy at point '*A*'(E_1) is given in the below expression.

$$E_1 = \frac{p_1}{\rho_w g} + \frac{V_1^2}{2g} + z_1 = \frac{196.2 \times 10^3}{10^3 \times 9.81} + \frac{2.55^2}{2 \times 9.81} + 0 = 20.331 \text{ m}$$

Total energy at point '*B*' (E_2) is given by,

$$E_2 = E_1 - h_{f1} = 20.331 - 10.605 = 9.726 \text{ m}$$

Procedure for drawing total energy line (T.E.L.):

(i) The point C lies at a height of $[(1/40) \times 200] = 5\ m$ from the datum line *x-x* passing through point A. Draw a vertical line AP equal to the total energy at point A, i.e., AP = 20.331 m.

(ii) Draw a dotted horizontal line through P which meets the vertical line through B at Q. From Q, take QR = 10.605 m in the vertical downward direction as frictional head loss in pipe AB.

(iii) From R, locate S such that RS = 0.186 m which represents the head loss due to sudden enlargement.

(iv) Through S, draw a dotted horizontal line which meets the vertical line through C at T. Below point T, locate a point U such that TU = 0.334 m which is equal to the frictional head loss in pipe BC.

(v) Join PRSU which represents the required total energy line.

Procedure for drawing hydraulic gradient line (H.G.L.) Hydraulic gradient line can be drawn by subtracting $V_1^2/(2g)$ and $V_2^2/(2g)$ from total energy line.

(i) Vertical depth of H.G.L. from *PR* is given by,

$$= \frac{V_1^2}{2g} = \frac{2.55^2}{2 \times 9.81} = 0.331 \text{ m}$$

(ii) Vertical depth of H.G.L. from *SU* is given by,

$$= \frac{V_2^2}{2g} = \frac{0.64^2}{2 \times 9.81} = 0.021 \text{ m}$$

(iii) Join *W* to *Y*. Thus, the line *VWYX* represents the hydraulic gradient line.

14.6 ❒ PIPES IN SERIES (COMPOUND PIPES)

The pipes of different diameters and lengths connected end to end to from a pipeline are termed as pipes in series or compound pipes as shown in Figure 14.13.

Let water flow from tank A to tank B through a compound pipe consisting of pipes 1, 2 and 3. Let D_1, D_2 and D_3 be the diameters of pipes 1, 2 and 3, respectively, L_1, L_2 and L_3 be their corresponding lengths, V_1, V_2 and V_3 be the corresponding velocity of flow, f_1, f_2, and f_3 be the corresponding coefficient of frictions and h be the difference of water level in the two tanks.

For a steady flow, the discharge through each pipe is same as given below.

$$Q = A_1V_1 = A_2V_2 = A_3V_3$$

Also the difference in water surface level will be equal to the sum of various head losses in the pipe as given below.

$$h = h_L = (h_L)_i + h_{f1} + (h_L)_c + h_{f2} + (h_L)_e + h_{f3} + (h_L)_o$$

Thus $$h = \frac{0.5V_1^2}{2g} + \frac{4f_1L_1V_1^2}{2gD_1} + \frac{0.5V_2^2}{2g} + \frac{4f_2L_2V_2^2}{2gD_2} + \frac{(V_2 - V_3)^2}{2g} + \frac{4f_3L_3V_3^2}{2gD_3} + \frac{V_3^2}{2g} \tag{14.18}$$

When minor losses are not considered then Equation (14.18) becomes,

$$h = \frac{4f_1L_1V_1^2}{2gD_1} + \frac{4f_2L_2V_2^2}{2gD_2} + \frac{4f_3L_3V_3^2}{2gD_3} \tag{14.19}$$

If $f_1 = f_2 = f_3 = f$, then Equation (14.19) is written as follows.

$$h = \frac{4f}{2g}\left[\frac{L_1V_1^2}{D_1} + \frac{L_2V_2^2}{D_2} + \frac{L_3V_3^2}{D_3}\right] \tag{14.20}$$

Example 14.23 Three pipes connected in series have diameters as 0.3 m, 0.2 m and 0.4 m and lengths as 400 m, 200 m and 300 m and coefficients of friction as 0.007, 0.0072 and 0.0074, respectively. If the pipes join two water reservoirs A and B having a difference in water surface levels as 15 m, then determine the discharge of water considering minor energy losses and neglecting minor energy losses.

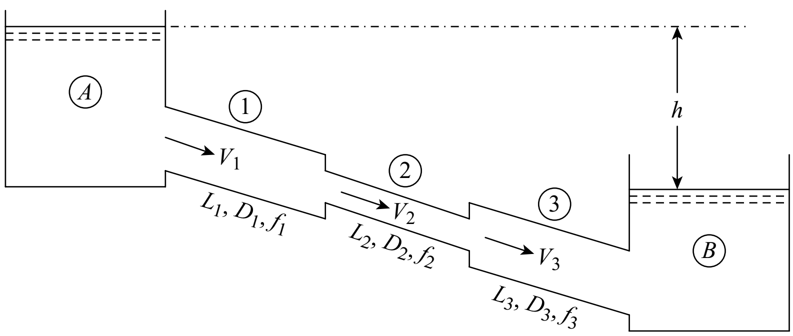


Figure 14.13 *Pipes in series (compound pipes)*

Solution

Refer Figure 14.13. Let $D_1 = 0.3$ m, $D_2 = 0.2$ m, $D_3 = 0.4$ m, $L_1 = 400$ m, $L_2 = 200$ m, $L_3 = 300$ m, $f_1 = 0.007$, $f_2 = 0.0072$, $f_3 = 0.0074$ and $h = 15$ m.

$$A_1V_1 = A_2V_2 = A_3V_3 \qquad \text{[Continuity equation]}$$

$$V_2 = \frac{A_1V_1}{A_2} = \frac{(\pi/4)D_1^2}{(\pi/4)D_2^2}V_1 = \frac{0.3^2}{0.2^2}V_1 = 2.25V_1$$

$$V_3 = \frac{A_1V_1}{A_3} = \frac{(\pi/4)D_1^2}{(\pi/4)D_3^2}V_1 = \frac{0.3^2}{0.4^2}V_1 = 0.5625V_1$$

(i) Considering minor energy losses, we get:

$$(h_L)_i = \frac{0.5V_1^2}{2g}$$

$$h_{f1} = \frac{4f_1L_1V_1^2}{2gD_1} = \frac{4\times 0.007\times 400\times V_1^2}{2g\times 0.3} = 37.33\frac{V_1^2}{2g}$$

$$(h_L)_c = \frac{0.5V_2^2}{2g} = \frac{0.5\times(2.25V_1)^2}{2g} = 2.53\frac{V_1^2}{2g}$$

$$h_{f2} = \frac{4f_2L_2V_2^2}{2gD_2} = \frac{4\times 0.0072\times 200\times(2.25V_1)^2}{2g\times 0.2} = 145.8\frac{V_1^2}{2g}$$

$$(h_L)_e = \frac{(V_2-V_3)^2}{2g} = \frac{(2.25V_1-0.5625V_1)^2}{2g} = 2.85\frac{V_1^2}{2g}$$

$$h_{f3} = \frac{4f_3L_3V_3^2}{2gD_3} = \frac{4\times 0.0074\times 300\times(0.5625V_1)^2}{2g\times 0.4} = 7.02\frac{V_1^2}{2g}$$

$$(h_L)_o = \frac{V_3^2}{2g} = \frac{(0.5625V_1)^2}{2g} = 0.32\frac{V_1^2}{2g}$$

Total head loss (h_L) will be equal to the difference in water levels of the tanks.

$$h_L = h = (h_L)_i + h_{f1} + (h_L)_c + h_{f2} + (h_L)_e + h_{f3} + (h_L)_o$$

Thus

$$15 = 0.5\frac{V_1^2}{2g} + 37.33\frac{V_1^2}{2g} + 2.53\frac{V_1^2}{2g} + 145.8\frac{V_1^2}{2g} + 2.85\frac{V_1^2}{2g} + 7.02\frac{V_1^2}{2g} + 0.32\frac{V_1^2}{2g}$$

$$15 = \frac{V_1^2}{2g}[0.5 + 37.33 + 2.53 + 145.8 + 2.85 + 7.02 + 0.32]$$

$$15\times 2g = 196.35V_1^2$$

$$\therefore V_1 = \sqrt{\frac{15\times 2\times 9.81}{196.35}} = 1.224 \text{ m/s}$$

$$Q = A_1V_1 = \frac{\pi}{4}D_1^2 \times V_1 = \frac{\pi}{4}\times 0.3^2 \times 1.224 = \mathbf{0.08652\ m^3/s}$$

(ii) Neglecting minor energy losses, we get:

$$h_L = h = h_{f1} + h_{f2} + h_{f3}$$

Thus
$$15 = 37.33\frac{V_1^2}{2g} + 145.8\frac{V_1^2}{2g} + 7.02\frac{V_1^2}{2g}$$

$$15 = \frac{V_1^2}{2g}\left[37.33 + 145.8 + 7.02\right]$$

$$15\times 2g = 190.15V_1^2$$

$$\therefore V_1 = \sqrt{\frac{15\times 2\times 9.81}{190.15}} = 1.244 \text{ m/s}$$

$$Q = A_1V_1 = \frac{\pi}{4}D_1^2 \times V_1 = \frac{\pi}{4}\times 0.3^2 \times 1.244 = \mathbf{0.08793\ m^3/s}$$

14.7 ❒ EQUIVALENT PIPE

A pipe of uniform diameter having loss of head and discharge equal to the loss of head and discharge of a compound pipe made of several pipes of different diameters and lengths is called an equivalent pipe. The uniform diameter of the equivalent pipe is known as the equivalent diameter (or size) of the compound pipe. The size of the equivalent pipe may be determined as follows.

Let a compound pipe consisting of pipes 1, 2 and 3 of diameters D_1, D_2 and D_3, respectively and lengths L_1, L_2 and L_3, velocities of flow V_1, V_2 and V_3, coefficient of frictions f_1, f_2 and f_3, and Q be the discharge through it.

Let D be the diameter of the equivalent pipe, L be its length, V be the velocity of flow through it, h be the total head loss and $f = f_1 = f_2 = f_3$ be its friction coefficient then it would carry the same discharge Q.

Neglecting minor losses, then for an equivalent pipe of a compound pipe, the total head loss due to friction remains the same.

Head loss in the equivalent pipe = Head loss in the compound pipe

Thus
$$\frac{4fLV^2}{2gD} = \frac{4f_1L_1V_1^2}{2gD_1} + \frac{4f_2L_2V_2^2}{2gD_2} + \frac{4f_3L_3V_3^2}{2gD_3}$$

$$\frac{LV^2}{D} = \frac{L_1V_1^2}{D_1} + \frac{L_2V_2^2}{D_2} + \frac{L_3V_3^2}{D_3} \qquad [\because f = f_1 = f_2 = f_3] \qquad \text{(i)}$$

$$Q = AV = A_1V_1 = A_2V_2 = A_3V_3 \qquad [\because Q \text{ remains same}]$$

From expression (i), we get:

$$\frac{L}{D}\left(\frac{Q}{A}\right)^2 = \frac{L_1}{D_1}\left(\frac{Q}{A_1}\right)^2 + \frac{L_2}{D_2}\left(\frac{Q}{A_2}\right)^2 + \frac{L_3}{D_3}\left(\frac{Q}{A_3}\right)^2$$

$$\frac{L}{D}\left[\frac{Q}{(\pi/4)D^2}\right]^2 = \frac{L_1}{D_1}\left[\frac{Q}{(\pi/4)D_1^2}\right]^2 + \frac{L_2}{D_2}\left[\frac{Q}{(\pi/4)D_2^2}\right]^2 + \frac{L_3}{D_3}\left[\frac{Q}{(\pi/4)D_3^2}\right]^2$$

$$\boxed{\frac{L}{D^5} = \frac{L_1}{D_1^5} + \frac{L_2}{D_2^5} + \frac{L_3}{D_3^5}} \tag{14.21}$$

The Equation (14.21) is known as Dupuit's equation that can be used to determine the size of the equivalent pipe, i.e., diameter D of the equivalent pipe can be determined when D_1, D_2, D_3 and $L = (L_1 + L_2 + L_3)$ are known.

Example 14.24 Three pipes connected in series have diameters of 0.6 m, 0.5 m and 0.4 m and are of lengths 400 m, 250 m and 200 m, respectively. If these pipes are to be replaced by an equivalent pipe of length 850 m, then determine its diameter.

Solution

Let $D_1 = 0.6$ m , $D_2 = 0.5$ m, $D_3 = 0.4$ m, $L_1 = 400$ m, $L_2 = 250$ m, $L_3 = 200$ m and $L = 850$ m.

Since
$$\frac{L}{D^5} = \frac{L_1}{D_1^5} + \frac{L_2}{D_2^5} + \frac{L_3}{D_3^5}$$

Thus
$$\frac{850}{D^5} = \frac{400}{0.6^5} + \frac{250}{0.5^5} + \frac{200}{0.4^5} = 32675.283$$

$$\therefore D = \left(\frac{850}{32675.283}\right)^{1/5} = \mathbf{0.482\ m}$$

14.8 ❐ PIPES IN PARALLEL

When a pipeline divides into two or more branches which again join together into a single pipe, the flow of liquid through the branch pipes is known as parallel flow as shown in Figure 14.14. The main pipe AB divides into two branches BCE and BDE. Let Q_1, D_1, L_1, V_1 and f_1 refer to branch pipe BCE and Q_2, D_2, L_2, V_2 and f_2 refer to branch pipe BDE.

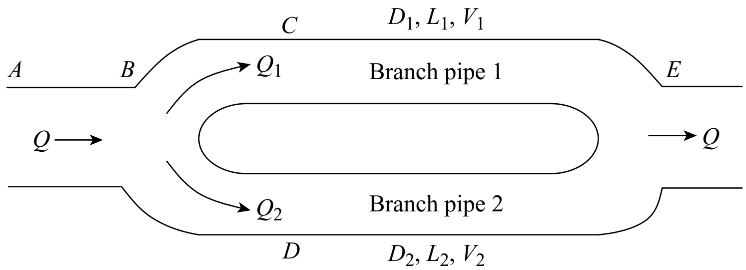


Figure 14.14 *Pipes in parallel*

The discharge in the main pipe (Q) is equal to the sum of discharges (Q_1 and Q_2) through the parallel pipes as given below.

$$Q = Q_1 + Q_2 \tag{14.22}$$

Since the flow of liquid in pipes BCE and BDE occurs under the difference of head between the sections B and E and hence, the head loss in each branch pipes must be same.

Thus $$h_{f1} = h_{f2}$$

$$\frac{4f_1L_1V_1^2}{2gD_1} = \frac{4f_2L_2V_2^2}{2gD_2} \tag{14.23}$$

$$\frac{L_1V_1^2}{D_1} = \frac{L_2V_2^2}{D_2} \qquad [\text{When } f_1 = f_2] \tag{14.23a}$$

Generally, the parallel arrangement is used in water supply system when discharge is to be required to increase through the main supply.

Example 14.25 A main pipe divides into two parallel branch pipes and again forms one pipeline. The diameter and length of parallel pipes are 0.8 m and 1200 m and 0.6 m and 1200 m, respectively. Calculate the flow rate in each parallel pipes if the total flow in the main pipe is 3 m^3/s and the friction coefficients for the first and second parallel pipes are 0.005 and 0.006, respectively.

Solution

Refer Figure 14.14. Let $D_1 = 0.8$ m, $L_1 = 1200$ m, $D_2 = 0.6$ m, $L_2 = 1200$ m, $Q = 3\ \text{m}^3/\text{s}$, $f_1 = 0.005$ and $f_2 = 0.006$.

Since $$\frac{4f_1L_1V_1^2}{2gD_1} = \frac{4f_2L_2V_2^2}{2gD_2}$$

Thus $$\frac{4\times0.005\times1200\times V_1^2}{2\times9.81\times0.8} = \frac{4\times0.006\times1200\times V_2^2}{2\times9.81\times0.6}$$

$$\therefore V_1 = \left(\frac{0.006\times0.8}{0.005\times0.6}\times V_2^2\right)^{1/2} = 1.265V_2$$

Since $$Q = Q_1 + Q_2 = \frac{\pi}{4}D_1^2V_1 + \frac{\pi}{4}D_2^2V_2$$

Thus $$3 = \frac{\pi}{4}\times0.8^2\times1.265V_2 + \frac{\pi}{4}\times0.6^2\times V_2 = 0.9186V_2$$

$$\therefore V_2 = \frac{3}{0.9186} = 3.266\ \text{m/s}$$

$$V_1 = 1.265V_2 = 1.265\times3.266 = 4.1315\ \text{m/s}$$

$$Q_1 = \frac{\pi}{4}D_1^2V_1 = \frac{\pi}{4}\times0.8^2\times4.1315 = \mathbf{2.077\ m^3/s}$$

$$Q_2 = Q - Q_1 = 3 - 2.08 = \mathbf{0.92\ m^3/s}$$

Example 14.26 Two pipes each of length 300 m are available for connecting to a reservoir from which a flow of 0.06 m^3/s is required. If the diameters of the two pipes are 0.15 m and 0.25 m, respectively, then find the ratio of the head lost when the pipes are connected in series to the head lost when they are connected in parallel. Neglect the minor losses and take the friction coefficient as $f = 0.0025$ in Darcy's formula.

Solution

Let $L_1 = L_2 = L = 300$ m, $Q = Q_1 = Q_2 = 0.06\ \text{m}^3/\text{s}$, $D_1 = 0.15$ m, $D_2 = 0.25$ m and $f = 0.0025$.

The head lost due to friction when the pipes are connected in series is given below.

$$(h_f)_{\text{series}} = h_{f1} + h_{f2} = \frac{32 fLQ^2}{\pi^2 g D_1^5} + \frac{32 fLQ^2}{\pi^2 g D_2^5} = \frac{32 fLQ^2}{\pi^2 g}\left[\frac{1}{D_1^5} + \frac{1}{D_2^5}\right]$$

$$\therefore (h_f)_{\text{series}} = \frac{32 \times 0.0025 \times 300 \times 0.06^2}{\pi^2 \times 9.81} \times \left[\frac{1}{0.15^5} + \frac{1}{0.25^5}\right] = 12.6652 \text{ m}$$

Since $\quad Q = Q_1 + Q_2 \quad$ (For parallel pipes)

Thus $$Q_1 + Q_2 = 0.06 \tag{i}$$

Since $\quad h_{f1} = h_{f2} \quad$ (For parallel pipes)

Thus $$\frac{32 fLQ_1^2}{\pi^2 g D_1^5} = \frac{32 fLQ_2^2}{\pi^2 g D_2^5}$$

$$\therefore Q_2 = \left(\frac{D_2}{D_1}\right)^{5/2} Q_1 = \left(\frac{0.25}{0.15}\right)^{5/2} Q_1 = 3.5861 Q_1 \tag{ii}$$

Solving expressions (i) and (ii), we get:

$$Q_1 + 3.5861 Q_1 = 0.06$$

$$\therefore Q_1 = \frac{0.06}{4.5861} = 0.0131 \text{ m}^3/\text{s}$$

$$Q_2 = Q - Q_1 = 0.06 - 0.0131 = 0.0469 \text{ m}^3/\text{s}$$

Head lost for the parallel pipes is given by,

$$(h_f)_{\text{parallel}} = h_{f1} = \frac{32 fLQ_1^2}{\pi^2 g D_1^5} = \frac{32 \times 0.0025 \times 300 \times 0.0131^2}{\pi^2 \times 9.81 \times 0.15^5} = 0.5602 \text{ m}$$

$$\frac{(h_f)_{\text{series}}}{(h_f)_{\text{parallel}}} = \frac{12.6652}{0.5602} = \mathbf{22.61}$$

Example 14.27 A pipeline of diameter 0.6 m and length 1500 m carries water from a tank in which the height of water is maintained at 0.3 m above the axis of the pipe. To increase the discharge, another pipe of the same diameter is connected parallel to original pipe in the second half of the length. If the coefficient of friction for both pipes is $f = 0.01$ and minor losses are neglected, then determine the increase in discharge.

Solution

Refer Figure 14.15. Let $D = D_1 = D_2 = 0.6$ m, $L = 1500$ m, $h = h_f = 0.3$ m, $L_1 = L_2 = L/2 = (1500/2) = 750$ m and $f = f_1 = f_2 = 0.01$.

Case I: When there is only one pipeline of diameter 0.6 m and length 1500 m. Refer Fig 14.15 (a). Let Q_s be the discharge through main single pipe.

$$\because h_f = \frac{32 fLQ_s^2}{\pi^2 gD^5}$$

Thus
$$0.3 = \frac{32\times 0.01\times 1500\times Q_s^2}{\pi^2\times 9.81\times 0.6^5}$$

$$\therefore Q_s = \sqrt{\frac{0.3\times\pi^2\times 9.81\times 0.6^5}{32\times 0.01\times 1500}} = 0.0686 \text{ m}^3/\text{s or } 68.6\ l/\text{s}$$

Case II: When another pipe of diameter 0.6 m is connected in parallel to the second half. Refer Figure 14.15(b). Let Q be the total discharge, Q_1 and Q_2 be the discharge through first and second parallel pipes, respectively.

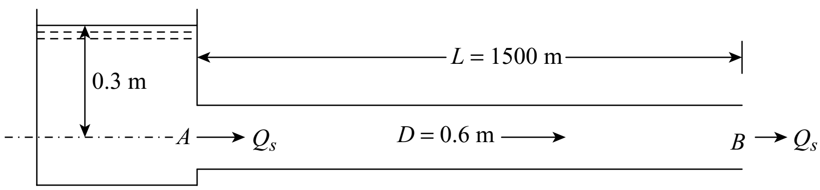
$$Q = Q_1 + Q_2$$

$$Q_1 = Q_2 = \frac{Q}{2} \qquad \text{(Since parallel pipes)}$$

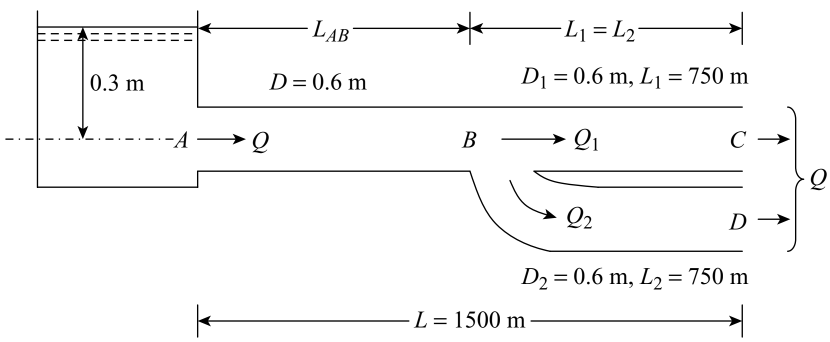
$$L_{AB} = L - L_1 = 1500 - 750 = 750 \text{ m}$$

The head lost due to friction for the pipe ABC (or pipe ABD) is given by,

$$h_f = (h_f)_{AB} + (h_f)_{BC} = (h_f)_{AB} + h_{f1} = \frac{32 fL_{AB}Q^2}{\pi^2 gD^5} + \frac{32 fL_1Q_1^2}{\pi^2 gD_1^5}$$



(a)



(b)

Figure 14.15

Thus
$$0.3 = \frac{32 \times 0.01 \times 750 \times Q^2}{\pi^2 \times 9.81 \times 0.6^5} + \frac{32 \times 0.01 \times 750 \times (Q/2)^2}{\pi^2 \times 9.81 \times 0.6^5}$$

$$0.3 = 31.878Q^2 + 7.9694Q^2 = 39.8474\, Q^2$$

$$\therefore Q = \sqrt{\frac{0.3}{39.8474}} = 0.0868 \text{ m}^3/\text{s or } 86.8\ l/\text{s}$$

The increase in discharge is given by,

$$Q_{\text{increase}} = Q - Q_s = 86.8 - 68.6 = \mathbf{18.2\ \mathit{l}/s}$$

Example 14.28 Two pipes of diameters D and d of equal length L are considered. When the pipes are arranged in parallel, the loss of head for either pipe when a total quantity of water Q flows through them is h but when the pipes are arranged in series the loss of head is H. If $d = (D/2)$, then find the percentage of total flow through each pipe when placed in parallel and the ratio of H to h. Neglect minor losses and assume friction coefficients to be constants.

Solution

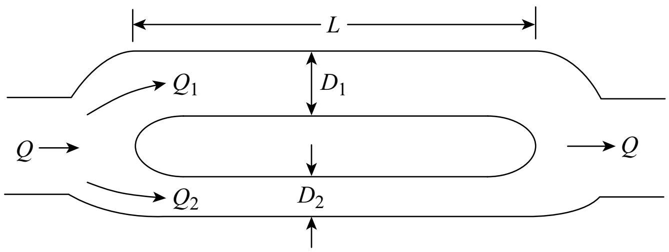
Let $D_1 = D$, $D_2 = d$, $L_1 = L_2 = L$, $(h_f)_{\text{parallel}} = h$, $(h_f)_{\text{series}} = H$, $d = (D/2)$ and $f = f_1 = f_2$.

Case I: Refer Figure 14.16(a), when the pipes are connected in parallel.

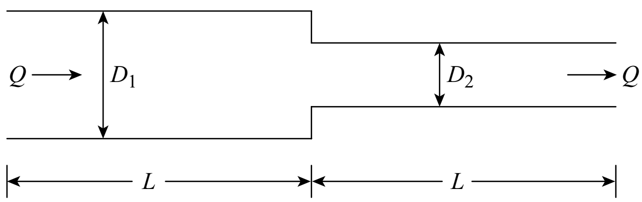
$$Q = Q_1 + Q_2 \tag{i}$$

and
$$h_{f1} = h_{f2}$$

$$\frac{32\, fLQ_1^{\,2}}{\pi^2 gD_1^{\,5}} = \frac{32\, fLQ_2^{\,2}}{\pi^2 gD_2^{\,5}}$$



(a)



(b)

Figure 14.16

$$\frac{Q_1}{Q_2} = \left(\frac{D_1}{D_2}\right)^{5/2} = \left(\frac{D}{D/2}\right)^{5/2} = 5.657$$

$$\therefore Q_1 = 5.657Q_2 \qquad \text{(ii)}$$

Solving expressions (i) and (ii), we get:

$$5.657Q_2 + Q_2 = Q$$

$$Q_2 = \frac{Q}{6.657} = \mathbf{0.1502}\boldsymbol{Q}$$

$$Q_1 = Q - Q_2 = Q - 0.1502Q = \mathbf{0.8498}\boldsymbol{Q}$$

$$h = (h_f)_{\text{parallel}} = h_{f1} = \frac{32fLQ_1^2}{\pi^2 gD_1^5} = \frac{32fL(0.8498Q)^2}{\pi^2 gD^5} = 0.7222\times\frac{32fLQ^2}{\pi^2 gD^5} \qquad \text{(iii)}$$

Case II: Refer Figure 14.16(b), when the pipes are connected in series.

$$H = (h_f)_{\text{series}} = h_{f1} + h_{f2} = \frac{32fLQ^2}{\pi^2 gD_1^5} + \frac{32fLQ^2}{\pi^2 gD_2^5} = \frac{32fLQ^2}{\pi^2 gD^5} + \frac{32fLQ^2}{\pi^2 gd^5}$$

Thus $$H = \frac{32fLQ^2}{\pi^2 gD^5} + \frac{32fLQ^2}{\pi^2 g(D/2)^5} = \frac{32fLQ^2}{\pi^2 g}\left[\frac{1}{D^5} + \frac{32}{D^5}\right] = 33\times\frac{32fLQ^2}{\pi^2 gD^5} \qquad \text{(iv)}$$

Dividing expression (iv) by expression (iii), we get:

$$\frac{H}{h} = 33\times\frac{32fLQ^2}{\pi^2 gD^5}\times\frac{1}{0.7222}\times\frac{\pi^2 gD^5}{32fLQ^2} = \frac{33}{0.7222} = \mathbf{45.694}$$

Example 14.29 A pumping plan forces water through a 0.6 m main, the friction head being 27 m. In order to reduce power consumption, it is proposed to lay another main of appropriate diameter alongside the existing one so that the two pipes may work in parallel for the entire length and reduce the friction head to 9 m only. Find the diameter of the new main if with the exception of the diameter it is similar to the existing one in every aspect.

Solution

Let $D_1 = 0.6$ m, $(h_f)_{\text{single}} = 27$ m and $(h_f)_{\text{parallel}} = 9$ m.

Case I: For single pipe conveying discharge, we get:

$$(h_f)_{\text{single}} = \frac{32fLQ^2}{\pi^2 gD_1^5}$$

$$\frac{32fLQ^2}{\pi^2 g\times 0.6^5} = 27 \qquad \text{(i)}$$

Case II: Refer Figure 14.16(a), when second pipe of diameter D_2 is connected in parallel.

$$Q = Q_1 + Q_2$$

and $$h_{f1} = h_{f2} = 9$$

$$\frac{32 fLQ_1^2}{\pi^2 g \times D_1^5} = \frac{32 fLQ_2^2}{\pi^2 gD_2^5} = 9$$

Thus $$\frac{32 fLQ_1^2}{\pi^2 g \times 0.6^5} = \frac{32 fLQ_2^2}{\pi^2 gD_2^5} = 9 \qquad [\because f \text{ and } L \text{ are same}] \qquad \text{(ii)}$$

$$\frac{Q_2}{Q_1} = \left(\frac{D_2}{0.6}\right)^{5/2} \qquad \text{(iii)}$$

Dividing expression (i) by expression (ii), we get:

$$\frac{32 fLQ^2}{\pi^2 g \times 0.6^5} \times \frac{\pi^2 g \times 0.6^5}{32 fLQ_1^2} = \frac{27}{9}$$

Thus $$\frac{Q^2}{Q_1^2} = 3 \text{ or } Q_1^2 = \frac{Q^2}{3}$$

$$\therefore Q_1 = \frac{Q}{\sqrt{3}} = 0.5773Q$$

$$Q_2 = Q - Q_1 = Q - 0.57735Q = 0.42265Q$$

Substituting the values of Q_1 and Q_2 in expression (iii), we get:

$$\frac{0.42265Q}{0.57735Q} = \left(\frac{D_2}{0.6}\right)^{5/2}$$

$$\therefore D_2 = 0.6 \times \left(\frac{0.42265}{0.57735}\right)^{2/5} = \mathbf{0.5296\ m}$$

Example 14.30 Two tanks having a difference in water level of 8.4 m are connected by pipelines. First a pipe of diameter 0.5 m and length 1500 m connects the first tank at one end and its other end connects to a junction from which two parallel pipes of lengths 750 m and diameter 0.25 m connects to another tank placed at lower level. If the coefficient of frictions 'f' for the first pipe is 0.01 and for both the parallel pipes are 0.006, then determine the total discharge through the pipe from the first to the second water tank.

Solution

Refer Figure 14.17. Let $h = h_f = 8.4$ m, $D_1 = 0.5$ m, $L_1 = 1500$ m, $L_2 = L_3 = 750$ m, $D_2 = D_3 = 0.25$ m, $f_1 = 0.01$ and $f_2 = f_3 = 0.006$.

Let Q_1 be the total discharge through the main pipe, Q_2 and Q_3 be the discharges through the parallel pipes.

Since $$h_{f2} = h_{f3} \qquad \text{(Parallel pipes)}$$

Thus $$\frac{4 f_2 L_2 V_2^2}{2gD_2} = \frac{4 f_3 L_3 V_3^2}{2gD_3}$$

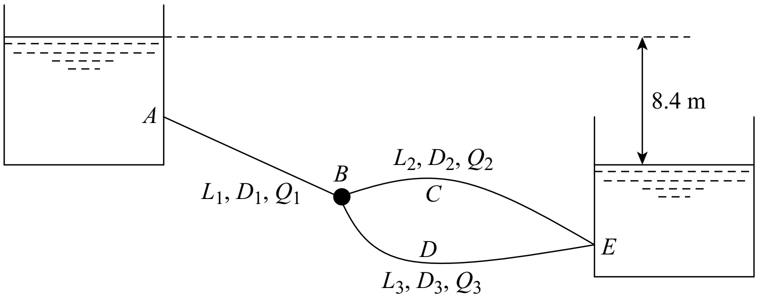


Figure 14.17

$$\therefore V_2 = V_3 \qquad [\because f, L \text{ and } D \text{ are equal}]$$

Thus $$Q_2 = Q_3$$

Now $$Q_1 = Q_2 + Q_3 = 2Q_2 \qquad [\because Q_2 = Q_3]$$

$$Q_2 = \frac{Q_1}{2}$$

The friction loss through pipe *ABCE* (or *ABDE*) is given by,

$$h = h_f = (h_f)_{AB} + (h_f)_{BCE} = h_{f1} + h_{f2} = \frac{32 f_1 L_1 Q_1^2}{\pi^2 g D_1^5} + \frac{32 f_2 L_2 Q_2^2}{\pi^2 g D_2^5}$$

$$8.4 = \frac{32 \times 0.01 \times 1500 \times Q_1^2}{\pi^2 \times 9.81 \times 0.5^5} + \frac{32 \times 0.006 \times 750 \times (Q_1/2)^2}{\pi^2 \times 9.81 \times 0.25^5} = 539.39 Q_1^2$$

$$\therefore Q_1 = \sqrt{\frac{8.4}{539.39}} = \mathbf{0.1248\ m^3/s}$$

Example 14.31 Two tanks having a difference of water level 20 m are connected by a pipe of diameter 0.5 m and length 3000 m. If the pipe is tapped at a distance of 1000 m from the beginning of pipe and water is drawn at a rate of 0.2 m^3/s, then determine the rate of flow of water into the lower tank. Take $f = 0.005$ and neglect minor losses.

Solution

Refer Figure 14.18. Let $h = h_f = 20$ m, $D_1 = D_2 = 0.5$ m, $L = 3000$ m, $L_1 = 1000$ m, $L_2 = 3000 - 1000 = 2000$ m, $q = 0.2$ m^3/s and $f_1 = f_2 = 0.005$.

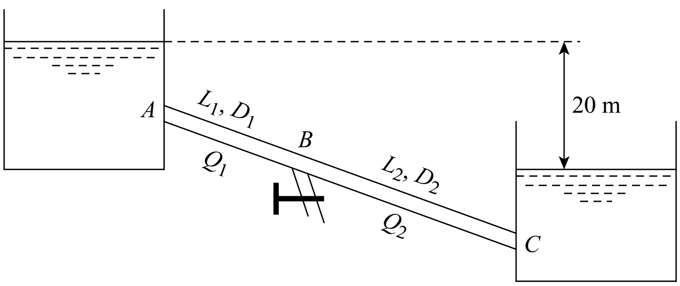


Figure 14.18

Let Q_1 and Q_2 be the discharges through main pipe before and after tapping.

$$Q_2 = Q_1 - q = (Q_1 - 0.2)\ \text{m}^3/\text{s}$$

The head lost due to friction for the pipe *ABC* is given by,

$$h_f = (h_f)_{AB} + (h_f)_{BC} = h_{f1} + h_{f2} = \frac{32 f_1 L_1 Q_1^2}{\pi^2 g D_1^5} + \frac{32 f_2 L_2 Q_2^2}{\pi^2 g D_2^5}$$

Thus
$$20 = \frac{32 \times 0.005 \times 1000 \times Q_1^2}{\pi^2 \times 9.81 \times 0.5^5} + \frac{32 \times 0.005 \times 2000 \times (Q_1 - 0.2)^2}{\pi^2 \times 9.81 \times 0.5^5}$$

$$20 = 52.8812 Q_1^2 + 105.7624 (Q_1 - 0.2)^2$$

$$20 = 52.8812 Q_1^2 + 105.7624 Q_1^2 + 4.2305 - 42.305 Q_1$$

$$158.6436 Q_1^2 - 42.305 Q_1 - 15.7695 = 0$$

$$\therefore Q_1 = \frac{42.305 \pm \sqrt{42.305^2 + 4 \times 158.6436 \times 15.7695}}{2 \times 158.6436} = 0.47565\ \text{m}^3/\text{s}$$

$$\therefore Q_2 = Q_1 - 0.2 = 0.47565 - 0.2 = \mathbf{0.27565\ m^3/s}$$

Example 14.32 For the distribution main of a city water supply, a pipe of diameter 0.3 m and length *L* metre is required. As pipes above 0.25 m diameter are not readily available, it is decided to lay two parallel pipes of the same diameter and of length *L* m. Determine the diameter of the parallel pipes if the total discharge in the parallel pipes is same as in the single main pipe and the coefficient of friction is same for all the pipes.

Solution

Let $D = 0.3$ m, $L_1 = L_2 = L$, $D_1 = D_2$, $Q = (Q_1 + Q_2)$ and $f_1 = f_2 = f$, where Q and V is the discharge and velocity, respectively, in the single main pipe, Q_1, Q_2 and V_1, V_2 are the discharges and velocities in the parallel pipes, respectively.

$$Q = Q_1 + Q_2 = 2Q_1 \qquad [\because Q_2 = Q_1]$$

Thus
$$AV = 2A_1 V_1$$

$$\frac{\pi}{4} D^2 \times V = 2 \times \frac{\pi}{4} D_1^2 \times V_1$$

$$\frac{V}{V_1} = 2\frac{D_1^2}{D^2} \qquad \text{(i)}$$

Now
$$h_f = h_{f1}$$

Thus
$$\frac{4fLV^2}{2gD} = \frac{4fLV_1^2}{2gD_1}$$

$$\left(\frac{V}{V_1}\right)^2 = \frac{D}{D_1} \qquad \text{(ii)}$$

Thus
$$\left(2\frac{D_1^2}{D^2}\right)^2 = \frac{D}{D_1} \qquad \text{[Substitute (i) in (ii)]}$$

$$4\frac{D_1^4}{D^4} = \frac{D}{D_1}$$

$$D_1^5 = \frac{D^5}{4}$$

$$\therefore D_1 = D \times \left(\frac{1}{4}\right)^{1/5} = 0.3 \times \left(\frac{1}{4}\right)^{1/5} = \mathbf{0.22736\ m}$$

14.9 ❐ BRANCHED PIPE SYSTEM

In a branched pipe system, three or more reservoirs having different free surface liquid levels are connected by means of pipes (main pipe and branches) which meet at a junction. The Figure 14.19 illustrates a branched pipe system in which three reservoirs *A*, *B* and *C* are connected to a junction *D* by means of three long pipes, namely 1, 2 and 3.

For solving branched pipe problems the assumptions are (i) the reservoirs are assumed large so that its liquid surface levels remain constant, (ii) the lengths, diameters and friction factors of pipes are known, (iii) flow is steady and (iv) minor losses are neglected. The three basic equations, namely continuity equation, Bernoulli's equation and Darcy–Weisbach equation are employed to solve branched pipes problems.

Let z_A, z_B and z_C be the heights of liquid free surface in the reservoirs *A*, *B* and *C*, respectively, z_D be the height of junction *D* from the datum and $p_D/(\rho g)$ be the pressure head at the junction *D*. Let L_1, L_2 and L_3 be the lengths, D_1, D_2 and D_3 be the diameters, V_1, V_2 and V_3 be the velocities, Q_1, Q_2 and Q_3 be the discharges, and h_{f1}, h_{f2} and h_{f3} be the loss of heads due to friction in the pipes 1, 2 and 3, respectively.

Considering that the flow is occurring from reservoir *A* to *B* and *C*. The flow from reservoir *A* occurs to junction *D* and then from junction *D* to reservoirs *C* and *B*. The flow from junction *D* to reservoir *B* will occur only if piezometric head at *D*, i.e., $[p_D/(\rho g) + z_D]$ is more than the piezometric head at *B*, i.e., z_B.

Applying Bernoulli's equation between the reservoir *A* and the junction *D*,

$$z_A = \left(\frac{p_D}{\rho g} + z_D\right) + h_{f1} \tag{14.24}$$

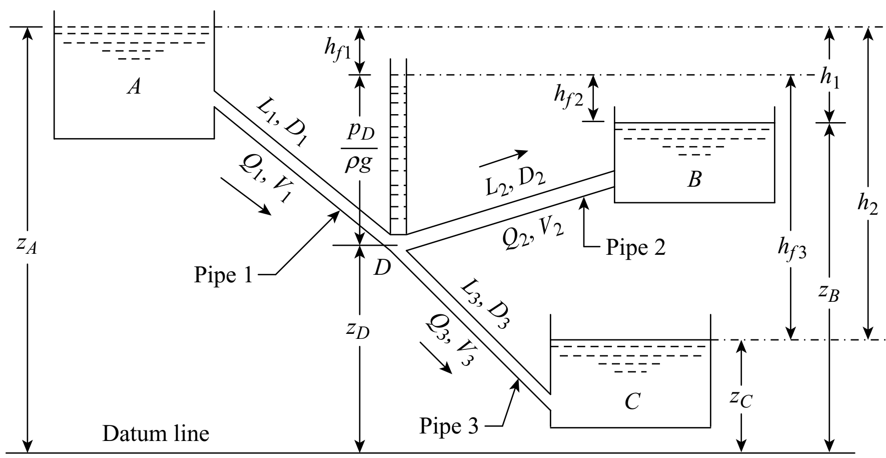


Figure 14.19 *Branched pipe system*

Applying Bernoulli's equation between junction *D* and the reservoir *B*,

$$\left(\frac{p_D}{\rho g}+z_D\right)=z_B+h_{f2} \tag{14.25}$$

Applying Bernoulli's equation between junction *D* and the reservoir *C*,

$$\left(\frac{p_D}{\rho g}+z_D\right)=z_C+h_{f3} \tag{14.26}$$

From continuity equation, we get:

$$Q_1=Q_2+Q_3 \tag{14.27}$$

Equations (14.24) to (14.27) are used for determining the four unknowns, namely Q_1, Q_2, Q_3 and $p_D/(\rho g)$.

The difference in levels of liquid surfaces between the tanks *A* and *B* is given as $h_1=h_{f1}+h_{f2}$ and the difference in levels of liquid surfaces between the tanks *A* and *C* is given as $h_2=h_{f1}+h_{f3}$.

It is to be noted that when the piezometric head at *D*, i.e., $[p_D/(\rho g)+z_D]$ is less than z_B then the liquid flows from reservoir *B* to junction *D*. In that case, both the reservoirs *A* and *B* would supply water to the reservoir *C* and hence, the equations will change accordingly.

Example 14.33 Three reservoirs *A*, *B* and *C* are connected by a piping system as shown in Figure 14.19. The lengths and diameters of pipes 1, 2 and 3 are (1250 m, 0.3 m), (650 m, 0.2 m) and (850 m, 0.3 m), respectively. Determine the discharge into or from reservoirs *B* and *C* if the discharge from reservoir A is 65 litres per second and the heights of liquid free surface from the datum in the reservoirs *A* and *B* are 40 m and 38 m, respectively. Take $f=0.006$ for all pipes and neglect minor losses. Also determine the height of water level in the reservoir *C*.

Solution

Refer Figure 14.19. Let $L_1=1250$ m, $D_1=0.3$ m, $L_2=650$ m, $D_2=0.2$ m, $L_3=850$ m, $D_3=0.3$ m, $Q_1=65$ *l*/s = $0.065\ \text{m}^3/\text{s}$, $z_A=40$ m, $z_B=38$ m and $f=0.006$.

$$V_1=\frac{Q_1}{A_1}=\frac{Q_1}{(\pi/4)D_1^2}=\frac{0.065}{(\pi/4)\times 0.3^2}=0.92\text{ m/s}$$

$$h_{f1}=\frac{4fL_1V_1^2}{2gD_1}=\frac{4\times 0.006\times 1250\times 0.92^2}{2\times 9.81\times 0.3}=4.314\text{ m}$$

Applying Bernoulli's equation between the reservoir *A* and the junction *D*, we get:

$$z_A=z_D+\frac{p_D}{\rho_w g}+h_{f1}$$

$$40=z_D+\frac{p_D}{\rho_w g}+4.314$$

Thus
$$z_D+\frac{p_D}{\rho_w g}=(40-4.314)=35.686\text{ m}$$

As piezometric head at *D*, i.e., $[z_D+p_D/(\rho_w g)]=35.686$ m is less than $z_B=38$ m and therefore, water flows from reservoir *B* to junction *D*.

Applying Bernoulli's equation between the reservoir *B* and the junction *D*, we get:

$$z_B=z_D+\frac{p_D}{\rho_w g}+h_{f2}$$

$$38 = 35.686 + h_{f2}$$

$$\therefore h_{f2} = (38 - 35.686) = 2.314 \text{ m}$$

But
$$h_{f2} = \frac{4fL_2V_2^2}{2gD_2}$$

Thus
$$2.314 = \frac{4\times 0.006\times 650\times V_2^2}{2\times 9.81\times 0.2}$$

$$\therefore V_2 = \sqrt{\frac{2.314\times 2\times 9.81\times 0.2}{4\times 0.006\times 650}} = 0.763 \text{ m/s}$$

$$Q_2 = A_2V_2 = \frac{\pi}{4}D_2^2\times V_2 = \frac{\pi}{4}\times 0.2^2\times 0.763 = \mathbf{0.02397\ m^3/s}$$

$$\therefore Q_3 = Q_1 + Q_2 = 0.065 + 0.02397 = \mathbf{0.08897\ m^3/s}$$

$$V_3 = \frac{Q_3}{A_3} = \frac{Q_3}{(\pi/4)D_3^2} = \frac{0.08897}{(\pi/4)\times 0.3^2} = 1.259 \text{ m/s}$$

Applying Bernoulli's equation between the junction D and reservoir C, we get:

$$z_D + \frac{p_D}{\rho_w g} = z_C + h_{f3}$$

$$z_D + \frac{p_D}{\rho_w g} = z_C + \frac{4fL_3V_3^2}{2gD_3}$$

Thus
$$35.686 = z_C + \frac{4\times 0.006\times 850\times 1.259^2}{2\times 9.81\times 0.3}$$

$$35.686 = z_C + 5.494$$

$$\therefore z_C = (35.686 - 5.494) = \mathbf{30.192\ m}$$

Example 14.34 If the data in a branched pipe system shown in Figure 14.19 are $L_1 = 1000$ m, $D_1 = 0.16$ m, $L_2 = 800$ m, $D_2 = 0.2$ m, $L_3 = 600$ m, $D_3 = 0.24$ m, $Q_2 = 0.32$ m^3/s, $Q_3 = 0.24$ m^3/s and for all the pipes $f = 0.01$, then determine the difference in levels of water surfaces between the tanks A and B, and A and C.

Solution

Refer Figure 14.19. Let $L_1 = 1000$ m, $D_1 = 0.16$ m, $L_2 = 800$ m, $D_2 = 0.2$ m, $L_3 = 600$ m, $D_3 = 0.24$ m, $Q_2 = 0.32$ m^3/s, $Q_3 = 0.24$ m^3/s and $f = 0.01$.

Let h_1 be the difference of water surface between tank A and B and h_2 be the difference of water surface levels between tank A and C.

$$Q_1 = Q_2 + Q_3 = 0.32 + 0.24 = 0.56 \text{ m}^3\text{/s}$$

$$\because h_1 = h_{f1} + h_{f2} = \frac{32fL_1Q_1^2}{\pi^2 gD_1^5} + \frac{32fL_2Q_2^2}{\pi^2 gD_2^5}$$

$$\therefore h_1 = \frac{32 \times 0.01 \times 1000 \times 0.56^2}{\pi^2 \times 9.81 \times 0.16^5} + \frac{32 \times 0.01 \times 800 \times 0.32^2}{\pi^2 \times 9.81 \times 0.2^5} = \mathbf{10730.66\ m}$$

$$\because h_2 = h_{f1} + h_{f3} = \frac{32 f L_1 Q_1^2}{\pi^2 g D_1^5} + \frac{32 f L_3 Q_3^2}{\pi^2 g D_3^5}$$

$$\therefore h_2 = \frac{32 \times 0.01 \times 1000 \times 0.56^2}{\pi^2 \times 9.81 \times 0.16^5} + \frac{32 \times 0.01 \times 600 \times 0.24^2}{\pi^2 \times 9.81 \times 0.24^5} = \mathbf{10028.01\ m}$$

Example 14.35 A pipeline of diameter 0.5 m and length 4000 m connects two reservoirs P and Q whose constant difference of water level is 10 m. At a distance of 1500 m from reservoir P, a 1000 m long branched pipe leads to reservoir R whose water level is 15 m below that of reservoir P. If the flow into both the reservoirs Q and R is same, then determine the diameter of the branched pipe. Take $f = 0.0075$ for all pipes and neglect minor losses.

Solution

Refer Figure 14.20. Let $D_1 = D_2 = 0.5$ m, $L = 4000$ m, $h_1 = 10$ m, $L_1 = 1500$ m, $L_2 = 4000 - 1500 = 2500$ m, $L_3 = 1000$ m, $h_2 = 15$ m, $Q_2 = Q_3$ and $f = 0.0075$.
Let D_3 be the diameter of the branched pipe.

$$Q_2 = Q_3 = \frac{Q_1}{2}$$

$$\because h_1 = h_{f1} + h_{f2} = \frac{32 f L_1 Q_1^2}{\pi^2 g D_1^5} + \frac{32 f L_2 Q_2^2}{\pi^2 g D_2^5}$$

Thus

$$10 = \frac{32 \times 0.0075 \times 1500 \times Q_1^2}{\pi^2 \times 9.81 \times 0.5^5} + \frac{32 \times 0.0075 \times 2500 \times (Q_1/2)^2}{\pi^2 \times 9.81 \times 0.5^5}$$

$$10 = 168.56 Q_1^2$$

$$\therefore Q_1 = \sqrt{\frac{10}{168.56}} = 0.2436\ \text{m}^3/\text{s}$$

$$\therefore Q_2 = Q_3 = \frac{0.2436}{2} = \mathbf{0.1218\ m^3/s}$$

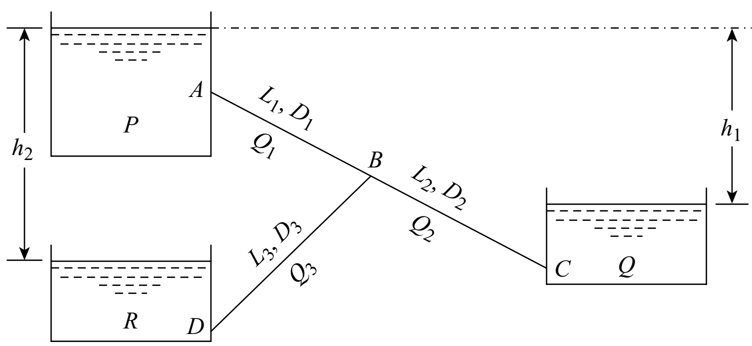


Figure 14.20

$$\because h_2 = h_{f1} + h_{f3} = \frac{32 f L_1 Q_1^2}{\pi^2 g D_1^5} + \frac{32 f L_3 Q_3^2}{\pi^2 g D_3^5}$$

Thus
$$15 = \frac{32 \times 0.0075 \times 1500 \times 0.2436^2}{\pi^2 \times 9.81 \times 0.5^5} + \frac{32 \times 0.0075 \times 1000 \times 0.1218^2}{\pi^2 \times 9.81 \times D_3^5}$$

$$15 = 7.0605 + \frac{0.036774}{D_3^5}$$

$$\therefore D_3 = \left(\frac{0.036774}{15 - 7.0605}\right)^{1/5} = \mathbf{0.34131\ m}$$

14.10 ❐ SIPHON

A siphon is a long bent pipe used to carry water from one reservoir at a higher level to another reservoir at a lower level when the two reservoirs are separated by a hill or high level ground. For some length, from the entrance section, a siphon will rise above the water level in the upper reservoir and then for the remaining length it gets down and connects the lower reservoir as shown in Figure 14.21.

The rising portion of the siphon is known as the inlet limb (or inlet leg). The highest point (*S*) of the siphon is known as summit. The portion between the summit and the lower reservoir is known as outlet limb (or outlet leg). The pressure at point S is less than the pressure at point A since it is above point *A*. Thus, pressure at summit is below the atmospheric pressure. Theoretically, the pressure at S may be reduced to –10.3 m of water. However, at –7.6 m of water head (i.e., 10.3 – 7.6 = 2.7 m absolute), the dissolved air and gases would come out from water and collect at summit to form airlock and thereby, obstruct the flow. Therefore, the siphon should be laid so that no section of the pipe will be more than 7.6 m above the hydraulic gradient line at the section. Moreover, to limit the reduction of the pressure at point S, the length of inlet limb is to be shortened which decreases the frictional losses.

Example 14.36 A siphon of diameter 0.2 m and length 1500 m connects two tanks having a difference in water level of 20 m. The summit is 4 m above the water level in the upper tank. Determine the maximum length of the siphon from the upper tank to the summit if separation occurs at 2.74 m of water absolute. Neglect minor losses, take the coefficient of friction $f = 0.005$ and atmospheric pressure = 10.33 m of water.

Solution

Refer Figure 14.21. Let $D = 0.2$ m, $L = 1500$ m, $h_1 = 20$ m, $h = 4$ m, $p_S/(\rho_w g) = 2.74$ m, $f = 0.005$ m, and $p_A/(\rho_w g) = 10.33$ m.

Let L_1 be the maximum length of the siphon from the upper tank to the summit.

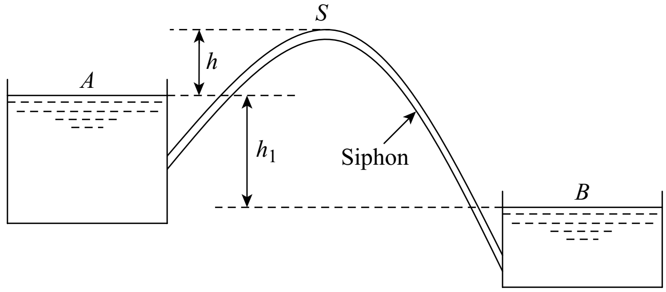


Figure 14.21 *Siphon*

Applying Bernoulli's equation to points A and B and assuming B as datum, we get:

$$\frac{p_A}{\rho_w g}+\frac{V_A^2}{2g}+z_A=\frac{p_B}{\rho_w g}+\frac{V_B^2}{2g}+z_B+(h_f)_{AB}$$

Thus $$10.33+0+20=10.33+0+0+(h_f)_{AB} \qquad [\because z_A=h_1]$$

$$\therefore (h_f)_{AB}=20$$

Thus $$\frac{4fLV^2}{2gD}=20$$

$$\frac{4\times 0.005\times 1500\times V^2}{2\times 9.81\times 0.2}=20$$

$$\therefore V=\sqrt{\frac{20\times 2\times 9.81\times 0.2}{4\times 0.005\times 1500}}=1.6174 \text{ m/s}$$

Now applying Bernoulli's equation to points A and S and assuming A as datum, we get:

$$\frac{p_A}{\rho_w g}+\frac{V_A^2}{2g}+z_A=\frac{p_S}{\rho_w g}+\frac{V^2}{2g}+z_S+(h_f)_{AS} \qquad [\because V_S=V]$$

Thus $$10.33+0+0=2.74+\frac{1.6174^2}{2\times 9.81}+4+\frac{4fL_1V^2}{2gD} \qquad [\because z_S=h]$$

$$10.33=6.8733+\frac{4\times 0.005\times L_1\times 1.6174^2}{2\times 9.81\times 0.2}$$

$$0.01333L_1=10.33-6.8733=3.4567$$

$$\therefore L_1=\frac{3.4567}{0.01333}=\mathbf{259.32\ m}$$

Example 14.37 Two reservoirs are connected by a siphon of diameter 0.2 m and length 2000 m whose surface level differs by 25 m. The pipeline crosses a ridge whose summit is 8 m above the level of water and 300 m distant from the higher reservoir. Determine the minimum depth of pipe below the summit of ridge in order that pressure at the apex does not fall 7.4 m below atmospheric pressure. Take coefficient of friction $f=0.007$ and neglect minor energy losses. Also determine the discharge through the syphon.

Solution

Refer Figure 14.22. Let $D=0.2$ m, $L=2000$ m, $h_1=h_f=25$ m, $H_S=8$ m, $L_1=300$ m, $p_c/(\rho_w g)=10.3-7.4=2.9$ m and $f=0.007$.

Let h_S be the minimum depth of pipe below the summit of ridge.

Since $$h_f=\frac{4fLV^2}{2gD}$$

Thus $$\frac{4\times 0.007\times 2000\times V^2}{2\times 9.81\times 0.2}=25$$

$$\therefore V=\sqrt{\frac{25\times 2\times 9.81\times 0.2}{4\times 0.007\times 2000}}=1.3235 \text{ m/s}$$

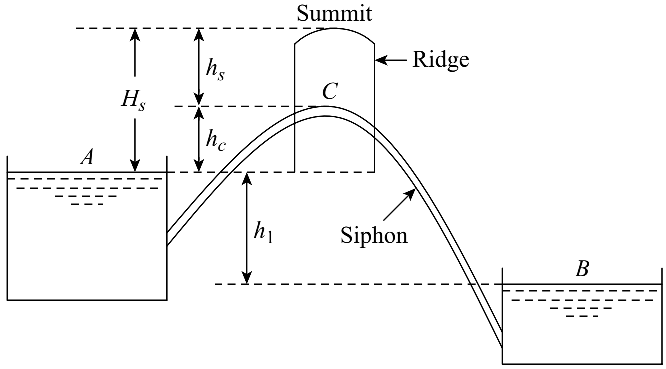


Figure 14.22

$$Q = AV = \frac{\pi}{4}D^2 \times V = \frac{\pi}{4} \times 0.2^2 \times 1.3235 = \mathbf{0.0416\ m^3/s}$$

Applying Bernoulli's equation to points A and C and assuming A as datum, we get:

$$\frac{p_A}{\rho_w g} + \frac{V_A{}^2}{2g} + z_A = \frac{p_c}{\rho_w g} + \frac{V^2}{2g} + z_c + (h_f)_{AC} \qquad [\because V_c = V]$$

Thus
$$10.3 + 0 + 0 = 2.9 + \frac{1.3235^2}{2 \times 9.81} + h_c + \frac{4 f L_1 V^2}{2gD} \qquad [\because z_c = h_c]$$

$$10.3 = 2.9 + 0.0893 + h_c + \frac{4 \times 0.007 \times 300 \times 1.3235^2}{2 \times 9.81 \times 0.2}$$

$$\therefore h_c = 10.3 - (2.9 + 0.0893 + 3.7497) = \mathbf{3.561\ m}$$

$$h_S = H_S - h_c = 8 - 3.561 = \mathbf{4.439\ m}$$

14.11 ❒ POWER TRANSMISSION THROUGH PIPES

The pipes carrying water under pressure may be utilized to transmit hydraulic power. This type of transmission is commonly used for the working of several hydraulic machines. The power transmitted depends on the discharge passing through the pipe and the total head available at the end of the pipe. When water flows along the pipe it is subjected to frictional resistance which causes loss of head due to friction. Consider that a pipe is connected to a high level storage tank as shown in Figure 14.23.

Let H be the total head at the source, h_f be the loss of head due to friction, f be the coefficient of friction, L be the length of the pipe, D be the diameter of the pipe, V be the velocity of flow in the pipe, P be the power available at the outlet of the pipe and neglecting minor head losses.

$$\text{Net head} = h = H - h_f = H - \frac{4 f L V^2}{2gD}$$

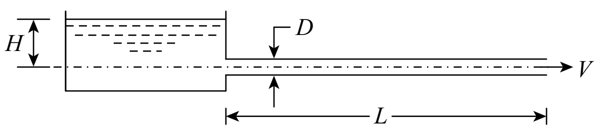


Figure 14.23

$$P = \text{Weight of water per second} \times \text{Net head} = \rho_w gQ\left(H - \frac{4fLV^2}{2gD}\right)$$

$$P = \rho_w g \times \frac{\pi}{4} D^2 V \times \left(H - \frac{4fLV^2}{2gD}\right) \tag{14.28}$$

The condition for maximum power transmitted may be obtained by differentiating Equation (14.28) with respect to V and equating it to zero, we get the following expression.

$$\frac{dP}{dV} = \frac{d}{dV}\left[\frac{\pi}{4}\rho_w gD^2 \times \left(VH - \frac{4fLV^3}{2gD}\right)\right] = 0$$

$$H - 3 \times \frac{4fLV^2}{2gD} = 0$$

$$H - 3 \times h_f = 0$$

$$\boxed{\therefore h_f = \frac{H}{3}} \tag{14.29}$$

The Equation (14.29) shows that the power transmitted through a pipe is maximum when the loss of head due to friction is one third of the total head at the inlet of the pipe.

The efficiency of power transmission (η) through the pipe is given by,

$$\eta = \frac{\text{Power available at the outlet}}{\text{Power available at the inlet}} = \frac{H - h_f}{H}$$

For maximum power transmission, $h_f = (H/3)$ and thus, from the above expression, maximum efficiency (η_{max}) is given below.

$$\boxed{\eta_{max} = \frac{H - (H/3)}{H} = \frac{2}{3} \text{ or } 66.7\%} \tag{14.30}$$

Example 14.38 Water is being supplied to a turbine through a 400 m long horizontal pipe for generating 860 kW of power at 80% efficiency. If water is available at a head of 120 m and the friction coefficient $f = 0.005$, then calculate the required discharge and minimum diameter of pipe to maintain the flow rate.

Solution

Let $L = 400$ m, $P_1 = 860$ kW, $\eta = 80\,\%$, $H = 120$ m and $f = 0.005$. Let Q be the required discharge and D be the minimum diameter of the pipe. For minimum diameter of pipe or for maximum transmission of power, we get:

$$h_f = \frac{H}{3} = \frac{120}{3} = 40 \text{ m}$$

$$h = H - h_f = 120 - 40 = 80 \text{ m}$$

$$P = \frac{P_1}{\eta} = \frac{860}{0.8} = 1075 \text{ kW}$$

Since

$$P = \frac{\rho_w gQh}{1000} \text{ kW}$$

Thus
$$1075 = \frac{1000 \times 9.81 \times Q \times 80}{1000}$$

$$\therefore Q = \frac{1075 \times 1000}{1000 \times 9.81 \times 80} = \mathbf{1.3698\ m^3/s}$$

Since
$$h_f = \frac{32 fLQ^2}{\pi^2 g D^5}$$

Thus
$$40 = \frac{32 \times 0.005 \times 400 \times 1.3698^2}{\pi^2 \times 9.81 D^5}$$

$$\therefore D = \left[\frac{32 \times 0.005 \times 400 \times 1.3698^2}{\pi^2 \times 9.81 \times 40}\right]^{1/5} = \mathbf{0.4992\ m}$$

Example 14.39 It is desired to transmit 130 kW power through a 2800 m long pipeline. If the pressure at the inlet of the pipe is 4512.6 kN/m^2, then the pressure drop over the full length of pipe is 833.85 kN/m^2 and the friction coefficient $f = 0.006$, calculate the diameter of the pipe and the efficiency of transmission.

Solution

Let $P = 130$ kW, $L = 2800$ m, $p = 4512.6$ kN/m^2, $p_d = 833.85$ kN/m^2 and $f = 0.006$. Let D be the diameter and η be the efficiency of transmission.

$$H = \frac{p}{\rho_w g} = \frac{4512.6 \times 10^3}{1000 \times 9.81} = 460 \text{ m}$$

$$h_f = \frac{p_d}{\rho_w g} = \frac{833.85 \times 10^3}{1000 \times 9.81} = 85 \text{ m}$$

$$h = H - h_f = 460 - 85 = 375 \text{ m}$$

Since
$$P = \frac{\rho_w g Q h}{1000} \text{ kW}$$

Thus
$$130 = \frac{1000 \times 9.81 \times Q \times 375}{1000}$$

$$\therefore Q = \frac{130 \times 1000}{1000 \times 9.81 \times 375} = \mathbf{0.03534\ m^3/s}$$

Since
$$h_f = \frac{32 fLQ^2}{\pi^2 g D^5}$$

Thus
$$85 = \frac{32 \times 0.006 \times 2800 \times 0.03534^2}{\pi^2 \times 9.81 \times D^5}$$

$$\therefore D = \left[\frac{32 \times 0.006 \times 2800 \times 0.03534^2}{\pi^2 \times 9.81 \times 85}\right]^{1/5} = \mathbf{0.1522\ m}$$

$$\eta = \frac{H - h_f}{H} = \frac{460 - 85}{460} = \mathbf{0.8152\ or\ 81.52\%}$$

14.12 ❐ FLOW THROUGH NOZZLES

A gradually converging short tube called nozzle fitted at the exit of the pipe converts the total energy of the flowing liquid into velocity energy. Nozzles are used where higher velocity of liquid flow is required such as in fire extinguisher and in Pelton turbine.

14.12.1 Discharge through Nozzle

Consider a nozzle fitted at the end of a pipe connected to a reservoir as shown in Figure 14.24.

Let H be the total head at the source, h_f be the loss of head due to friction, f be the coefficient of friction, L be the length of the pipe, D be the diameter of the pipe, d be the diameter of the nozzle, V be the velocity of flow in the pipe , V_o be the velocity of flow at the exit of the nozzle , A be the area of pipe or nozzle inlet, a be the area of the nozzle exit, P be the power transmitted by the jet issued from the nozzle and neglecting minor head losses.

$$\text{Head at inlet of nozzle} = h_i = H - h_f = H - \frac{4fLV^2}{2gD}$$

$$\text{Head at outlet of nozzle} = h_o = \frac{V_o^2}{2g}$$

When head losses in the nozzle are neglected, $h_i = h_o$ and the expressions are given below.

$$H - \frac{4fLV^2}{2gD} = \frac{V_o^2}{2g} \tag{14.31}$$

$$H = \frac{V_o^2}{2g} + \frac{4fLV^2}{2gD} \tag{14.31a}$$

$$V = \frac{aV_o}{A} \qquad \text{[From continuity equation]}$$

Substituting this value of V in expression (14.31a), we get:

$$H = \frac{V_o^2}{2g} + \frac{4fL}{2gD}\left(\frac{aV_o}{A}\right)^2 = \frac{V_o^2}{2g}\left[1 + \frac{4fLa^2}{DA^2}\right]$$

$$\therefore V_o = \sqrt{\frac{2gH}{1 + \frac{4fL}{D}\left(\frac{a}{A}\right)^2}} \tag{14.32}$$

Discharge through the nozzle is given by,

$$q = aV_o = \frac{\pi}{4}d^2 \times \sqrt{\frac{2gH}{1 + \frac{4fL}{D}\left(\frac{a}{A}\right)^2}} \tag{14.33}$$

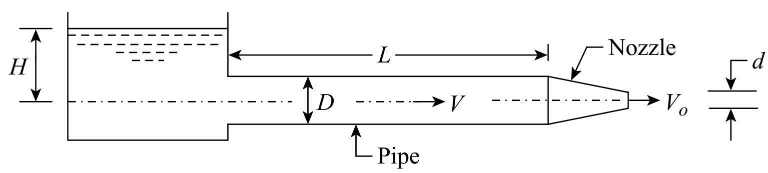


Figure 14.24

14.12.2 Efficiency of Power Transmission through Nozzle

The kinetic energy (K.E.) of the jet coming out of the nozzle is equal to the power available at its exit.

$$\text{Power available at nozzle exit: } P_o = \frac{1}{2}mV_o^2 = \frac{1}{2}\rho aV_o \times V_o^2$$

$$\text{Power available at nozzle inlet: } P_i = \rho gqH \text{ Watts}$$

Efficiency of power transmission through the nozzle is given by,

$$\eta = \frac{P_o}{P_i} = \frac{(1/2)\rho aV_o \times V_o^2}{\rho gqH} = \frac{(1/2)\rho aV_o \times V_o^2}{\rho g \times aV_o \times H} = \frac{V_o^2}{2gH}$$

Substituting the value of V_o from Equation (14.32) in the above expression, we get:

$$\eta = \frac{1}{2gH}\left(\sqrt{\frac{2gH}{1+\frac{4fL}{D}\left(\frac{a}{A}\right)^2}}\right)^2 = \frac{1}{1+\frac{4fL}{D}\left(\frac{a}{A}\right)^2} \tag{14.34}$$

14.12.3 Condition for Maximum Power through Nozzle

From Equation (14.31), we get:

$$V_o^2 = 2g\left[H - \frac{4fLV^2}{2gD}\right] = 2g\left[H - \frac{4fL}{2gD}\left(\frac{aV_o}{A}\right)^2\right] \qquad [\because V = aV_o/A]$$

The power transmitted through the nozzle is given by,

$$P = \frac{1}{2}\rho aV_o \times V_o^2 = \frac{1}{2}\rho aV_o \times 2g\left[H - \frac{4fL}{2gD}\left(\frac{aV_o}{A}\right)^2\right]$$

Thus

$$P = \rho ag\left(V_oH - \frac{4fLa^2V_o^3}{2gDA^2}\right)$$

For maximum power transmission, $(dP/dV_o) = 0$ and the expression is as follows.

$$\frac{d}{dV_o}\left[\rho ag\left(V_oH - \frac{4fLa^2V_o^3}{2gDA^2}\right)\right] = 0$$

$$H - 3\times\frac{4fL}{2gD}\times\frac{a^2V_o^2}{A^2} = 0$$

$$H - 3\times\frac{4fL}{2gD}\times V^2 = 0 \qquad [\because V = aV_o/A]$$

$$H - 3\times h_f = 0$$

$$\boxed{\therefore h_f = \frac{H}{3}} \tag{14.35}$$

The Equation (14.35) shows that the power transmitted by a nozzle is maximum when the head lost due to friction in pipe is one third to the total head supplied at the inlet of the pipe.

14.12.4 Diameter of Nozzle for Maximum Power Transmission through Nozzle

From Equation (14.35), we get:

$$H = 3 \times h_f = 3 \times \frac{4fLV^2}{2gD}$$

Also
$$H = \frac{V_o^2}{2g} + \frac{4fLV^2}{2gD} \qquad \text{[Equation (14.31a)]}$$

Thus
$$3 \times \frac{4fLV^2}{2gD} = \frac{V_o^2}{2g} + \frac{4fLV^2}{2gD}$$

$$\frac{8fLV^2}{2gD} = \frac{V_o^2}{2g}$$

$$\frac{8fL}{2gD} \times \frac{a^2V_o^2}{A^2} = \frac{V_o^2}{2g} \qquad [\because V = aV_o / A]$$

$$\left(\frac{A}{a}\right)^2 = \frac{8fL}{D} \tag{14.36}$$

$$\left[\frac{(\pi/4)D^2}{(\pi/4)d^2}\right]^2 = \frac{8fL}{D}$$

$$\frac{D^4}{d^4} = \frac{8fL}{D}$$

$$\boxed{\therefore d = \left(\frac{D^5}{8fL}\right)^{1/4}} \tag{14.37}$$

Thus, from Equation (14.37), the diameter of the nozzle can be determined.

Example 14.40 Find the maximum power transmitted by a jet of water discharging freely out of nozzle fitted to a pipe 300 m long and 100 mm diameter with coefficient of friction as 0.01. The available head at the base of nozzle is 90 m.

Solution

Let $L = 300$ m, $D = 100$ mm $= 0.1$ m, $f = 0.01$ and $h = 90$ m.

$$d = \left(\frac{D^5}{8fL}\right)^{1/4} = \left(\frac{0.1^5}{8 \times 0.01 \times 300}\right)^{1/4} = 0.02541 \text{ m}$$

$$h = \frac{V_o^2}{2g} = 90$$

$$\therefore V_o = \sqrt{90 \times 2g} = \sqrt{90 \times 2 \times 9.81} = 42.02 \text{ m/s}$$

$$q = \frac{\pi}{4} d^2 V_o = \frac{\pi}{4} \times 0.02541^2 \times 42.02 = 0.02131 \text{ m}^3/\text{s}$$

$$P = \frac{\rho_w g q h}{1000} = \frac{1000 \times 9.81 \times 0.02131 \times 90}{1000} = \mathbf{18.8146 \text{ kW}}$$

Example 14.41 A nozzle with effective diameter 40 mm and coefficient of velocity 0.97 is fitted at the discharge end of a pipe of diameter 0.2 m and length 1000 m. If the coefficient of friction for the pipe is 0.006 and the head of water at its inlet is 150 m, then determine (i) the actual velocity of the jet (ii) discharge and (iii) power produced by the jet.

Solution

Let $d = 40$ mm $= 0.04$ m, $C_v = 0.97$, $D = 0.2$ m, $L = 1000$ m, $f = 0.006$ and $h = H = 150$ m.

(i) $$\left(\frac{a}{A}\right)^2 = \left[\frac{(\pi/4)d^2}{(\pi/4)D^2}\right]^2 = \left[\frac{(\pi/4) \times 0.04^2}{(\pi/4) \times 0.2^2}\right]^2 = 0.0016$$

$$V_o = \sqrt{\frac{2gH}{1 + \frac{4fL}{D}\left(\frac{a}{A}\right)^2}} = \sqrt{\frac{2 \times 9.81 \times 150}{1 + \frac{4 \times 0.006 \times 1000}{0.2} \times 0.0016}} = 49.69 \text{ m/s}$$

$$V_{\text{actual}} = C_v V_o = 0.97 \times 49.69 = \mathbf{48.2 \text{ m/s}}$$

(ii) $$q = aV_{\text{actual}} = \frac{\pi}{4} d^2 \times V_{\text{actual}} = \frac{\pi}{4} \times 0.04^2 \times 48.2 = \mathbf{0.06057 \text{ m}^3/\text{s}}$$

(iii) $$P = \frac{\rho_w g q h}{1000} = \frac{1000 \times 9.81 \times 0.06057 \times 150}{1000} = \mathbf{89.129 \text{ kW}}$$

Example 14.42 Find the maximum power transmitted by a jet of water discharging freely out of a nozzle fitted to a pipe of length 200 m and diameter 100 mm with coefficient of friction as 0.008. Also determine the diameter of the nozzle if the available head at the inlet of the pipe is 126 m. Take the coefficient of velocity as 0.97.

Solution

Let $L = 200$ m, $D = 100$ mm $= 0.1$ m, $f = 0.008$, $H = 126$ m and $C_v = 0.97$.

$$d = \left(\frac{D^5}{8fL}\right)^{1/4} = \left(\frac{0.1^5}{8 \times 0.008 \times 200}\right)^{1/4} = \mathbf{0.02973 \text{ m}}$$

$$h = H - h_f = H - \frac{H}{3} = 126 - \frac{126}{3} = 84 \text{ m} \qquad \text{[For maximum power]}$$

$$V_o = C_v\sqrt{2gh} = 0.97 \times \sqrt{2 \times 9.81 \times 84} = 39.38 \text{ m/s}$$

$$q = \frac{\pi}{4} d^2 \times V_o = \frac{\pi}{4} \times 0.02973^2 \times 39.38 = 0.02734 \text{ m}^3/\text{s}$$

$$P = \frac{\rho_w g q h}{1000} = \frac{1000 \times 9.81 \times 0.02734 \times 84}{1000} = \mathbf{22.529 \text{ kW}}$$

14.13 ❐ WATER HAMMER

The phenomenon of sudden rise in pressure in the pipe is called water hammer. It happens when the flowing water is suddenly brought to rest by closing the valve (or by any similar cause) and thereby, momentum of the moving water gets destroyed. Thus, a wave of high pressure transmits along the pipe and has the effect of hammering action on the wall of the pipe. In some cases, the rise in pressure may be so large that the pipe may even burst. The magnitude of the pressure rise depends on (i) speed at which the valve is closed, (ii) velocity of flow, (iii) length of the pipe and (iv) elastic properties of the pipe material as well as that of the flowing fluid. In this section, water hammer during gradual closure of valve, sudden closure of valve in a rigid pipe and sudden closure of valve in an elastic pipe have been discussed.

14.13.1 Gradual Closure of Valve

Consider a long pipe of length L through which water is flowing at a uniform velocity V and a valve is fitted at its end (Figure 14.25). Let A be the cross-sectional area of the pipe, p be the increase in pressure, t be the time in seconds required to close the valve and C be the velocity of pressure wave.

Since the valve is closed gradually, the velocity of water decreases from V to zero in time t seconds.

Retardation (or acceleration) of water = $\frac{V-0}{t} = \frac{V}{t}$

Retarding force = Mass × Retardaion = $\rho_w AL \times \frac{V}{t}$

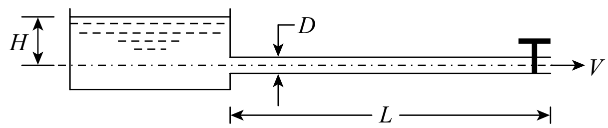


Figure 14.25

Force due to increase in pressure due to pressure wave = pA

Equating the two forces, we get:

$$\rho_w AL \times \frac{V}{t} = pA$$

$$p = \frac{\rho_w LV}{t} \tag{14.38}$$

The pressure head is given by,

$$H = \frac{p}{\rho_w g} = \frac{\rho_w LV}{\rho_w gt} = \frac{LV}{gt} \tag{14.39}$$

The valve closure is said to be gradual when

$$\boxed{t > \frac{2L}{C}} \tag{14.40}$$

The valve closure is said to be sudden when

$$\boxed{t < \frac{2L}{C}} \tag{14.41}$$

14.13.2 Sudden Closure of Valve in a Rigid Pipe

In a perfectly long rigid pipe (Figure 14.25) when the valve is closed suddenly, the kinetic energy of the flowing water converts into its strain energy if the effect of friction is neglected. Let A be the cross-sectional area of the pipe, p be the increase in pressure, C be the velocity of pressure wave and K be the bulk modulus of elasticity of water.

$$\text{Loss of K.E. of water} = \text{Gain of strain energy of water}$$

$$\frac{1}{2}mV^2 = \frac{1}{2}\left(\frac{p^2}{K}\right) \times \text{Volume}$$

$$\frac{1}{2}\rho_w ALV^2 = \frac{1}{2}\left(\frac{p^2}{K}\right)AL$$

$$p^2 = \rho_w KV^2$$

$$p = V\sqrt{\rho_w K} = V\sqrt{\frac{\rho_w{}^2 K}{\rho_w}} \qquad \text{[Multiply and divide by } \rho_w\text{]}$$

But $$\sqrt{\frac{K}{\rho_w}} = C$$

$$\therefore\ p = \rho_w VC \tag{14.42}$$

14.13.3 Sudden Closure of Valve in an Elastic Pipe

Due to sudden closure of the valve, the longitudinal stress (σ_l) and circumferential stress (σ_c) are produced in the walls of an elastic pipe. Thus, some of the kinetic energy of water is absorbed by the pipe as strain energy. Let D be the diameter of pipe, A be its cross-sectional area, t_p be the thickness of the pipe wall, E be the modulus of elasticity, v_p be the Poisson's ratio of the pipe material, p be the increase in pressure, C be the velocity of pressure wave and K be the bulk modulus of elasticity of water.

The strain energy per unit volume stored in the pipe material is given by,

$$e_s = \frac{1}{2E}[\sigma_l{}^2 + \sigma_c{}^2 - 2v_p\sigma_l\sigma_c]$$

But $\sigma_l = (pD)/(4t_p)$ and $\sigma_c = (pD)/(2t_p)$, the above expression becomes,

$$e_s = \frac{1}{2E}\left[\left(\frac{pD}{4t_p}\right)^2 + \left(\frac{pD}{2t_p}\right)^2 - 2v_p \times \frac{pD}{4t_p} \times \frac{pD}{2t_p}\right]$$

$$e_s = \frac{1}{2E}\left[\frac{p^2D^2}{16t_p{}^2} + \frac{p^2D^2}{4t_p{}^2} - 2\times 0.25 \times \frac{p^2D^2}{8t_p{}^2}\right] \qquad \text{[Take } v_p = 0.25\text{]}$$

Thus $$e_s = \frac{p^2D^2}{8Et_p{}^2}$$

Volume of the pipe material = πDLt_p

The total strain energy of the pipe material is given by,

$$E_s = \frac{p^2D^2}{8Et_p{}^2} \times \pi DLt_p = \frac{p^2DL}{2Et_p} \times \frac{\pi}{4}D^2 = \frac{p^2DLA}{2Et_p} \qquad [\because A = (\pi/4)D^2]$$

Since loss of kinetic energy of water is equal to the sum of gain of strain energy in water and strain energy stored in pipe material.

Thus $$\frac{1}{2}\rho_w ALV^2 = \frac{1}{2}\left(\frac{p^2}{K}\right)AL + \frac{p^2DLA}{2Et_p}$$

Diving both sides by $(1/2)AL$, we get:

$$\rho_w V^2 = \frac{p^2}{K} + \frac{p^2 D}{Et_p} = p^2\left[\frac{1}{K} + \frac{D}{Et_p}\right]$$

$$\therefore p = \sqrt{\frac{\rho_w V^2}{\frac{1}{K} + \frac{D}{Et_p}}} = V\sqrt{\frac{\rho_w}{\frac{1}{K} + \frac{D}{Et_p}}} \tag{14.43}$$

For a rigid pipe, $E = \infty$ and thus, Equation (14.43) becomes $p = V\sqrt{\rho_w K}$, i.e., increase in pressure due to sudden closure of the valve in a rigid pipe.

If poisons ratio (ν_p) is also considered, then Equation (14.43) is written as follows.

$$p = V\sqrt{\frac{\rho_w}{\frac{1}{K} + \frac{D}{Et_p}(1.25 - \nu_p)}} \tag{14.44}$$

14.13.4 Time Taken by Pressure Wave to Travel from Valve to the Tank and from Tank to Valve

Let L be the length of the pipe and C be the velocity of pressure wave. Time taken (t) by the pressure wave to travel from the valve to the tank and from the tank to the valve is given below.

$$t = \frac{\text{Distance travelled from valve to tank and back}}{\text{Velocity of pressure wave}} = \frac{L+L}{C} = \frac{2L}{C} \tag{14.45}$$

Example 14.43 The water flows through a pipe of diameter 0.5 m and length 3200 m with a velocity of 2.4 m/s. If a valve is provided at the end of the pipe and the velocity of pressure wave is 1600 m/s, then determine the rise in pressure when valve is closed in 20 seconds.

Solution

Let $D = 0.5$ m, $L = 3200$ m, $V = 2.4$ m/s and $C = 1600$ m/s.

$$t = 20 \text{ s}$$

$$\frac{2L}{C} = \frac{2 \times 3200}{1600} = 4 \text{ s}$$

Since $t > \frac{2L}{C}$, the valve closure is gradual. Therefore, the expression for rise in pressure is given below.

$$p = \frac{\rho_w LV}{t} = \frac{1000 \times 3200 \times 2.4}{20} = \mathbf{384000\ N/m^2}$$

Example 14.44 A water main of diameter 0.3 m and length 3500 m discharges water into a reservoir at a rate of 0.1414 m^3/s. If the pipeline is gradually closed in 24 seconds by operating a valve at the reservoir end and test pressure for concrete main is 30.73 m, then state whether there is any chance of bursting of the pipe.

Solution

Let $D = 0.3$ m, $L = 3500$ m, $Q = 0.1414$ m^3/s, $t = 24$ s and $H_t = 30.73$ m.

$$V = \frac{Q}{A} = \frac{Q}{(\pi/4)D^2} = \frac{0.1414}{(\pi/4)\times 0.3^2} = 2 \text{ m/s}$$

Since the valve closure is gradual, the rise in pressure is given by,

$$p = \frac{\rho_w LV}{t} = \frac{1000\times 3500\times 2}{24} = 291666.67 \text{ N/m}^2$$

$$H = \frac{p}{\rho_w g} = \frac{291666.67}{1000\times 9.81} = 29.73 \text{ m}$$

Since $H < H_t$, there is no chance of bursting of the pipe

Example 14.45 The water flows through a steel pipe of diameter 0.5 m, length 2000 m and wall thickness 10 mm with a velocity of 2.1 m/s. A valve provided at the end of the pipe is suddenly closed, the time of closure is 1.6 s. Determine the rise in pressure if the pipe is considered elastic and its modulus of elasticity is 200 GN/m^2. Take bulk modulus of elasticity of water as 2 GN/m^2.

Solution

Let $D = 0.5$ m, $L = 2000$ m, $t_p = 10$ mm $= 0.01$ m, $V = 2.1$ m/s, $t = 1.6$ s, $E = 200$ GN/m^2 $= 200\times 10^9$ N/m^2 and $K = 2$ GN/m^2 $= 2\times 10^9$ N/m^2.

$$C = \sqrt{\frac{K}{\rho_w}} = \sqrt{\frac{2\times 10^9}{1000}} = 1414.21 \text{ m/s}$$

$$\frac{2L}{C} = \frac{2\times 2000}{1414.21} = 2.83 \text{ s}$$

Since $t < \frac{2L}{C}$, the valve closure is sudden and therefore, the expression for rise in pressure is given below.

$$p = V\sqrt{\frac{\rho_w}{\frac{1}{K}+\frac{D}{Et_p}}} = 2.1\times\sqrt{\frac{1000}{\frac{1}{2\times 10^9}+\frac{0.5}{200\times 10^9\times 0.01}}} = \mathbf{2424871.13\ N/m^2}$$

Summary

1. The major loss of energy which is caused by friction is measured by the following formulae:
   (i) **Darcy–Weisbach formula:**
   $$h_f = \frac{4fLV^2}{2gD} = \frac{32fLQ^2}{\pi^2 gD^5},$$
   here L is the length of the pipe, D is the diameter of the pipe, V is the average flow velocity, Q is the discharge and f is the coefficient of friction.
   (ii) **Chezy's formula:** $V = C\sqrt{m\,i}$, here $m = (A/P)$ and $i = (h_f/L)$
   (iii) **Manning's formula:** $V = (1/n)m^{2/3}i^{1/2}$, here n is the Manning's roughness.
   (iv) **Hazen William's formula:** $V = 0.848\,k\,m^{0.63}\,i^{0.54}$, here k is the coefficient.
2. Minor energy losses which occur due to change in the velocity of flowing fluid in magnitude or direction in a pipe is measured by the following formulae.
   (i) **Loss of head due to sudden enlargement:**
   $$(h_L)_e = (V_1 - V_2)^2/(2g)$$
   (ii) **Loss of head due to sudden contraction:**
   $$(h_L)_c = \frac{V_2^2}{2g}\left(\frac{1}{C_c}-1\right)^2$$
   (iii) **Loss of head at the inlet of a pipe:** $(h_L)_i = 0.5V^2/(2g)$

(iv) **Loss of head at the outlet of a pipe:** $(h_L)_o = V^2/(2g)$

(v) **Loss of head due to obstruction:**

$$(h_L)_{\text{obs}} = \frac{V^2}{2g}\left[\frac{A}{C_c(A-a)} - 1\right]^2,$$

here A is the area of cross section of the main pipe and a is the maximum area of obstruction.

(vi) **Loss of head due to bend:** $(h_L)_b = kV^2/(2g)$

(vii) **Loss of head in various pipe fittings:**

$$(h_L)_f = kV^2/(2g)$$

3. Hydraulic gradient line is the line joining the piezometric heads while total energy line is the line joining the total heads at various points in a flow along the length of the pipe with respect to some reference line.

4. The pipes of different diameters and lengths connected end to end to form a pipeline are termed as pipes in series or compound pipes.

5. A pipe of uniform diameter having loss of head and discharge equal to the loss of head and discharge of a compound pipe made of several pipes of different diameters and lengths is called an equivalent pipe.

6. **Dupuit's equation:** $\dfrac{L}{D^5} = \dfrac{L_1}{D_1^5} + \dfrac{L_2}{D_2^5} + \dfrac{L_3}{D_3^5}$

7. When a pipeline divides into two or more branches which again join together into a single pipe, the flow of liquid through the branch pipes is known as parallel flow.

8. In a branched pipe system, three or more reservoirs having different free surface levels are connected by means of pipes having one or more junctions between them.

9. A siphon is a long bent pipe used to carry water from one reservoir at a higher level to another reservoir at a lower level when the two reservoirs are separated by a hill or high level ground.

10. The power transmitted through a pipe is maximum when the loss of head (h_f) due to friction is one third of the total head (H) at the inlet of the pipe, i.e., $h_f = H/3$.

11. Efficiency of power transmission (η) through the pipe:

$$\eta = \frac{H - h_f}{H}$$

12. Maximum efficiency through a pipe: $\eta_{\max} = (2/3)$ or 66.7%

13. Velocity of water at the nozzle exit:

$$V_o = \sqrt{(2gH)\Big/\left[1 + \frac{4fL}{D}\left(\frac{a}{A}\right)^2\right]}$$

Here, H is the total head, f is the coefficient of friction, D is the diameter of the pipe, A is the area of pipe or nozzle inlet and a is the area of the nozzle exit.

14. **Efficiency of power transmission through the nozzle:**

$$\eta = 1\Big/\left[1 + \frac{4fL}{D}\left(\frac{a}{A}\right)^2\right]$$

15. Power transmitted by a nozzle is maximum when the head lost due to friction in pipe is one third to the total head supplied at the inlet of the pipe, i.e., $h_f = H/3$.

16. Diameter of the nozzle: $d = \left[D^5/(8fL)\right]^{1/4}$

17. The phenomenon of sudden rise in pressure in the pipe is called water hammer.

18. The valve closure is said to be gradual when $t > (2L/C)$.

19. The valve closure is said to be sudden when $t < (2L/C)$.

20. The increase in pressure due to gradual valve closure is $p = (\rho_w LV)/t$.

21. **Velocity of pressure wave:** $C = \sqrt{K/\rho_w}$, here K is bulk modulus of elasticity.

22. Increase in pressure due to sudden valve closure for rigid pipe is $p = \rho_w VC$.

23. Increase in pressure due to sudden valve closure for an elastic pipe is

$$p = V\sqrt{\rho_w\Big/\left[\frac{1}{K} + \frac{D}{Et_p}\right]}$$

24. Time taken by pressure wave to travel from valve to the tank and from tank to valve is $t = 2L/C$.

Multiple-choice Questions

1. The head loss in turbulent flow in pipes varies
(a) Directly as velocity.
(b) Inversely as square of the velocity.
(c) Inversely as square of diameter.
(d) Approximately as the square of the velocity.

2. In networks of pipes
(a) Head loss in all the circuits is the same.
(b) Head loss around each elementary circuit must be zero.
(c) Elementary circuits are replaced by equivalent pipes.
(d) None of the above

3. Discharge in laminar flow through pipes varies
(a) Inversely as pressure drop.
(b) Directly with viscosity.
(c) Inversely with viscosity.
(d) As the square of radius.

4. When the pipes are connected in parallel, the total head loss
(a) Is equal to the inverse of the sum of head in each pipe.
(b) Is equal to the sum of loss of head in each pipe.
(c) Is same in each pipe.
(d) Is half that of in each pipe.

5. Maximum efficiency of power transmission through a pipe is
(a) 56.67%.
(b) 66.67%.
(c) 76.67%.
(d) 86.67%.

6. In a siphon, to avoid interruption, an air vessel is used
(a) At the summit.
(b) At the inlet.
(c) At the outlet.
(d) None of the above.

7. Which of the following relation may be used to replace a pipe of diameter *D* by *n* parallel pipes of diameter *d*?
(a) $d = nD$.
(b) $d = n/D$.
(c) $d = D/n^{0.25}$.
(d) $d = D/n^{0.4}$.

8. The hydraulic mean depth for a circular pipe of diameter *D* is
(a) 2*D*.
(b) *D*/2.
(c) *D*/3.
(d) *D*/4.

9. The total energy line lies over the hydraulic gradient line by an amount equal to
(a) Velocity head.
(b) Pressure head.
(c) Sum of velocity and pressure heads.
(d) None of the above.

10. The hydraulic gradient line for a flow through a pipe is always
(a) Below the total energy line.
(b) Above the total energy line.
(c) Above the axis of the pipe.
(d) None of the above.

11. The magnitude of water hammer depends on
(a) Speed with which the valve is closed.
(b) Diameter of the pipe.
(c) Elastic properties of pipe and the liquid.
(d) All the above.

12. The speed of pressure wave depends on
(a) Diameter of pipe.
(b) Density of liquid.
(c) Viscosity of liquid.
(d) None of the above.

13. The frictional resistance in a pipe during flow of liquid varies approximately with
(a) Velocity.
(b) Pressure.
(c) Square of velocity.
(d) Square of pressure.

14. Maximum head loss occurs in
(a) U-bend.
(b) 30° bend.
(c) 60° bend.
(d) 90° bend.

15. Water hammer is caused due to
(a) Friction.
(b) Incompressibility.
(c) Sudden enlargement.
(d) Sudden closure of the valve.

16. Loss of head at entrance of a pipe in terms of mean velocity of liquid (V) in the pipe is
(a) $(0.5V^2)/(2g)$.
(b) $V^2/(2g)$.
(c) V^2/g.
(d) $(0.2V^2)/(2g)$.

17. Loss of head at the exit of a pipe in terms of mean velocity of liquid (V) at the outlet of pipe is
(a) $(0.5V^2)/(2g)$.
(b) $V^2/(2g)$.
(c) V^2/g.
(d) $(0.2V^2)/(2g)$.

18. A pipe is said to be equivalent to another pipe if
(a) Length and discharge is same.
(b) Length and diameter are same.
(c) Velocity and diameter are same.
(d) Discharge and pressure head loss are same.

Review Questions

1. Define major and minor energy losses in pipes.
2. Derive expressions for the calculation of loss of head due to (i) sudden enlargement and (ii) sudden contraction.
3. What is a compound pipe? What will be the loss of head when pipes are connected in series?
4. Briefly explain an equivalent pipe.
5. What is a siphon? Where it is used and how it works?
6. Derive an expression for power transmission through pipes. Also derive the condition for maximum efficiency and corresponding efficiency of transmission.
7. Derive expressions for discharge through the nozzle and efficiency of power transmission through the nozzle. Also obtain expression for the condition for maximum power through the nozzle.
8. Derive an expression for diameter of nozzle for maximum power transmission through it.
9. What do you mean by water hammer? Derive an expression for the rise of pressure when the valve is closed gradually.
10. Derive an expression for the rise of pressure when the valve is closed suddenly and the pipe is rigid.
11. Derive an expression for the rise of pressure when the valve is closed suddenly and the pipe is an elastic one.

Problems

1. In a pipe of diameter 30 cm and length 50 m water flows with a mean velocity of 3.2 m/s. Determine the head loss due to friction using (i) Darcy-Weisbach and (ii) Chezy's formula for which $C = 60$. Take $\nu = 0.01$ stoke.

   [**Ans.** 0.88 m, 1.89 m]
2. Using Chezy's formula, find the loss of head due to friction in a pipe of diameter 0.3 m and length 100 m , if the average velocity of flow is 3 m/s and $C = 55$.

   [**Ans.** 3.97 m]
3. An oil of specific gravity 0.72 flows through a pipe of diameter 0.2 m and length 800 m. If the oil flow rate is 0.2 m^3/s and kinematic viscosity of oil is 0.26 stokes, then find the head lost due to friction and the power required to maintain this flow.

   [**Ans.** 176.04 m, 248.68 kW]
4. Water is flowing with a velocity of 5.2 m/s through a pipe of diameter of 0.3 m and length 10 m. Determine the head lost due to friction if the coefficient of friction is given by $f = 0.015 + (0.08/\mathrm{Re}^{0.3})$, where Re is the Reynolds number. Take kinematic viscosity of water as 0.01 stoke.

   [**Ans.** 2.96 m]
5. Water is supplied to a city of population 4×10^5. The reservoir is 6.5 km away from the city and friction head loss in the pipeline is limited to 15 m. Each person consumes 0.18 m^3 of water per day. If coefficient of friction for pipeline is 0.0075 and half of the daily supply is pumped in 8 hours, then determine the size of the supply main.

   [**Ans.** 1.11 m]
6. A pipeline of diameter 0.24 m and length 1000 m carries water from one end to the other. Determine the flow rate through the pipe if coefficient of friction $f = 0.009$ and the pressures measured at the inlet and outlet of the pipeline are 15 kN/m^2 and 3 kN/m^2, respectively.

   [**Ans.** 0.0181 m^3/s]
7. Water flows through a 2000 m long pipe with a velocity of 1.5 m/s. If the head loss through the pipe is 15 m of water and coefficient of friction $f = 0.008$, then determine the minimum diameter of the pipe.

   [**Ans.** 0.49 m]
8. Determine the loss of head when a pipe of diameter 0.2 m is suddenly enlarged to a diameter of 0.4 m and the rate of flow of water through the pipe is 300 litres per second.

   [**Ans.** 2.613 m]
9. The diameter of a horizontal pipe 0.2 m is suddenly enlarged to 0.4 m. The rate of flow of water through this pipe is 250 litres per second. If the intensity of pressure in the smaller pipe is 120 kN/m^2, then calculate (i) the loss of head due to sudden enlargement, (ii) power lost due to enlargement and (ii) pressure intensity in the large pipe.

   [**Ans.** 1.816 m, 4.4537 kW, 131.886 kN/m^2]
10. At a sudden enlargement of water main from 200 mm to 400 mm diameter, the hydraulic gradient rises by 8 mm. Determine the rate of flow of water through the pipe.

    [**Ans.** 0.02036 m^3/s]
11. The water flow through a horizontal pipe at a rate of 60 litres per second. If the diameter of the pipe which is 15 cm is suddenly enlarged to 22.5 cm, then determine (i) the loss of head due to sudden expansion, (ii) pressure difference in the two pipes and (iii) change in pressure if the change of section is gradual without any loss.

    [**Ans.** 0.1814 m, 0.2901 m, 0.4716 m]

12. In a pipe of diameter 100 mm an oil of specific gravity 0.78 flows at a rate of 20 litres per second. If a sudden enlargement occurs in the second pipe so that maximum pressure rise is obtained, then determine (i) the loss of energy in sudden expansion and (ii) differential gauge length indicated by an oil-mercury manometer connected between the two pipes.
[**Ans.** 0.0825 m, 0.01 m]

13. A horizontal pipe of diameter 0.6 m is suddenly contracted to a diameter of 0.3 m. The pressure intensities in the large and smaller pipe is measured as 100 kPa and 80 kPa, respectively. If the coefficient of contraction is 0.62, then determine the discharge through the pipe.
[**Ans.** 0.39 m^3/s]

14. A pipe of diameter 0.15 m reduces in diameter to 0.1 m suddenly and it carries water at a rate of 40 litres per second. If the coefficient of contraction is 0.6, then find the pressure loss across the contraction.
[**Ans.** 16.171 kN/m^2]

15. Two pipes of diameters 0.2 m and 0.125 m are connected by means of a flange such that the axes of the two pipes are in straight line. Water flows from the larger pipe to the smaller pipe at a rate of 50 litres per second. The differential pressure reading on a water-mercury manometer between the two pipes read 7.8 cm. Determine the loss of head due to contraction and the coefficient of contraction.
[**Ans.** 0.2658 m, 0.641]

16. When a sudden contraction is introduced in a horizontal pipeline from 0.5 m diameter to 0.25 m diameter, the pressure changes from 105 kPa to 69 kPa. If the coefficient of contraction is 0.65, then find the water flow rate. Following this if there is a sudden enlargement from 0.25 m to 0.5 m and if the pressure at the 0.25 m section is 69 kPa, then what is the pressure at the 0.5 m enlarged portion?
[**Ans.** 0.376 m^3/s, 80 kPa]

17. The one end of a horizontal pipe of diameter 25 cm and length 50 m is connected to a large reservoir and the other end is open to the atmosphere. If the height of water in the tank is maintained constant at 3 m above the centre line of the pipe and friction coefficient $f = 0.009$, then determine the discharge through the pipe in litres per second.
[**Ans.** 127.6 *l*/s]

18. In a horizontal pipe of diameter 0.1 m, water flows at a velocity of 2.5 m/s. If a solid plate of diameter 75 mm obstruct the flow and the coefficient of contraction is 0.62, then determine the loss of head due to obstruction.
[**Ans.** 2.31 m]

19. The two tanks are connected by a horizontal pipe of diameter 0.2 m and length 100 m. If the discharge through the pipe is 200 litres per second and friction coefficient $f = 0.008$, then determine the difference in elevations between the water surfaces in the tanks.
[**Ans.** 36.19 m]

20. A pipe of length 40.5 m connected to a large reservoir at one end discharges water to the atmosphere at the other end. The diameter of the pipe is 0.15 m for the first 25 m length and then its diameter is suddenly enlarged to 0.3 m. If the height of water level in the reservoir is maintained at 8 m level and friction coefficient $f = 0.01$, then determine the rate of flow through the pipe.
[**Ans.** 78.7 *l*/s]

21. A pipe of diameter 0.2 m and length 1.6 km connecting two water reservoirs has a slope of 1 in 100. The level of water in the first reservoir is 12 m above the inlet pipe and in the second tank 3 m above the outlet of the pipe. If the friction coefficient $f = 0.005$, then determine the rate of flow through the pipe.
[**Ans.** 54. 76 *l*/s]

22. A pipe of diameter 40 mm takes off abruptly from a large reservoir and runs 6 m, then expands abruptly to 80 mm diameter and runs 45 m, and next discharges directly into the open air with a velocity of 2 m/s. Determine the required height of water surface above the point of discharge if the friction coefficient $f = 0.009$.
[**Ans.** 25.413 m]

23. Determine the water level to be maintained in the tank for discharging 0.75 litres per second of water through a pipe connected to a tank. The diameter and length of the horizontal pipe are respectively 30 mm and 30 m, and the friction coefficient $f = 0.005$.
[**Ans.** 1.231 m]

24. Two tanks with a difference in elevation of 15 m are connected by three pipes in series. The pipes are 300 m long of diameter 0.3 m, 150 m long of diameter 0.2 m, and 200 m long of diameter 0.25 m, respectively. The friction factor f_f in the relation $h_f = f_f LV^2/(2gD)$ for the three pipes is 0.018, 0.020, and 0.019, respectively, which account for friction losses. If the sudden contraction coefficient is 0.24, then determine the flow rate.
[**Ans.** 0.1064 m^3/s]

25. Three pipes of diameters 0.4 m, 0.2 m and 0.3 m of lengths of 400 m, 200 m and 300 m, respectively, are connected in series. The compound pipe connects two water tanks having a difference of water level of 25 m. Determine the discharge through the compound pipe neglecting minor losses when the coefficient of friction for the compound pipe is 0.006.
[**Ans.** 0.1257 m^3/s]

26. A compound piping system consists of 1.8 km of 0.5 m diameter, 1.2 km of 0.4 m and 0.6 km of 0.3 m diameter pipes of the same material connected in series. Determine the equivalent length of a 0.4 m diameter pipe.
[**Ans.** 4.318 km]

27. A main pipe divides into two parallel pipes which again forms one pipe. The lengths and diameters for the first and

second parallel pipes are (2000 m, 1 m) and (2000 m, 0.8 m), respectively, then find the rate of flow in litres per second in each parallel pipe, if total flow in the main is 2500 litres per second and the coefficient of friction 'f' for each parallel pipe is 0.006.

[**Ans.** 1590 *l*/s, 910 *l*/s]

28. Two pipes each of length 300 m are available for connecting to a tank from which a flow of 85 litres per second is required. If the diameters of the two pipes are 0.3 m and 0.15 m, respectively, then find the ratio of the head lost when the pipes are connected in series to the head lost when they are connected in parallel. Neglect minor losses.

[**Ans.** 45.7]

29. If a pipeline of diameter 0.3 m and length 1540 m connects two tanks having a difference of water level 20 m, then determine the discharge through the pipe. If an additional pipe of the same diameter and length 540 m is attached parallel to the last 540 m length of the existing pipe then determine the increase in discharge. Take $f = 0.01$ and neglect minor losses.

[**Ans.** 97.72 *l*/s, 16.1 *l*/s]

30. Two pipes of diameters 50 mm and 100 mm and each 200 m in length are connected parallel between two reservoirs having a water level difference of 12 m. If the two pipes are to be replaced by a single pipe supplying the same quantity of water, then determine the required diameter. Take $f = 0.01$ for all pipes and neglect minor losses.

[**Ans.** 106.7 mm]

31. Two water tanks are connected by three parallel pipes of equal length L and of diameters d, $3d$ and $5d$. Determine the flow through the largest pipe when the flow through the smallest pipe is 40 litres per second and the coefficient of friction 'f' is same for all the pipes. Neglect minor losses.

[**Ans.** 2236.4 *l*/s]

32. A compound piping system consists of 1800 m of 0.6 m, 1200 m of 0.5 m and 600 m of 0.4 m connected in series. Convert this system into (i) an equivalent length of 0.4 m pipe and (ii) equivalent size pipe of length 3600 m.

[**Ans.** 1230.25 m, 0.496 m]

33. If 220.725 kW power is to be transmitted through a 2000 m long pipe in which a pressure of 9810 kN/m^2 is maintained at its inlet. If there is a pressure drop of 1962 kN/m^2 over the entire length of the pipe and coefficient of friction is 0.0065, then determine the diameter of the pipe and the efficiency of power transmission.

[**Ans.** 0.1112 m, 80%]

34. Power is to be transmitted to a hydraulic accumulator along a distance of 5000 m through a number of 0.1 m diameter horizontal pipes laid in parallel with an efficiency of 92%. If the pressure at the discharge end is maintained constant at 6670.8 kN/m^2, the power transmitted is 140 kW and the coefficient of friction is 0.008, then determine the minimum number of pipes required.

[**Ans.** 3]

35. If the pressure of water at the inlet to a pipe of length 300 m and diameter 0.2 is 10^4 kN/m^2, then determine the maximum rate at which power can be transmitted at the outlet from the pipe. Take coefficient of friction $f = 0.01$.

[**Ans.** 2206.668 kW]

36. For maximum power transmission, find the diameter of a nozzle fitted at the end of a pipe of diameter 0.1 m and length 300 m if $f = 0.01$.

[**Ans.** 0.0254 m]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (d)	**2.** (b)	**3.** (c)	**4.** (c)	**5.** (b)
6. (a)	**7.** (d)	**8.** (d)	**9.** (a)	**10.** (a)
11. (d)	**12.** (b)	**13.** (c)	**14.** (d)	**15.** (d)
16. (a)	**17.** (b)	**18.** (d)		

Chapter 15

Boundary Layer Theory

15.1 ❒ INTRODUCTION

A real fluid (viscous fluid) consists of adjacent layers piled on top of each other. When it flows over a solid surface, the velocity of the particles in the first fluid layer adjacent to the surface becomes zero due to no-slip condition. This motionless layer slows down the particles of the adjacent layer of the fluid due to friction. This layer then slows down the molecules of the next layer and so on. Therefore, in the immediate vicinity of the boundary surface, a small region develops in which the velocity of flowing fluid increases gradually from zero at the boundary surface to the velocity of mainstream. This region is known as boundary layer.

The concept of boundary layer introduced by Ludwig Prandtl (a German engineer) in 1904 permits the solution of viscous flow problems that would have been impossible through the application of Navier-Stokes equations to the complete flow field. According to Prandtl, the flow may be considered of two regions, namely thin boundary layer region and freestream region. In thin boundary layer region, appreciable viscous forces are produced due to large velocity gradient even if viscosity may be small. In freestream region (i.e., outside the boundary layer zone) viscous forces are negligible and thus, the flow may be treated as non-viscous and the theory of ideal flow may be used for analysing the problems.

The velocity gradient in boundary layer is considerably large and the fluid flowing over the surface exerts a large shear stress along the direction of motion. As a result, the shear force acts on the solid surface and it is known as viscous shear force or drag force. The theory dealing with this phenomenon is called boundary layer theory. In this chapter, the concepts regarding estimation of boundary layer thickness parameters, the shear stress and the associated drag on the flat plate surface is briefly discussed.

15.2 ❒ DESCRIPTION OF BOUNDARY LAYER

Boundary layer is a narrow region near the solid surface over which velocity gradients and shear stresses are large. Consider the parallel flow of a fluid over a thin stationary flat plate as shown in Figure 15.1. The x-coordinate is measured along the plate surface from the leading edge of the plate in the direction of flow and y is measured from the surface in the normal direction.

The fluid approaches the plate in the x-direction with a uniform upstream velocity u which is nearly equal to the freestream velocity U over the plate away from the surface. At the leading edge of the plate, the thickness of the boundary layer is zero, but its thickness increases with distance from the leading edge. The fluid in contact with the boundary has zero velocity and at some distance δ from the boundary, the velocity is nearly U. Thus, a velocity gradient is set up which develops shear resistance to the flow and thus, it slows down the motion of the fluid. Due to continued action of shear resistance, a large group of fluid particles is retarded when this retarded layer of fluid moves downstream. Thus, the

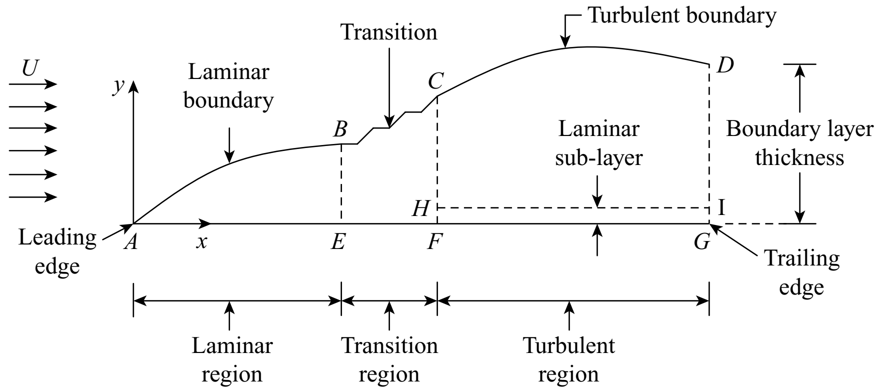


Figure 15.1 *Boundary layer for flow over a flat plate and different flow regime*

thickness of the boundary layer δ goes on increasing in the downstream direction. This is also referred to as the growth of the boundary layer. The shear resistance acting in between the adjacent flowing layers is responsible for rotational flow within the boundary layer.

The parameters which affect the boundary layer thickness are (i) increase in distance x from the leading edge increases δ, (ii) increase in kinematic viscosity (ν) of the fluid increases δ, (iii) increase in the velocity of flow (U) of the approaching fluid decreases δ and (iv) the presence of negative pressure gradient ($-\partial p/\partial x$) decreases δ and vice-versa.

The wall shear stress (τ) decreases along the length of the plate as the velocity gradient at the wall decreases downstream.

15.2.1 Laminar Boundary Layer

Whether the flow of the incoming fluid stream is laminar or turbulent, up to a certain distance from the leading edge; the flow in the boundary layer is always laminar. This is known as laminar boundary layer which is shown by *AB* in Figure 15.1 and the length along the plate up to which it exists is called the laminar region which is shown by *AE*. For the flow over a flat plate, the length of the laminar region is obtained from the fact that for laminar flow the Reynolds number is equal to 5×10^5, which is also known as critical Reynolds number $(\text{Re}_x)_c$. The mathematical expression for critical Reynolds number is given below.

$$\boxed{(\text{Re}_x)_c = \frac{Ux}{\nu} = 5\times10^5}$$

Thus, if the values of U and ν are known, then the distance x can be evaluated up to which the laminar boundary layer exists. When the length of the plate is less than x, then the boundary layer will be fully laminar. The velocity distribution in a laminar boundary layer is parabolic.

15.2.2 Transition Region

If the plate is long enough (i.e., more than x), then beyond some distance from the leading edge, the laminar boundary layer becomes unstable and the flow in the boundary layer shows the characteristics between those of laminar and turbulent flow. In other words, the laminar flow undergoes a change in its flow structure at certain point, which is known as the transition point (point *B* in Figure 15.1) and referred to as transition flow. Generally, this region of the boundary layer is small and is known as transition region which is shown by *EF* in Figure 15.1.

15.2.3 Turbulent Boundary Layer

After the transition region, the flow in the boundary layer becomes turbulent and there is a rapid increase in its thickness. This is known as turbulent boundary layer which is shown by *CD* in Figure 15.1 and the length along the plate up to which it exists is called the turbulent region which is shown by *FG*. The velocity distribution in a turbulent boundary layer is either logarithmic or it follows one-seventh power law. The boundary changes from laminar to turbulent when $\text{Re}_x > 5\times10^5$, i.e., when $\text{Re}_x > (\text{Re}_x)_c$. In real engineering flows, the transition to turbulent flow occurs more abruptly. Here, transition is ignored by treating the first part of transition as laminar and the remaining part as turbulent.

15.2.4 Laminar Sublayer

In turbulent boundary layer region, there is a very thin layer just adjacent to the boundary in which the flow is laminar. This thin layer is known as laminar sublayer and its thickness is denoted by δ' (*HFGI* in Figure 15.1). The velocity distribution in a laminar sublayer is parabolic, but its thickness is very small and thus, a linear distribution is considered, i.e., $(du/dy) = (u/y)$. Therefore, velocity gradient is considered to be constant and the shear stress would be equal to the boundary shear stress (τ_o), where its expression is given below.

$$\tau_o = \mu\left(\frac{du}{dy}\right)_{y=0} = \mu\frac{u}{y} \tag{15.1}$$

According to Nikuradse, the expression for thickness of laminar sublayer (δ') is given below.

$$\boxed{\delta' = \frac{11.6\nu}{\sqrt{\tau_o/\rho}} = \frac{11.6\nu}{u_s}} \tag{15.2}$$

Here, $u_s = \sqrt{\tau_o/\rho}$ is the shear velocity (friction velocity) and in some books, it is denoted by u_*.

15.3 ❐ BOUNDARY LAYER PARAMETERS

15.3.1 Boundary Layer Thickness

The velocity within the boundary layer increases from zero at the boundary surface to the velocity of the mainstream asymptotically. Therefore, the thickness of the boundary layer, δ is arbitrarily defined as the distance from the boundary surface in which the velocity (u) reaches 99% of the velocity of the mainstream (U). Thus, δ is defined as the distance y from the surface at which $u = 0.99U$. This definition provides the approximate value of the boundary layer thickness and hence, δ is generally known as nominal thickness.

To accurately measure the effect of boundary layer on the flow, the boundary layer can be defined in terms of the displacement thickness (δ_d), the momentum thickness (δ_m) and the energy thickness (δ_e).

15.3.2 Displacement Thickness (δ_d)

The retardation of flow in the boundary layer causes decrease in mass flow rate as compared to the flow which would have been in the absence of the boundary layer. Displacement thickness can be defined as the distance perpendicular to the boundary surface to which the boundary surface has to be displaced into the flow to compensate for reduction in the discharge due to the formation of boundary layer. Displacement thickness is denoted by (δ_d). In some books, it is denoted by δ^*.

Consider a freestream of an incompressible fluid of density ρ flowing with a velocity U over a thin smooth plate as shown in Figure 15.2. Consider an elementary strip of thickness dy at a distance y from the surface of the plate and assuming unit width of the plate.

The mass of fluid per second flowing through the strip with the velocity of the mainstream (i.e., when the plate is not there) is given below.

$$m_1 = \rho\times U\times dy\times 1 = \rho U dy$$

The mass of fluid per second flowing through the strip is given by,

$$m_2 = \rho\times u\times dy\times 1 = \rho u dy$$

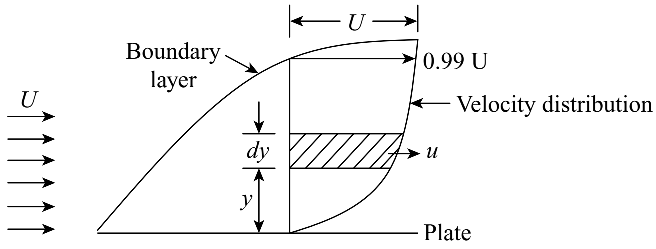


Figure 15.2 *Displacement thickness*

The reduction in mass flow rate is given by,

$$m_1 - m_2 = \rho U dy - \rho u dy = \rho(U - u)dy$$

Total reduction in mass flow rate (m_R) is given by integrating the above expression.

$$m_R = \int_0^\delta \rho(U - u)dy = \rho\int_0^\delta (U - u)dy$$

When the plate is displaced by an amount δ_d, the flow through the displaced volume must be equal to the reduction in mass flow rate as given below.

$$\rho U \delta_d = \rho\int_0^\delta (U - u)dy = \rho U\int_0^\delta \left(1 - \frac{u}{U}\right)dy$$

Thus $$\boxed{\delta_d = \int_0^\delta \left(1 - \frac{u}{U}\right)dy} \tag{15.3}$$

15.3.3 Momentum Thickness (δ_m)

Due to retardation of flow in the boundary layer, there is a reduction of the momentum flux. The momentum thickness may be defined as the perpendicular distance by which the boundary should be displaced to compensate for the reduction in momentum of the flowing fluid on account of the boundary layer formation. The momentum thickness is denoted by δ_m. In some books, it is denoted by θ. Assuming unit width of the plate and let δ_m be the distance by which the plate is displaced when the fluid is moving with freestream velocity U.

The reduction of momentum per second of fluid flowing through δ_m with freestream velocity U is given by,

$$M_{R1} = \text{Mass} \times \text{Velocity} = (\rho U \delta_m) \times U = \rho \delta_m U^2 \tag{i}$$

The mass flow per second through the elementary strip (Refer Figure 15.2) is given by,

$$m = \rho u dy$$

Momentum per second of the above mass of fluid before entering the boundary layer is given by,

$$M_1 = \rho u dy \times U = \rho U u dy$$

Momentum per second of the fluid inside the boundary layer is given by,

$$M_2 = \rho u dy \times u = \rho u^2 dy$$

Reduction in momentum per second is given by,

$$M_1 - M_2 = \rho U u dy - \rho u^2 dy = \rho u(U - u)dy$$

Total reduction in momentum per second (M_{R2}) is given by,

$$M_{R2} = \int_0^\delta M_1 - M_2 = \int_0^\delta \rho u(U - u)dy = \rho U \int_0^\delta u\left(1 - \frac{u}{U}\right)dy \quad \text{(ii)}$$

Equating expressions (i) and (ii), we get:

$$\rho \delta_m U^2 = \rho U \int_0^\delta u\left(1 - \frac{u}{U}\right)dy$$

$$\boxed{\therefore \delta_m = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy} \quad (15.4)$$

The ratio of displacement thickness to momentum thickness is known as shape factor. It is denoted by H and the expression is given below.

$$\boxed{H = \frac{\delta_d}{\delta_m}} \quad (15.5)$$

The shape factor for laminar flow is approximately 2.6 and for turbulent flow, it is about 1.4. Thus, higher the value of shape factor, smoother is the velocity profile.

15.3.4 Energy Thickness (δ_e)

The energy thickness may be defined as the perpendicular distance by which the boundary should be displaced to compensate for the reduction in energy of the flowing fluid on account of the boundary layer formation. It is denoted by δ_e. In some books, it is denoted by δ^{**}. Assuming unit width of the plate and let δ_e be the distance by which the plate is displaced when the fluid is moving with freestream velocity U.

The reduction in kinetic energy per second of fluid flowing through δ_e with freestream velocity U is given by,

$$(\text{K.E.})_{R1} = \frac{1}{2}mv^2 = \frac{1}{2}(\rho U \delta_e)U^2 \quad \text{(i)}$$

The mass flow per second through the elementary strip (Refer Figure 15.2) is given by,

$$m = \rho u dy$$

Kinetic energy per second of the above mass of fluid before entering the boundary layer is given by,

$$(\text{K.E.})_1 = \frac{1}{2}(\rho u dy)U^2$$

Kinetic energy per second of the fluid inside the boundary layer is given by,

$$(\text{K.E.})_2 = \frac{1}{2}mv^2 = \frac{1}{2}(\rho u dy)u^2$$

Reduction in kinetic energy per second is given by,

$$(\text{K.E.})_1 - (\text{K.E.})_2 = \frac{1}{2}\rho u(U^2 - u^2)dy$$

Total reduction in K.E. per second $[(\text{K.E.})_{R2}]$ may be obtained by integrating the above expression as given below.

$$(\text{K.E.})_{R2} = \int_0^\delta \frac{1}{2}\rho u(U^2 - u^2)dy = \frac{1}{2}\rho U^2 \int_0^\delta u\left(1 - \frac{u^2}{U^2}\right)dy \quad \text{(ii)}$$

Equating expressions (i) and (ii), we get:

$$\frac{1}{2}(\rho U \delta_e) \times U^2 = \frac{1}{2}\rho U^2 \int_0^\delta u\left(1 - \frac{u^2}{U^2}\right)dy$$

$$\boxed{\therefore \delta_e = \int_0^\delta \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right)dy} \quad (15.6)$$

The reduction in mass, momentum and energy takes place because the streamlines are effectively displaced outwards due to flow retardation near solid surface.

Example 15.1 The velocity distribution in the boundary layer is given by $(u/U) = (y/\delta)$, where u is the velocity at a distance y from the plate, δ is the boundary layer thickness and $u = U$ at $y = \delta$. Determine (i) the displacement thickness (δ_d), (ii) momentum thickness (δ_m), (iii) energy thickness (δ_e) and (iv) value of (δ_d/δ_m).

Solution

Let $(u/U) = (y/\delta)$ and $u = U$ at $y = \delta$.

(i) $$\delta_d = \int_0^\delta \left(1 - \frac{u}{U}\right)dy = \int_0^\delta \left(1 - \frac{y}{\delta}\right)dy = \left[y - \frac{y^2}{2\delta}\right]_0^\delta = \left[\delta - \frac{\delta^2}{2\delta}\right] = \boldsymbol{\frac{\delta}{2}}$$

(ii) $$\delta_m = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy = \int_0^\delta \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)dy = \int_0^\delta \left(\frac{y}{\delta} - \frac{y^2}{\delta^2}\right)dy$$

$$\therefore \delta_m = \left[\frac{y^2}{2\delta} - \frac{y^3}{3\delta^2}\right]_0^\delta = \left[\frac{\delta^2}{2\delta} - \frac{\delta^3}{3\delta^2}\right] = \frac{\delta}{2} - \frac{\delta}{3} = \boldsymbol{\frac{\delta}{6}}$$

(iii) $$\delta_e = \int_0^\delta \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right)dy = \int_0^\delta \frac{y}{\delta}\left(1 - \frac{y^2}{\delta^2}\right)dy = \int_0^\delta \left(\frac{y}{\delta} - \frac{y^3}{\delta^3}\right)dy$$

$$\therefore \delta_e = \left[\frac{y^2}{2\delta} - \frac{y^4}{4\delta^3}\right]_0^\delta = \left[\frac{\delta^2}{2\delta} - \frac{\delta^4}{4\delta^3}\right] = \frac{\delta}{2} - \frac{\delta}{4} = \boldsymbol{\frac{\delta}{4}}$$

(iv) $$H = \frac{\delta_d}{\delta_m} = \frac{(\delta/2)}{(\delta/6)} = \mathbf{3}$$

Example 15.2 If the velocity distribution in the boundary layer is given by $(u/U) = [2(y/\delta) - (y/\delta)^2]$, where δ is the boundary layer thickness, determine (i) the displacement thickness (δ_d), (ii) momentum thickness (δ_m), (iii) energy thickness (δ_e) and (iv) shape factor (δ_d/δ_m).

Solution

Let $(u/U) = [2(y/\delta) - (y/\delta)^2]$ and $u = U$ at $y = \delta$.

(i) $$\delta_d = \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left[1 - \left\{2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right\}\right] dy$$

$$\therefore \delta_d = \left[y - \frac{2y^2}{2\delta} + \frac{y^3}{3\delta^2}\right]_0^\delta = \left[\delta - \frac{\delta^2}{\delta} + \frac{\delta^3}{3\delta^2}\right] = \left(\delta - \delta + \frac{\delta}{3}\right) = \mathbf{\frac{\delta}{3}}$$

(ii) $$\delta_m = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left[1 - \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\right] dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left(1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2}\right) dy = \int_0^\delta \left(\frac{2y}{\delta} - \frac{4y^2}{\delta^2} + \frac{2y^3}{\delta^3} - \frac{y^2}{\delta^2} + \frac{2y^3}{\delta^3} - \frac{y^4}{\delta^4}\right) dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta} - \frac{5y^2}{\delta^2} + \frac{4y^3}{\delta^3} - \frac{y^4}{\delta^4}\right) dy = \left[\frac{2y^2}{2\delta} - \frac{5y^3}{3\delta^2} + \frac{4y^4}{4\delta^3} - \frac{y^5}{5\delta^4}\right]_0^\delta$$

$$\therefore \delta_m = \left[\frac{\delta^2}{\delta} - \frac{5\delta^3}{3\delta^2} + \frac{\delta^4}{\delta^3} - \frac{\delta^5}{5\delta^4}\right] = \left[\delta - \frac{5\delta}{3} + \delta - \frac{\delta}{5}\right] = \mathbf{\frac{2}{15}}\delta$$

(iii) $$\delta_e = \int_0^\delta \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right) dy = \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left[1 - \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)^2\right] dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left(1 - \frac{4y^2}{\delta^2} - \frac{y^4}{\delta^4} + \frac{4y^3}{\delta^3}\right) dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta} - \frac{8y^3}{\delta^3} - \frac{2y^5}{\delta^5} + \frac{8y^4}{\delta^4} - \frac{y^2}{\delta^2} + \frac{4y^4}{\delta^4} + \frac{y^6}{\delta^6} - \frac{4y^5}{\delta^5}\right) dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta} - \frac{8y^3}{\delta^3} - \frac{6y^5}{\delta^5} + \frac{12y^4}{\delta^4} - \frac{y^2}{\delta^2} + \frac{y^6}{\delta^6}\right) dy$$

$$= \left[\frac{2y^2}{2\delta} - \frac{8y^4}{4\delta^3} - \frac{6y^6}{6\delta^5} + \frac{12y^5}{5\delta^4} - \frac{y^3}{3\delta^2} + \frac{y^7}{7\delta^6}\right]_0^\delta$$

$$= \left[\frac{2\delta^2}{2\delta} - \frac{8\delta^4}{4\delta^3} - \frac{6\delta^6}{6\delta^5} + \frac{12\delta^5}{5\delta^4} - \frac{\delta^3}{3\delta^2} + \frac{\delta^7}{7\delta^6}\right]$$

$$\therefore \delta_e = \left[\delta - 2\delta - \delta + \frac{12\delta}{5} - \frac{\delta}{3} + \frac{\delta}{7}\right] = \mathbf{\frac{22}{105}}\delta$$

(iv) $$H = \frac{\delta_d}{\delta_m} = \frac{(\delta/3)}{(2\delta/15)} = \mathbf{\frac{5}{2}}$$

Example 15.3 The velocity distribution in the boundary layer is given by $(u/U)=(y/\delta)^{0.22}$. At a section, the freestream velocity (U) is measured to be 20 m/s and the boundary layer thickness (δ) was estimated as 30 mm. If the discharge passing over the spillway was 2 m^3/s per metre length of spillway, then calculate (i) the displacement thickness, (ii) momentum thickness, (iii) energy thickness and (iv) loss of energy up to the section under consideration in terms of metres of head.

Solution

Let $(u/U)=(y/\delta)^{0.22}$, $U=20$ m/s, $\delta=30$ mm and $Q=2$ m^3/s.

$$\text{(i) } \delta_d=\int_0^\delta\left(1-\frac{u}{U}\right)dy=\int_0^\delta\left[1-\left(\frac{y}{\delta}\right)^{0.22}\right]dy=\left[y-\frac{y^{1.22}}{1.22\delta^{0.22}}\right]_0^\delta=\left[\delta-\frac{\delta^{1.22}}{1.22\delta^{0.22}}\right]$$

$$\therefore \delta_d=\left[1-\frac{1}{1.22}\right]\delta=0.1803\delta=0.1803\times 30=\mathbf{5.41\ mm}$$

$$\text{(ii) } \delta_m=\int_0^\delta\frac{u}{U}\left(1-\frac{u}{U}\right)dy=\int_0^\delta\left(\frac{y}{\delta}\right)^{0.22}\left[1-\left(\frac{y}{\delta}\right)^{0.22}\right]dy=\int_0^\delta\left(\frac{y^{0.22}}{\delta^{0.22}}-\frac{y^{0.44}}{\delta^{0.44}}\right)dy$$

$$=\left[\frac{y^{1.22}}{1.22\delta^{0.22}}-\frac{y^{1.44}}{1.44\delta^{0.44}}\right]_0^\delta=\left[\frac{\delta^{1.22}}{1.22\delta^{0.22}}-\frac{\delta^{1.44}}{1.44\delta^{0.44}}\right]$$

$$\therefore \delta_m=\left[\frac{1}{1.22}-\frac{1}{1.44}\right]\delta=0.1252\delta=0.1252\times 30=\mathbf{3.756\ mm}$$

$$\text{(iii) } \delta_e=\int_0^\delta\frac{u}{U}\left(1-\frac{u^2}{U^2}\right)dy=\int_0^\delta\left(\frac{y}{\delta}\right)^{0.22}\left[1-\left(\frac{y}{\delta}\right)^{0.44}\right]dy=\int_0^\delta\left[\left(\frac{y}{\delta}\right)^{0.22}-\left(\frac{y}{\delta}\right)^{0.66}\right]dy$$

$$=\left[\frac{y^{1.22}}{1.22\delta^{0.22}}-\frac{y^{1.66}}{1.66\delta^{0.66}}\right]_0^\delta=\left[\frac{\delta^{1.22}}{1.22\delta^{0.22}}-\frac{\delta^{1.66}}{1.66\delta^{0.66}}\right]$$

$$\therefore \delta_e=\left[\frac{1}{1.22}-\frac{1}{1.66}\right]\delta=0.2173\delta=0.2173\times 30=\mathbf{6.519\ mm}$$

(iv) The energy loss per metre length of spillway is given by,

$$E_{\text{loss}}=\frac{1}{2}\times m\times U^2=\frac{1}{2}\times(\rho_w v)\times U^2=\frac{1}{2}\times\rho_w\times(AU)\times U^2$$

Thus
$$E_{\text{loss}}=\frac{1}{2}\times\rho_w\times(\delta_e\times 1)\times U^3=\frac{1}{2}\rho_w\delta_e U^3 \qquad [\because A=\delta_e\times 1]$$

$$\therefore E_{\text{loss}}=\frac{1}{2}\times 1000\times\frac{6.519}{1000}\times 20^3=26076\ \text{Nm/s}$$

Energy loss per metre of length in terms of metres of head is given by,

$$E_{\text{loss}}=\rho_w gQh$$

Thus $$26076 = 1000 \times 9.81 \times 2 \times h$$

$$\therefore h = \frac{26076}{1000 \times 9.81 \times 2} = \mathbf{1.329\ m}$$

15.4 ❐ DRAG FORCE ON A FLAT PLATE (VON KARMAN MOMENTUM INTEGRAL EQUATION)

Consider the parallel flow of a fluid with velocity U over a thin stationary flat plate as shown in Figure 15.3. Consider a small length dx of the plate at a distance x from the leading edge as shown in Figure 15.3(a) whose enlarged view is shown in Figure 15.3(b). Assume unit width of the plate as perpendicular to the direction of flow.

Let $ABCD$ be the control volume of the small element of the boundary layer and u be the velocity at any point within the boundary layer.

Mass flow rate of fluid entering through AD is given by,

$$m_{AD} = \int_0^\delta \rho \times u \times dy \times 1 = \int_0^\delta \rho u dy$$

Mass flow rate of fluid leaving through BC is given by,

$$m_{BC} = m_{AD} + \frac{\partial (m_{AD})}{\partial x} dx = \int_0^\delta \rho u dy + \frac{\partial}{\partial x}\left[\int_0^\delta \rho u dy\right] dx$$

Since $(m_{AD} + m_{DC}) = m_{BC}$ [Mass conservation law]

Therefore, mass flow rate of fluid entering through DC is given by,

$$m_{DC} = m_{BC} - m_{AD} = \int_0^\delta \rho u dy + \frac{\partial}{\partial x}\left[\int_0^\delta \rho u dy\right] dx - \int_0^\delta \rho u dy = \frac{\partial}{\partial x}\left[\int_0^\delta \rho u dy\right] dx$$

Momentum flux entering through AD is given by,

$$M_{AD} = m_{AD} \times u = \int_0^\delta \rho u^2 dy$$

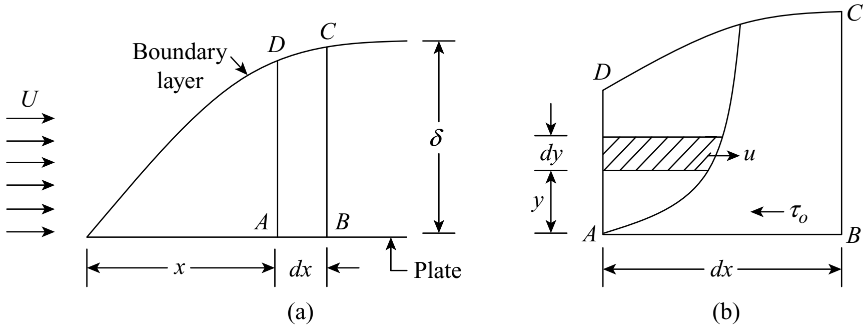


Figure 15.3 *Drag force on a flat plate*

Momentum flux leaving through *BC* is given by,

$$M_{BC} = m_{BC} \times u = \int_0^\delta \rho u^2 dy + \frac{\partial}{\partial x}\left[\int_0^\delta \rho u^2 dy\right]dx$$

The fluid enters through side *DC* with a uniform velocity U and thus, momentum flux entering through *DC* is given by,

$$M_{DC} = m_{DC} \times U = \frac{\partial}{\partial x}\left[\int_0^\delta \rho u dy\right]dx \times U = \frac{\partial}{\partial x}\left[\int_0^\delta \rho U dy\right]dx$$

Therefore, the rate of change of momentum of control volume *ABCD* is given by,

$$= M_{BC} - M_{AD} - M_{DC} = \int_0^\delta \rho u^2 dy + \frac{\partial}{\partial x}\left[\int_0^\delta \rho u^2 dy\right]dx - \int_0^\delta \rho u^2 dy - \frac{\partial}{\partial x}\left[\int_0^\delta \rho uU dy\right]dx$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho u^2 dy\right]dx - \frac{\partial}{\partial x}\left[\int_0^\delta \rho uU dy\right]dx = \frac{\partial}{\partial x}\left[\int_0^\delta (\rho u^2 dy - \rho uU dy\right]dx$$

$$= \rho\frac{\partial}{\partial x}\left[\int_0^\delta (u^2 - uU)dy\right]dx \qquad [\because \rho = \text{Constant}] \qquad (15.7)$$

According to the principle of momentum, the total force on the control volume *ABCD* must be equal to the rate of change of momentum in the same direction. The only external force on the control volume is the shear force on the boundary surface *AB* in the opposite direction, i.e., from *B* to *A*. The value of this force is given in the below expression.

$$\Delta F_D = \tau_o \times dx \times 1 = \tau_o dx$$

Thus, external force in the direction of rate of change in momentum is given by,

$$= -\Delta F_D = -\tau_o dx \qquad (15.8)$$

Now simplifying Equations (15.7) and (15.8), we get:

$$-\tau_o dx = \rho\frac{\partial}{\partial x}\left[\int_0^\delta (u^2 - uU)dy\right]dx$$

$$\tau_o = -\rho\frac{\partial}{\partial x}\left[\int_0^\delta (u^2 - uU)dy\right] = \rho\frac{\partial}{\partial x}\left[\int_0^\delta (uU - u^2)dy\right] = \rho U^2\frac{\partial}{\partial x}\left[\int_0^\delta \left(\frac{u}{U} - \frac{u^2}{U^2}\right)dy\right]$$

Thus $$\frac{\tau_o}{\rho U^2} = \frac{\partial}{\partial x}\left[\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy\right] \qquad (15.9)$$

Since $$\delta_m = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy$$

$$\boxed{\therefore \frac{\tau_o}{\rho U^2} = \frac{\partial \delta_m}{\partial x}} \qquad (15.9a)$$

The Equation (15.9a) is known as von Kármán momentum integral equation for boundary layer flow which is applicable to the following parameters.

1. Steady laminar boundary layers
2. Time averaged turbulent layers
3. Two-dimensional incompressible flows
4. For flows in which $(dp/dx) = 0$, i.e., pressure gradient is zero in the direction of flow.

From this equation, the value of shear stress (τ_o) can be determined for a given velocity profile in laminar, transition or turbulent region of a boundary layer. If b is the width of the plate, then drag force on a small length dx can be determined by the following expression.

$$\Delta F_D = \tau_o b dx$$

Thus, total drag on the plate of length l on one side can be given by,

$$F_D = \int_0^l \Delta F_D = \int_0^l \tau_o b dx \tag{15.10}$$

1. **Boundary conditions:** The following boundary conditions must be satisfied by any velocity profile.

   (i) At $y = 0$, $u = 0$ and $\frac{du}{dy}$ has some finite value.

   (ii) At $y = \delta$, $u = U$ and $\frac{du}{dy} = 0$.

2. **Local coefficient of drag (or skin friction coefficient):** It is defined as the ratio of the local wall shear stress (τ_o) to the dynamic pressure of the uniform flow stream. It is denoted by C_f. The mathematical expression for local coefficient of drag is given below.

$$\boxed{C_f = \frac{\tau_o}{(1/2)\rho U^2}} \tag{15.11}$$

   Here, ρ is the mass density of fluid and U is the freestream velocity.

3. **Average coefficient of drag:** It is defined as the ratio of the total drag force (F_D) to the quantity $(1/2)\rho A U^2$. It is also known as coefficient of drag and it is denoted by C_D. The mathematical expression for average coefficient of drag is given below.

$$\boxed{C_D = \frac{F_D}{(1/2)\rho A U^2}} \tag{15.12}$$

   Here, A is the surface area of the plate surface.

15.5 ❐ PRANDTL'S BOUNDARY LAYER EQUATIONS

Introducing the following simplifying assumptions:

(i) The body force is neglected, i.e., B_x, B_y and B_z are negligible,

(ii) The boundary is two-dimensional, i.e., $w = 0$ and $\frac{\partial(\,)}{\partial z} = 0$.

(iii) The flow is steady, i.e., $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = 0$.

By applying the above assumptions, the Navier-Stokes equations [i.e., Equation (12.3)] for constant properties derived in Chapter 12 are given below.

$$\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{15.13}$$

$$\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{15.14}$$

The Equations (15.13) and (15.14) can be further simplified by considering the relative order of magnitude of each term and dropping the small order terms.

(i) u and x are of the order of magnitude one and thus, $(\partial u/\partial x)$ and $(\partial^2 u/\partial x^2)$ are also of the order of magnitude unity.

(ii) y is of the order of magnitude δ and thus, $(\partial u/\partial y) \sim (1/\delta)$ and $(\partial^2 u/\partial y^2) \sim (1/\delta^2)$.

(iii) The continuity equation is $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$. Since $(\partial u/\partial x)$ is of the order of unity, the term $(\partial v/\partial y)$ is also of the order of magnitude one. Therefore, $(\partial v/\partial y) \sim (\delta/\delta) \sim 1$ and $(\partial^2 v/\partial y^2) \sim (\delta/\delta^2) \sim (1/\delta)$. Also $(\partial v/\partial x)$ and $(\partial^2 v/\partial x^2)$ are of the order of magnitude δ.

(iv) Since $\delta \propto \sqrt{\nu}$, ν is of the order of magnitude δ^2.

Now
$$\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$1 \quad 1 \qquad \delta \quad 1/\delta \qquad\qquad\qquad \delta^2 \quad 1 \qquad 1/\delta^2$$

In the above equation, the term $\nu\frac{\partial^2 u}{\partial x^2}$ is small when compared to other terms and it can be dropped.

Thus
$$\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2} \tag{15.15}$$

Now
$$\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$1 \quad \delta \qquad \delta \quad 1 \qquad\qquad\qquad \delta^2 \quad \delta \qquad 1/\delta$$

In Equation (15.14), all terms except $-\frac{1}{\rho}\frac{\partial p}{\partial y}$ are small and it can be neglected.

Thus
$$-\frac{1}{\rho}\frac{\partial p}{\partial y}=0$$

or
$$\frac{\partial p}{\partial y}=0 \tag{15.16}$$

It means $p = f(x)$ only and therefore, $\dfrac{\partial p}{\partial x} = \dfrac{dp}{dx}$.

From Equation (15.15), we get:

$$\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{15.17}$$

The Equations (15.16) and (15.17) are referred to as Prandtl's boundary layer equations for two-dimensional steady flow of incompressible fluids.

15.6 ❒ BLASIUS SOLUTION FOR LAMINAR BOUNDARY LAYER FLOWS

Blasius has obtained a solution for the Prandtl's boundary layer equations by exact analytical method. In the laminar boundary layer, the expression for thickness δ is given below.

$$\boxed{\delta = \frac{5x}{\sqrt{(Ux)/\nu}} = \frac{5x}{\sqrt{\text{Re}_x}}} \tag{15.18}$$

Skin friction coefficient or local drag coefficient (C_f) is given by,

$$\boxed{C_f = \frac{0.664}{\sqrt{\text{Re}_x}}} \tag{15.19}$$

Average drag coefficient or coefficient of drag (C_D) is given by,

$$\boxed{C_D = \frac{1.328}{\sqrt{\text{Re}_l}}, \text{ where } \text{Re}_l = (Ul)/\nu} \tag{15.20}$$

Thus, the total drag force (F_D) on one side of the plate can be obtained by the below expression.

$$\boxed{F_D = \frac{1}{2}C_D\rho U^2 A = \frac{1}{2}C_D\rho U^2 bl} \tag{15.21}$$

From the exact analytical solution of the boundary layer equations by Blasius, the following expressions for displacement thickness and momentum thickness have been obtained.

$$\delta_d = \frac{1.729x}{\sqrt{\text{Re}_x}} \tag{15.22}$$

$$\delta_m = \frac{0.664x}{\sqrt{\text{Re}_x}} \tag{15.23}$$

15.7 ❒ VELOCITY PROFILES FOR LAMINAR BOUNDARY LAYER

The equations for the boundary layer along a flat plate can also be derived by using Equation (15.9). For this, it is essential to assume a suitable velocity distribution for the boundary layer. The velocity distribution within the laminar boundary layer is of the form $(u/U) = f(\eta)$, where $\eta = (y/\delta)$ and $f(\eta)$ is a polynomial. This may be approximated by the parabolic equation given in Table 15.1 for various velocity profiles along with the values of boundary layer thickness (δ), local coefficient of drag (C_f) and drag coefficient (C_D) in terms of Reynolds number.

Table 15.1 *Values of δ, C_f and C_D in terms of Reynolds number*

S. no.	Velocity profile	δ	C_f	C_D
1.	$\frac{u}{U}=2\left(\frac{y}{\delta}\right)-\left(\frac{y}{\delta}\right)^2$	$\frac{5.48x}{\sqrt{\text{Re}_x}}$	$\frac{0.73}{\sqrt{\text{Re}_x}}$	$\frac{1.46}{\sqrt{\text{Re}_l}}$
2.	$\frac{u}{U}=\frac{3}{2}\left(\frac{y}{\delta}\right)-\frac{1}{2}\left(\frac{y}{\delta}\right)^3$	$\frac{4.64x}{\sqrt{\text{Re}_x}}$	$\frac{0.646}{\sqrt{\text{Re}_x}}$	$\frac{1.292}{\sqrt{\text{Re}_l}}$
3.	$\frac{u}{U}=2\left(\frac{y}{\delta}\right)-2\left(\frac{y}{\delta}\right)^3+\left(\frac{y}{\delta}\right)^4$	$\frac{5.84x}{\sqrt{\text{Re}_x}}$	$\frac{0.686}{\sqrt{\text{Re}_x}}$	$\frac{1.372}{\sqrt{\text{Re}_l}}$
4.	$\frac{u}{U}=\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$	$\frac{4.795x}{\sqrt{\text{Re}_x}}$	$\frac{0.654}{\sqrt{\text{Re}_x}}$	$\frac{1.31}{\sqrt{\text{Re}_l}}$
5.	Blasius solution $(\text{Re} < 3.2\times10^5)$	$\frac{5x}{\sqrt{\text{Re}_x}}$	$\frac{0.664}{\sqrt{\text{Re}_x}}$	$\frac{1.328}{\sqrt{\text{Re}_l}}$

Example 15.4 Determine the expressions for boundary layer thickness (δ), shear stress (τ_o), local coefficient of drag (C_f) and coefficient of drag (C_D) in terms of Reynolds number for the velocity profile of laminar boundary layer given by $(u/U) = 2(y/\delta) - (y/\delta)^2$.

Solution

Let $\frac{u}{U}=2\frac{y}{\delta}-\left(\frac{y}{\delta}\right)^2$ or $\frac{u}{U}=\frac{2y}{\delta}-\frac{y^2}{\delta^2}$ (i)

(i) $$\frac{\tau_o}{\rho U^2}=\frac{\partial}{\partial x}\left[\int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy\right]=\frac{\partial}{\partial x}\left[\int_0^\delta\left\{\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right\}\left\{1-\left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\right\}dy\right]$$

$$=\frac{\partial}{\partial x}\left[\int_0^\delta\left\{\frac{2y}{\delta}-\frac{4y^2}{\delta^2}+\frac{2y^3}{\delta^3}-\frac{y^2}{\delta^2}+\frac{2y^3}{\delta^3}-\frac{y^4}{\delta^4}\right\}dy\right]$$

$$=\frac{\partial}{\partial x}\left[\int_0^\delta\left\{\frac{2y}{\delta}-\frac{5y^2}{\delta^2}+\frac{4y^3}{\delta^3}-\frac{y^4}{\delta^4}\right\}dy\right]=\frac{\partial}{\partial x}\left[\frac{2y^2}{2\delta}-\frac{5y^3}{3\delta^2}+\frac{4y^4}{4\delta^3}-\frac{y^5}{5\delta^4}\right]_0^\delta$$

Thus $$\frac{\tau_o}{\rho U^2}=\frac{\partial}{\partial x}\left[\frac{\delta^2}{\delta}-\frac{5\delta^3}{3\delta^2}+\frac{\delta^4}{\delta^3}-\frac{\delta^5}{5\delta^4}\right]=\frac{\partial}{\partial x}\left[\delta-\frac{5\delta}{3}+\delta-\frac{\delta}{5}\right]=\frac{2}{15}\frac{\partial\delta}{\partial x}$$

$$\therefore \tau_o=\frac{2}{15}\rho U^2\frac{\partial\delta}{\partial x} \quad \text{(ii)}$$

Now $$u=U\left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right) \quad \text{[From expression (i)]}$$

$$\frac{du}{dy}=U\left(\frac{2}{\delta}-\frac{2y}{\delta^2}\right) \quad [\because U = \text{Constant}]$$

$$\left(\frac{du}{dy}\right)_{y=0} = U\left[\frac{2}{\delta} - \frac{2(0)}{\delta^2}\right] = \frac{2U}{\delta}$$

Thus $$\tau_o = \mu\left(\frac{du}{dy}\right)_{y=0} = \mu\frac{2U}{\delta} = \frac{2\mu U}{\delta} \qquad \text{(iii)}$$

Simplifying expressions (ii) and (iii), we get:

$$\frac{2}{15}\rho U^2 \frac{\partial\delta}{\partial x} = \frac{2\mu U}{\delta}$$

$$\delta d\delta = \frac{15\mu}{\rho U}dx \qquad [\because \delta = f(x) \text{ only}]$$

Integrating on both sides, we get:

$$\frac{\delta^2}{2} = \frac{15\mu}{\rho U}x + k \qquad [\because k = \text{Constant}]$$

Applying boundary condition: At $x = 0$, $\delta = 0$ and thus, $k = 0$.

Thus $$\frac{\delta^2}{2} = \frac{15\mu x}{\rho U}$$

$$\therefore \delta = \sqrt{\frac{30\mu x}{\rho U}} = \sqrt{\frac{30\mu x \times x}{\rho U \times x}} = \sqrt{\frac{30x^2}{(\rho U x)/\mu}} = \frac{\mathbf{5.48}\boldsymbol{x}}{\sqrt{\mathbf{Re}_{\boldsymbol{x}}}}$$

(ii) $$\tau_o = \frac{2\mu U}{\delta} = \frac{2\mu U}{(5.48x)/\sqrt{\text{Re}_x}} = \mathbf{0.365}\frac{\boldsymbol{\mu U}}{\boldsymbol{x}}\sqrt{\mathbf{Re}_{\boldsymbol{x}}}$$

(iii) $$C_f = \frac{\tau_o}{(1/2)\rho U^2} = \frac{0.365\{(\mu U)/x\}\sqrt{\text{Re}_x}}{(1/2)\rho U^2} = 0.73\frac{\sqrt{\text{Re}_x}}{(\rho U x)/\mu}$$

$$\therefore C_f = 0.73\frac{\sqrt{\text{Re}_x}}{\text{Re}_x} = \frac{\mathbf{0.73}}{\sqrt{\mathbf{Re}_{\boldsymbol{x}}}} \qquad [\because \text{Re}_x = (\rho U x)/\mu]$$

(iv) $$F_D = \int_0^l \tau_o b dx = \int_0^l 0.365\frac{\mu U}{x}\sqrt{\text{Re}_x}\, bdx$$

$$= \int_0^l 0.365\frac{\mu U}{x}\sqrt{\frac{\rho U x}{\mu}}\, bdx = 0.365\mu U b\sqrt{\frac{\rho U}{\mu}}\int_0^l x^{-1/2}dx$$

Thus $$F_D = 0.365\mu U b\sqrt{\frac{\rho U}{\mu}}\left[\frac{x^{1/2}}{(1/2)}\right]_0^l = 0.73\mu U b\sqrt{\frac{\rho U}{\mu}}\sqrt{l} = 0.73\mu U b\sqrt{\frac{\rho U l}{\mu}}$$

Now $$C_D = \frac{F_D}{(1/2)\rho A U^2} = \frac{0.73\mu U b\sqrt{(\rho U l)/\mu}}{(1/2)\rho(lb)U^2} = \frac{1.46\sqrt{\mu}}{\sqrt{\rho}\sqrt{l}\sqrt{U}}$$

$$\therefore C_D = \frac{1.46}{\sqrt{(\rho U l)/\mu}} = \frac{\mathbf{1.46}}{\sqrt{\mathbf{Re}_{\boldsymbol{l}}}} \qquad [\because \text{Re}_l = \rho U l/\mu]$$

Example 15.5 Determine the expressions for boundary layer thickness (δ), shear stress (τ_o), local coefficient of drag (C_f) and coefficient of drag (C_D) in terms of Reynolds number for the velocity profile of laminar boundary layer given by $(u/U) = (3/2)(y/\delta) - (1/2)(y/\delta)^3$.

Solution

Let $(u/U) = (3/2)(y/\delta) - (1/2)(y/\delta)^3$ or $\dfrac{u}{U} = \dfrac{3y}{2\delta} - \dfrac{y^3}{2\delta^3}$ (i)

(i) $$\frac{\tau_o}{\rho U^2} = \frac{\partial}{\partial x}\left[\int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy\right] = \frac{\partial}{\partial x}\left[\int_0^\delta \left\{\frac{3y}{2\delta} - \frac{y^3}{2\delta^3}\right\}\left\{1-\left(\frac{3y}{2\delta} - \frac{y^3}{2\delta^3}\right)\right\}dy\right]$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left\{\frac{3y}{2\delta} - \frac{9y^2}{4\delta^2} + \frac{3y^4}{4\delta^4} - \frac{y^3}{2\delta^3} + \frac{3y^4}{4\delta^4} - \frac{y^6}{4\delta^6}\right\}dy\right]$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left\{\frac{3y}{2\delta} - \frac{9y^2}{4\delta^2} + \frac{6y^4}{4\delta^4} - \frac{y^3}{2\delta^3} - \frac{y^6}{4\delta^6}\right\}dy\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{3y^2}{4\delta} - \frac{9y^3}{12\delta^2} + \frac{6y^5}{20\delta^4} - \frac{y^4}{8\delta^3} - \frac{y^7}{28\delta^6}\right]_0^\delta$$

$$= \frac{\partial}{\partial x}\left[\frac{3\delta^2}{4\delta} - \frac{9\delta^3}{12\delta^2} + \frac{6\delta^5}{20\delta^4} - \frac{\delta^4}{8\delta^3} - \frac{\delta^7}{28\delta^6}\right]$$

Thus $$\frac{\tau_o}{\rho U^2} = \frac{\partial}{\partial x}\left[\frac{3\delta}{4} - \frac{3\delta}{4} + \frac{3\delta}{10} - \frac{\delta}{8} - \frac{\delta}{28}\right] = \frac{39}{280}\frac{\partial \delta}{\partial x}$$

$$\therefore \tau_o = \frac{39}{280}\rho U^2 \frac{\partial \delta}{\partial x} \quad \text{(ii)}$$

$$u = U\left[\frac{3y}{2\delta} - \frac{y^3}{2\delta^3}\right] \quad \text{[From expression (i)]}$$

$$\frac{du}{dy} = U\left(\frac{3}{2\delta} - \frac{3y^2}{2\delta^3}\right) \quad [\because U = \text{Constant}]$$

$$\left(\frac{du}{dy}\right)_{y=0} = U\left(\frac{3}{2\delta} - \frac{3(0)^2}{2\delta^3}\right) = \frac{3U}{2\delta}$$

Thus $$\tau_o = \mu\left(\frac{du}{dy}\right)_{y=0} = \mu\frac{3U}{2\delta} = \frac{3\mu U}{2\delta} \quad \text{(iii)}$$

Simplifying expressions (ii) and (iii), we get:

$$\frac{39}{280}\rho U^2 \frac{\partial \delta}{\partial x} = \frac{3\mu U}{2\delta}$$

$$\delta d\delta = \frac{140\mu}{13\rho U}dx \quad [\because \delta = f(x) \text{ only}]$$

Integrating on both sides, we get:

$$\frac{\delta^2}{2} = \frac{140\mu}{13\rho U}x + k \qquad [\because k = \text{Constant}]$$

Applying boundary condition: At $x = 0$, $\delta = 0$ and thus, $k = 0$.

Thus
$$\frac{\delta^2}{2} = \frac{140\mu x}{13\rho U}$$

$$\therefore \delta = \sqrt{\frac{280\,\mu x}{13\,\rho U}} = \sqrt{\frac{280}{13}\frac{\mu x \times x}{\rho U \times x}} = \sqrt{\frac{280}{13}\frac{x^2}{(\rho U x)/\mu}} = \frac{\mathbf{4.64}\boldsymbol{x}}{\sqrt{\mathbf{Re}_{\boldsymbol{x}}}}$$

(ii) $\tau_o = \dfrac{3\mu U}{2\delta} = \dfrac{3\mu U}{2\times[(4.64x)/\sqrt{\text{Re}_x})]} = \mathbf{0.323}\dfrac{\boldsymbol{\mu U}}{\boldsymbol{x}}\sqrt{\mathbf{Re}_{\boldsymbol{x}}}$

(iii) $C_f = \dfrac{\tau_o}{(1/2)\rho U^2} = \dfrac{0.323\{(\mu U)/x\}\sqrt{\text{Re}_x}}{(1/2)\rho U^2} = 0.646\dfrac{\sqrt{\text{Re}_x}}{(\rho U x)/\mu}$

$$\therefore C_f = 0.646\frac{\sqrt{\text{Re}_x}}{\text{Re}_x} = \frac{\mathbf{0.646}}{\sqrt{\mathbf{Re}_{\boldsymbol{x}}}} \qquad [\because \text{Re}_x = (\rho U x)/\mu]$$

(iv) $F_D = \displaystyle\int_0^l \tau_o b\,dx = \int_0^l 0.323\frac{\mu U}{x}\sqrt{\text{Re}_x}\times b\,dx = \int_0^l 0.323\frac{\mu U}{x}\sqrt{\frac{\rho U x}{\mu}}\,b\,dx$

$$= 0.323\mu Ub\sqrt{\frac{\rho U}{\mu}}\int_0^l x^{-1/2}dx = 0.323\mu Ub\sqrt{\frac{\rho U}{\mu}}\left[\frac{x^{1/2}}{(1/2)}\right]_0^l$$

Thus
$$F_D = 0.646\mu Ub\sqrt{\frac{\rho U}{\mu}}\sqrt{l} = 0.646\mu Ub\sqrt{\frac{\rho Ul}{\mu}}$$

Now
$$C_D = \frac{F_D}{(1/2)\rho AU^2} = \frac{0.646\,\mu Ub\sqrt{(\rho Ul)/\mu}}{(1/2)\,\rho(lb)U^2} = \frac{1.292\sqrt{\mu}}{\sqrt{\rho}\sqrt{l}\sqrt{U}}$$

$$\therefore C_D = \frac{1.292}{\sqrt{(\rho Ul)/\mu}} = \frac{\mathbf{1.292}}{\sqrt{\mathbf{Re}_{\boldsymbol{l}}}} \qquad [\because \text{Re}_l = \rho Ul/\mu]$$

Example 15.6 Determine the expressions for boundary layer thickness (δ), shear stress (τ_o), local coefficient of drag (C_f) and coefficient of drag (C_D) in terms of Reynolds number for the velocity profile of laminar boundary layer given by $(u/U) = 2(y/\delta) - 2(y/\delta)^3 + (y/\delta)^4$.

Solution

Let $(u/U) = 2(y/\delta) - 2(y/\delta)^3 + (y/\delta)^4$ or $\dfrac{u}{U} = \dfrac{2y}{\delta} - \dfrac{2y^3}{\delta^3} + \dfrac{y^4}{\delta^4}$ (i)

$$\text{(i)}\quad \frac{\tau_o}{\rho U^2} = \frac{\partial}{\partial x}\left[\int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy\right] = \frac{\partial}{\partial x}\left[\int_0^\delta \left\{\frac{2y}{\delta}-\frac{2y^3}{\delta^3}+\frac{y^4}{\delta^4}\right\}\left\{1-\left(\frac{2y}{\delta}-\frac{2y^3}{\delta^3}+\frac{y^4}{\delta^4}\right)\right\}dy\right]$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left\{\frac{2y}{\delta}-\frac{4y^2}{\delta^2}+\frac{9y^4}{\delta^4}-\frac{4y^5}{\delta^5}-\frac{2y^3}{\delta^3}-\frac{4y^6}{\delta^6}+\frac{4y^7}{\delta^7}-\frac{y^8}{\delta^8}\right\}dy\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{2y^2}{2\delta}-\frac{4y^3}{3\delta^2}+\frac{9y^5}{5\delta^4}-\frac{4y^6}{6\delta^5}-\frac{2y^4}{4\delta^3}-\frac{4y^7}{7\delta^6}+\frac{4y^8}{8\delta^7}-\frac{y^9}{9\delta^8}\right]_0^\delta$$

$$= \frac{\partial}{\partial x}\left[\frac{2\delta^2}{2\delta}-\frac{4\delta^3}{3\delta^2}+\frac{9\delta^5}{5\delta^4}-\frac{4\delta^6}{6\delta^5}-\frac{2\delta^4}{4\delta^3}-\frac{4\delta^7}{7\delta^6}+\frac{4\delta^8}{8\delta^7}-\frac{\delta^9}{9\delta^8}\right]$$

Thus
$$\frac{\tau_o}{\rho U^2} = \frac{\partial}{\partial x}\left[\delta-\frac{4\delta}{3}+\frac{9\delta}{5}-\frac{4\delta}{6}-\frac{2\delta}{4}-\frac{4\delta}{7}+\frac{4\delta}{8}-\frac{\delta}{9}\right] = \frac{37}{315}\frac{\partial\delta}{\partial x}$$

$$\therefore \tau_o = \frac{37}{315}\rho U^2\frac{\partial\delta}{\partial x} \qquad \text{(ii)}$$

$$u = U\left[\frac{2y}{\delta}-\frac{2y^3}{\delta^3}+\frac{y^4}{\delta^4}\right] \qquad \text{[From expression (i)]}$$

$$\frac{du}{dy} = U\left(\frac{2}{\delta}-\frac{6y^2}{\delta^3}+\frac{4y^3}{\delta^4}\right) \qquad [\because U = \text{Constant}]$$

$$\left(\frac{du}{dy}\right)_{y=0} = U\left(\frac{2}{\delta}-\frac{6(0)^2}{\delta^3}+\frac{4(0)^3}{\delta^4}\right) = \frac{2U}{\delta}$$

Thus
$$\tau_o = \mu\left(\frac{du}{dy}\right)_{y=0} = \mu\frac{2U}{\delta} = \frac{2\mu U}{\delta} \qquad \text{(iii)}$$

Simplifying expressions (ii) and (iii), we get:

$$\frac{37}{315}\rho U^2\frac{\partial\delta}{\partial x} = \frac{2\mu U}{\delta}$$

$$\delta\, d\delta = \frac{630\mu}{37\rho U}dx \qquad [\because \delta = f(x) \text{ only}]$$

Integrating on both sides, we get:

$$\frac{\delta^2}{2} = \frac{630\mu}{37\rho U}x + k \qquad [\because k = \text{Constant}]$$

Applying boundary condition: At $x = 0$, $\delta = 0$ and thus, $k = 0$.

Thus
$$\frac{\delta^2}{2} = \frac{630\mu x}{37\rho U}$$

$$\delta = \sqrt{\frac{1260\mu x}{37\rho U}} = \sqrt{\frac{1260}{37}\frac{\mu x\times x}{\rho U\times x}} = \sqrt{\frac{1260}{37}\frac{x^2}{(\rho Ux)/\mu}} = \mathbf{\frac{5.84x}{\sqrt{Re_x}}}$$

(ii) $\tau_o = \dfrac{2\mu U}{\delta} = \dfrac{2\mu U}{(5.84x)/\sqrt{\text{Re}_x}} = \mathbf{0.343}\dfrac{\boldsymbol{\mu U}}{\boldsymbol{x}}\sqrt{\mathbf{Re}_{\boldsymbol{x}}}$

(iii) $C_f = \dfrac{\tau_o}{(1/2)\rho U^2} = \dfrac{0.343\{(\mu U)/x\}\sqrt{\text{Re}_x}}{(1/2)\rho U^2} = 0.686\dfrac{\sqrt{\text{Re}_x}}{(\rho Ux)/\mu}$

$$\therefore C_f = 0.686\frac{\sqrt{\text{Re}_x}}{\text{Re}_x} = \frac{\mathbf{0.686}}{\sqrt{\mathbf{Re}_{\boldsymbol{x}}}} \qquad [\because \text{Re}_x = (\rho Ux)/\mu]$$

(iv) $F_D = \int_0^l \tau_o b dx = \int_0^l 0.343\dfrac{\mu U}{x}\sqrt{\text{Re}_x}\times bdx = \int_0^l 0.343\dfrac{\mu U}{x}\sqrt{\dfrac{\rho Ux}{\mu}}\,bdx$

$$= 0.343\mu Ub\sqrt{\frac{\rho U}{\mu}}\int_0^l x^{-1/2}dx = 0.343\mu Ub\sqrt{\frac{\rho U}{\mu}}\left[\frac{x^{1/2}}{(1/2)}\right]_0^l = 0.686\mu Ub\sqrt{\frac{\rho U}{\mu}}\sqrt{l}$$

Thus $$F_D = 0.686\mu Ub\sqrt{\frac{\rho Ul}{\mu}}$$

Now $$C_D = \frac{F_D}{(1/2)\rho AU^2} = \frac{0.686\mu Ub\sqrt{(\rho Ul)/\mu}}{(1/2)\rho(lb)U^2} = \frac{1.372\sqrt{\mu}}{\sqrt{\rho}\sqrt{l}\sqrt{U}} = \frac{1.372}{\sqrt{(\rho Ul)/\mu}}$$

$$\therefore C_D = \frac{\mathbf{1.372}}{\sqrt{\mathbf{Re}_{\boldsymbol{l}}}} \qquad [\because \text{Re}_l = \rho Ul/\mu]$$

Example 15.7 Determine the expressions for boundary layer thickness (δ), shear stress (τ_o), local coefficient of drag (C_f) and coefficient of drag (C_D) in terms of Reynolds number for the velocity profile of laminar boundary layer given by $(u/U) = \sin[(\pi y)/2\delta]$.

Solution

Let $$\frac{u}{U} = \sin\frac{\pi y}{2\delta} \qquad \text{(i)}$$

(i) $\dfrac{\tau_o}{\rho U^2} = \dfrac{\partial}{\partial x}\left[\int_0^\delta \dfrac{u}{U}\left(1-\dfrac{u}{U}\right)dy\right] = \dfrac{\partial}{\partial x}\left[\int_0^\delta \sin\left(\dfrac{\pi y}{2\delta}\right)\left\{1-\sin\left(\dfrac{\pi y}{2\delta}\right)\right\}dy\right]$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left\{\sin\left(\frac{\pi y}{2\delta}\right) - \sin^2\left(\frac{\pi y}{2\delta}\right)\right\}dy\right]$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left\{\sin\left(\frac{\pi y}{2\delta}\right) - \frac{1-\cos\{(2\pi y)/(2\delta)\}}{2}\right\}dy\right]$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left\{\sin\left(\frac{\pi y}{2\delta}\right) - \frac{1}{2} + \frac{\cos\{(\pi y)/\delta\}}{2}\right\}dy\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{-\cos\{(\pi y)/(2\delta)\}}{\pi/(2\delta)} - \frac{y}{2} + \frac{\sin\{(\pi y)/\delta)\}}{(2\pi)/\delta}\right]_0^\delta$$

$$= \frac{\partial}{\partial x}\left[\frac{-\cos\{(\pi \delta)/(2\delta)\}}{\pi/(2\delta)} - \frac{\delta}{2} + \frac{\sin\{(\pi \delta)/\delta\}}{(2\pi)/\delta} + \frac{\cos(0)}{\pi/(2\delta)} - \frac{\sin(0)}{(2\pi)/\delta}\right]$$

Thus $$\frac{\tau_o}{\rho U^2} = \frac{\partial}{\partial x}\left[0 - \frac{\delta}{2} + 0 + \frac{1}{(\pi/2\delta)} - 0\right] = \frac{\partial}{\partial x}\left(\frac{2\delta}{\pi} - \frac{\delta}{2}\right) = \left(\frac{4-\pi}{2\pi}\right)\frac{\partial \delta}{\partial x}$$

$$\therefore \tau_o = \left(\frac{4-\pi}{2\pi}\right)\rho U^2 \frac{\partial \delta}{\partial x} \qquad \text{(ii)}$$

$$u = U \sin\left(\frac{\pi y}{2\delta}\right) \qquad \text{[From expression (i)]}$$

$$\frac{du}{dy} = U\left[\cos\left(\frac{\pi y}{2\delta}\right)\right] \times \frac{\pi}{2\delta} \qquad [\because U = \text{Constant}]$$

$$\left(\frac{du}{dy}\right)_{y=0} = U\left[\cos\left(\frac{0}{2\delta}\right)\right] \times \frac{\pi}{2\delta} = \frac{\pi U}{2\delta}$$

Thus $$\tau_o = \mu\left(\frac{du}{dy}\right)_{y=0} = \mu\frac{\pi U}{2\delta} = \frac{\mu \pi U}{2\delta} \qquad \text{(iii)}$$

Simplifying expressions (ii) and (iii), we get:

$$\left(\frac{4-\pi}{2\pi}\right)\rho U^2 \frac{\partial \delta}{\partial x} = \frac{\mu \pi U}{2\delta}$$

$$\delta d\delta = \frac{\mu \pi}{2\rho U}\left(\frac{2\pi}{4-\pi}\right)dx = 11.5\frac{\mu}{\rho U}dx \qquad [\because \delta = f(x) \text{ only}]$$

Integrating on both sides, we get:

$$\frac{\delta^2}{2} = 11.5\frac{\mu}{\rho U}x + k \qquad [\because k = \text{Constant}]$$

Applying boundary condition: At $x = 0$, $\delta = 0$ and thus, $k = 0$.

Thus $$\frac{\delta^2}{2} = \frac{11.5 \mu x}{\rho U}$$

$$\delta = \sqrt{\frac{23\mu x}{\rho U}} = \sqrt{\frac{23\mu x \times x}{\rho U \times x}} = \sqrt{\frac{23x^2}{(\rho U x)/\mu}} = \frac{\mathbf{4.795}\boldsymbol{x}}{\sqrt{\mathbf{Re}_{\boldsymbol{x}}}}$$

(ii) $$\tau_o = \frac{\mu \pi U}{2\delta} = \frac{\mu \pi U}{2 \times \{(4.795x)/\sqrt{\text{Re}_x}\}} = \mathbf{0.327}\frac{\boldsymbol{\mu U}}{\boldsymbol{x}}\sqrt{\mathbf{Re}_{\boldsymbol{x}}}$$

(iii) $C_f = \dfrac{\tau_o}{(1/2)\rho U^2} = \dfrac{0.327\{(\mu U)/x\}\sqrt{\text{Re}_x}}{(1/2)\rho U^2} = 0.654\dfrac{\sqrt{\text{Re}_x}}{(\rho Ux)/\mu}$

$$\therefore C_f = 0.654\frac{\sqrt{\text{Re}_x}}{\text{Re}_x} = \frac{\mathbf{0.654}}{\sqrt{\mathbf{Re}_x}} \qquad [\because \text{Re}_x = (\rho Ux)/\mu]$$

(iv) $F_D = \int_0^l \tau_o b\,dx = \int_0^l 0.327\dfrac{\mu U}{x}\sqrt{\text{Re}_x} \times b\,dx = \int_0^l 0.327\dfrac{\mu U}{x}\sqrt{\dfrac{\rho Ux}{\mu}}\, b\,dx$

$$= 0.327\mu Ub\sqrt{\frac{\rho U}{\mu}}\int_0^l x^{-1/2}dx = 0.327\mu Ub\sqrt{\frac{\rho U}{\mu}}\left[\frac{x^{1/2}}{(1/2)}\right]_0^l$$

Thus $\qquad F_D = 0.654\mu Ub\sqrt{\dfrac{\rho U}{\mu}}\sqrt{l} = 0.654\mu Ub\sqrt{\dfrac{\rho Ul}{\mu}}$

Now $\qquad C_D = \dfrac{F_D}{(1/2)\rho AU^2} = \dfrac{0.654\,\mu Ub\sqrt{(\rho Ul)/\mu}}{(1/2)\rho(lb)U^2} = \dfrac{1.31\sqrt{\mu}}{\sqrt{\rho}\sqrt{l}\sqrt{U}} = \dfrac{1.31}{\sqrt{(\rho Ul)/\mu}}$

$$\therefore C_D = \frac{\mathbf{1.31}}{\sqrt{\mathbf{Re}_l}} \qquad [\because \text{Re}_l = \rho Ul/\mu]$$

Example 15.8 A plate of length 0.5 m and width 0.25 m is placed longitudinally in a fluid of specific gravity 0.9 and of kinematic viscosity one stoke. If the fluid is moving with a velocity of 5 m/s, then determine (i) friction drag on the plate, (ii) thickness of boundary layer and (iii) shear stress at the trailing edge of the plate. Use Blasius solution.

Solution

Let $l = 0.5$ m, $b = 0.25$ m, $S_{\text{fluid}} = 0.9$, $\nu = 1$ stoke $= 10^{-4}$ m^2/s and $U = 5$ m/s.

(i) $\rho = S_{\text{fluid}}\,\rho_w = 0.9 \times 1000 = 900$ kg/m^3

$\text{Re}_l = \dfrac{Ul}{\nu} = \dfrac{5 \times 0.5}{10^{-4}} = 2.5 \times 10^4$

Since $\text{Re}_l < 5 \times 10^5$, the flow over the plate is entirely laminar.

Using Blasius solution, we get:

$$C_D = \frac{1.328}{\sqrt{\text{Re}_l}} = \frac{1.328}{\sqrt{2.5 \times 10^4}} = 0.0084$$

Drag on one side of the plate is given by,

$$F_D = \frac{C_D \rho U^2 bl}{2} = \frac{0.0084 \times 900 \times 5^2 \times 0.25 \times 0.5}{2} = 11.8125\ N$$

Since the plate is wetted on both sides, the total drag is given by,

$$\therefore (F_D)_{\text{total}} = 2F_D = 2 \times 11.8125 = \mathbf{23.625\ N}$$

(ii) $\delta = \frac{5x}{\sqrt{Re_x}} = \frac{5 \times 0.5}{\sqrt{2.5 \times 10^4}} = \mathbf{0.0158\ m\ or\ 15.8\ mm}$

(iii) $\tau_o = C_f \frac{\rho U^2}{2} = \frac{0.664}{\sqrt{Re_l}} \frac{\rho U^2}{2} = \frac{0.664}{\sqrt{2.5 \times 10^4}} \times \frac{900 \times 5^2}{2} = \mathbf{47.244\ N/m^2}$

Example 15.9 Atmospheric air with kinematic viscosity of 15.5×10^{-6} m^2/s flows parallel to a flat plate at a velocity of 2.5 m/s. Determine the boundary layer thickness and the local skin friction coefficient at $x = 1.2$ m from the leading edge of the plate using Blasius solution. Also compare the corresponding values obtained for these parameters by von Kármán integral technique for cubic velocity profile.

Solution

Let $\nu = 15.5 \times 10^{-6}$ m^2/s, $U = 2.5$ m/s and $x = 1.2$ m.

(i) $Re_x = \frac{Ux}{\nu} = \frac{2.5 \times 1.2}{15.5 \times 10^{-6}} = 193548.4$

$$\delta = \frac{5x}{\sqrt{Re_x}} = \frac{5 \times 1.2}{\sqrt{193548.4}} = \mathbf{0.01364\ m\ or\ 13.64\ mm}$$

$$C_f = \frac{0.664}{\sqrt{Re_x}} = \frac{0.664}{\sqrt{193548.4}} = \mathbf{1.5093 \times 10^{-3}}$$

(ii) Approximate solution for cubic profile is given by,

$$\delta = \frac{4.64x}{\sqrt{Re_x}} = \frac{4.64 \times 1.2}{\sqrt{193548.4}} = 0.01266 \text{ m or } 12.66 \text{ mm}$$

$$C_f = \frac{0.646}{\sqrt{Re_x}} = \frac{0.646}{\sqrt{193548.4}} = 1.4684 \times 10^{-3}$$

Let Δ_δ and Δ_{C_f} be the deviations for δ and C_f, respectively.

$$\therefore \Delta_\delta = \frac{13.64 - 12.66}{13.64} \times 100 = \mathbf{7.18\%}$$

$$\therefore \Delta_{C_f} = \frac{(1.5093 - 1.4684) \times 10^{-3}}{1.5093 \times 10^{-3}} \times 100 = \mathbf{2.71\%}$$

Example 15.10 Atmospheric air with kinematic viscosity of 15×10^{-6} m^2/s flows parallel to a flat plate at a velocity of 8 m/s. The length and width of the plate are 1 m and 0.75 m, respectively. If the laminar boundary layer exist up to a value of $Re_x = 2 \times 10^5$ and the velocity profile is given by the relation $(u/U) = 2(y/\delta) - (y/\delta)^2$, then determine the maximum distance from the leading edge up to which the laminar boundary layer exists. Also determine the maximum thickness of laminar boundary layer.

Solution

Let $\nu = 15 \times 10^{-6}$ m^2/s, $U = 8$ m/s, $l = 1$ m, $b = 0.75$ m, $Re_x = 2 \times 10^5$ and $(u/U) = 2(y/\delta) - (y/\delta)^2$.

If $Re_x = 2 \times 10^5$, then x is the distance from leading edge up to which laminar boundary layer exists.

$$Re_x = \frac{Ux}{\nu}$$

Thus $$2\times10^5 = \frac{8\times x}{15\times10^{-6}}$$

$$\therefore x = \frac{2\times10^5\times15\times10^{-6}}{8} = \mathbf{0.375\ m\ or\ 375\ mm}$$

Maximum thickness of the boundary layer is given by,

$$\delta = \frac{5.48x}{\sqrt{Re_x}} = \frac{5.48\times0.375}{\sqrt{2\times10^5}} = \mathbf{0.00459\ m\ or\ 4.59\ mm}$$

Example 15.11 Atmospheric air with kinematic viscosity of 15×10^{-6} m²/s flows parallel to a flat plate at a velocity of 3 m/s. The length and width of the plate are 0.6 m and 0.5 m, respectively. If $\rho = 1.24$ kg/m³ and the velocity profile is given by the relation $(u/U) = \sin[(\pi y)/(2\delta)]$, determine (i) the boundary layer thickness at the end of the plate, (ii) shear stress at 0.2 m from the leading edge and (iii) drag force on one side of the plate.

Solution

Let $\nu = 15\times10^{-6}$ m²/s, $U = 3$ m/s, $l = 0.6$ m, $b = 0.5$ m, $\rho = 1.24$ kg/m³, $(u/U) = \sin[(\pi y)/(2\delta)]$ and $x = 0.2$ m.

(i) $$Re_l = \frac{Ul}{\nu} = \frac{3\times0.6}{15\times10^{-6}} = 1.2\times10^5$$

Since $Re_l < 5\times10^5$, the flow over the plate is entirely laminar.

$$\therefore \delta = \frac{4.795l}{\sqrt{Re_l}} = \frac{4.795\times0.6}{\sqrt{1.2\times10^5}} = \mathbf{0.0083\ m\ or\ 8.3\ mm}$$

(ii) $$Re_x = \frac{Ux}{\nu} = \frac{3\times0.2}{15\times10^{-6}} = 4\times10^4$$

$$C_f = \frac{0.654}{\sqrt{Re_x}} = \frac{0.654}{\sqrt{4\times10^4}} = 0.00327$$

$$\tau_o = C_f\times\frac{1}{2}\rho U^2 = 0.00327\times\frac{1}{2}\times1.24\times3^2 = \mathbf{0.01825\ N/m^2}$$

(iii) $$C_D = \frac{1.31}{\sqrt{Re_l}} = \frac{1.31}{\sqrt{1.2\times10^5}} = 0.00378$$

$$F_D = \frac{C_D\rho U^2 bl}{2} = \frac{0.00378\times1.24\times3^2\times0.5\times0.6}{2} = \mathbf{0.00633\ N}$$

Example 15.12 Calculate the thickness of the boundary layer and the shear stress at 1.3 m from the leading edge of a plate for which the velocity profile is given by the relation $(u/U) = (3/2)(y/\delta) - (1/2)(y/\delta)^3$. The plate is 1.8 m long and 1.2 m wide and it is placed in water which is moving with a velocity of 0.18 metres per second. Also find the total drag force on the plate if dynamic viscosity for water is 0.01 poise.

Solution

Let $x = 1.3$ m, $(u/U) = (3/2)(y/\delta) - (1/2)(y/\delta)^3$, $l = 1.8$ m, $b = 1.2$ m, $U = 0.18$ m/s and $\mu = 0.01$ poise $= 0.001$ Ns/m².

$$\mathrm{Re}_x = \frac{\rho_w U x}{\mu} = \frac{1000 \times 0.18 \times 1.3}{0.001} = 2.34 \times 10^5$$

Since $\mathrm{Re}_x < 5 \times 10^5$, the flow over the plate is entirely laminar.

$$\therefore \delta = \frac{4.64x}{\sqrt{\mathrm{Re}_x}} = \frac{4.64 \times 1.3}{\sqrt{2.34 \times 10^5}} = \mathbf{0.01247\ m\ or\ 12.47\ mm}$$

$$C_f = \frac{0.646}{\sqrt{\mathrm{Re}_x}} = \frac{0.646}{\sqrt{2.34 \times 10^5}} = 0.001335$$

$$\tau_o = C_f \times \frac{1}{2}\rho_w U^2 = 0.001335 \times \frac{1}{2} \times 1000 \times 0.18^2 = \mathbf{0.02163\ N/m^2}$$

$$\mathrm{Re}_l = \frac{\rho_w U l}{\mu} = \frac{1000 \times 0.18 \times 1.8}{0.001} = 3.24 \times 10^5$$

$$C_D = \frac{1.292}{\sqrt{\mathrm{Re}_l}} = \frac{1.292}{\sqrt{3.24 \times 10^5}} = 0.00227$$

$$F_D = \frac{C_D \rho_w U^2 bl}{2} = \frac{0.00227 \times 1000 \times 0.18^2 \times 1.2 \times 1.8}{2} = 0.079432\ \mathrm{N}$$

$$\therefore (F_D)_{\text{total}} = 2F_D = 2 \times 0.079432 = \mathbf{0.158864\ N}$$

Example 15.13 Find the ratio of friction drag on the front two-third and rear one-third of a flat plate kept in a uniform stream at zero incidence, if the boundary layer is laminar over the whole plate.

Solution

Let $(F_D)_{\text{front}}$ and $(F_D)_{\text{rear}}$ be the skin friction drag on the front two-third and rear one-third of the plate, respectively.

For the front two-third portion of the plate, we get:

$$(\mathrm{Re}_x)_{\text{front}} = \frac{Uxl}{\nu} = \frac{U(2/3)l}{\nu}$$

$$(C_D)_{\text{front}} = \frac{1.328}{\sqrt{\mathrm{Re}_x}} = \frac{1.328}{\sqrt{(2/3)\{(Ul)/\nu\}}} = \frac{1.6265}{\sqrt{(Ul)/\nu}}$$

Since $$F_D = C_D \times \frac{\rho U^2}{2} \times bl$$

Thus, drag for the front two-third portion of the plate is given by,

$$(F_D)_{\text{front}} = \frac{1.6265}{\sqrt{(Ul)/\nu}} \times \frac{\rho U^2}{2} \times b \times \frac{2l}{3} = \frac{0.5422\,\rho U^2 bl}{\sqrt{(Ul/\nu)}} \qquad \text{(i)}$$

Similarly, the drag for the entire plate is given below.

$$F_D = \frac{1.328}{\sqrt{(Ul)/\nu}} \times \frac{\rho U^2}{2} bl = \frac{0.664\,\rho U^2 bl}{\sqrt{(Ul)/\nu}}$$

Thus $$(F_D)_{\text{rear}} = F_D - (F_D)_{\text{front}}$$

$$\therefore (F_D)_{\text{rear}} = \frac{0.664\,\rho U^2 bl}{\sqrt{(Ul)/\nu}} - \frac{0.5422\,\rho U^2 bl}{\sqrt{(Ul)/\nu}} = \frac{0.1218\,\rho U^2 bl}{\sqrt{(Ul)/\nu}} \quad \text{(ii)}$$

Now dividing (i) by (ii), we get:

$$\frac{(F_D)_{\text{front}}}{(F_D)_{\text{rear}}} = \frac{0.5422\,\rho U^2 bl}{\sqrt{(Ul)/\nu}} \times \frac{\sqrt{(Ul)/\nu}}{0.1218\,\rho U^2 bl} = \mathbf{4.45}$$

Example 15.14 A smooth flat plate is placed in a uniform fluid stream. Find the fraction of the length from the leading edge where the drag force is equal to half of the total drag force on one side of the plate if the boundary layer is laminar.

Solution

Let $(F_D)_x$ and $(F_D)_l$ be the skin friction drags over the lengths x and l, respectively.

$$(F_D)_x = \frac{(F_D)_l}{2}$$

Substituting the values of $(F_D)_x$ and $(F_D)_l$, we get:

$$\frac{1.328}{\sqrt{(Ux)/\nu}} \times \frac{\rho U^2}{2} \times b \times x = \frac{1}{2} \times \frac{1.328}{\sqrt{(Ul)/\nu}} \times \frac{\rho U^2}{2} \times b \times l$$

$$\sqrt{x} = \frac{1}{2}\sqrt{l}$$

$$\therefore x = \frac{\boldsymbol{l}}{\mathbf{4}}$$

15.8 ❐ TURBULENT BOUNDARY LAYER

The turbulent boundary layers are thicker than laminar boundary layer. The velocity distribution in turbulent boundary layers is more uniform than in laminar ones. The velocity profile for a turbulent boundary layer is given below.

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^n \quad (15.24)$$

Here, $n = (1/7)$ for $5 \times 10^5 < \text{Re} < 10^7$ and therefore, Equation (15.24) is written as follows.

$$\boxed{\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7}} \quad (15.25)$$

The Equation (15.25) is known as one-seventh power law which satisfactorily describes the velocity distribution for most of the region of turbulent boundary layer but it cannot be applied at the boundary itself. This is because $(\partial u/\partial y) = \infty$ at $y = 0$. There is a laminar sublayer just immediately adjacent to the boundary which is so thin that its velocity profile is taken as linear.

The viscous shear stress (τ_o) for the flat plate was given by Blausius as follows.

$$\tau_o = 0.0225\rho U^2 \left(\frac{\mu}{\rho U \delta}\right)^{1/4} \quad (15.26)$$

The values of δ, τ_o, C_f, F_D and C_D for the velocity profile according to Equation (15.25) in the turbulent boundary layer are evaluated in terms of Reynolds number as given below.

(i) **Boundary layer thickness** δ

$$\frac{\tau_o}{\rho U^2} = \frac{\partial}{\partial x}\left[\int_0^{\delta} \frac{u}{U}\left(1-\frac{u}{U}\right)dy\right] \qquad \text{[Equation (15.9)]}$$

Substituting $(u/U) = (y/\delta)^{1/7}$ in the above equation, we get:

$$\frac{\tau_o}{\rho U^2} = \frac{\partial}{\partial x}\left[\int_0^{\delta}\left(\frac{y}{\delta}\right)^{1/7}\left\{1-\left(\frac{y}{\delta}\right)^{1/7}\right\}dy\right] = \frac{\partial}{\partial x}\left[\int_0^{\delta}\left\{\left(\frac{y}{\delta}\right)^{1/7}-\left(\frac{y}{\delta}\right)^{2/7}\right\}dy\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{7}{8}\frac{y^{8/7}}{\delta^{1/7}}-\frac{7}{9}\frac{y^{9/7}}{\delta^{2/7}}\right]_0^{\delta} = \frac{\partial}{\partial x}\left[\frac{7}{8}\frac{\delta^{8/7}}{\delta^{1/7}}-\frac{7}{9}\frac{\delta^{9/7}}{\delta^{2/7}}\right] = \frac{\partial}{\partial x}\left[\frac{7}{8}\delta-\frac{7}{9}\delta\right]$$

Thus

$$\frac{\tau_o}{\rho U^2} = \frac{7}{72}\frac{\partial \delta}{\partial x}$$

$$\therefore \tau_o = \frac{7}{72}\rho U^2\frac{\partial \delta}{\partial x} \qquad (15.27)$$

Also

$$\tau_o = 0.0225\rho U^2\left(\frac{\mu}{\rho U\delta}\right)^{1/4} \qquad \text{[Eq. (15.26)]}$$

Simplifying Equations (15.27) and (15.26), we get:

$$\frac{7}{72}\rho U^2\frac{\partial \delta}{\partial x} = 0.0225\rho U^2\left(\frac{\mu}{\rho U\delta}\right)^{1/4}$$

$$\frac{7}{72}\frac{\partial \delta}{\partial x} = 0.0225\left(\frac{\mu}{\rho U\delta}\right)^{1/4}$$

$$\delta^{1/4}\partial\delta = 0.0225\times\frac{72}{7}\left(\frac{\mu}{\rho U}\right)^{1/4}\partial x$$

Integrating both sides and simplifying, we get:

$$\frac{4}{5}\delta^{5/4} = 0.2314\left(\frac{\mu}{\rho U}\right)^{1/4}x+k$$

Here, k is a constant and applying boundary condition at $x = 0$, $\delta = 0$ and thus, $k = 0$.

Thus

$$\frac{4}{5}\delta^{5/4} = 0.2314\left(\frac{\mu}{\rho U}\right)^{1/4}x$$

$$\delta = \left[0.2314\times\frac{5}{4}\left(\frac{\mu}{\rho U}\right)^{1/4}x\right]^{4/5} = 0.371\left(\frac{\mu}{\rho U}\right)^{1/5}x^{4/5}$$

$$\therefore \delta = 0.371\left[\frac{1}{(\rho U x)/\mu}\right]^{1/5}x^{4/5}x^{1/5} = \frac{0.371x}{(\text{Re}_x)^{1/5}} \qquad (15.28)$$

(ii) **Shear stress (τ_o)** Substituting the value of δ from Equation (15.28) in Equation (15.26), we get:

$$\tau_o = 0.0225\rho U^2\left[\frac{\mu}{\rho U}\frac{1}{\{(0.371x)/\mathrm{Re}_x^{1/5}\}}\right]^{1/4} = \frac{0.0225}{0.371^{1/4}}\rho U^2\left[\frac{\mu}{\rho Ux}(\mathrm{Re}_x)^{1/5}\right]^{1/4}$$

Multiplying and dividing the above expression by 2 and substituting $\mathrm{Re}_x = (\rho Ux)/\mu$, we get:

$$\tau_o = \frac{2}{2}\times 0.0288\rho U^2\left[\frac{(\mathrm{Re}_x)^{1/5}}{\mathrm{Re}_x}\right]^{1/4} = \frac{\rho U^2}{2}\frac{0.0576}{(\mathrm{Re}_x)^{1/5}} \tag{15.29}$$

(iii) **Local coefficient of drag (C_f)**

$$\tau_o = C_f\frac{\rho U^2}{2} \qquad \text{[From Equation (15.11)]}$$

Simplifying the above equation with Equation (15.29), we get:

$$C_f\frac{\rho U^2}{2} = \frac{\rho U^2}{2}\frac{0.0576}{(\mathrm{Re}_x)^{1/5}}$$

$$\boxed{\therefore C_f = \frac{0.0576}{(\mathrm{Re}_x)^{1/5}}} \tag{15.30}$$

(iv) **Drag force (F_D)** The total drag force on one side of the plate is given by,

$$F_D = \int_0^l \tau_o b\,dx = \int_0^l \frac{\rho U^2}{2}\frac{0.0576}{(\mathrm{Re}_x)^{1/5}}b\,dx$$

$$= \int_0^l \frac{\rho U^2}{2}\frac{0.0576}{\{(\rho Ux)/\mu\}^{1/5}}b\,dx = \frac{\rho U^2}{2}\frac{0.0576}{\{(\rho U)/\mu\}^{1/5}}b\int_0^l x^{-1/5}dx$$

$$= \frac{\rho U^2}{2}\frac{0.0576}{\{(\rho U)/\mu\}^{1/5}}b\left[\frac{x^{4/5}}{4/5}\right]_0^l = \frac{\rho U^2}{2}\frac{0.0576}{\{(\rho U)/\mu\}^{1/5}}b\times\frac{5}{4}l^{4/5}$$

$$= \frac{\rho U^2}{2}\frac{0.072}{\{(\rho Ul)/\mu\}^{1/5}}bl \qquad \text{[Multiply and divide by } l^{1/5}\text{]}$$

$$\boxed{\therefore F_D = \frac{\rho U^2}{2}\frac{0.072}{(\mathrm{Re}_l)^{1/5}}bl} \tag{15.31}$$

(v) **Coefficient of drag (C_D)** The coefficient of drag is given by,

$$C_D = \frac{F_D}{(1/2)\rho AU^2} = \frac{F_D}{(1/2)\rho lbU^2} \qquad \text{[Eq. (15.12)]}$$

Substituting Equation (15.31) in the above equation, we get:

$$\boxed{C_D = \frac{\rho U^2}{2}\frac{0.072}{(\mathrm{Re}_l)^{1/5}}bl\times\frac{1}{(1/2)\rho lbU^2} = \frac{0.072}{(\mathrm{Re}_l)^{1/5}}} \tag{15.32}$$

The Equation (15.32) is valid for $5\times10^5 < \text{Re}_l < 10^7$.

If $10^7 < \text{Re}_l < 10^9$, then the relation given by Schlichting is used for determining C_D as follows.

$$\boxed{C_D = \frac{0.455}{(\log_{10}\text{Re}_l)^{2.58}}} \tag{15.33}$$

Example 15.15 A smooth flat plate of length 5 m and width 2 m moves with a velocity of 5 m/s in a uniform stationary air stream. Find (i) the thickness of the boundary layer at the trailing edge of the plate if kinematic viscosity and density of air are 15×10^{-6} m^2/s and 1.23 kg/m^3, respectively. Also determine (ii) the coefficient of drag, (iii) total drag on one side of the plate if the boundary layer is turbulent from the very beginning, (iv) shear stress and (v) the thickness of laminar sublayer.

Solution

Let $l = 5$ m, $b = 2$ m, $U = 5$ m/s, $\nu = 15\times10^{-6}$ m^2/s and $\rho = 1.23$ kg/m^3.

(i) $\text{Re}_l = \dfrac{Ul}{\nu} = \dfrac{5\times5}{15\times10^{-6}} = 16.67\times10^5$

Since $\text{Re}_l > 5\times10^5$, the boundary layer is turbulent at the trailing edge. The expression for boundary layer thickness for turbulent boundary layer at the trailing edge of the plate is given below.

$$\delta = \frac{0.371l}{(\text{Re}_l)^{1/5}} = \frac{0.371\times5}{(16.67\times10^5)^{1/5}} = \mathbf{0.1057\ m\ or\ 105.7\ mm}$$

(ii) $C_D = \dfrac{0.072}{(\text{Re}_l)^{1/5}} = \dfrac{0.072}{(16.67\times10^5)^{1/5}} = \mathbf{0.0041}$

(iii) $F_D = \dfrac{\rho U^2 C_D bl}{2} = \dfrac{1.23\times5^2\times0.0041\times2\times5}{2} = \mathbf{0.6304\ N}$

(iv) $\tau_o = \dfrac{\rho U^2}{2}\dfrac{0.0576}{(\text{Re}_x)^{1/5}} = \dfrac{1.23\times5^2}{2}\times\dfrac{0.0576}{(16.67\times10^5)^{1/5}} = \mathbf{0.05045\ N/m^2}$

(v) $\delta' = \dfrac{11.6\nu}{\sqrt{\tau_o/\rho}} = \dfrac{11.6\times15\times10^{-6}}{\sqrt{0.05045/1.23}} = \mathbf{0.00086\ m\ or\ 0.86\ mm}$

Example 15.16 Find the ratio of friction drag on the front half and rear half of a flat plate kept in a uniform stream at zero incidence, if the boundary layer is turbulent over the whole plate.

Solution

Let $(F_D)_{\text{front}}$ and $(F_D)_{\text{rear}}$ be the skin friction drag on the front half and rear half of the plate, respectively.

For the front half portion of the plate, we get:

$$(C_D)_{\text{front}} = \frac{0.072}{\text{Re}_x^{1/5}} = \frac{0.072}{[(Ux)/\nu]^{1/5}} = \frac{0.072}{[\{U(l/2)\}/\nu]^{1/5}} = \frac{0.0827}{[(Ul)/\nu]^{1/5}}$$

Thus, drag for the front half portion of the plate is given by,

$$(F_D)_{\text{front}} = \frac{C_D\rho U^2 b}{2}\frac{l}{2} = \frac{0.0827\rho U^2 b}{2[(Ul)/\nu]^{1/5}}\frac{l}{2} = \frac{0.0207\rho U^2 bl}{[(Ul)/\nu]^{1/5}}$$

Similarly, drag for the entire plate is given by,

$$F_D = \frac{1}{2}\frac{0.072}{[(Ul)/\nu]^{1/5}}\rho U^2 bl = \frac{0.036\rho U^2 bl}{[(Ul)/\nu]^{1/5}}$$

Thus $$(F_D)_{\text{rear}} = F_D - (F_D)_{\text{front}} = \frac{0.036\rho U^2 bl}{[(Ul)/\nu]^{1/5}} - \frac{0.0207\rho U^2 bl}{[(Ul)/\nu]^{1/5}} = \frac{0.0153\rho U^2 bl}{[(Ul)/\nu]^{1/5}}$$

$$\therefore \frac{(F_D)_{\text{front}}}{(F_D)_{\text{rear}}} = \frac{0.0207\rho U^2 bl}{[(Ul)/\nu]^{1/5}} \times \frac{[(Ul)/\nu]^{1/5}}{0.0153\rho U^2 bl} = \mathbf{1.353}$$

Example 15.17 A ship of 5 m draft and 125 m long needs 500 kW of power at a speed of 20 km per hour. If the density and kinematic viscosity of sea water is 1025 kg/m^3 and 10^{-6} m^2/s, respectively, then find the combined wave and form resistance of the ship. Assume that boundary layer is turbulent from the very beginning.

Solution

Let $b = 5$ m, $l = 125$ m, $P = 500$ kW, $U = 20$ km/hr, $\rho = 1025$ kg/m^3 and $\nu = 10^{-6}$ m^2/s.

$$U = \frac{20\times 1000}{3600} = 5.555 \text{ m/s}$$

$$\text{Re}_l = \frac{Ul}{\nu} = \frac{5.555\times 125}{10^{-6}} = 6.944\times 10^8$$

Since $$10^7 < \text{Re}_l < 10^9$$

Thus $$C_D = \frac{0.455}{(\log_{10}\text{Re}_l)^{2.58}} = \frac{0.455}{[\log_{10}(6.944\times 10^8)]^{2.58}} = 1.644\times 10^{-3}$$

Since the ship is wetted on both sides, the total friction drag is given by,

$$(F_D)_{\text{friction}} = 2\times\frac{C_D\rho lbU^2}{2} = C_D\rho lbU^2$$

$$\therefore (F_D)_{\text{friction}} = 1.644\times 10^{-3}\times 1025\times 125\times 5\times 5.555^2 = 32499.29 \text{ N}$$

Total drag is given by,

$$F_D = \frac{P}{U} = \frac{500\times 10^3}{5.555} = 90009 \text{ N}$$

Also $$F_D = F_{\text{form}} + F_{\text{wave}} + F_{\text{friction}}$$

$$\therefore (F_{\text{form}} + F_{\text{wave}}) = F_D - F_{\text{friction}} = 90009 - 32499.29 = \mathbf{57509.71\ N}$$

Example 15.18 A cylindrical shaped submarine with rounded nose is 4 m in diameter and 40 m long. If the density and kinematic viscosity of sea water is 1025 kg/m^3 and 10^{-6} m^2/s, respectively, then find the power required to overcome the boundary friction if the submarine moves at a speed of 25.2 km/h and the boundary layer is turbulent from the very beginning.

Solution

Let $d = 4$ m, $l = 40$ m, $\rho = 1025$ kg/m^3, $\nu = 10^{-6}$ m^2/s and $U = 25.2$ km/hr.

$$U = \frac{25.2 \times 1000}{3600} = 7 \text{ m/s}$$

$$\text{Re}_l = \frac{Ul}{\nu} = \frac{7 \times 40}{10^{-6}} = 2.8 \times 10^8$$

Since $$10^7 < \text{Re}_l < 10^9$$

Thus $$C_D = \frac{0.455}{(\log_{10} \text{Re}_l)^{2.58}} = \frac{0.455}{[\log_{10}(2.8 \times 10^8)]^{2.58}} = 0.00185$$

$$A = \pi dl = \pi \times 4 \times 40 = 502.655 \text{ m}^2$$

Drag force is given by,

$$F_D = C_D \frac{\rho A U^2}{2} = 0.00185 \times \frac{1025 \times 502.655 \times 7^2}{2} = 23352.41 \text{ N}$$

Total power required to overcome boundary friction is given by,

$$P = \frac{F_D U}{1000} = \frac{23352.41 \times 7}{1000} = \mathbf{163.467\ kW}$$

15.9 ❒ TOTAL DRAG DUE TO LAMINAR AND TURBULENT LAYERS

Combined relations are required when for some distance from the leading edge of the plate, the boundary layer is laminar and it becomes turbulent for the remaining portion of the plate. For such cases, Prandtl proposed the following relation for computing the average drag coefficient.

$$C_D = \frac{0.074}{\text{Re}_l^{1/5}} - \frac{A_1}{\text{Re}_l} \tag{15.34}$$

The constant A_1 in Equation (15.34) depends on the critical Reynolds number $(\text{Re}_x)_c$ at which the laminar boundary layer becomes turbulent. In most of the cases, the value of $(\text{Re}_x)_c$ is taken as 5×10^5 for which $A_1 = 1700$ and thus Equation (15.34) is written as follows.

$$\boxed{C_D = \frac{0.074}{\text{Re}_l^{1/5}} - \frac{1700}{\text{Re}_l}} \tag{15.35}$$

For $(\text{Re}_x)_c = 3 \times 10^5$, $A_1 = 1050$ and for $(\text{Re}_x)_c = 10^6$, $A_1 = 3300$. The Equation (15.35) is applicable for values of Re_l up to 10^7.

The Prandtl-Schlichting equation given below is used for Re_l ranging from 10^7 to 10^9.

$$C_D = \frac{0.455}{(\log_{10} \text{Re}_l)^{2.58}} - \frac{A_1}{\text{Re}_l} \tag{15.36}$$

In Equation (15.36), the value of A_1 again depends on the value of $(\text{Re}_x)_c$ and it is the same as discussed above. Thus, for $(\text{Re}_x)_c = 5 \times 10^5$, $A_1 = 1700$ and therefore, Equation (15.36) is written as follows.

$$\boxed{C_D = \frac{0.455}{(\log_{10} \text{Re}_l)^{2.58}} - \frac{1700}{\text{Re}_l}} \tag{15.37}$$

The Equation (15.36) is valid in the entire range of $5 \times 10^5 < \text{Re}_l < 10^9$ and it agrees with the equation (15.34) up to $\text{Re}_l = 10^7$.

Example 15.19 Determine the power required to propel a submarine at a speed of 16.2 km/hour in water having density and kinematic viscosity as 1000 kg/m^3 and 10^{-6} m^2/s, respectively. The length of the hull of a submarine is 80 m and its surface area is 2000 m^2. The boundary layer at the leading edge is laminar and the critical Reynolds number at which the flow in boundary layer changes from laminar to turbulent is 5×10^5.

Solution

Let $U = 16.2$ km/hr, $\rho_w = 1000$ kg/m^3, $\nu = 10^{-6}$ m^2/s, $l = 80$ m, $A = 2000$ m^2 and $(\text{Re}_x)_c = 5\times 10^5$.

$$U = \frac{16.2\times 1000}{3600} = 4.5 \text{ m/s}$$

$$\text{Re}_l = \frac{Ul}{\nu} = \frac{4.5\times 80}{10^{-6}} = 3.6\times 10^8$$

Since at leading edge, the boundary layer is laminar which changes from laminar to turbulent on the surface of the submarine.

Thus
$$C_D = \frac{0.074}{\text{Re}_l^{1/5}} - \frac{1700}{\text{Re}_l} = \frac{0.074}{(3.6\times 10^8)^{1/5}} - \frac{1700}{3.6\times 10^8} = 1.434\times 10^{-3}$$

$$F_D = \frac{C_D \rho_w A U^2}{2} = \frac{1.434\times 10^{-3}\times 1000\times 2000\times 4.5^2}{2} = 29038.5 \text{ N}$$

Power required to propel the submarine is given by,

$$P = \frac{F_D U}{1000} = \frac{29038.5\times 4.5}{1000} = \mathbf{130.67325\ kW}$$

Example 15.20 A small submarine moves in fresh water at a speed of 10.8 km/hour and experiences a total drag of 65 N. If the length of its hull is 3 m and its surface area is 3 m^2, then find (i) the skin friction drag, (ii) wave drag and (iii) wave drag coefficient. Take the density and kinematic viscosity of water as 1000 kg/m^3 and 10^{-6} m^2/s, respectively. The boundary layer at the leading edge is laminar and the critical Reynolds number at which the flow in boundary layer changes from laminar to turbulent is 5 × 10^5. Neglect the form resistance.

Solution

Let $U = 10.8$ km/hr, $(F_D)_{\text{total}} = 65$ N, $l = 3$ m, $A = 3$ m^2, $\rho_w = 1000$ kg/m^3, $\nu = 10^{-6}$ m^2/s and $(\text{Re}_x)_c = 5\times 10^5$.

(i) $U = \dfrac{10.8\times 1000}{3600} = 3$ m/s

$$\text{Re}_l = \frac{Ul}{\nu} = \frac{3\times 3}{10^{-6}} = 9\times 10^6$$

Since at leading edge, the boundary layer is laminar which changes from laminar to turbulent on the surface of the submarine.

Thus
$$C_D = \frac{0.074}{\text{Re}_l^{1/5}} - \frac{1700}{\text{Re}_l} = \frac{0.074}{(9\times 10^6)^{1/5}} - \frac{1700}{9\times 10^6} = 2.82\times 10^{-3}$$

$$(F_D)_{\text{skin}} = \frac{C_D \rho_w A U^2}{2} = \frac{2.82\times 10^{-3}\times 1000\times 3\times 3^2}{2} = \mathbf{38.07\ N}$$

(ii) $(F_D)_{\text{wave}} = (F_D)_{\text{total}} - (F_D)_{\text{skin}} = 65 - 38.07 = \mathbf{26.93\ N}$

(iii) $$(C_D)_{\text{wave}} = \frac{(F_D)_{\text{wave}}}{(1/2)\rho_w A U^2} = \frac{26.93}{(1/2)\times 1000\times 3\times 3^2} = \mathbf{1.995\times 10^{-3}}$$

Example 15.21 A barge with bottom surface in rectangular shape is 25 m long and 10 m wide, which is travelling down a river with a velocity of 0.7 m/s. A laminar boundary layer is existed up to a Reynolds number equivalent to 5 × 10^5 and subsequently, abrupt transition occurs to turbulent boundary layer. If the density and kinematic viscosity of water are 998 kg/m^3 and 10^{-6} m^2/s, respectively, then determine (i) the maximum distance from the leading edge up to which the laminar boundary persists and the maximum boundary layer thickness at that point, (ii) the total drag force on the flat bottom surface of the barge and (iii) power required to push the bottom surface through water at the given velocity.

Solution

Let $l = 25$ m, $b = 10$ m, $U = 0.7$ m/s, $(\text{Re}_x)_c = 5\times10^5$, $\rho_w = 998$ kg/m^3 and $\nu = 10^{-6}$ m^2/s.

(i) Let x_c be the distance from the leading edge up to which the laminar boundary persists and δ be the maximum boundary layer thickness.

$$A = lb = 25\times10 = 250 \text{ m}^2$$

$$(\text{Re}_x)_c = \frac{Ux_c}{\nu} = 5\times10^5$$

$$\therefore x_c = \frac{5\times10^5\times\nu}{U} = \frac{5\times10^5\times10^{-6}}{0.7} = \mathbf{0.7143\ m}$$

$$\delta = \frac{5x_c}{\sqrt{\text{Re}_x}} = \frac{5\times0.7143}{\sqrt{5\times10^5}} = \mathbf{5.05\times10^{-3}\ m\ or\ 5.05\ mm}$$

(ii) $\text{Re}_l = \dfrac{Ul}{\nu} = \dfrac{0.7\times25}{10^{-6}} = 1.75\times10^7$

Since at leading edge, the boundary layer is laminar which changes from laminar to turbulent on the surface of the barge, we get the following result.

$$C_D = \frac{0.455}{(\log_{10}\text{Re}_l)^{2.58}} - \frac{1700}{\text{Re}_l} = \frac{0.455}{(\log_{10}1.75\times10^7)^{2.58}} - \frac{1700}{1.75\times10^7} = 2.653\times10^{-3}$$

$$F_D = \frac{C_D\rho_w AU^2}{2} = \frac{2.653\times10^{-3}\times998\times250\times0.7^2}{2} = \mathbf{162.17\ N}$$

(iii) The power required to push the bottom surface through water is given by,

$$P = F_D U = 162.17\times0.7 = \mathbf{113.519\ W}$$

Example 15.22 A streamlined train is 240 m long with a typical cross section having a perimeter of 8.5 m above the wheels. If the boundary layer changes from laminar to turbulent on the train surface and the density and kinematic viscosity of air are 1.24 kg/m^3 and 1.5×10^{-5} m^2/s, respectively, then determine the approximate surface drag (friction drag) of the train when running at 84 km/hr.

Solution

Let $l = 240$ m, $p = 8.5$ m, $\rho = 1.24$ kg/m^3, $\nu = 1.5\times10^{-5}$ m^2/s and $U = 84$ km/hr.

$$U = \frac{84\times1000}{3600} = 23.33 \text{ m/s}$$

$$\text{Re}_l = \frac{Ul}{\nu} = \frac{23.33\times240}{1.5\times10^{-5}} = 3.73\times10^8$$

Thus, the boundary layer is turbulent and assuming abrupt transition from laminar to turbulent on the surface of the train, we get the following value.

$$C_D = \frac{0.455}{(\log_{10} \text{Re}_l)^{2.58}} - \frac{1700}{\text{Re}_l} = \frac{0.455}{(\log_{10} 3.73\times10^8)^{2.58}} - \frac{1700}{3.73\times10^8} = 1.776\times10^{-3}$$

Frictional drag force on the train surface is given by,

$$F_D = C_D \times \frac{1}{2}\rho A U^2 = \frac{C_D \rho (pl) U^2}{2}$$

$$\therefore F_D = \frac{1.776\times10^{-3}\times1.24\times(8.5\times240)\times23.33^2}{2} = \mathbf{1222.63\ N}$$

15.10 ❒ BOUNDARY LAYER SEPARATION, ITS EFFECTS, AND CONTROL

The pressure gradient ($\partial p/\partial x$) in the direction of flow greatly affects the boundary layer thickness. If the pressure gradient is zero, then the boundary layer continues to grow in thickness along a flat plate. If the pressure gradient is negative (i.e., pressure decreases in the direction of flow), then the boundary layer reduces in thickness and is held in place. The transfer of momentum from the main flow to the boundary layer will sustain the flow in the boundary layer. However, if the pressure gradient is positive or adverse (i.e., pressure increases in the direction of flow), then the boundary layer thickens rapidly, the flow decelerates and the velocity pattern reverses or back flow sets in. The flow near the boundary is continuously retarded and a point is reached when the flow starts separating from the boundary. The point at which the flow separates from the boundary is called separation point. The positive pressure gradient reduces the momentum of flow of the fluid within the boundary layer due to higher shear stresses and causes separation from the solid surface.

Consider the flow past a curved surface (i.e., converging-diverging boundary surface) as shown in Figure 15.4.

As fluid flows in the region of converging boundary *ABC*, it is accelerated and the velocity becomes maximum at *C* where pressure is minimum. Thus, the pressure decreases in the direction of flow (i.e., pressure gradient becomes negative) which pushes high pressure region to the low pressure region. Therefore, the boundary layer remains thin and is held in place on the solid surface.

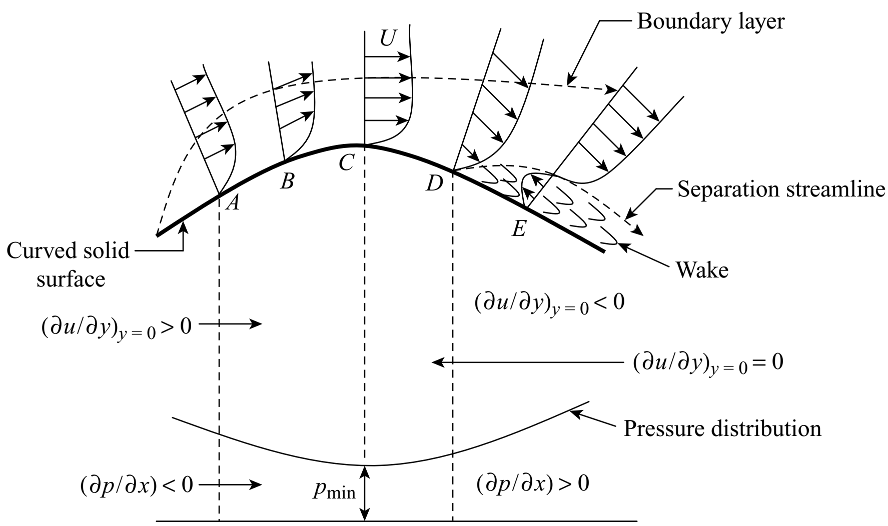


Figure 15.4 *Separation of boundary layer*

In the diverging portion *CDE*, the flow is decelerated and pressure increases in the direction of flow (i.e., pressure gradient becomes positive). As a result, the net pressure force in an element of fluid in the boundary layer opposes the forward flow. Thus, at a certain distance, on the downstream of point *C*, the fluid near the boundary surface is soon brought to a standstill. The value of the velocity gradient ($\partial u/\partial y$) at the boundary surface is then zero (at point *D*) and the fluid no longer follows the contour of the curved surface and it separates from it. This point *D* is known as separation point. On downstream of separation point, a further retardation of the fluid close to the boundary has a reverse or back flow in the separated region known as wake. If all the points below which a reverse flow occurs are joined by a smooth curve, then a line dividing the forward and reverse flows is obtained which is known as separation streamline. Large irregular eddies and turbulence developed in the wake causes loss of energy and decrease in efficiency.

The flow separation depends on the curvature of the surface, the Reynolds number of the flow and the roughness of the boundary surface. The following conditions determine the boundary layer separation.

(i) If $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ is positive, then there is no separation and thus, the flow is attached.

(ii) If $\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0$, then the flow is on the verge of separation.

(iii) If $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ is negative, then the flow is separated.

15.10.1 Effects of Boundary Layer Separation

The boundary layer separation is unstable and it is an inefficient process. For external boundary layer separation, it leads to increase in pressure drag which is much more than frictional drag. For internal flows, boundary layer separation causes increase in flow losses. It occurs in diffusers, turbine blades, fans, pumps, aerofoils, etc.

15.10.2 Methods of Controlling Separation

The methods which are generally adopted to retard the flow separation is as follows.

1. Streamlining of body shapes shifts the point of separation downstream and thereby, reduces the wake region.
2. Suction of the retarded layers by suction slots.
3. Artificial roughening of the boundary surface produces an early onset of turbulence which resists the separation.
4. Providing small divergence in a diffuser.
5. Acceleration of the fluid in the boundary layer.
6. Motion of solid boundary.
7. Guidance of flow in a confined passage.
8. Providing a rotating cylinder near the leading edge which induces Magnus effect and the fluid remains attached to the upper surface of the body for its full length.

Example 15.23 State whether the flow is separated or on the verge of separation or not separated for the following velocity profiles in the boundary layer.

(i) $(u/U) = 2(y/\delta) - (y/\delta)^2$

(ii) $(u/U) = 2(y/\delta)^2 - (y/\delta)^3$

(iii) $(u/U) = -2(y/\delta) + (y/\delta)^2$

Solution

(i) $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \Rightarrow u = U\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]$

$$\frac{\partial u}{\partial y} = U\left[\frac{2}{\delta} - \frac{2y}{\delta^2}\right]$$

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = U\left[\frac{2}{\delta} - \frac{2(0)}{\delta^2}\right] = \frac{2U}{\delta}$$

Since $(\partial u/\partial y)_{y=0}$ is positive, the flow is not separated.

(ii) $\frac{u}{U} = 2\left(\frac{y}{\delta}\right)^2 - \left(\frac{y}{\delta}\right)^3 \Rightarrow u = U\left[2\left(\frac{y}{\delta}\right)^2 - \left(\frac{y}{\delta}\right)^3\right]$

$$\frac{\partial u}{\partial y} = U\left[\frac{4y}{\delta^2} - \frac{3y^2}{\delta^3}\right]$$

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = U\left[\frac{4(0)}{\delta^2} - \frac{3(0)}{\delta^3}\right] = 0$$

Since $(\partial u/\partial y)_{y=0}$ is zero, the flow is on the verge of separation.

(iii) $\frac{u}{U} = -2\left(\frac{y}{\delta}\right) + \left(\frac{y}{\delta}\right)^2 \Rightarrow u = U\left[-2\left(\frac{y}{\delta}\right) + \left(\frac{y}{\delta}\right)^2\right]$

$$\frac{\partial u}{\partial y} = U\left[-\frac{2}{\delta} + \frac{2y}{\delta^2}\right]$$

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = U\left[-\frac{2}{\delta} + \frac{2(0)}{\delta^2}\right] = -\frac{2U}{\delta}$$

Since $(\partial u/\partial y)_{y=0}$ is negative, the flow is separated.

Summary

1. Boundary layer is a narrow region near the solid surface over which velocity gradients and shear stresses are large.
2. For a flat plate, the length of the laminar zone is obtained for laminar flow from the critical Reynolds number $(\text{Re}_x)_c = [(Ux)/\nu] = 5 \times 10^5$, here U is the freestream velocity, x is the distance from leading edge and ν is the kinematic viscosity of fluid.
3. In turbulent boundary layer region, there is a very thin layer just adjacent to the boundary in which the flow is laminar. This thin layer is known as laminar sublayer and its thickness is given as $\delta' = \frac{11.6\nu}{\sqrt{\tau_o/\rho}} = \frac{11.6\nu}{u_s}$.
4. The thickness of the boundary layer (δ) is arbitrarily defined as the distance from the boundary surface in which the velocity reaches 99% of the velocity of the mainstream.

Thus, δ is defined as the distance y from the surface at which $u = 0.99U$.

5. Displacement thickness (δ_d) can be defined as the distance perpendicular to the boundary surface to which the boundary surface has to be displaced into the flow to compensate for reduction in the discharge due to the formation of the boundary layer.

6. The momentum thickness (δ_m) may be defined as the perpendicular distance by which the boundary should be displaced to compensate for the reduction in momentum of the flowing fluid on account of the boundary layer formation.

7. The energy thickness (δ_e) may be defined as the perpendicular distance by which the boundary should be displaced to compensate for the reduction in energy of the flowing fluid on account of the boundary layer formation.

8. $\delta_d = \int_0^\delta \left(1 - \frac{u}{U}\right) dy;\quad \delta_m = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy;$

$$\delta_e = \int_0^\delta \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right) dy$$

9. Shape factor: $H = \delta_d / \delta_m$

10. The von Karman momentum integral equation for boundary layer flow which is applicable for both laminar and turbulent flows is $\frac{\tau_o}{\rho U^2} = \frac{\partial \delta_m}{\partial x}$.

11. The local coefficient of drag or skin friction coefficient (C_f) is defined as the ratio of the local wall shear stress (τ_o) to the dynamic pressure of the uniform flow stream and is given by $C_f = \tau_o/[(1/2)\rho U^2]$, here ρ is the mass density of fluid and U is the freestream velocity.

12. The average coefficient of drag (C_D) is defined as the ratio of the total drag force (F_D) to the quantity $(1/2)\rho A U^2$. It is also known as coefficient of drag and is given by $C_D = F_D/[(1/2)\rho A U^2]$, here A is the surface area of the plate surface.

13. The Prandtl's boundary layer equations for two-dimensional steady flow of incompressible fluids is $\frac{\partial p}{\partial y} = 0$ and $\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$.

14. **Blasius solution:** $\delta = \frac{5x}{\sqrt{\mathrm{Re}_x}}$, $C_f = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$, and $C_D = \frac{1.328}{\sqrt{\mathrm{Re}_l}}$

15. The values of δ, τ_o, C_f and C_D for the velocity profile according to one-seventh power law for the turbulent boundary layer evaluated in terms of Reynolds number are:

$$\delta = \frac{0.371x}{\mathrm{Re}_x^{1/5}},\ \tau_o = \frac{\rho U^2}{2}\frac{0.0576}{\mathrm{Re}_x^{1/5}},\ C_f = \frac{0.0576}{\mathrm{Re}_x^{1/5}} \text{ and } C_D = \frac{0.072}{\mathrm{Re}_l^{1/5}}$$

16. Prandtl proposed the following relation for computing the average drag coefficient due to laminar and turbulent layers when the value of $(\mathrm{Re}_x)_c = 5\times10^5$: $C_D = \frac{0.074}{\mathrm{Re}_l^{1/5}} - \frac{1700}{\mathrm{Re}_l}$.

17. **Prandtl-Schlichting equation for $10^7 < \mathrm{Re} < 10^9$:**

$$C_D = \frac{0.455}{(\log_{10}\mathrm{Re}_l)^{2.58}} - \frac{1700}{\mathrm{Re}_l}$$

18. The conditions for boundary layer separation is (i) if $(\partial u/\partial y)_{y=0}$ is positive, then there is no separation and thus, the flow is attached, (ii) if $(\partial u/\partial y)_{y=0} = 0$, then the flow is on the verge of separation and (iii) if $(\partial u/\partial y)_{y=0}$ is negative, then the flow is separated.

Multiple-choice Questions

1. In laminar boundary layer, the nominal thickness (δ) varies with longitudinal distance (x) as
   (a) x^2.
   (b) $1/x^2$.
   (c) $x^{1/2}$.
   (d) $x^{-1/2}$.

2. The laminar sublayer exists
   (a) Only in smooth turbulent boundary layers.
   (b) Only in laminar boundary layer.
   (c) Only in rough fully developed turbulent boundary layers.
   (d) In all turbulent boundary layers.

3. The thickness of laminar sublayer (δ') in terms of kinematic viscosity (ν) and friction velocity (u_s) is equal to
   (a) $(11.6\nu)/u_s$.
   (b) $(11.6u_s)/\nu$.
   (c) $11.6\nu/\sqrt{u_s}$.
   (d) $(11.6\sqrt{u_s})/\nu$.

4. The boundary layer exists due to
   (a) Gravitational force.
   (b) Surface tension.
   (c) Density of fluid.
   (d) Viscosity of fluid.

5. The local thickness of turbulent boundary layer varies with longitudinal distance (x) as
   (a) $x^{1/7}$.
   (b) $x^{1/5}$.

(c) $x^{4/5}$.
(d) None of the above

6. The von Karman momentum integral equation is applicable to
(a) Laminar, transition and turbulent boundary layer flows.
(b) Only laminar boundary layer flow.
(c) Only turbulent boundary layer flow.
(d) None of the above.

7. The velocity (u) in turbulent boundary layer varies as one-seventh power law and the growth of the boundary layer thickness (δ/x) varies as
(a) $Re_x^{1/2}$.
(b) $Re_x^{-1/2}$.
(c) $Re_x^{1/5}$
(d) $Re_x^{-1/5}$.

Review Questions

1. Explain the boundary layer for flow over a flat plate. Also discuss the different flow regimes.
2. Define and derive expressions for (i) displacement thickness, (ii) momentum thickness and (iii) energy thickness.
3. Derive von Karman momentum integral equation for boundary layer.
4. Derive Prandtl's boundary layer equations for two-dimensional steady flow of incompressible fluids.
5. What do you mean by boundary layer separation? What is the effect of pressure gradient on boundary layer separation? Also state the different methods of preventing the separation of boundary layer.
6. What is meant by boundary layer? How will you decide whether a boundary layer flow is attached flow, detached flow or on the verge of separation?
7. Derive the expressions for (i) boundary layer thickness, (ii) shear stress, (iii) local coefficient of drag, (iv) drag force and (v) coefficient of drag for one-seventh power law velocity profile for turbulent boundary layer flow.
8. Explain the characteristics of laminar and turbulent boundary layers.

Problems

1. The velocity distribution in the boundary layer is $(u/U) = (1.5y)/\delta - y^2/(2\delta^2)$, where δ is the boundary layer thickness. Determine (i) the ratio of displacement thickness to boundary layer thickness and (ii) ratio of momentum thickness to boundary layer thickness.

[**Ans.** (5/12), (19/120)]

2. The velocity distribution in the boundary layer is $(u/U) = (y/\delta)^{1/7}$, where δ is the boundary layer thickness. Determine (i) the displacement thickness, (ii) momentum thickness, (iii) shape factor, (iv) energy thickness and (v) energy loss due to boundary layer if at a section the boundary layer thickness is 40 mm and the freestream velocity is 20 m/s. If the discharge through the boundary layer region is 5 m^3/s per metre width, then express this energy loss in terms of metres of head when density is 1.2 kg/m^3.

[**Ans.** $\delta/8, (7\delta)/72, 1.286, (7\delta)/40, 0.571$ m]

3. The velocity distribution in the boundary layer over the face of a spillway is $(u/U) = (y/\delta)^{11/50}$, where δ is the boundary layer thickness. Determine the displacement thickness, energy thickness and energy loss up to a particular section if the freestream velocity U is 20 m/s, boundary layer thickness is 50 mm and the discharge through the boundary layer region is 5 m^3/s per metre length of spillway.

[**Ans.** 9.016 mm, 10.86 mm, 0.8856 m]

4. The boundary layer thickness at a distance of 1 m from the leading edge of a flat plate kept over zero angle of incidence to the flow direction is 1 mm. If the velocity outside the boundary layer is 25 m/s, then determine the boundary layer thickness at a distance of 4 m.

[**Ans.** 2 mm]

5. A smooth plate of length 2.5 m and width 2 m immersed in oil of specific gravity 0.8 moves with a velocity of 1.5 m/s along its length. If the kinematic viscosity of oil is 10^{-4} m^2/s, then determine the thickness of boundary layer and shear stress at the trailing edge of the plate.

[**Ans.** 64.5 mm, 3.086 N/m^2]

6. A thin plate of length 0.6 m and width 0.5 m moves in still atmospheric air at a velocity of 7.5 m/s. If the density and kinematic viscosity of air is 1.24 kg/m^3 and 0.15 stokes, respectively, then find the thickness of the boundary layer at the end of the plate and drag force on one side of the plate.

[**Ans.** 5.48 mm, 0.02532 N]

7. A plate of size 0.45 m × 0.15 m placed longitudinally in a stream of crude oil of specific gravity 0.925 and kinematic viscosity of 0.9 stoke flows with a velocity of 6 m/s. Determine (i) the friction drag on the plate, (ii) thickness of the boundary layer at the trailing edge and (iii) shear stress at the trailing edge.

[**Ans.** 8.62 N, 13 mm, 63.829 N/m^2]

8. Find the thickness of boundary layer at the end of the plate and the drag force on one side of a plate of length 1.5 m and width 1 m when immersed in water flowing with a velocity of 0.1 m/s. If the velocity profile for laminar boundary is given by the relation $(u/U) = 2(y/\delta) - (y/\delta)^2$ and the viscosity of water is 0.001 Ns/m^2, then calculate the value of coefficient of drag.

[**Ans.** 21.2 mm, 0.0283 N, 0.00377]

9. Find the thickness of boundary layer and the shear stress 1.5 m from the leading edge of the plate. A plate of length 2.2 m and width 1.5 m is immersed in water flowing with a velocity of 200 mm/s. If the velocity profile for laminar boundary is given by the relation $(u/U) = (3/2)(y/\delta) - (1/2)(y/\delta)^3$ and the viscosity of water is 0.001 Ns/m^2, then find the total drag force on the plate.

[**Ans.** 12.7 mm, 0.0235 N/m^2, 0.257 N]

10. Find the ratio of friction drag on the front half and rear half of the plate kept at zero incidence in a stream of uniform velocity, if the boundary layer is laminar over the whole plate.

[**Ans.** 2.414]

11. Find the ratio of friction drag on the front two-third and rear one-third of the flat plate kept at zero incidence in a stream of uniform velocity, if the boundary layer is turbulent over the entire plate.

[**Ans.** 2.61]

12. A plate of width 0.5 m and length 5 m is kept parallel to the flow of water with freestream velocity 3 m/s. Determine the drag force on both sides of the plate if the boundary layer is turbulent from the very beginning and kinematic viscosity of water is 0.01 stokes.

[**Ans.** 59.175 N]

13. Air flows past a 0.15 m long flat plate in a wind tunnel at a speed of 207.07 m/s. If the boundary layer is turbulent over the entire length of the plate and the density and kinematic viscosity of air are 1.18 kg/m^3 and 15.53×10^{-6} m^2/s, respectively, then determine (i) the boundary layer thickness, (ii) wall shear stress at the trailing edge and total drag per unit width of the plate.

[**Ans.** 3.06 mm, 80.04 N/m^2, 30.02 N]

14. Determine the power required to propel a submarine at a speed of 18 km/hour in water having the density and kinematic viscosity as 1000 kg/m^3 and 10^{-6} m^2/s, respectively. The length of the hull of a submarine is 100 m and its surface area is 5000 m^2. The boundary layer at the leading edge is laminar and the critical Reynolds number at which the flow in boundary layer changes from laminar to turbulent is 5×10^5.

[**Ans.** 419.9375 kW]

15. A barge of a rectangular bottom 25 m long and 8 m wide moves with a velocity of 2 m/s. If the density and dynamic viscosity of water are 1025 kg/m^3 and 0.001 Ns/m^2, respectively and the boundary is smooth, then determine the frictional drag on the bottom of the barge.

[**Ans.** 947.1 N]

16. A streamlined train is 200 m long with a typical cross section having a perimeter of 9 m above the wheels. If the boundary layer changes from laminar to turbulent on the train surface and the density and kinematic viscosity of water are 1.24 kg/m^3 and 1.5×10^{-5} m^2/s, respectively, then determine the approximate surface drag (friction drag) of the train when running at 90 km/hr.

[**Ans.** 1255.5 N]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (c) **2.** (d) **3.** (a) **4.** (d) **5.** (c)
6. (a) **7.** (d)

Chapter **16**

Drag and Lift on Submerged Bodies

16.1 ❐ INTRODUCTION

We often encounter problems related to completely submerged (immersed) bodies in a fluid and having motion relative to the fluid. In several engineering problems, either the body moves through a stationary fluid or the fluid moves over the stationary body or both the body and fluid remains in motion. For example, aeroplanes, submarines, automobiles, ships, all kinds of turbines, etc., move through air or water, and structures like buildings, bridges, etc., remains submerged in air or water. For designing such objects, the knowledge of the forces exerted on them by the fluid is of significant importance. The design of turbine blades, compressor blades and pump impeller blades are some of the other areas that requires knowledge of the phenomenon involved. When a body moves through the stationary fluid or the fluid moves over the stationary body, a force is exerted on the body and it is known as the drag force.

Generally, the force exerted by the fluid on the moving body can be made to incline to the direction of motion. Thus, this force has one of the components in the direction of motion and another perpendicular to the direction of motion. The component of the force in the direction of motion is called the drag force and the component perpendicular to the direction of motion is called lift force. However, if the lift force is zero, then only drag force acts on the body. Analytical methods for the determination of drag are limited to a few simple shaped bodies. Thus, experiments are to be performed in the wind tunnels to determine the drag. This chapter sheds light on the simple approach of analysing such forces acting on the submerged moving bodies, such as plates, circular cylinders, spheres and airfoils.

16.2 ❐ DRAG AND LIFT

A body submerged in a real fluid and having relative motion may be subjected to two types of forces, namely drag force and lift force. For example, an airplane moving at a constant velocity U is shown in Figure 16.1(a).

1. **Drag Force:** The component of force in the direction of flow on the submerged body is called drag force, which is denoted by F_D.
2. **Lift Force:** The component of force perpendicular to the direction of flow is called lift force, which is denoted by F_L.

However, for a submerged symmetrical body like sphere or a cylinder, facing the flow symmetrically, there is no lift and thus, the total force exerted by the fluid is equal to the drag on the body.

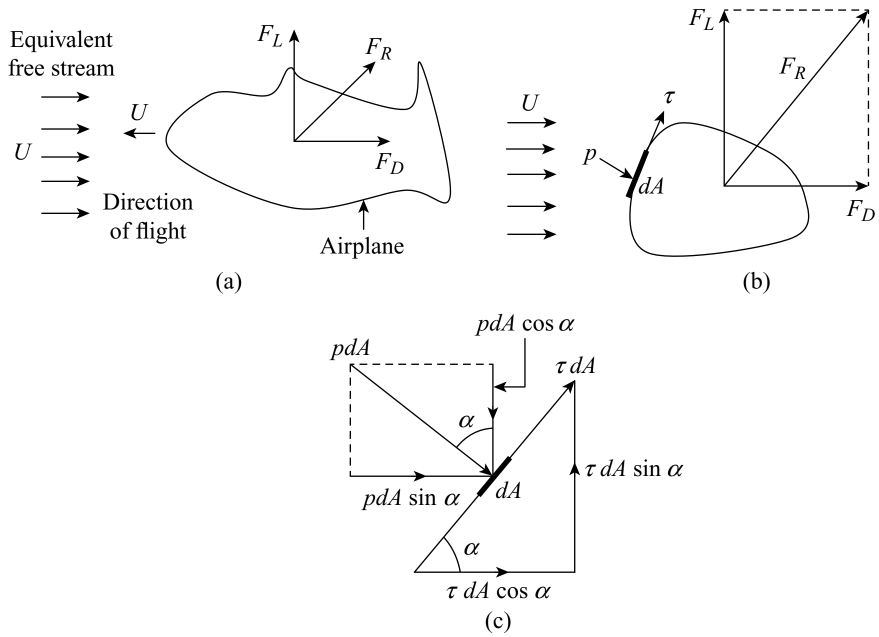


Figure 16.1 *Drag and lift forces*

16.2.1 Types of Drag

1. **Shear drag (surface or skin friction drag):** In the boundary layer zone, considerable shearing stresses are caused due to large velocity gradients. These shear stresses exerts a tangential force on the body which is termed as shear or skin friction drag.
2. **Pressure drag (form drag):** When fluid flows past a submerged body, it causes a region of low pressure (wake) on the downstream side of the body. On account of separation of flow, there exists a pressure difference between the upstream and downstream sides. The pressure difference so created results in producing a drag on the body, which is known as pressure drag or form drag.
3. **Wave drag:** The drag resulted from the waves set up due to the motion of a ship or a boat on the surface of water is called wave drag.
4. **Induced drag:** The drag force produced by the lift force in an airfoil (blade) of finite span is called induced drag.
5. **Profile drag or boundary drag:** If the wave drag and induced drag are neglected, then the total drag is equal to the sum of skin friction drag and pressure drag. The sum of skin friction drag and pressure drag is called profile drag or boundary drag.

16.2.2 Expression for Drag and Lift

Consider a randomly shaped body held stationary in a fluid moving with a uniform velocity U as shown in Figure 16.1(b). Consider an elemental area dA on the surface of the body. Let p be the pressure intensity and τ be the shear stress acting on the area dA. The force due to pressure intensity pdA acts normal to the surface and is inclined at an angle α with the vertical axis (Figure 16.1(c)). The force due to the shear stress τdA acts tangentially to the surface. The components of these two forces in the direction of motion and perpendicular to the direction of motion gives rise to drag force and lift force on the area dA.

The drag force dF_D acting on the small area dA is the sum of pressure force and shear force in the direction of fluid motion and the expression is given below.

$$dF_D = pdA\sin\alpha + \tau dA\cos\alpha \tag{16.1}$$

The total drag force (F_D) acting on the body can be obtained by integrating Equation (16.1) as given below.

$$F_D = \int_A (p \sin\alpha + \tau \cos\alpha) dA \quad (16.2)$$

The first term in Equation (16.2) is called pressure drag and the second term is called friction drag and the respective expressions are given below.

$$\text{Pressure drag} = \int_A p \sin\alpha dA \quad (16.3)$$

$$\text{Friction drag} = \int_A \tau \cos\alpha dA \quad (16.4)$$

The relative magnitude of the pressure drag and friction drag depends upon the shape and position of the submerged body. If the body is a thin plate and if it is held parallel to the flow, then the pressure drag will be zero and the total drag will be only the frictional drag. However, if the plate is held perpendicular to the flow direction, then the friction drag will be zero and the total drag will be the pressure drag.

Similarly, the lift force dF_L acting on the small area dA is given by,

$$dF_L = -pdA\cos\alpha + \tau dA \sin\alpha$$

Thus, the total lift force (F_L) is given by,

$$F_L = \int_A (-p\cos\alpha + \tau \sin\alpha) dA \quad (16.5)$$

For a body moving through a fluid of mass density ρ at a uniform velocity U, the mathematical expressions for the calculation of the drag and the lift may be given as follows.

$$\boxed{F_D = C_D A \frac{\rho U^2}{2}} \quad (16.6)$$

$$\boxed{F_L = C_L A \frac{\rho U^2}{2}} \quad (16.7)$$

In Equations (16.6) and (16.7), C_D and C_L are the drag coefficient and lift coefficient, respectively, A is the characteristic area which is usually taken as either the largest projected area of the submerged body or the projected area of the submerged body on a plane perpendicular to the direction of fluid flow and the term $[(\rho U^2)/2]$ is the dynamic pressure of the flowing fluid.

The resultant force (F_R) on the body is given by,

$$F_R = \sqrt{F_D^2 + F_L^2} \quad (16.8)$$

The drag and lift coefficients can be defined as the ratios of corresponding forces to the dynamic forces on the projected area. The mathematical expressions for these coefficients are given below.

$$C_D = \frac{F_D}{(1/2)\rho A U^2} \quad (16.9)$$

$$C_L = \frac{F_L}{(1/2)\rho A U^2} \quad (16.10)$$

From Equations (16.9) and (16.10), we get:

$$\boxed{\frac{C_L}{C_D} = \frac{F_L}{F_D}} \quad (16.11)$$

The Equation (16.11) shows that the ratio of lift force to drag force is same as the ratio of the coefficient of lift to the coefficient of drag.

16.2.3 Dimensional Analysis of Drag and Lift

Usually, it is not possible to predict the total drag on a body merely by analytical method. Thus, various experiments are conducted to determine these forces and the analysis of the results may be obtained on the basis of dimensional analysis. Consider a body of characteristic length L, moving with a velocity U, through a fluid of mass density ρ, viscosity μ, modulus of elasticity E and acceleration due to gravity g. Let η be the dimensionless shape factor which represents the effect of the shape of the body on the force acting on it. The expression for force exerted on the body is given below.

$$F = f(L, U, \rho, \mu, E, g, \eta) \tag{16.12}$$

Using Buckingham pi theorem, Equation (16.12) may be rewritten as follows.

$$\frac{F}{(1/2)\rho U^2 L^2} = f\left(\frac{\rho UL}{\mu}, \frac{U}{\sqrt{E/\rho}}, \frac{U}{\sqrt{gL}}, \eta\right) \tag{16.13}$$

The non-dimensional parameters in the above equation are (i) Reynolds number, $\mathrm{Re} = (\rho UL)/\mu$, (ii) Mach number, $Ma = U/(\sqrt{E/\rho})$ and (iii) Froude number, $Fr = U/\sqrt{gL}$.

Thus
$$\frac{F}{(1/2)\rho U^2 L^2} = f(\mathrm{Re}, Ma, Fr, \eta) \tag{16.14}$$

The Equation (16.14) is applicable to the drag as well as the lift force as given below.

$$C_D = f(\mathrm{Re}, Ma, Fr, \eta) \tag{16.15}$$

$$C_L = f(\mathrm{Re}, Ma, Fr, \eta) \tag{16.16}$$

The Equations (16.15) and (16.16) indicate that the drag and lift coefficients depends upon certain parameters, such as Reynolds number, Mach number, Froude number and shape factor. The shape factor describes the geometry of the body. The Reynolds number represents the effect of viscosity and it is predominant if the body is completely submerged in a fluid and the fluid is incompressible. Mach number represents the effect of elasticity of the fluid and it is predominant if the body is completely submerged in a fluid and the fluid is compressible. Froude number represents the effect of gravity and is predominant if the body remains partly submerged in the liquid and partly outside the liquid. For most of the problems, the gravity and compressibility effects are not significant. Thus, Equations (16.15) and (16.16) can be written as follows.

$$C_D = f(\mathrm{Re}, \eta) \tag{16.17}$$

$$C_L = f(\mathrm{Re}, \eta) \tag{16.18}$$

Example 16.1 Air of density 1.2 kg/m^3 moves at a speed of 40 km/hr in a stationary flat plate of size 1 m × 1 m. If the drag and lift coefficients are 0.16 and 0.8, respectively, then determine (i) the drag force, (ii) lift force, (iii) resultant force and its direction and (iv) power required to hold the plate stationary.

Solution

Let $\rho = 1.2$ kg/m^3, $U = 40$ km/hr, size $= 1\text{ m} \times 1\text{ m}$, $C_D = 0.16$ and $C_L = 0.8$.

$$A = 1 \times 1 = 1\text{ m}^2$$

$$U = \frac{40 \times 1000}{3600} = 11.11\text{ m/s}$$

(i) $F_D = \dfrac{C_D A\rho U^2}{2} = \dfrac{0.16 \times 1 \times 1.2 \times 11.11^2}{2} = \mathbf{11.85\ N}$

(ii) $F_L = \dfrac{C_L A\rho U^2}{2} = \dfrac{0.8 \times 1 \times 1.2 \times 11.11^2}{2} = \mathbf{59.25\ N}$

(iii) $F_R = \sqrt{F_D^2 + F_L^2} = \sqrt{11.85^2 + 59.25^2} = \mathbf{60.42\ N}$

$$\alpha = \tan^{-1}\left(\frac{F_L}{F_D}\right) = \tan^{-1}\left(\frac{59.25}{11.85}\right) = \mathbf{78.69°}$$

(iv) $P = F_D U = 11.85 \times 11.11 = \mathbf{131.6535\ W}$

Example 16.2 A truck having a projected area of 6.5 m^2 travelling at 70 km/hr has a total resistance of 2000 N. Of this 20% is due to rolling friction and 10% is due to surface friction. The rest is due to form drag. If the density of air is 1.24 kg/m^3, then determine the coefficient of form drag.

Solution

Let $A = 6.5$ m^2, $U = 70$ km/hr, $F_{total} = 2000$ N, $F_{rolling} = 20\%$ of F_{total}, $F_{surface} = 10\%$ of F_{total}, $F_{form} = F_{total} - F_{rolling} - F_{surface}$ and $\rho = 1.24$ kg/m^3.

$$U = \frac{70 \times 1000}{3600} = 19.444 \text{ m/s}$$

$$F_{rolling} = \frac{20}{100} \times 2000 = 400 \text{ N}$$

$$F_{surface} = \frac{10}{100} \times 2000 = 200 \text{ N}$$

$$F_{form} = F_{total} - F_{rolling} - F_{surface} = 2000 - 400 - 200 = 1400 \text{ N}$$

Also
$$F_{form} = \frac{1}{2} C_D A \rho U^2$$

Thus
$$1400 = \frac{1}{2} C_D \times 6.5 \times 1.24 \times 19.444^2$$

$$\therefore C_D = \frac{1400 \times 2}{6.5 \times 1.24 \times 19.444^2} = \mathbf{0.919}$$

Example 16.3 A man weighing 640 N jumps out of an aeroplane due to emergency with the help of a hemispherical shaped parachute. Find the diameter of the parachute if the man comes down with a velocity of 16 m/s, the coefficient of drag is 0.54 and the density of air is 1.2 kg/m^3.

Solution

Refer Figure 16.2. Let $W_{man} = F_D = 640$ N, $U = 16$ m/s, $C_D = 0.54$ and $\rho = 1.2$ kg/m^3.

Let D be the diameter of the parachute.

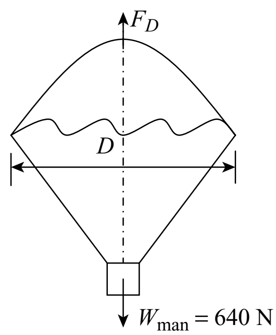


Figure 16.2

Since
$$F_D = \frac{1}{2} C_D A \rho U^2 = \frac{1}{2} C_D \left(\frac{\pi}{4} D^2\right) \rho U^2$$

Thus
$$640 = \frac{1}{2} \times 0.54 \times \frac{\pi}{4} \times D^2 \times 1.2 \times 16^2$$

$$\therefore D = \sqrt{\frac{640 \times 2 \times 4}{0.54 \times \pi \times 1.2 \times 16^2}} = \mathbf{3.1344\ m}$$

Example 16.4 A man descends down from an aeroplane under emergency with the help of a hemispherical shaped parachute of diameter 3 m and weight 30 N. Find the weight of the man if the man comes down with a velocity of 20 m/s, the coefficient of drag is 0.6 and the density of air is 1.21 kg/m^3.

Solution

Let $D = 3$ m, $W_p = 30$ N, $U = 20$ m/s, $C_D = 0.6$ and $\rho = 1.21$ kg/m^3.

Let W_m be the weight of the descending man.

$$F_D = W_p + W_m = \frac{1}{2}C_D A\rho U^2 = \frac{1}{2}C_D\left(\frac{\pi}{4}D^2\right)\rho U^2$$

Thus
$$30 + W_m = \frac{1}{2}\times 0.6\times\frac{\pi}{4}\times 3^2\times 1.21\times 20^2$$

$$\therefore W_m = 1026.36 - 30 = \mathbf{996.36\ N}$$

Example 16.5 A jet plane of weight 25 kN has a wing area of 20.4 m^2. The plane flies at a velocity of 900 km/hr and its engine delivers 7500 kW power. If 64% of the power is used to overcome the drag resistance of the wing, then determine the coefficients of drag and lift for the wing. Take density of air as 1.2 kg/m^3.

Solution

Let $W_{\text{plane}} = 25$ kN, $A = 20.4$ m^2, $U = 900$ km/hr, $P = 7500$ kW, $P_{\text{drag}} = 64\%$ of P and $\rho = 1.2$ kg/m^3.

$$U = \frac{900\times 1000}{3600} = 250 \text{ m/s}$$

$$P_{\text{drag}} = \frac{64}{100}\times 7500 = 4800 \text{ kW}$$

Also
$$P_{\text{drag}} = \frac{F_D U}{1000}$$

Thus
$$4800 = \frac{F_D\times 250}{1000}$$

$$\therefore F_D = \frac{4800\times 1000}{250} = 19200 \text{ N}$$

Also
$$F_D = \frac{1}{2}C_D A\rho U^2$$

$$19200 = \frac{1}{2}C_D\times 20.4\times 1.2\times 250^2$$

$$\therefore C_D = \frac{19200\times 2}{20.4\times 1.2\times 250^2} = \mathbf{0.0251}$$

The lift force should be equal to the weight of the plane and it is calculated as follows.

$$F_L = W_{\text{plane}} = \frac{1}{2}C_L A\rho U^2$$

$$25\times 10^3 = \frac{1}{2}C_L\times 20.4\times 1.2\times 250^2$$

$$\therefore C_L = \frac{25\times 10^3\times 2}{20.4\times 1.2\times 250^2} = \mathbf{0.0327}$$

Example 16.6 A kite weighing 9 N and an area of 0.9 m^2 makes an angle of 7° to the horizontal when it flies in a wind speed of 30 km/hr. Calculate the drag and lift coefficients if pull on the string attached to the kite is 35 N and it is inclined to the horizontal at 45°. Take density of air as 1.2 kg/m^3.

Solution

Refer Figure 16.3. Let $W = 9\text{ N},\ A = 0.9\text{ m}^2, \alpha = 7°, U = 30\text{ km/hr}, P = 35\text{ N}, \beta = 45°$ and $\rho = 1.2\text{ kg/m}^3$.

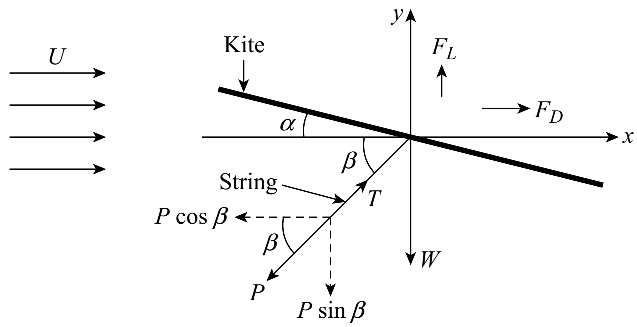


Figure 16.3

$$U = \frac{30 \times 1000}{3600} = 8.333\text{ m/s}$$

F_D = Force exerted by wind in direction of motion, i.e., x-direction

F_D = Component of pull along x-direction

Thus $F_D = P\cos\beta = 35\cos 45° = 24.75\text{ N}$

Also $F_D = \frac{1}{2}C_D A\rho U^2$

$$24.75 = \frac{1}{2}C_D \times 0.9 \times 1.2 \times 8.333^2$$

$$\therefore C_D = \frac{24.75 \times 2}{0.9 \times 1.2 \times 8.333^2} = \mathbf{0.66}$$

Now F_L = Force exerted by wind $\perp$ to direction of motion, i.e., y-direction

F_L = Component of pull in vertically downward direction + W

Thus $F_L = P\sin\beta + W = 35\sin 45° + 9 = 33.75\text{ N}$

Also $F_L = \frac{1}{2}C_L A\rho U^2$

$$33.75 = \frac{1}{2}C_L \times 0.9 \times 1.2 \times 8.333^2$$

$$\therefore C_L = \frac{33.75 \times 2}{0.9 \times 1.2 \times 8.333^2} = \mathbf{0.9}$$

Example 16.7 A kite weighing 7 N and surface area of 0.81 m^2 makes an angle at 10° to the horizontal when flying in the wind. Calculate the wind speed and the tension in the string if it is inclined to the horizontal at 45° and the drag and lift coefficients are 0.6 and 0.8, respectively. Take density of air as 1.2 kg/m^3.

Solution

Refer Figure 16.3. Let $W = 7$ N, $A = 0.81$ m^2, $\alpha = 10°$, $\beta = 45°$, $C_D = 0.6$, $C_L = 0.8$ and $\rho = 1.2$ kg/m^3.

Let the speed of wind be U and T be the tension in string.

$$F_D = \text{Force exerted by wind in direction of motion, i.e., } x\text{-direction}$$

Thus $$F_D = \text{Component of pull along } x\text{-direction} = P\cos\beta = P\cos 45°$$

Also $$F_D = \frac{1}{2}C_D A\rho U^2$$

$$P\cos 45° = \frac{1}{2}\times 0.6\times 0.81\times 1.2\times U^2 = 0.2916U^2 \quad \text{(i)}$$

Now $$F_L = \text{Force exerted by wind} \perp \text{to direction of motion, i.e., } y\text{-direction}$$

$$F_L = \text{Component of pull in vertically downward direction} + W$$

Thus $$F_L = P\sin\beta + W = P\sin 45° + 7$$

Also $$F_L = \frac{1}{2}C_L A\rho U^2$$

Thus $$P\sin 45° + 7 = \frac{1}{2}\times 0.8\times 0.81\times 1.2\times U^2 = 0.3888U^2$$

Thus $$P\sin 45° = 0.3888U^2 - 7 \quad \text{(ii)}$$

$$P\cos 45° = P\sin 45°$$

Thus $$0.2916U^2 = 0.3888U^2 - 7$$

$$(0.3888 - 0.2916)U^2 = 7$$

$$\therefore U = \sqrt{\frac{7}{0.3888 - 0.2916}} = 8.486 \text{ m/s}$$

or $$U = \frac{8.486\times 3600}{1000} = \mathbf{30.5496\ km/hr}$$

Substituting the value of $U = 8.486$ m/s in expression (i), we get:

$$P\cos 45° = 0.2916\times 8.486^2$$

$$P = \frac{0.2916\times 8.486^2}{\cos 45°} = 29.697 \text{ N}$$

$$\therefore T = P = \mathbf{29.697\ N}$$

Example 16.8 Air with a velocity of 0.8 m/s flows over a cylinder of 60 mm diameter. If the length of the cylinder is 1 m, $(C_D)_{\text{total}} = 1.5$, $(C_D)_{\text{shear}} = 0.2$ and $\rho = 1.2$ kg/m^3, then determine the total drag, shear drag and pressure drag.

Solution

Let $U = 0.8$ m/s, $D = 60$ mm $= 0.06$ m, $L = 1$ m, $(C_D)_{\text{total}} = 1.5$, $(C_D)_{\text{shear}} = 0.2$ and $\rho = 1.2$ kg/m^3.

$$A = LD = 1 \times 0.06 = 0.06 \text{ m}^2$$

$$(F_D)_{\text{total}} = \frac{(C_D)_{\text{total}} A\rho U^2}{2} = \frac{1.5 \times 0.06 \times 1.2 \times 0.8^2}{2} = \mathbf{0.03456\ N}$$

$$(F_D)_{\text{shear}} = \frac{(C_D)_{\text{shear}} A\rho U^2}{2} = \frac{0.2 \times 0.06 \times 1.2 \times 0.8^2}{2} = \mathbf{0.00461\ N}$$

$$(F_D)_{\text{pressure}} = (F_D)_{\text{total}} - (F_D)_{\text{shear}} = 0.03456 - 0.00461 = \mathbf{0.02995\ N}$$

Example 16.9 The total aerodynamic force acting on the rectangular wing of a small aeroplane is 24000 N when it flies horizontally at a speed of 210 km/hr. If the lift-drag ratio is 10, the span and chord of the wing are 10 m and 1.6 m, respectively and $\rho = 1.2$ kg/m^3, then determine (i) the lift and drag coefficients, (ii) total weight of the aeroplane and (iii) power required for flight.

Solution

Let $F_L = 24000$ N, $U = 210$ km/hr, $(F_L/F_D) = 10$, $L = 10$ m, $C = 1.6$ m and $\rho = 1.2$ kg/m^3.

Let W be the weight of the plane and P be the power.

(i) $U = \dfrac{210 \times 1000}{3600} = 58.333$ m/s

$$A = LC = 10 \times 1.6 = 16 \text{ m}^2$$

Since
$$F_L = \frac{1}{2} C_L A\rho U^2$$

Thus
$$24000 = \frac{1}{2} C_L \times 16 \times 1.2 \times 58.333^2$$

$$\therefore C_L = \frac{24000 \times 2}{16 \times 1.2 \times 58.333^2} = \mathbf{0.735}$$

Since
$$\frac{C_L}{C_D} = \frac{F_L}{F_D}$$

$$\therefore C_D = \frac{C_L}{(F_L/F_D)} = \frac{0.735}{10} = \mathbf{0.0735}$$

(ii) Since the aeroplane can carry a maximum weight equal to the lift force, we get the following value.

$$\therefore W = F_L = \mathbf{24000\ N}$$

(iii) $F_D = \dfrac{F_L}{10} = \dfrac{24000}{10} = 2400$ N

$$P = \frac{F_D U}{1000} = \frac{2400 \times 58.333}{1000} = \mathbf{139.9992\ kW}$$

Example 16.10 A car travelling with a speed of 60 km/hr has a projected area of 1.6 m^2. (i) Determine the power required to overcome wind resistance by the car if the drag coefficient is 0.36. (ii) For using the same power, also determine the percentage change in speed of the car when the drag coefficient is reduced to 0.3 by streamlining the car body. Take density of air as $\rho = 1.2$ kg/m^3.

Solution

Let $U = 60$ km/hr, $A = 1.6$ m^2, $C_D = 0.36$ and 0.3 and $\rho = 1.2$ kg/m^3.

(i) $U = \dfrac{60 \times 1000}{3600} = 16.67$ m/s

$$F_D = \frac{C_D A \rho U^2}{2} = \frac{0.36 \times 1.6 \times 1.2 \times 16.67^2}{2} = 96.04 \text{ N}$$

$$P = F_D U = 96.04 \times 16.67 = \mathbf{1600.9868\ W}$$

(ii) $P = F_D U = \dfrac{1}{2} C_D A \rho U^2 \times U$

Thus
$$1600.9868 = \frac{1}{2} \times 0.3 \times 1.6 \times 1.2 \times U^3$$

$$\therefore U = \left(\frac{1600.9868 \times 2}{0.3 \times 1.6 \times 1.2}\right)^{1/3} = 17.715 \text{ m/s}$$

$$\text{Percentage increase in speed} = \frac{17.715 - 16.67}{16.67} \times 100 = \mathbf{6.27\%}$$

16.3 ❒ STREAMLINED AND BLUFF BODIES

16.3.1 Streamlined Body

A streamlined body is a body whose surface coincides with the streamlines when the body is held in the flow. Some of the examples of streamlined bodies are thin airfoil, aeroplane, submarine and spaceship. In this case, flow separation takes place only at the trailing edge, the wake formation zone will be very small and consequently, the pressure drag will be very small. Therefore, the total drag will only be due to friction. Streamlined bodies (airfoil and cylinder) are shown in Figure 16.4(a). The streamlined bodies are employed to provide lift. Since the drag on a streamlined body is low, the ratio of lift force to drag force (F_L/F_D) is high. The flight of birds is attributed to the generation of high value of F_L/F_D by virtue of their wings.

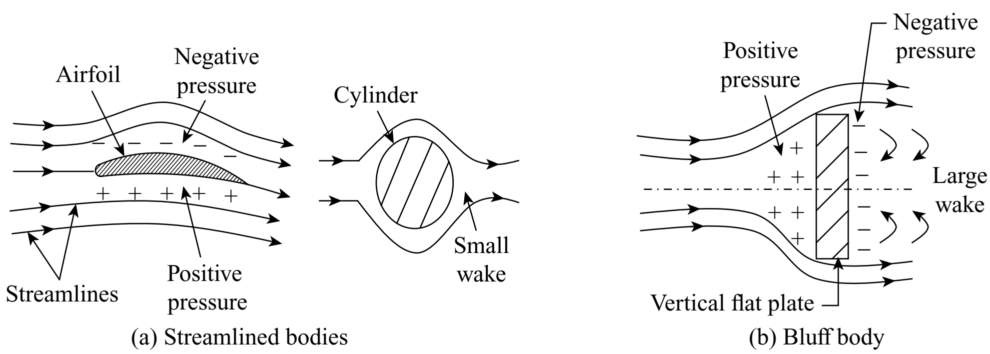


Figure 16.4 *Streamlined and bluff bodies*

16.3.2 Bluff Body

A bluff body is a body whose surface does not coincide with the streamlines and the flow is separated from the beginning of the leading edge itself. Some examples of bluff bodies are chimneys, tall buildings and advertising boards. In this case, there is a large wake zone due to which the pressure drag is very large in comparison to friction drag. The drag on a bluff body is mainly due to the eddy formation and wake effect. Thus, a body for which pressure drag is very large as compared to friction drag is termed as bluff bodies. A bluff body (vertical flat plate) is shown in Figure 16.4(b). A bluff body is used to promote turbulence and the mixing of flow to accelerate the diffusion process in a combustion chamber.

16.4 ❒ DRAG ON A SPHERE (STOKES' LAW)

The drag force on the sphere is a function of Reynolds number which is defined as the ratio of inertia force to the viscous force of the fluid and is given by $\text{Re} = (\rho UD)/\mu$. Here, D is the diameter of the sphere, ρ and μ are the density and viscosity of the flowing fluid respectively with a velocity U. When the velocity of flow is very small and Re < 0.2, Stokes theoretically analysed the flow around a sphere and found that the total drag force acting on the sphere is given as follows.

$$\boxed{F_D = 3\pi\mu DU} \tag{16.19}$$

Although Equation (16.19) for drag force on a sphere was derived for a case with $\text{Re} < 0.2$, but it turns out that the approximation is reasonable up to $\text{Re} \cong 1$.

Stokes further mentioned that out of the total drag given by Equation (16.19), two-thirds of the drag is contributed by skin friction and the remaining one-third is due to pressure difference. The expressions for skin friction drag, $(F_D)_{\text{skin}}$ and pressure drag, $(F_D)_{\text{pressure}}$ are given below.

$$(F_D)_{\text{skin}} = \frac{2}{3} \times 3\pi\mu DU = 2\pi\mu DU$$

$$(F_D)_{\text{pressure}} = \frac{1}{3} \times 3\pi\mu DU = \pi\mu DU$$

$$C_D = \frac{F_D}{(1/2)\rho AU^2} = \frac{3\pi\mu DU}{(1/2)\rho(\pi/4)D^2U^2} = \frac{24\mu}{\rho UD} = \frac{24}{(\rho UD/\mu)} = \frac{24}{\text{Re}} \tag{16.20}$$

Generally, the Equation (16.20) is designated as Stokes law. Oseen, a Swedish physicist improved Stokes' analysis by partly taking into account the inertia terms and gave the following relation which is valid for $0.2 < \text{Re} < 2$.

$$C_D = \frac{24}{\text{Re}}\left(1 + \frac{3}{16\,\text{Re}}\right) \tag{16.21}$$

The values of C_D for different ranges of Reynolds number are (i) $C_D = 0.4$ when $5 < \text{Re} < 10^3$, (ii) $C_D = 0.5$ when $10^3 < \text{Re} < 10^5$ and (ii) $C_D = 0.2$ when $\text{Re} > 10^5$.

16.5 ❒ TERMINAL VELOCITY OF A BODY

If a body falls down from rest into a fluid, then it starts accelerating in the fluid due to gravitational force. As velocity increases, the drag on the body also increases. At one stage, the sum of drag force (F_D) and buoyant force (F_B) acting upwards becomes equal to the weight of the body (W) acting downwards and the net force acting on the body becomes zero. In this condition, while the body comes down, it does not accelerate any more. It starts moving down with a constant maximum velocity which is known as terminal velocity.

The Stokes law is also applicable to spherical particles of diameter D settling under gravity in viscous fluids. The terminal velocity (U) can be calculated as follows.

$$W = F_D + F_B$$

Let w_s and w_f be the specific weights of the particles and the fluid, respectively.

$$\frac{\pi}{6}D^3 w_s = 3\pi\mu DU + \frac{\pi}{6}D^3 w_f$$

$$3\pi\mu DU = \frac{\pi}{6}D^3(w_s - w_f)$$

$$\boxed{\therefore U = \frac{(w_s - w_f)D^2}{18\mu}} \quad (16.22)$$

Example 16.11 A metallic ball of diameter 3 mm having a specific gravity of 12 falls in a fluid of specific gravity 0.85 and viscosity 15 poise. Calculate (i) drag on the ball, (ii) pressure drag and the skin drag, and (iii) terminal velocity of the ball in the fluid.

Solution

Let $D = 3\text{ mm} = 0.003\text{ m}$, $S_b = 12$, $S_f = 0.85$ and $\mu = 15\text{ poise} = 1.5\text{ Ns/m}^2$.

(i) $w_b = S_b \rho_w g = 12 \times 10^3 \times 9.81 = 117720\text{ N/m}^3$

$$w_f = S_f \rho_w g = 0.85 \times 10^3 \times 9.81 = 8338.5\text{ N/m}^3$$

Weight of the ball (W) is given by the product of volume (v) and specific weight as given below.

$$W = v \times w_b = \frac{\pi}{6}D^3 \times w_b = \frac{\pi}{6} \times 0.003^3 \times 117720 = 1.6642 \times 10^{-3}\text{ N}$$

Buoyant force (F_B) is given by,

$$F_B = \text{Volume} \times w_f = \frac{\pi}{6}D^3 w_f = \frac{\pi}{6} \times 0.003^3 \times 8338.5 = 0.1179 \times 10^{-3}\text{ N}$$

$$F_D = W - F_B = 1.6642 \times 10^{-3} - 0.1179 \times 10^{-3} = \mathbf{1.5463 \times 10^{-3}\ N}$$

(ii) $$(F_D)_{\text{pressure}} = \frac{F_D}{3} = \frac{1.5463 \times 10^{-3}}{3} = \mathbf{5.1543 \times 10^{-4}\ N}$$

$$(F_D)_{\text{skin}} = \frac{2}{3}F_D = \frac{2}{3} \times 1.5463 \times 10^{-3} = \mathbf{1.0309 \times 10^{-3}\ N}$$

(iii) $F_D = 3\pi\mu DU$

Thus $$1.5463 \times 10^{-3} = 3\pi \times 1.5 \times 0.003U$$

$$\therefore U = \frac{1.5463 \times 10^{-3}}{3\pi \times 1.5 \times 0.003} = \mathbf{0.03646\ m/s}$$

$$\text{Re} = \frac{\rho UD}{\mu} = \frac{S_l \rho_w UD}{\mu} = \frac{0.85 \times 1000 \times 0.03646 \times 0.003}{1.5} = 0.062$$

Since Re < 0.2, use of the expression $F_D = 3\pi\mu DU$ is valid.

Example 16.12 If the density and kinematic viscosity of air are 1.2 kg/m^3 and 0.15×10^{-4} m^2/s, respectively, then using Stokes' law, determine the velocity of raindrops of diameter 0.25 mm falling from the atmosphere.

Solution

Let $\rho = 1.2$ kg/m^3, $\nu = 0.15 \times 10^{-4}$ m^2/s and $D = 0.25$ mm = 0.00025 m.

$$w_s = \rho_w g = 10^3 \times 9.81 = 9810 \text{ N/m}^3$$

$$w_f = \rho g = 1.2 \times 9.81 = 11.772 \text{ N/m}^3$$

$$\mu = \rho \nu = 1.2 \times 0.15 \times 10^{-4} = 1.8 \times 10^{-5} \text{ Ns/m}^2$$

$$U = \frac{(w_s - w_f)D^2}{18\mu} = \frac{(9810 - 11.772) \times 0.00025^2}{18 \times 1.8 \times 10^{-5}} = \mathbf{1.89 \text{ m/s}}$$

Example 16.13 A ball of diameter 80 mm is supported in the air when air is flowing vertically up with a velocity of 12 m/s. If the density and kinematic viscosity of air are 1.2 kg/m^3 and 0.15×10^{-4} m^2/s, respectively and the buoyancy force of air is neglected, then determine (i) the weight of the ball and (ii) density of its material.

Solution

Let $D = 80$ mm = 0.08 m, $U = 12$ m/s, $\rho = 1.2$ kg/m^3 and $\nu = 0.15 \times 10^{-4}$ m^2/s.

Let W be the weight of the ball and ρ_b be its density.

(i) Since the ball is suspended in equilibrium, we get the below value.

$$W = F_D = \frac{1}{2} C_D A \rho U^2 = \frac{1}{2} C_D \left(\frac{\pi}{4} D^2\right) \rho U^2$$

$$\text{Re} = \frac{UD}{\nu} = \frac{12 \times 0.08}{1.5 \times 10^{-4}} = 6400$$

Since $10^3 < 6400 < 10^5$, $C_D = 0.5$.

$$\therefore W = \frac{1}{2} \times 0.5 \times \frac{\pi}{4} \times 0.08^2 \times 1.2 \times 12^2 = \mathbf{0.21715 \text{ N}}$$

(ii) $W = \text{Volume} \times \rho_b g = \dfrac{\pi}{6} D^3 \rho_b g$

$$0.21715 = \frac{\pi}{6} \times 0.08^3 \times \rho_b \times 9.81$$

$$\therefore \rho_b = \frac{0.21715 \times 6}{\pi \times 0.08^3 \times 9.81} = \mathbf{82.57 \text{ kg/m}^3}$$

Example 16.14 Determine the viscosity of the liquid when a metallic ball of diameter 75 mm having a specific gravity of 7.8 falls in a liquid of specific gravity 0.85 with a velocity of 0.06 m/s.

Solution

Let $D = 75$ mm = 0.075 m, $S_b = 7.8$, $S_l = 0.85$ and $U = 0.06$ m/s. Let W be the weight of the ball, F_D be the drag force, F_B be the buoyancy force and μ be the viscosity of liquid.

$$\rho_b = S_b\rho_w = 7.8\times10^3 = 7800 \text{ kg/m}^3$$

$$\rho_l = S_l\rho_w = 0.85\times10^3 = 850 \text{ kg/m}^3$$

Since $W = F_D + F_B$ [Equilibrium condition]

Thus
$$\frac{\pi}{6}D^3\rho_b g = 3\pi\mu DU + \frac{\pi}{6}D^3\rho_l g$$

$$\therefore \mu = \frac{D^3 g(\rho_b - \rho_l)}{18DU} = \frac{0.075^3\times9.81\times(7800-850)}{18\times0.075\times0.06} = \mathbf{355.1\ Ns/m^2}$$

Example 16.15 A metallic ball of diameter 40 mm and of specific gravity 8 is dropped in water. If the coefficient of drag on the ball in water is 0.52, then determine the terminal velocity of the ball moving through the water.

Solution

Let $D = 40$ mm $= 0.04$ m, $S_b = 8$ and $C_D = 0.52$. Let U be the terminal velocity of the ball, F_D be the drag force, F_B be the buoyancy force and ρ_w be the density of water.

$$\rho_b = S_b\rho_w = 8\times10^3 = 8000 \text{ kg/m}^3$$

$$W = F_D + F_B \quad \text{[Equilibrium condition]}$$

$$\frac{\pi}{6}D^3\rho_b g = \frac{1}{2}C_D A\rho_w U^2 + \frac{\pi}{6}D^3\rho_w g$$

Thus
$$U = \left[\frac{(\pi/6)D^3 g(\rho_b-\rho_w)}{(1/2)C_D A\rho_w}\right]^{1/2} = \left[\frac{\pi D^3 g(\rho_b-\rho_w)}{3C_D(\pi/4)D^2\rho_w}\right]^{1/2}$$

$$\therefore U = \left[\frac{\pi\times0.04^3\times9.81\times(8000-1000)}{3\times0.52\times(\pi/4)\times0.04^2\times1000}\right]^{1/2} = \mathbf{2.654\ m/s}$$

Example 16.16 A metallic spherical ball of diameter 0.003 m falls in a fluid at a terminal velocity of 0.04 m/s. If the specific weights of steel and fluid are 75000 N/m^3 and 12500 N/m^3, respectively, then using Stokes' law calculate (i) the dynamic viscosity of fluid, (ii) drag force and (iii) drag coefficient for the ball.

Solution

Let $D = 0.003$ m, $U = 0.04$ m/s, $w_b = 75000$ N/m^3 and $w_f = 12500$ N/m^3.

(i) Weight of the ball is given by,

$$W = \text{Volume}\times w_b = \frac{\pi}{6}D^3 w_b = \frac{\pi}{6}\times0.003^3\times75000 = 0.00106 \text{ N}$$

Buoyant force is given by,

$$F_B = \text{Volume}\times w_f = \frac{\pi}{6}D^3 w_f = \frac{\pi}{6}\times0.003^3\times12500 = 0.000177 \text{ N}$$

$$F_D = 3\pi\mu DU = 3\pi\times\mu\times0.003\times0.04 = 0.001131\mu$$

Since $W = F_D + F_B$

Thus $$0.00106 = 0.001131\mu + 0.000177$$

$$\therefore \mu = \frac{0.00106 - 0.000177}{0.001131} = \mathbf{0.781\ Ns/m^2}$$

$$\text{Re} = \frac{\rho UD}{\mu} = \frac{(w_f/g)UD}{\mu} = \frac{(12500/9.81)\times 0.04\times 0.003}{0.781} = 0.196$$

Since Re < 0.2, use of the expression $F_D = 3\pi\mu DU$ is valid.

(ii) $F_D = 3\pi\mu DU = 3\pi \times 0.781\times 0.003\times 0.04 = \mathbf{0.00088\ N}$

(iii) $C_D = \dfrac{24}{\text{Re}} = \dfrac{24}{0.196} = \mathbf{122.45}$

16.6 ❐ DRAG ON A CYLINDER

When a real fluid moving with velocity U flows past a cylinder of diameter D and length L placed such that its length is perpendicular to the direction of flow, the followings observations are made.

(i) For Re < 1, $F_D \propto U$ and $C_D \propto \dfrac{1}{\text{Re}}$.

(ii) For Re = 1 to 2000, C_D decreases and attains a minimum value of 0.95 at Re = 2000.

(iii) For Re = 2000 to 3×10^4, C_D increases and attains a maximum value of 1.2 at Re = 3×10^4.

(iv) For Re = 3×10^4 to 3×10^5, C_D decreases and attains a minimum value of 0.3 at Re = 3×10^5.

(v) For Re > 3×10^6, C_D increases and attains a value of 0.7 in the end.

The flow pattern during the flow of a real fluid around an infinitely long circular cylinder placed perpendicular to the direction of flow changes significantly with the Reynolds number.

(i) For Re < 0.5, the inertia forces are negligible relative to the viscous forces and therefore, the flow is wholly laminar. The flow pattern around the cylinder will be symmetrical about the horizontal and vertical axes as shown in Figure 16.5(a) and the drag on the cylinder is due to the viscous shearing at its surface.

(ii) For Re in the range of 2 to 30, the laminar flow separates symmetrically at stagnation points S_1 and S_2 and two weak eddies (vortices) are formed which rotates in opposite direction as shown in Figure 16.5(b). It is the initial stage for the development of the wake. The separated boundary layers come close to each other and join, thereby limiting the size of the wake.

(iii) For Re > 40, the eddying motion is no more stable. For Re ranging about 40 to 70, the pair of vortices as well as the wake becomes quite distinct. The size of eddies enlarges and eventually, breaks off from the cylinder. At Re ≈ 90, the periodic oscillation of the wake is observed as shown in Figure 16.5(c).

(iv) For Re > 90, the alternate breaking and washing away of the eddies can be noticed. This process of the formation of the vortices and their washing away from the two sides of the cylinder continues. Consequently, two rows of vortices moving in the downward direction with very small velocity are formed in the wake as shown in Figure 16.5(d). The configuration or arrangement of vortices in two rows rotating in opposite direction is called Karman vortex trails or Karman vortex street named after von Karman. As per Karman, the configuration of vortices may be symmetrical or staggered. The staggered configuration of vortices is shown in Figure 16.5(d). The alternate shedding of the vortices starts at Re ≈ 45 and it can be seen only at Re ≈ 120, but it continues up to Re ≈ 10^5. Karman proved that the configuration of vortices is stable only when the ratio of transverse spacing (h) to the longitudinal spacing (l) is equal to 0.281, i.e., $h/l = 0.281$.

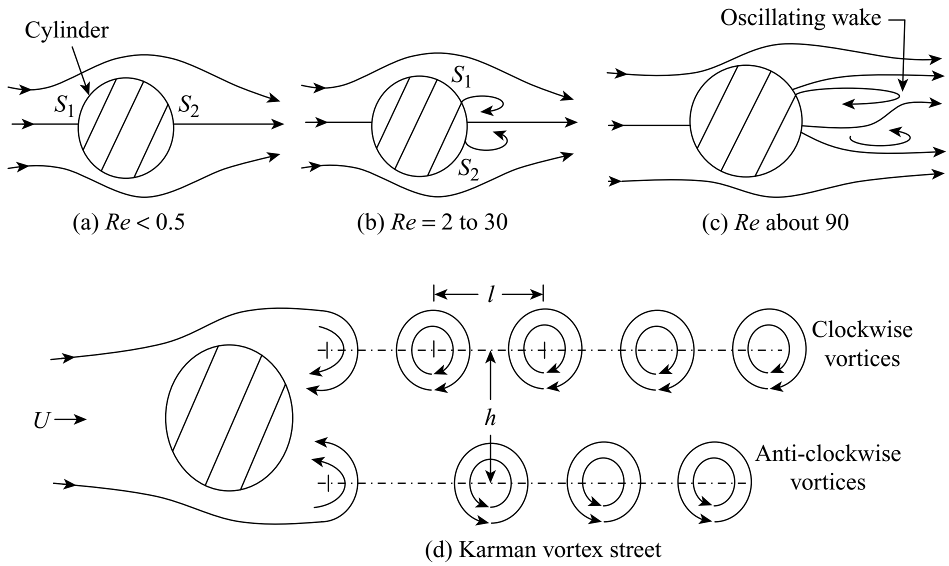


Figure 16.5 *Flow past an infinitely long circular cylinder*

(v) Strouhal number (Sn) can be given by $Sn = (fD)/U$, here f is the frequency with which the vortices are shed alternately from the cylinder on the downstream side, D is the diameter of cylinder and U is the velocity of flow. The frequency f can be determined by the relation $Sn = 0.198[1-(19.7/\text{Re})]$.

(vi) The alternate shedding of the vortices causes periodic transverse forces on the cylinder and thus, it produces transverse oscillations. The closeness of natural frequency of vibration of the cylinder to the frequency of the vortex shedding can cause damage due to resonance. This consideration is very important to the design of elastic structures that are exposed to high winds, like cylindrical chimneys, towers, suspension bridges, etc. Alternate shedding of vortices from the two sides of a cylinder give rise to the phenomenon of singing of telephone or power lines in the wind, fluttering of wires and poles.

16.7 ❒ CIRCULATION AND LIFT ON A CYLINDER

When an ideal fluid flows over a stationary cylinder of radius R with a uniform velocity U, the streamlines will be symmetrical on the front and the rear sides of the cylinder (Figure 16.6(a)). The flow first stagnates at S_1, accelerates to a maximum velocity at the top and bottom and decelerates to a second stagnation point S_2. Again the flow from S_2 accelerates away from the cylinder and merges with the main flow.

Let α be the angle made by the point on the circumference of the cylinder with the direction of flow. The expression for velocity u_α at any point on the surface of the cylinder is given below.

$$u_\alpha = 2U \sin\alpha \tag{16.23}$$

The pressure distribution at the front and rear of the cylinder is symmetrical and equal on the upper and lower halves of the cylinder. Thus, no unbalanced force act on the cylinder and eventually, the drag and the lift acting on the cylinder is zero.

When a constant circulation (Γ) is imparted to this cylinder, the streamlines takes the form of concentric circles and the flow pattern so formed is illustrated in Figure 16.6(b). The expression for peripheral velocity (u_c) on the surface of cylinder is given below.

$$\boxed{u_c = \frac{\Gamma}{2\pi R}} \tag{16.24}$$

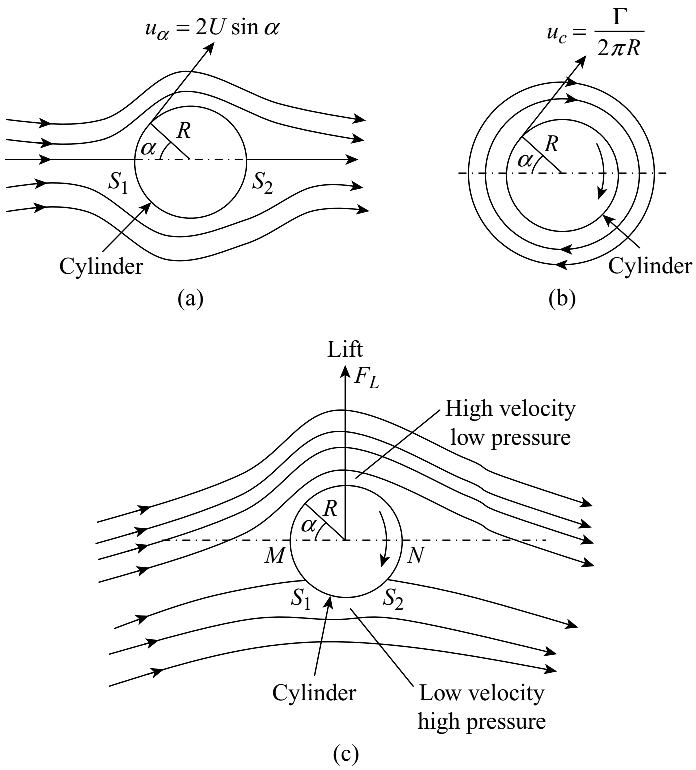


Figure 16.6 *Flow pattern over a stationary and rotating cylinder*

A composite flow pattern shown in Figure 16.6(c) is obtained by superimposing the above two flow patterns. The resultant velocity (u) at any point on the surface of the cylinder is obtained by adding Equations (16.23) and (16.24) as given below.

$$u = 2U \sin \alpha + \frac{\Gamma}{2\pi R} \tag{16.25}$$

The flow pattern so obtained is unsymmetrical about the horizontal axis. The circulation is taken clockwise and thus, the superimposition causes higher velocity around the upper half portion of the cylinder than that of the lower one. Since α varies from 0° to 180° for the upper half portion of the cylinder, $2U \sin \alpha$ has a positive value. However, for lower half of the cylinder, α varies from 180° to 360° for which the value of $2U \sin \alpha$ would be negative. Thus, the value of resultant velocity given by Equation (16.25) will be higher around the upper half portion of the cylinder than the lower half. Thus, according to Bernoulli's theorem, the pressure on the lower half portion of the cylinder will be more than the pressure on the upper half portion of the cylinder. Due to this pressure difference on the two halves of the cylinder, a force acts on the cylinder in the direction perpendicular to the flow direction. This upward force is known as lift force (F_L). However, since the flow is symmetrical about the vertical axis, no unbalanced forces act on the cylinder and hence, the drag is zero. Such a phenomenon of generation of lift by a spinning cylinder in a fluid stream is known as Magnus effect. This effect was named after the German scientist Heinrich Magnus (1802–1870), who was the first to study the lift of rotating bodies (Figure 16.6(c)). This effect can be observed in bullets fired from rifle. Magnus effect can be used in 'rotor sails' of ships (Flettner design) by employing rotating cylinder to produce a propulsive force (Figure 16.7(a)). The Flettner design is not so popular, but it is of considerable scientific interest. The Magnus effect can also be productively used in changing the flight of balls in different ball games, like cricket, table tennis, golf and tennis. The Figure 16.7(b) shows the type of spin given to the ball to achieve a drop, rise and curve in their trajectories.

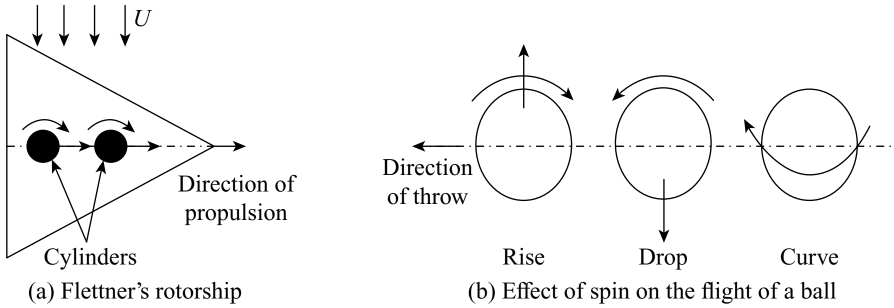


Figure 16.7 *Examples of Magnus effect*

Since at the stagnation points $u = 0$, the Equation (16.25) can be written as follows.

$$2U \sin\alpha + \frac{\Gamma}{2\pi R} = 0$$

$$\boxed{\therefore \sin\alpha = -\frac{\Gamma}{4\pi RU}} \tag{16.26}$$

The location of stagnation points on the surface of the cylinder can be known from Equation (16.26) which has been discussed in Section 9.6.6 (Chapter 9).

16.8 ❒ EXPRESSION FOR LIFT ON A ROTATING CYLINDER

Let a cylinder rotate in a uniform flow field. Consider a small elemental length ds on the surface of the cylinder which makes an angle $d\alpha$ at the centre of the cylinder as shown in Figure 16.8.

Let p_o and U be the pressure and the velocity of the fluid far away from the cylinder, respectively, p and u be the pressure and the velocity on the surface of the element, respectively, L and R be the length and radius of the cylinder, respectively and $ds = Rd\alpha$ be the length of the element.

Applying Bernoulli's equation between any point far away from the cylinder and at any point on the surface of the cylinder, we get the below expression.

$$p_o + \frac{1}{2}\rho U^2 = p + \frac{1}{2}\rho u^2$$

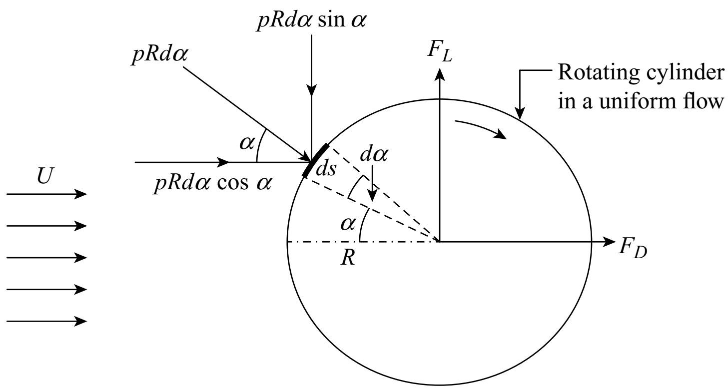


Figure 16.8 *Lift on a rotating cylinder*

Substituting $u = 2U\sin\alpha + \Gamma/(2\pi R)$ from Equation (16.25) in the above equation, we get:

$$p_o + \frac{1}{2}\rho U^2 = p + \frac{1}{2}\rho\left(2U\sin\alpha + \frac{\Gamma}{2\pi R}\right)^2$$

Thus
$$p = p_o + \frac{1}{2}\rho U^2\left[1 - \left(2\sin\alpha + \frac{\Gamma}{2\pi UR}\right)^2\right] \tag{16.27}$$

Assuming unit length of the cylinder, the force acting on the element dF is given by,

$$dF = pRd\alpha$$

By resolving this force into horizontal and vertical directions, the drag force dF_D and lift force dF_L can be respectively obtained as follows.

$$dF_D = pRd\alpha\cos\alpha$$

$$dF_L = -pRd\alpha\sin\alpha$$

The total drag and lift on the cylinder can be obtained by integrating the above expressions over the entire surface of the cylinder.

Since the flow pattern for a rotating cylinder in a uniform flow is symmetrical with respect to vertical axis, there is no drag on the cylinder, i.e., $\int_0^{2\pi} pR\cos\alpha d\alpha = F_D = 0$. This concept of zero drag on bodies immersed in a steady flow of ideal fluid is called D'Alembert's paradox.

Now
$$F_L = -\int_0^{2\pi} pR\sin\alpha d\alpha$$

Substituting the value of p from Equation (16.27) in the above expression, we get:

$$F_L = -R\left[\int_0^{2\pi} p_o\sin\alpha d\alpha + \int_0^{2\pi}\frac{1}{2}\rho U^2\sin\alpha d\alpha - \int_0^{2\pi}\frac{1}{2}\rho U^2\left(2\sin\alpha + \frac{\Gamma}{2\pi UR}\right)^2\sin\alpha d\alpha\right]$$

$$F_L = -R\left[\begin{array}{l}\int_0^{2\pi} p_o\sin\alpha d\alpha + \int_0^{2\pi}\frac{1}{2}\rho U^2\sin\alpha d\alpha \\ \quad -\frac{1}{2}\rho U^2\left\{\int_0^{2\pi} 4\sin^3\alpha d\alpha + \int_0^{2\pi}\frac{2\Gamma}{\pi UR}\sin^2\alpha d\alpha + \int_0^{2\pi}\frac{\Gamma^2}{4\pi^2 U^2 R^2}\sin\alpha d\alpha\right\}\end{array}\right]$$

If n is an odd number, then $\int_0^{2\pi}\sin^n\alpha d\alpha = 0$, and the above expression reduces to the following expression.

$$F_L = R\left(\frac{1}{2}\rho U^2\frac{2\Gamma}{\pi UR}\right)\int_0^{2\pi}\sin^2\alpha d\alpha = \frac{\rho U\Gamma}{\pi}\int_0^{2\pi}\left(\frac{1-\cos 2\alpha}{2}\right)d\alpha$$

$$\therefore F_L = \frac{\rho U\Gamma}{\pi}\left[\frac{1}{2}\alpha - \frac{\sin 2\alpha}{2}\right]_0^{2\pi} = \rho U\Gamma \text{ per unit length of cylinder}$$

Thus, the total lift of a cylinder of length L is given by,

$$\boxed{F_L = \rho L U \Gamma} \tag{16.28}$$

The Equation (16.28) is known as Kutta-Joukowski equation which is applicable to all types of bodies of any shape including airfoils.

16.8.1 Expression for Lift Coefficient for a Rotating Cylinder

The lift force can also be given by Equation (16.7) as follows.

$$F_L = C_L A \frac{\rho U^2}{2}$$

Substituting the values of F_L from Equation (16.28) and $A = 2RL$ in the above expression, we get:

$$\rho L U \Gamma = C_L (2RL) \frac{\rho U^2}{2}$$

$$\therefore C_L = \frac{\rho L U \Gamma}{RL\rho U^2} = \frac{\Gamma}{RU} \tag{16.29}$$

Now $$\frac{\Gamma}{R} = 2\pi u_c \qquad [\because u_c = \Gamma/(2\pi R)]$$

$$\boxed{\therefore C_L = \frac{2\pi u_c}{U}} \tag{16.30}$$

Example 16.17 Find the rotational speed of a cylinder of diameter 1.4 m and length 10 m rotating in the air stream. The cylinder axis is perpendicular to the air stream flowing with a velocity of 55 m/s and the lift produced is 7000 N per metre length of the cylinder. Take density of air as 1.2 kg/m^3.

Solution

Let $D = 1.4$ m, $L = 10$ m, $U = 55$ m/s, $(F_L/L) = 7000$ N/m and $\rho = 1.2$ kg/m^3.

$$R = \frac{D}{2} = \frac{1.4}{2} = 0.7 \text{ m}$$

Since $$\frac{F_L}{L} = \rho U \Gamma = \rho U (2\pi R u_c) = \rho U (2\pi R) \times \frac{2\pi R N}{60}$$

$$\therefore N = \frac{(F_L/L) \times 60}{4\rho U \pi^2 R^2} = \frac{7000 \times 60}{4 \times 1.2 \times 55 \times \pi^2 \times 0.7^2} = \mathbf{328.965 \text{ rpm}}$$

Example 16.18 A cylinder of diameter 1.6 m and length 10 m rotates at 350 rpm with its axis normal to the flow direction. Determine the theoretical circulation around the cylinder, lift coefficient and lift force if air flows with a velocity of 10 m/s and its density is 1.2 kg/m^3.

Solution

Let $D = 1.6$ m, $L = 10$ m, $N = 350$ rpm, $U = 10$ m/s and $\rho = 1.2$ kg/m^3.

$$u_c = \frac{\pi D N}{60} = \frac{\pi \times 1.6 \times 350}{60} = 29.32 \text{ m/s}$$

$$\Gamma = \pi D u_c = \pi \times 1.6 \times 29.32 = 147.38 \text{ m}^2/\text{s}$$

$$C_L = \frac{2\pi u_c}{U} = \frac{2\pi \times 29.32}{10} = 18.422$$

$$F_L = \rho L U \Gamma = 1.2 \times 10 \times 10 \times 147.38 = \mathbf{17685.6\ N}$$

Example 16.19 As an application of Magnus effect, a ship is built having two vertical rotors 10 m high and 3 m in diameter. The rotors are spun at 250 rpm. On a day when the air temperature is 20°C and relative motion of the air to ship results in 54 km per hour wind, calculate the force emitted by the spinning rotors on the ship. Take density of air as 1.25 kg/m^3.

Solution

Let $n = 2$, $L = 10$ m, $D = 3$ m, $N = 250$ rpm, $U = 54$ km/hr and $\rho = 1.25$ kg/m^3.

$$U = \frac{54 \times 1000}{3600} = 15 \text{ m/s}$$

$$u_c = \frac{\pi D N}{60} = \frac{\pi \times 3 \times 250}{60} = 39.27 \text{ m/s}$$

$$\Gamma = \pi D u_c = \pi \times 3 \times 39.27 = 370.11 \text{ m}^2/\text{s}$$

$$F_L = n \rho L U \Gamma = 2 \times 1.25 \times 10 \times 15 \times 370.11 = \mathbf{138791.25\ N}$$

Example 16.20 Air having a velocity of 40 m/s is flowing over a cylinder of radius 0.75 m and length 10 m, when the axis of the cylinder is perpendicular to the air stream. (i) Calculate the speed at which the cylinder is to be rotated about its axis so that a lift force of 7 kN/m length of the cylinder is developed. (ii) Also find the location of the stagnation points. Take density of air as 1.25 kg/m^3.

Solution

Let $U = 40$ m/s, $R = 0.75$ m, $L = 10$ m, $(F_L/L) = 7$ kN/m $= 7000$ N/m and $\rho = 1.25$ kg/m^3.

(i) $$\frac{F_L}{L} = \rho U \Gamma = \rho U (2\pi R u_c) = \rho U (2\pi R) \times \frac{2\pi R N}{60}$$

$$\therefore N = \frac{(F_L/L) \times 60}{4 \rho U \pi^2 R^2} = \frac{7000 \times 60}{4 \times 1.25 \times 40 \times \pi^2 \times 0.75^2} = \mathbf{378.266\ rpm}$$

(ii) $$\Gamma = \frac{(F_L/L)}{\rho U} = \frac{7000}{1.25 \times 40} = 140 \text{ m}^2/\text{s}$$

$$\sin\alpha = -\frac{\Gamma}{4\pi R U} = -\frac{140}{4\pi \times 0.75 \times 40} = -0.3714$$

or $$\sin\alpha = -\sin(21.8°)$$

Thus $$\sin\alpha = \sin(180 + 21.8°) \text{ and } \sin(360 - 21.8°)$$

$$\therefore \alpha = \mathbf{201.8° \text{ and } 338.2°}$$

Example 16.21 A cylinder whose axis is perpendicular to the stream of air having a velocity of 20 m/s rotates at 300 rpm. The cylinder is 2 m in diameter and 10 m long. Find (i) the circulation, (ii) theoretical lift force per unit length, (iii) position of stagnation points, (iv) actual lift, drag and direction of resultant force. For determining actual drag and lift, assume $(u_c/U) = 1.57$, $C_L = 3.4$ and $C_D = 0.65$. Take density of air as 1.24 kg/m^3. (v) Also determine the speed of rotation of the cylinder which yields only a single stagnation point.

Solution

Let $U = 20$ m/s, $N = 300$ rpm, $D = 2$ m, $L = 10$ m, $(u_c/U) = 1.57$, $C_L = 3.4$, $C_D = 0.65$ and $\rho = 1.24$ kg/m^3.

$$R = \frac{D}{2} = \frac{2}{2} = 1 \text{ m}$$

(i) $u_c = \dfrac{\pi DN}{60} = \dfrac{\pi \times 2 \times 300}{60} = 31.416$ m/s

$$\Gamma = \pi D u_c = \pi \times 2 \times 31.416 = \mathbf{197.3925\ m^2/s}$$

(ii) $\dfrac{F_L}{L} = \rho U \Gamma = 1.24 \times 20 \times 197.3925 = \mathbf{4895.334\ N}$

(iii) $\sin\alpha = -\dfrac{\Gamma}{4\pi RU} = -\dfrac{197.3925}{4\pi \times 1 \times 20} = -0.7854$

or $\qquad \sin\alpha = -\sin(51.76°)$

Thus $\qquad \sin\alpha = \sin(180 + 51.76°)$ and $\sin(360 - 51.76°)$

$$\therefore \alpha = \mathbf{231.76° \text{ and } 308.24°}$$

(iv) $F_L = C_L \times \dfrac{1}{2}\rho U^2 LD = 3.4 \times \dfrac{1}{2} \times 1.24 \times 20^2 \times 10 \times 2 = \mathbf{16864\ N}$

$$F_D = C_D \times \frac{1}{2}\rho U^2 LD = 0.65 \times \frac{1}{2} \times 1.24 \times 20^2 \times 10 \times 2 = \mathbf{3224\ N}$$

Thus, the resultant force (F) is given by,

$$F = \sqrt{F_L^2 + F_D^2} = \sqrt{16864^2 + 3224^2} = \mathbf{17169.411\ N}$$

If the inclination of the resultant force with the horizontal is α, then its value is given below.

$$\alpha = \tan^{-1}\left(\frac{F_L}{F_D}\right) = \tan^{-1}\left(\frac{16864}{3224}\right) = \mathbf{79.18°}$$

(v) For a single stagnation point, we get:

$$\Gamma = 4\pi RU = 4\pi \times 1 \times 20 = 251.33 \text{ m}^2/\text{s}$$

$$u_c = \frac{\Gamma}{2\pi R} = \frac{251.33}{2\pi \times 1} = 40 \text{ m/s}$$

Also $\qquad u_c = \dfrac{\pi DN}{60}$

$$\therefore N = \frac{60 u_c}{\pi D} = \frac{60 \times 40}{\pi \times 2} = \mathbf{381.972\ rpm}$$

16.9 ❒ BASIC TERMINOLOGY FOR AN AIRFOIL

An airfoil (aerofoil) is a streamlined body which may be either symmetrical or unsymmetrical as shown in Figure 16.9. Since a considerable power is wasted in overcoming the drag force, the drag should be reduced. A streamlined body has either no separation or the separation is limited to a small section near the rear part of the body and thus, it reduces the drag. The applications of airfoils are widely used in airplanes and turbomachinery.

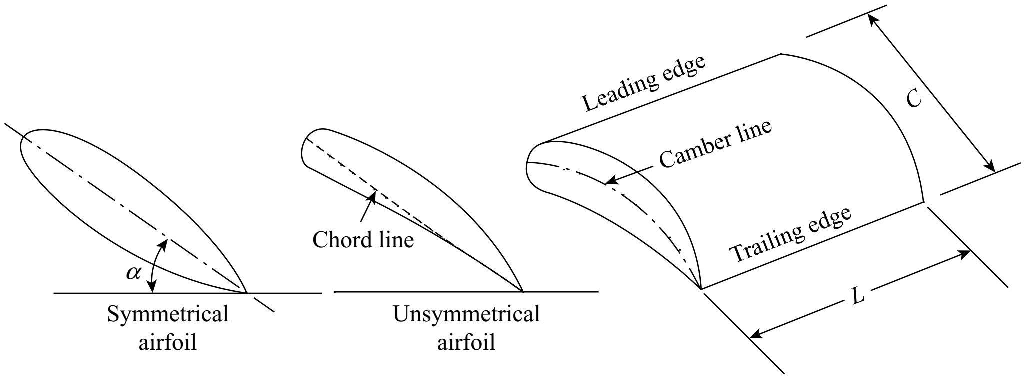


Figure 16.9 *Types of airfoils*

Some of the important terms related to an airfoil are defined below.

1. **Leading edge:** The front edge of the blade on thicker side is termed as leading edge.
2. **Trailing edge:** The back edge of the blade on thinner side is termed as trailing edge.
3. **Chord:** The line joining the leading (front) and trailing (rear) edges in a direction perpendicular to the two edges is known as chord and it is denoted by C.
4. **Span:** The overall length of the airfoil is known as span and it is denoted by L.
5. **Aspect ratio:** It is the ratio of the span to the chord, i.e., (L/C).
6. **Camber line:** The line joining the midpoints of the profile of the airfoil is known as camber line.
7. **Angle of attack:** The angle between the chord line and direction of the fluid stream is called the angle of attack and it is denoted by α.
8. **Stall:** If the angle of attack of an airfoil is greater than the angle of attack for maximum lift, then the airfoil is said to be operating under the stall condition. Under stall condition, the flow separates from the airfoil and eddies are formed. Therefore, the drag coefficient increases considerably.

16.10 ❐ CIRCULATION AND LIFT ON AN AIRFOIL

As airfoil is a streamlined body, the drag force on it is always small. The lift on an airfoil is due to negative pressure generated on its upper side. Theoretically, the expression for circulation (Γ) developed on the airfoil is given below.

$$\Gamma = \pi C U \sin\alpha \tag{16.31}$$

$$F_L = \rho L U \Gamma = \rho L U \times \pi C U \sin\alpha = \pi\rho C L U^2 \sin\alpha \tag{16.32}$$

Also
$$F_L = \frac{1}{2} C_L A \rho U^2 = \frac{1}{2} C_L (CL) \rho U^2 \tag{16.33}$$

Simplifying Equations (16.32) and (16.33), we get:

$$\frac{1}{2} C_L (CL) \rho U^2 = \pi\rho C L U^2 \sin\alpha$$

$$\therefore C_L = \frac{2\pi\rho C L U^2 \sin\alpha}{C L \rho U^2} = 2\pi \sin\alpha \tag{16.34}$$

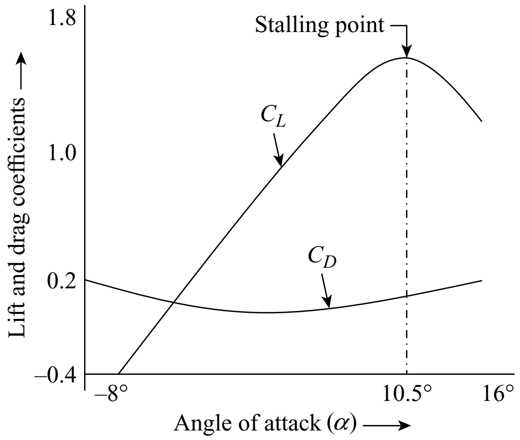


Figure 16.10 *Variation of lift and drag coefficients with angle of attack*

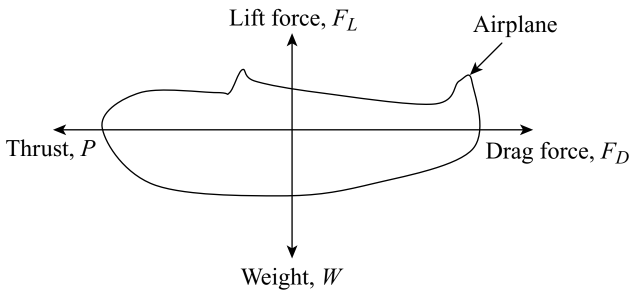


Figure 16.11 *Equilibrium position of an airplane*

It can be seen from Equation (16.34) that the lift of an airfoil increases as the angle of attack (α) increases. Practically, the lift coefficient (C_L) increases up to a certain angle of attack, beyond which it decreases drastically. This point is known as the stalling point. In practice, the operating point of airfoil always lies below the stalling point. Figure 16.10 shows the variation trend of lift and drag coefficients for varying values of angle of attack.

For equilibrium, the weight (W) of an airplane (a flying object) is equal to the lift force (F_L) in its airfoils, whereas the thrust (P) developed by its engine is equal to the drag force (F_D) as schematically shown in Figure 16.11.

Example 16.22 An aeroplane flying in a horizontal direction at 324 km/hr weighs 35 kN. Determine the lift coefficient, circulation and power required to drive the plane if it spans 16 m, its wing surface area is 36 m^2, drag coefficient is 0.04 and density of air is 1.2 kg/m^3.

Solution

Let $U = 324$ km/hr, $W = 35$ kN $= 35\times10^3$ N, $L = 16$ m, $A = 36$ m^2, $C_D = 0.04$ and $\rho = 1.2$ kg/m^3.

$$U = \frac{324\times1000}{60\times60} = 90 \text{ m/s}$$

For equilibrium in vertical direction, the lift equals the weight of aeroplane, where the expression is given below.

$$W = \frac{1}{2}C_L\rho AU^2$$

$$35\times10^3 = \frac{1}{2}\times C_L \times1.2\times36\times90^2$$

$$\therefore C_L = \frac{35\times10^3\times2}{1.2\times36\times90^2} = \mathbf{0.2}$$

$$\Gamma = \frac{F_L}{\rho LU} = \frac{35\times10^3}{1.2\times16\times90} = \mathbf{20.255\ m^2/s}$$

$$F_D = \frac{1}{2}C_D\rho AU^2 = \frac{1}{2}\times0.04\times1.2\times36\times90^2 = 6998.4\ \text{N}$$

$$\text{Power} = \frac{F_DU}{1000} = \frac{6998.4\times90}{1000} = \mathbf{629.856\ kW}$$

Example 16.23 An aeroplane flying in a horizontal direction at 720 km/hr having a wing area of 26 m^2 weighs 25 kN. Determine the lift and drag coefficients, if its engine delivers 6000 kW and 60% of its power is used to overcome the drag resistance of the wing. Take density of air as 1.2 kg/m^3.

Solution

Let $U = 720$ km/hr, $A = 26$ m^2, $W = 25\ kN = 25\times10^3$ N, $P_t = 6000$ kW, $P_{\text{drag}} = 60\%$ of P_t and $\rho = 1.2$ kg/m^3.

$$U = \frac{720\times1000}{60\times60} = 200\ \text{m/s}$$

For equilibrium in vertical direction, the lift equals the weight of aeroplane, where the expression is given below.

$$W = \frac{1}{2}C_L\rho AU^2$$

$$25\times10^3 = \frac{1}{2}\times C_L\times1.2\times26\times200^2$$

$$\therefore C_L = \frac{25\times10^3\times2}{1.2\times26\times200^2} = \mathbf{0.04}$$

$$P_{\text{drag}} = \frac{60}{100}\times6000 = 3600\ \text{kW}$$

Also
$$P_{\text{drag}} = F_DU = \frac{1}{2}C_D\rho AU^2\times U = \frac{1}{2}C_D\rho AU^3$$

Thus
$$3600\times10^3 = \frac{1}{2}\times C_D\times1.2\times26\times200^3$$

$$\therefore C_D = \frac{3600\times10^3\times2}{1.2\times26\times200^3} = \mathbf{0.029}$$

Summary

1. A body submerged in a real fluid and having relative motion may be subjected to two types of forces, namely drag and lift forces.
2. The component of force in the direction of flow on the submerged body is called drag force, which is given by $F_D = (1/2)C_D A\rho U^2$, here C_D is the drag coefficient, A is the projected area of the body, ρ is the density of fluid and U is freestream velocity.
3. The component of force perpendicular to the direction of flow is called lift force and it is given by $F_L = (1/2)C_L A\rho U^2$, here C_L is the lift coefficient.
4. A streamlined body is a body whose surface coincides with the streamlines when the body is held in the flow.
5. A bluff body is a body whose surface does not coincide with the streamlines and the flow is separated from the beginning of the leading edge itself.
6. For Re < 0.2, Stokes found that the total drag force acting on the sphere is given by $F_D = 3\pi\mu DU$ out of which two-thirds of the drag is contributed by skin friction and remaining one-third is due to pressure difference.
7. The Stokes law is given by $C_D = 24/\text{Re}$, here Re is the Reynolds number.
8. Terminal velocity is the maximum constant velocity attained by a falling body. At the terminal velocity, the weight of the body (W) is equal to the drag force (F_D) plus the buoyant force (F_B), i.e., $W = F_D + F_B$.
9. The velocity u_α at any point on the surface of the cylinder is given by $u_\alpha = 2U\sin\alpha$, here α is the angle made by the point on the circumference of the cylinder with the direction of flow.
10. The peripheral velocity (u_c) on the surface of cylinder is $u_c = \Gamma/(2\pi R)$, here Γ is the circulation provided to the cylinder and R is its radius.
11. The resultant velocity at any point on the surface of cylinder is $u = 2U\sin\alpha + \Gamma/(2\pi R)$, here U is the uniform velocity of the fluid.
12. The position of stagnation point on the surface of cylinder is $\sin\alpha = -\Gamma/(4\pi RU)$.
13. The total lift on a rotating cylinder of length L is given by $F_L = \rho LU\Gamma$ which is also known as Kutta-Joukowski equation.
14. The lift coefficient for a rotating cylinder is $C_L = (2\pi u_c)/U$.
15. The line joining the leading (front) and trailing (rear) edges in a direction perpendicular to the two edges are known as chord (C).
16. The angle between the chord line and direction of the fluid stream is called the angle of attack (α).
17. The circulation (Γ) developed on the airfoil is $\Gamma = \pi CU\sin\alpha$.
18. The lift coefficient for an airfoil is $C_L = 2\pi\sin\alpha$.
19. For equilibrium, the weight of an airplane (a flying object) is equal to the lift force in its airfoils, whereas the thrust developed by its engine is equal to the drag force.

Multiple-choice Questions

1. The terminal velocity of a falling body is equal to
   (a) Half of maximum velocity.
   (b) Maximum constant velocity with which body falls.
   (c) Maximum velocity with which body falls.
   (d) None of the above.
2. Total drag on a body is the sum of
   (a) Velocity drag and pressure drag.
   (b) Velocity drag and friction drag.
   (c) Pressure drag and friction drag.
   (d) None of the above.
3. The lift force produced on the cylinder due to its rotation in a uniform flow is caused by
   (a) Symmetrical streamline patterns.
   (b) Shear stress.
   (c) Pressure difference between the two halves, the bottom-half being subjected to a higher pressure.
   (d) None of the above.
4. A body is called bluff body if the surface of the body
   (a) Is rough.
   (b) Is smooth.
   (c) Coincides with the streamlines.
   (d) Does not coincide with the streamlines.
5. A body is called streamlined body when it is placed in a flow and the surface of the body
   (a) Is rough.
   (b) Is smooth.
   (c) Coincides with the streamlines.
   (d) Does not coincide with the streamlines.
6. The lift coefficient (C_L) for an airfoil in terms of angle of attack (α) is equal to
   (a) $2\pi^2\sin\alpha$.
   (b) $2\pi\sin\alpha$.
   (c) $2\pi\sin^2\alpha$.
   (d) $2\sin\alpha$.

7. The velocity at the top of a spinning ball is
(a) Equal to that at the bottom.
(b) Independent of spinning.
(c) Less than at the bottom.
(d) Greater than at the bottom.

8. The tangential velocity (u_α) of ideal fluid at any point on the surface of the cylinder in terms of velocity (U) and angle (α) is equal to
(a) $(U \sin \alpha)/2$.
(b) $U^2 \sin \alpha$.
(c) $2U \sin \alpha$.
(d) None of the above.

9. Pressure drag results from
(a) Breakdown of flow near the forward stagnation point.
(b) Skin friction.
(c) Occurrence of waves setup during motion.
(d) Occurrence of wake.

10. For a sphere falling at terminal velocity in the Stokes law range, the drag coefficient (C_D) in terms of Reynolds number (Re) is equal to
(a) $24\,\text{Re}$.
(b) $\text{Re}/24$.
(c) $24/\text{Re}$.
(d) None of the above.

Review Questions

1. What do you mean by drag force and lift force of an object submerged in a fluid?

2. Briefly discuss the types of drag.

3. Derive expressions for the drag and lift force acting on a stationary object immersed in a moving fluid with a velocity U.

4. Differentiate between a streamlined body and a bluff body.

5. Give the expression for drag on a sphere when the Reynolds number (Re) is up to 0.2. Also derive the expression for coefficient of drag for the sphere as $C_D = (24/\text{Re})$ for the given range of Reynolds number.

6. Define terminal velocity of a body.

7. Discuss the variation of drag coefficient for a cylinder over a wide range of Reynolds number.

8. Derive an expression for the lift produced on a rotating cylinder placed in a uniform flow field such that the axis of the cylinder is perpendicular to the direction of flow. Also derive an expression for the lift coefficient.

9. Define stagnation points. How its position for a rotating cylinder in a uniform flow is determined? Also give the conditions for single stagnation point.

10. Discuss the basic terminology for an airfoil. Also derive an expression for the lift coefficient of an airfoil.

Problems

1. The experiments were performed in a wind tunnel with a wind speed of 40 km/hr on a flat plate of dimensions 2 m long and 1 m wide. If the coefficients of drag and lift are 0.15 and 0.75, respectively and the density of air is 1.15 kg/m^3, then determine (i) the drag force, (ii) lift force, (iii) resultant force and its direction and (iv) power exerted by air on the plate.

[**Ans.** 21.29 N, 106.46 N, 108.57 N, 78.69°, 236.53 W]

2. If the density of air is 1.2 kg/m^3 and drag coefficient is 1.25, then find the diameter of a hemispherical parachute used for dropping an object of mass 90 kg with a speed of 5 m/s.

[**Ans.** 7.743 m]

3. A car having frontal projected area of 2 m^2 travels at a speed of 72 km/hr. If the coefficient of drag is 0.38, then determine the power required to overcome wind resistance by the car. Also determine the change in speed of the car if the power developed remains same and the drag coefficient is reduced from 0.38 to 0.32 by streamlining the car. Take density of air as 1.2 kg/m^3.

[**Ans.** 3.648 kW, 76.248 km/hr]

4. A kite weighing 9.8 N and having an area of 1 m^2 makes an angle of 7° to the horizontal when flying in a wind speed of 36 km/hr. If the pull on the string attached to the kite is 49 N and it is inclined to the horizontal at 45°, then determine the lift and drag coefficients when the density of air is 1.2 kg/m^3.

[**Ans.** 0.577, 0.741]

5. A small jet plane engine has a mechanical efficiency of 0.6 and develops 5.1 MW power. The plane has a wing area of 20 m^2 and it weighs 25 kN. If the specific weight of the air is 12 N/m^3 and the plane flies at 900 km/hr speed, then determine the lift and drag coefficients.

[**Ans.** 0.0327, 0.0267]

6. A light plane has a wing span of 10 m and a chord of 2 m for the aerofoil section. What are the lift and drag forces during take-off at a speed of 194.4 km/hr, if the coefficients of lift and drag are 0.9 and 0.07, respectively. Take density of air as 1.25 kg/m^3.

[**Ans.** 32.805 kN, 2.5515 kN]

7. A ball of diameter 6 cm is supported in a vertical air stream which is flowing at a velocity of 6 m/s. If the density and kinematic viscosity of air are 1.25 kg/m^3 and 1.4 stokes, respectively, then determine the weight of the ball.
[**Ans.** 0.03184 N]

8. A metallic ball of diameter 0.004 m drops in a fluid of specific gravity 0.9 and viscosity 15 poise. If the density of ball material is 12000 kg/m^3, then determine (i) the drag force on the ball, (ii) pressure drag and skin drag and (iii) terminal velocity of the ball.
[**Ans.** 0.003649 N, 0.00122 N, 0.00243 N, 0.0645 m/s]

9. A steel ball of diameter 4 cm and of specific gravity 8.5 is dropped in water. If the coefficient of drag is 0.5, then determine the terminal velocity of the ball in water.
[**Ans.** 2.8 m/s]

10. A cylinder of diameter 1.6 m and length 12 m rotates at 242 rpm with its axis perpendicular to the stream of water flowing at a velocity of 16 m/s. Find the circulation, theoretical lift, position of stagnation points and the rpm of the cylinder for a single stagnation point.
[**Ans.** 101.89 m^2/s, 19.56×10^6 N, 219.3° and 320.7°, 381.97 rpm]

11. The air flows with a velocity of 40 m/s over a cylinder of diameter 1.4 m and length 10 m. The cylinder develops a lift of 6.8 kN/m when it rotates about its axis. If the cylinder is perpendicular to the air stream and the density of air is 1.2 kg/m^3, then determine the speed of rotation and the location of stagnation points.
[**Ans.** 439.4 rpm, 203.74°, 336.26°]

12. The span and chord of a rectangular wing of an aeroplane are 10 m and 1.6 m, respectively. The lift force acting on the wing during a horizontal flight at 198 km/hr is 25 kN. Determine the coefficients of drag and lift, weight of the aeroplane and power required for the flight if the lift drag ratio is 10 and density of air is 1.2 kg/m^3.
[**Ans.** 0.086, 0.86, 25 kN, 137.5 kW]

13. The angle of attack for an airfoil of chord length 2.2 m and span 16 m is 7°. Find the weight of the airfoil and the power required to drive if it moves with a velocity of 75 m/s. Take the density of air as 1.24 kg/m^3, the coefficients of lift and drag as 0.6 and 0.04, respectively.
[**Ans.** 73656 N, 368.28 kW]

14. A jet plane weighing 29500 N having a wing area of 20 m^2 flies at a velocity of 252 km/hr. If the engine delivers 7500 kW and 65% of the power is used to overcome the drag resistance of the wing, then determine the coefficients of lift and drag. Take density of air as 1.2 kg/m^3.
[**Ans.** 0.502, 1.184]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (b)	**2.** (c)	**3.** (c)	**4.** (d)	**5.** (c)
6. (b)	**7.** (d)	**8.** (c)	**9.** (c)	**10.** (c)

Chapter 17

Compressible Fluid Flow

17.1 ❒ INTRODUCTION

Compressible flow is defined as the flow in which the density of fluid does not remain constant. In earlier chapters, the density of the fluid flow was assumed to be constant and this is true for all liquids which are considered incompressible. However, in the case of flow of gases, the density changes with pressure and temperature. Therefore, the basic equations which are developed earlier on the assumption of constant density cannot be applied to compressible fluid flow. Some of the examples where compressible flow occurs are flow of gases in orifices and nozzles, gas flow in turbines, compressors and in rockets, flight of aircraft and projectiles with very high speed at high altitude, water hammer problems, etc.

In compressible flow, the thermodynamic behaviour of fluid is to be considered, since the change in density of fluid is always accompanied by changes in pressure and temperature. Compressibility becomes predominant when velocity becomes equal to or more than the velocity of sound in the fluid medium. In this chapter, the basic equations have been developed by considering the change of density in fluid flow and it is applied for the analysis of the fluid flowing through many devices. The effect of changing density of the flowing fluid in the analysis of fluid through nozzles, rotary machines and projectiles is also discussed in this chapter.

17.2 ❒ CONTINUITY EQUATION

According to the principle of conservation of mass, in one-dimensional steady flow, the mass flow rate is constant and the expression is given below.

$$\rho AV = C \tag{17.1}$$

Here, ρ be the mass density of fluid, A be the area of cross section of passage, V be the velocity of fluid and C be the constant.

Differentiating Equation (17.1), we get:

$$d(\rho AV) = 0$$

$$\rho d(AV) + AVd\rho = 0$$

$$\rho AdV + \rho VdA + AVd\rho = 0$$

Dividing by ρAV, we get:

$$\boxed{\frac{dV}{V} + \frac{dA}{A} + \frac{d\rho}{\rho} = 0} \tag{17.2}$$

The Equation (17.2) is the differential form of the continuity equation in compressible flow.

17.3 ❒ BERNOULLI'S EQUATION (ENERGY EQUATION)

For steady compressible fluid flow, the Euler's equation can be obtained as derived for incompressible flow in Chapter 7 and it is given below.

$$\frac{dp}{\rho} + gdz + VdV = 0$$

Integrating the above equation, we get:

$$\int \frac{dp}{\rho} + \int gdz + \int VdV = C, \text{ here } C \text{ is a constant}$$

$$\boxed{\int \frac{dp}{\rho} + gz + \frac{V^2}{2} = C} \tag{17.3}$$

In the case of compressible flow, ρ is not a constant and thus, it cannot be taken outside the integration sign of Equation (17.3). For compressible fluids, the pressure (p) changes with change in density (ρ) and it depends upon the type of process. Therefore, the Bernoulli's equation will be different for isothermal process and for adiabatic process as derived below.

17.3.1 Bernoulli's Equation for Isothermal Process

For an isothermal process, we have,

$$\frac{p}{\rho} = k \text{ or } \rho = \frac{p}{k}, \text{ here } k \text{ is a constant}$$

Thus
$$\int \frac{dp}{\rho} = \int \frac{dp}{(p/k)} = k \ln p = \frac{p}{\rho} \ln p$$

Substituting the value $\int \frac{dp}{\rho}$ in Equation (17.3), we get:

$$\frac{p}{\rho} \ln p + gz + \frac{V^2}{2} = C$$

Dividing both sides by g, we get:

$$\frac{p}{\rho g} \ln p + z + \frac{V^2}{2g} = C_1, \text{ here } C_1 = \text{Constant} \tag{17.4}$$

The Equation (17.4) is the Bernoulli's equation for compressible flow undergoing isothermal process.

17.3.2 Bernoulli's Equation for Adiabatic Process

For an adiabatic process, we have,

$$\frac{p}{\rho^\gamma} = k \text{ or } \rho = \left(\frac{p}{k}\right)^{1/\gamma}, \text{ here } k = \text{Constant}$$

Thus
$$\int \frac{dp}{\rho} = \int \frac{dp}{(p/k)^{1/\gamma}} = k^{1/\gamma} \int p^{-1/\gamma} dp = \left(\frac{p}{\rho^\gamma}\right)^{1/\gamma} \times \frac{p^{(-1/\gamma)+1}}{(-1/\gamma)+1}$$

$$\int \frac{dp}{\rho} = \left(\frac{p}{\rho^\gamma}\right)^{1/\gamma} \times \frac{\gamma}{\gamma - 1} p^{(\gamma-1)/\gamma} = \left(\frac{\gamma}{\gamma - 1}\right) \frac{p}{\rho}$$

Substituting the value $\int \frac{dp}{\rho}$ in Equation (17.3), we get:

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p}{\rho}+gz+\frac{V^2}{2}=C$$

Dividing both sides by g, we get:

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p}{\rho g}+z+\frac{V^2}{2g}=C_1,\text{ here } C_1=\text{Constant} \tag{17.5}$$

The Equation (17.5) is the Bernoulli's equation for compressible flow undergoing adiabatic process.

Example 17.1 A gas flows isothermally through a horizontal pipe with a mass flow rate of 0.5 kg/s. The cross-sectional area, temperature and pressure at section 1–1 is observed to be 0.004 m^2, 20°C and 4 bar (abs), respectively. If at section 2–2, the area of cross-section and pressure are measured as 0.002 m^2 and 3 bar (abs), respectively, then determine the velocities of the gas at the given sections assuming $R = 287$ J/kg K.

Solution

Let $m = 0.5$ kg/s, $A_1 = 0.004$ m^2, $T_1 = T_2 = 20°C = 20 + 273.15 = 293.15$ K , $p_1 = 4$ bar, $A_2 = 0.002$ m^2, $p_2 = 3$ bar and $R = 287$ J/kg K.

$$\rho_1=\frac{p_1}{RT_1}=\frac{4\times10^5}{287\times293.15}=4.754\text{ kg/m}^3$$

$$V_1=\frac{m}{\rho_1 A_1}=\frac{0.5}{4.754\times0.004}=\mathbf{26.294\ m/s}$$

$$\rho_2=\frac{p_2}{RT_2}=\frac{3\times10^5}{287\times293.15}=3.566\text{ kg/m}^3$$

$$V_2=\frac{m}{\rho_2 A_2}=\frac{0.5}{3.566\times0.002}=\mathbf{70.11\ m/s}$$

Example 17.2 Air flows with a velocity of 320 m/s adiabatically through a horizontal pipe. The pressure and temperature at section 1–1 are observed to be 0.78 bar (abs) and 40°C, respectively. The pipe changes in diameter and at this section 2–2, the pressure is measured as 1.17 bar (abs), respectively. Determine the velocity of air at section 2–2 assuming $R = 287$ J/kg K and $\gamma = 1.4$.

Solution

Let $V_1 = 320$ m/s, $p_1 = 0.78$ bar, $T_1 = 40°C = 40 + 273.15 = 313.15$ K, $p_2 = 1.17$ bar , $R = 287$ J/kg K and $\gamma = 1.4$.

$$\frac{p_1}{\rho_1}=RT_1=287\times313.15=89874.05$$

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1 g}+z_1+\frac{V_1^2}{2g}=\left(\frac{\gamma}{\gamma-1}\right)\frac{p_2}{\rho_2 g}+z_2+\frac{V_2^2}{2g}\quad\text{[Bernoulli's equation]}$$

$$\frac{V_2^2}{2g}=\left(\frac{\gamma}{\gamma-1}\right)\left(\frac{p_1}{\rho_1 g}-\frac{p_2}{\rho_2 g}\right)+\frac{V_1^2}{2g}\qquad[\because z_1=z_2]$$

$$\frac{V_2^2}{2}=\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left(1-\frac{p_2}{p_1}\frac{\rho_1}{\rho_2}\right)+\frac{V_1^2}{2} \quad \text{(i)}$$

Since $$\frac{\rho_1}{\rho_2}=\left(\frac{p_1}{p_2}\right)^{1/\gamma} \quad \text{(ii)}$$

Thus $$\frac{V_2^2}{2}=\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-\frac{p_2}{p_1}\left(\frac{p_1}{p_2}\right)^{1/\gamma}\right]+\frac{V_1^2}{2} \quad \text{[Substitute (ii) in (i)]}$$

$$V_2^2=2\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{1-\frac{1}{\gamma}}\right]+V_1^2$$

Thus $$V_2^2=2\times\left(\frac{1.4}{1.4-1}\right)\times 89874.05\times\left[1-\left(\frac{1.17}{0.78}\right)^{1-\frac{1}{1.4}}\right]+320^2$$

$$\therefore V_2=\sqrt{25129}=\mathbf{158.52\ m/s}$$

17.4 ❐ VELOCITY OF SOUND IN A FLUID MEDIUM

Each molecule of a fluid has to travel through a small distance to transmit the disturbance to the next molecule. The distance between the molecules depends upon the elastic properties (density and pressure) of the fluid. When pressure at a point of any fluid is changed, the new pressure of the fluid is transmitted to the rest of the fluid with a velocity called pressure wave velocity and it travels with the velocity of sound (or sonic velocity). In order to find an expression for the pressure wave velocity or velocity of sound in the fluid medium, consider a long rigid tube of cross section area A fitted with a piston at one end and containing a compressible fluid initially at rest as illustrated in Figure 17.1. If the piston suddenly moves, then a pressure wave propagates through the fluid with sonic velocity.

Let V_p be the velocity of the piston, C be the pressure wave velocity or velocity of sound in the fluid medium in time dt, $dx = V_p\,dt$ be the distance travelled by the piston, $dL = Cdt$ be the distance travelled by the pressure wave, p be the pressure of the fluid at rest, $(p + dp)$ be the pressure after compression, ρ be the density of the fluid at rest and $(\rho + d\rho)$ be

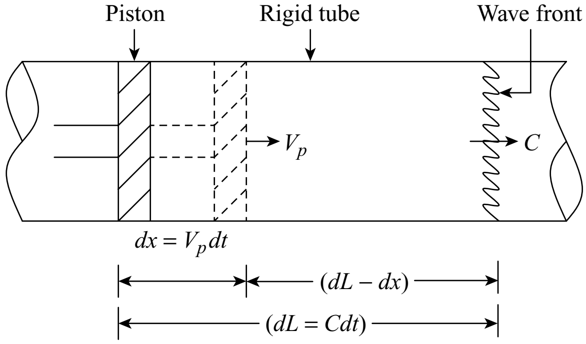


Figure 17.1 *Pressure wave propagation*

the density after compression. Applying continuity equation, i.e., mass of the fluid in the tube before compression will be equal to the mass of fluid after compression.

$$\rho(AdL) = (\rho + d\rho)[A(dL - dx)]$$

$$\rho dL = (\rho + d\rho)(dL - dx)$$

$$\rho Cdt = (\rho + d\rho)(Cdt - V_p dt) = (\rho + d\rho)(C - V_p)dt$$

$$\rho C = \rho C - \rho V_p + Cd\rho - V_p d\rho$$

$$Cd\rho - \rho V_p - V_p d\rho = 0 \qquad \text{(i)}$$

Since $C >> V_p$, neglecting the term $(V_p d\rho)$ and expression (i) becomes,

$$Cd\rho = \rho V_p$$

$$C = \frac{\rho V_p}{d\rho} \qquad \text{(ii)}$$

When piston moves with velocity V_p for time dt, the fluid which was initially at rest also moves with the same velocity V_p and pressure in the fluid increases from p to $(p + dp)$. Applying impulse-momentum equation, i.e., force on the compressed fluid will be equal to the rate of change of momentum.

$$(p + dp)A - pA = \text{Mass per second} \times \text{Change of velocity}$$

$$dpA = \frac{\rho AdL}{dt}(V_p - 0)$$

$$dp = \frac{\rho dL}{dt}V_p = \frac{\rho(Cdt)}{dt}V_p = \rho CV_p \qquad [\because dL = Cdt]$$

$$C = \frac{dp}{\rho V_p} \qquad \text{(iii)}$$

Multiplying expressions (ii) and (iii), we get:

$$C^2 = \frac{\rho V_p}{d\rho} \times \frac{dp}{\rho V_p} = \frac{dp}{d\rho}$$

Thus

$$\boxed{C = \sqrt{\frac{dp}{d\rho}}} \qquad (17.6)$$

Thus, Equation (17.6) gives the velocity of sound wave in a fluid medium.

17.4.1 Velocity of Sound in Terms of Bulk Modulus

The mathematical expression for bulk modulus of elasticity (K) is given by Equation (1.22) as follows.

$$K = \frac{\text{Change in pressure}}{\left(\frac{\text{Change in volume}}{\text{Original volume}}\right)} = -\frac{dp}{\left(\frac{dv}{v}\right)} \qquad \text{(iv)}$$

We know that mass of a fluid is constant and thus,

$$\rho v = k, \text{ here } k = \text{constant}$$

Differentiating on both sides, we get:

$$\rho dv + v d\rho = 0 \Rightarrow \rho dv = -v d\rho$$

$$-\frac{dv}{v} = \frac{d\rho}{\rho} \qquad \text{(v)}$$

Substituting expression (v) in expression (iv), we get:

$$K = \frac{dp}{(d\rho/\rho)} \Rightarrow \frac{dp}{d\rho} = \frac{K}{\rho}$$

From Equation (17.6), we get:

$$\boxed{C = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{K}{\rho}}} \qquad (17.7)$$

The Equation (17.7) gives the velocity of sound wave in terms of bulk modulus of elasticity and density.

17.4.2 Velocity of Sound for Isothermal Process

For isothermal process, we know that,

$$\frac{p}{\rho} = k \text{ or } p\rho^{-1} = k, \text{ here } k = \text{Constant}$$

Differentiating, we get:

$$p(-1)\rho^{-2} d\rho + \rho^{-1} dp = 0$$

$$-p\rho^{-1} d\rho + dp = 0$$

$$\frac{dp}{d\rho} = \frac{p}{\rho}$$

$$\frac{dp}{d\rho} = \frac{p}{\rho} = RT \qquad \text{[Using Equation 1.11(a)]}$$

Substituting the above value in Equation (17.7), we get:

$$\boxed{C = \sqrt{\frac{p}{\rho}} = \sqrt{RT}} \qquad (17.8)$$

17.4.3 Velocity of Sound for Adiabatic Process

For adiabatic process, we know that,

$$\frac{p}{\rho^{\gamma}} = k \text{ or } p\rho^{-\gamma} = k, \text{ here } k = \text{Constant}$$

Differentiating, we get:

$$p(-\gamma)\rho^{-\gamma-1} d\rho + \rho^{-\gamma} dp = 0$$

$$-p\gamma\rho^{-1} d\rho + dp = 0$$

$$\frac{dp}{d\rho} = \frac{p\gamma}{\rho}$$

$$\frac{dp}{d\rho} = \gamma RT \qquad \text{[Using Equation 1.11(a)]}$$

Substituting this value $(dp/d\rho) = \gamma RT$ in Equation (17.7), we get:

$$\boxed{C = \sqrt{\gamma RT}} \tag{17.9}$$

Generally, for sound wave propagation in air, the process is considered to be adiabatic, since the velocity of the pressure wave is very high and there is no appreciable heat transfer.

17.5 ❐ MACH NUMBER

Mach number (M) is defined as the square root of the ratio of the inertia force (F_i) to the elastic force (F_e). The mathematical expression for Mach number is given below.

$$M = \sqrt{\frac{F_i}{F_e}} = \sqrt{\frac{\rho L^2 V^2}{KL^2}} = \sqrt{\frac{V^2}{K/\rho}} = \frac{V}{\sqrt{K/\rho}} = \frac{V}{C} = \frac{\text{Speed of flow}}{\text{Speed of sound}} \tag{17.10}$$

Here, $C = \sqrt{K/\rho}$ represents the velocity of sound in that fluid medium whose properties K and ρ are being taken. The value of the sound in air at room temperature at sea level is 346 m/s. At a smaller value of Mach number (less than 0.3 for gas flows), the compressibility effect is neglected. Therefore, the compressibility effects of air can be neglected at speeds below 100 m/s. For analysis of systems that involve high speed gas flows, such as rockets and spacecrafts, the flow speed is generally expressed in terms of Mach number. A flow is called sonic when $M = 1$, subsonic when $M < 1$, supersonic when $M > 1$ and hypersonic when $M >> 1$.

Example 17.3 An aeroplane is flying at a height of 15 km where temperature is −43°C. Find the speed of the plane corresponding to Mach number equal to 1.8. Assuming $R = 287$ J/kg K and $\gamma = 1.4$.

Solution

Let $h = 15$ km, $T = -43°\text{C} = -43 + 273.15 = 230.15$ K, $M = 1.8$, $R = 287$ J/kg K and $\gamma = 1.4$.

Assume adiabatic process, we get:

$$C = \sqrt{\gamma RT} = \sqrt{1.4 \times 287 \times 230.15} = 304.09 \text{ m/s}$$

Since
$$M = \frac{V}{C}$$

Thus
$$1.8 = \frac{V}{304.09}$$

$$\therefore V = 1.8 \times 304.09 = 547.362 \text{ m/s}$$

$$V = \frac{547.362 \times 3600}{1000} = \mathbf{1970.5032 \text{ km/hr}}$$

17.6 ❐ PROPAGATION OF PRESSURE WAVE IN A COMPRESSIBLE FLUID

A disturbance is created when an object moves in a stationary compressible fluid. This disturbance generates elastic or pressure waves which are transmitted radially in all directions with a velocity equal to that of sound in the compressible fluid. In order to study the pattern of pressure waves, consider a tiny projectile which moves in a straight line with a velocity V through the stationary compressible fluid. Let at time $t = 0$, the object is at position A, then in time t it will move through a distance AB equal to Vt. Let C be the velocity of sound in the compressible fluid medium. Thus, during time t, the wave

(1st disturbance) which originated at position A will grow into the surface of sphere of radius $Ct = a$ (assume) as shown in Figure 17.2. The growth of the other waves (2nd disturbance, 3rd disturbance and 4th disturbance) originated from the object at every $(t/4)$ interval of time for radii $(3/4)Ct = b$ (assume), $(1/2)Ct = c$ (assume) and $(1/4)Ct = d$ (assume) as the object moves from A to B is also illustrated in Figure 17.2.

Different pressure wave patterns will develop depending upon the magnitude of the Mach number (M) as discussed below.

Case I: When $M < 1$ (or $V < C$): In this case as $V < C$, the pressure waves travel ahead of the object towards position B $(AB < Ct)$. Thus, the object at point B is inside the sphere of radius a and also inside the other spheres formed by the waves started at intermediate positions of radii b, c and d as shown in Figure 17.2(a).

Case II: When $M = 1$ (or $V = C$): In this case as $V = C$, both the pressure wave and the object reaches point B at the same instant of time $(AB = Ct)$. Thus, the object at point B lies on the periphery of the sphere of radius Ct. Also for the waves started at intermediate positions, the point B will lie on the periphery of the other spheres of radii b, c and d as shown in Figure 17.2(b).

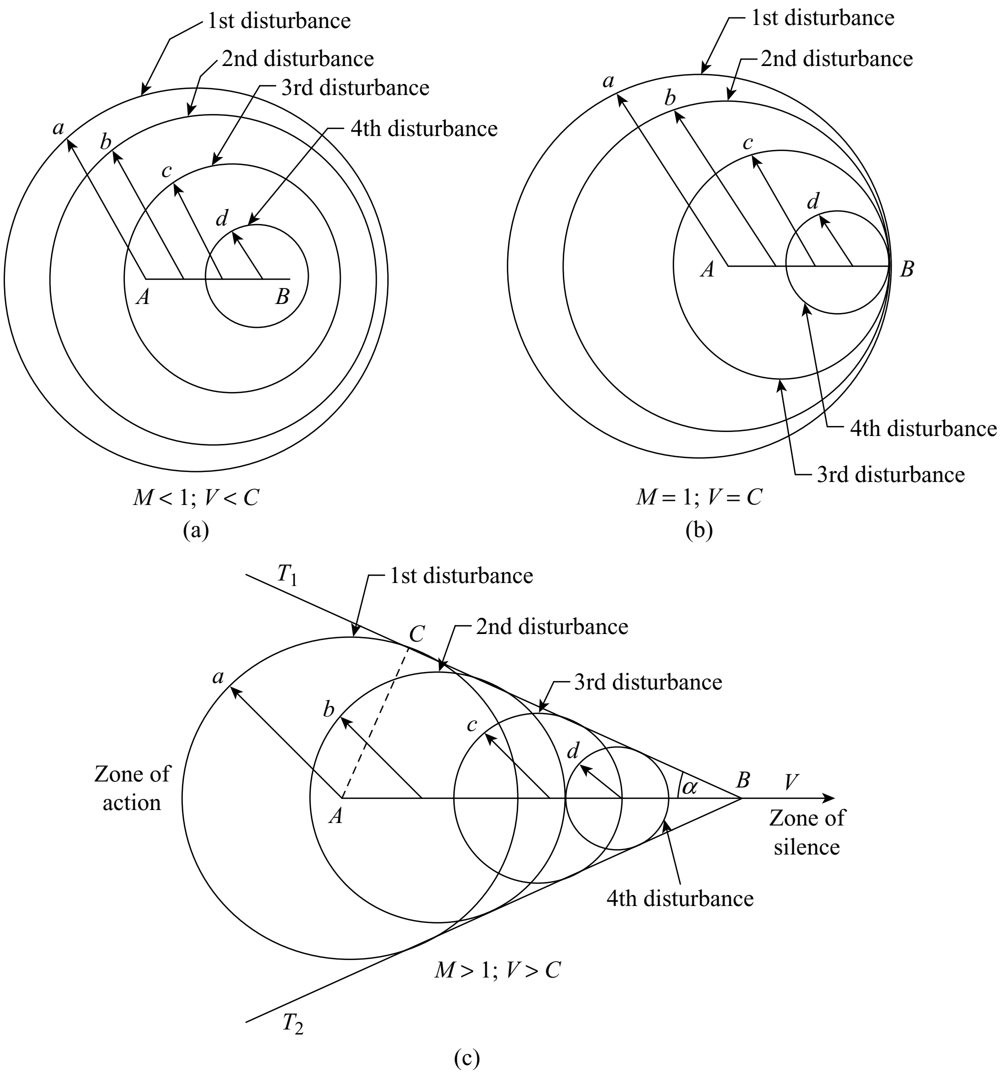


Figure 17.2 *Pressure wave propagation in a compressible fluid*

Case III: When $M > 1$ (or $V > C$): In this case as $V > C$, the object moves faster than the pressure wave. Hence, the distance $AB > Ct$ and thus, the point B remains outside the spheres of radii a, b, c and d at any instant of time as shown in Figure 17.2(c). When common tangents BT_1 and BT_2 are drawn, they become tangents to all the spheres of radii a, b, c and d. These common tangents form a cone (conical wave front) at point B which is termed as Mach cone. The semi-vertex angle (α) of this cone is called the Mach angle and the expression is given below.

$$\sin\alpha = \frac{AC}{AB} = \frac{Ct}{Vt} = \frac{C}{V} = \frac{1}{(V/C)} = \frac{1}{M} \tag{17.11}$$

The lines BT_1 and BT_2 are called as Mach line. The region inside the Mach cone is known as zone of action. The region (or atmosphere) outside the cone is termed as zone of silence and thus, the sound of jet plane is heard after it passes forward. In this case, the body travels faster than the message and reaches at point B unannounced. Fluid ahead of the Mach cone is undisturbed, but it suddenly undergoes changes in pressure, temperature and density as it passes through the cone. This sudden change in pressure, temperature and density is known as shock wave.

Example 17.4 Determine the velocity of bullet fired in standard air if the Mach angle is 35°. Take other data for air as temperature is 12°C, $R = 287$ J/kg K and $\gamma = 1.4$.

Solution

Let $\alpha = 35°$, $T = 12°\text{C} = 12 + 273.15 = 285.15\ \text{K}$, $R = 287$ J/kg K and $\gamma = 1.4$.

Assume adiabatic process, we get:

$$C = \sqrt{\gamma RT} = \sqrt{1.4 \times 287 \times 285.15} = 338.49 \text{ m/s}$$

Since $$\sin\alpha = \frac{C}{V}$$

Thus $$\sin 35° = \frac{338.4}{V}$$

$$\therefore V = \frac{338.4}{\sin 35°} = \mathbf{589.9824\ m/s}$$

Example 17.5 A projectile travels in air having pressure and temperature as 88.3 kPa and –2°C. If the Mach angle is 40°, then find the velocity of the projectile. Take $R = 287$ J/kg K and $\gamma = 1.4$.

Solution

Let $p = 88.3$ kPa, $T = -2°\text{C} = -2 + 273.15 = 271.15\ \text{K}$, $\alpha = 40°$, $R = 287$ J/kg K and $\gamma = 1.4$.

Assume adiabatic process, we get:

$$C = \sqrt{\gamma RT} = \sqrt{1.4 \times 287 \times 271.15} = 330.073 \text{ m/s}$$

Since $$\sin\alpha = \frac{C}{V}$$

Thus $$\sin 40° = \frac{330.073}{V}$$

$$\therefore V = \frac{330.073}{\sin 40°} = \mathbf{513.5\ m/s}$$

17.7 ❐ STAGNATION PROPERTIES

The point in a fluid stream where the velocity of flow becomes zero and kinetic energy is converted into pressure energy is called stagnation point. At this point, the values of pressure, density and temperature are called stagnation pressure (p_s), stagnation density (ρ_s) and stagnation temperature (T_s), respectively.

17.7.1 Stagnation Pressure

Let a compressible fluid flow past an immersed body under frictionless adiabatic (isentropic) conditions and two points 1 and 2 lie on a streamline as shown in Figure 17.3.

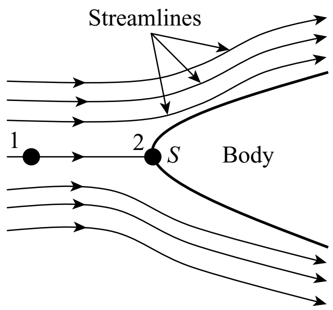


Figure 17.3 *Stagnation properties*

Let us consider that point 1 is far away from the body and point S is the stagnation point where the flow velocity is zero. Let V_1, p_1 and ρ_1 be the velocity, pressure and density, respectively, of the approaching fluid at point 1 and V_2, p_2 and ρ_2 be the corresponding quantities at the stagnation point, i.e., at point 2. Applying Bernoulli's equation for adiabatic flow at point 1 and 2, we get the below expression.

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1 g}+z_1+\frac{V_1^2}{2g}=\left(\frac{\gamma}{\gamma-1}\right)\frac{p_2}{\rho_2 g}+z_2+\frac{V_2^2}{2g}$$

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}+\frac{V_1^2}{2}=\left(\frac{\gamma}{\gamma-1}\right)\frac{p_2}{\rho_2}+\frac{V_2^2}{2} \quad [\because z_1 = z_2]$$

At stagnation point 2, $V_2 = 0$, $p_2 = p_s$, $\rho_2 = \rho_s$ and thus, the above expression becomes,

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}+\frac{V_1^2}{2}=\left(\frac{\gamma}{\gamma-1}\right)\frac{p_s}{\rho_s}$$

$$\left(\frac{\gamma}{\gamma-1}\right)\left(\frac{p_1}{\rho_1}-\frac{p_s}{\rho_s}\right)=-\frac{V_1^2}{2}$$

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left(1-\frac{p_s}{p_1}\frac{\rho_1}{\rho_s}\right)=-\frac{V_1^2}{2}$$

Now $$\frac{p_1}{\rho_1^{\gamma}}=\frac{p_s}{\rho_s^{\gamma}} \text{ or } \frac{\rho_1}{\rho_s}=\left(\frac{p_1}{p_s}\right)^{1/\gamma} \qquad [\because (p/\rho^{\gamma})=k]$$

Thus $$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-\frac{p_s}{p_1}\left(\frac{p_1}{p_s}\right)^{1/\gamma}\right]=-\frac{V_1^2}{2}$$

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-\frac{p_s}{p_1}\left(\frac{p_s}{p_1}\right)^{-1/\gamma}\right]=-\frac{V_1^2}{2}$$

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-\left(\frac{p_s}{p_1}\right)^{1-\frac{1}{\gamma}}\right]=-\frac{V_1^2}{2}$$

$$\left[1-\left(\frac{p_s}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right] = -\frac{V_1^2}{2}\left(\frac{\gamma-1}{\gamma}\right)\frac{\rho_1}{p_1}$$

$$\left(\frac{p_s}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = 1+\frac{V_1^2}{2}\left(\frac{\gamma-1}{\gamma}\right)\frac{\rho_1}{p_1}$$

Now $$C = \sqrt{\gamma RT} = \sqrt{\gamma \frac{p}{\rho}} \qquad [\because RT = p/\rho]$$

Thus $$C_1 = \sqrt{\gamma \frac{p_1}{\rho_1}} \text{ or } C_1^2 = \gamma \frac{p_1}{\rho_1}$$

Thus $$\left(\frac{p_s}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = 1+\frac{V_1^2}{2}(\gamma-1)\frac{1}{C_1^2}$$

$$\left(\frac{p_s}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = 1+\frac{M_1^2}{2}(\gamma-1) \qquad [\because (V_1^2/C_1^2) = M_1^2]$$

$$\frac{p_s}{p_1} = \left[1+\frac{(\gamma-1)}{2}M_1^2\right]^{\frac{\gamma}{\gamma-1}} \tag{17.12}$$

Thus $$\boxed{p_s = p_1\left[1+\frac{(\gamma-1)}{2}M_1^2\right]^{\frac{\gamma}{\gamma-1}}} \tag{17.13}$$

In Equation (17.13), for Mach number less than one, the term $[(\gamma-1)M_1^2]/2$ will be less than one and thus, the R.H.S. of this equation can be expanded by using the Binomial theorem as given below.

$$p_s = p_1\left[1+\left(\frac{\gamma}{\gamma-1}\right)\frac{(\gamma-1)}{2}M_1^2+\frac{1}{2}\left(\frac{\gamma}{\gamma-1}\right)\left(\frac{\gamma}{\gamma-1}-1\right)\left(\frac{\gamma-1}{2}M_1^2\right)^2+\ldots\right]$$

$$p_s = p_1\left[1+\frac{\gamma}{2}M_1^2+\frac{\gamma}{8}M_1^4+\ldots\right] = p_1 + p_1\left[\frac{\gamma}{2}M_1^2+\frac{\gamma}{8}M_1^4+\ldots\right]$$

$$\frac{p_s-p_1}{p_1} = \left[\frac{\gamma}{2}M_1^2+\frac{\gamma}{8}M_1^4+\ldots\right] = \frac{\gamma}{2}M_1^2\left[1+\frac{1}{4}M_1^2+\ldots\right]$$

If one more term is considered in expansion, then we have,

$$\frac{p_s-p_1}{p_1} = \frac{\gamma}{2}M_1^2\left[1+\frac{1}{4}M_1^2+\frac{(2-\gamma)}{24}M_1^4+\ldots\right]$$

$$\frac{p_s-p_1}{p_1} = \frac{\gamma}{2}\frac{V_1^2}{C_1^2}\left[1+\frac{1}{4}M_1^2+\frac{(2-\gamma)}{24}M_1^4+\ldots\right] \qquad [\because M_1^2 = V_1^2/C_1^2]$$

$$\frac{p_s - p_1}{p_1} = \frac{\gamma}{2}\frac{V_1^2}{(\gamma p_1)/\rho_1}\left[1 + \frac{1}{4}M_1^2 + \frac{(2-\gamma)}{24}M_1^4 + \dots\right] \quad [\because C_1^2 = (\gamma p_1)/\rho_1]$$

$$p_s - p_1 = \frac{1}{2}\rho_1 V_1^2\left[1 + \frac{1}{4}M_1^2 + \frac{(2-\gamma)}{24}M_1^4 + \dots\right] \tag{17.14}$$

$$\therefore p_s = p_1 + \frac{1}{2}\rho_1 V_1^2\left[1 + \frac{1}{4}M_1^2 + \frac{(2-\gamma)}{24}M_1^4 + \dots\right] \tag{17.15}$$

In Equation (17.15), the bracketed term may be considered as a compressibility factor. In this equation for small value of V_1 compared to C_1, the value of M_1 will be very small and thus, the bracketed term (i.e., compressibility factor) will be nearly equal to 1. Therefore, this equation reduces as given below.

$$\boxed{p_s = p_1 + \frac{1}{2}\rho_1 V_1^2} \tag{17.16}$$

The velocity at a point in an incompressible fluid measured by a pitot tube can be given by $V = \sqrt{2gh}$, here h is the difference in two heads. However, if we have to measure the velocity at any point in a compressible fluid, then the actual pressure difference has to be multiplied by a compressibility correction factor or compressibility factor for obtaining the correct velocity at that point. The expression for compressibility correction factor (CCF) is given below.

$$\boxed{CCF = \left[1 + \frac{1}{4}M_1^2 + \frac{(2-\gamma)}{24}M_1^4 + \dots\right]}$$

17.7.2 Stagnation Density

$$\frac{p_1}{\rho_1^\gamma} = \frac{p_s}{\rho_s^\gamma} \quad \text{or} \quad \frac{\rho_s}{\rho_1} = \left(\frac{p_s}{p_1}\right)^{1/\gamma} \qquad [\because (p/\rho^\gamma) = k]$$

$$\rho_s = \rho_1\left(\frac{p_s}{p_1}\right)^{1/\gamma}$$

Substituting the value of (p_s/p_1) from Equation (17.12) in the above expression, we get:

$$\rho_s = \rho_1\left[\left(1 + \frac{(\gamma-1)}{2}M_1^2\right)^{\frac{\gamma}{\gamma-1}}\right]^{1/\gamma} = \rho_1\left[1 + \frac{(\gamma-1)}{2}M_1^2\right]^{\frac{1}{\gamma-1}} \tag{17.17}$$

17.7.3 Stagnation Temperature

For stagnation point, the equation of state is as follows.

$$\frac{p_s}{\rho_s} = RT_s \qquad [\because (p/\rho) = RT]$$

$$T_s = \frac{p_s}{R\rho_s}$$

Substituting the value of p_s and ρ_s from Equations (17.13) and (17.17) in the above expression, we get:

$$T_s = \frac{1}{R}\frac{p_1\left[1+\frac{(\gamma-1)}{2}M_1^2\right]^{\frac{\gamma}{\gamma-1}}}{\rho_1\left[1+\frac{(\gamma-1)}{2}M_1^2\right]^{\frac{1}{\gamma-1}}} = \frac{1}{R}\frac{p_1}{\rho_1}\left[1+\frac{(\gamma-1)}{2}M_1^2\right]^{\frac{\gamma}{\gamma-1}-\frac{1}{\gamma-1}}$$

$$= \frac{1}{R}\frac{p_1}{\rho_1}\left[1+\frac{(\gamma-1)}{2}M_1^2\right] = \frac{1}{R}RT_1\left[1+\frac{(\gamma-1)}{2}M_1^2\right] \quad [\because (p_1/\rho_1) = RT_1]$$

Thus
$$\boxed{T_s = T_1\left[1+\frac{(\gamma-1)}{2}M_1^2\right]} \tag{17.18}$$

Example 17.6 For an aeroplane flying at 820.8 km/hour through still air having a pressure of 80 kPa and temperature −9°C, determine the stagnation pressure, temperature and density at the nose of the plane if $R = 287$ J/kg K and $\gamma = 1.4$.

Solution

Let $V_1 = 820.8$ km/hr, $p = 80$ kPa, $T = T_1 = -9°C = -9 + 273.15 = 264.15$ K, $R = 287$ J/kg K and $\gamma = 1.4$.

$$V_1 = \frac{820.8\times1000}{3600} = 228 \text{ m/s}$$

Assume adiabatic process, we get:

$$C_1 = \sqrt{\gamma RT} = \sqrt{1.4\times287\times264.15} = 325.784 \text{ m/s}$$

$$M_1 = \frac{V_1}{C_1} = \frac{228}{325.784} = 0.7$$

$$p_s = p_1\left[1+\frac{(\gamma-1)}{2}M_1^2\right]^{\frac{\gamma}{\gamma-1}} = 80\times\left[1+\frac{(1.4-1)}{2}\times0.7^2\right]^{\frac{1.4}{1.4-1}} = \mathbf{110.97\ kPa}$$

$$T_s = T_1\left[1+\frac{(\gamma-1)}{2}M_1^2\right] = 264.15\times\left[1+\frac{(1.4-1)}{2}\times0.7^2\right] = \mathbf{290.04\ K}$$

$$\rho_s = \frac{p_s}{RT_s} = \frac{110.97\times10^3}{287\times290.04} = \mathbf{1.333\ kg/m^3}$$

Example 17.7 Determine the stagnation pressure for air at a pressure of 215 kPa and temperature 25°C moving at a velocity of 200 m/s if (i) compressibility is considered and (ii) compressibility is neglected. Take $R = 287$ J/kg K and $\gamma = 1.4$.

Solution

Let $p_1 = 215$ kPa, $T_1 = 25°C = 25 + 273.15 = 298.15$ K, $V_1 = 200$ m/s, $R = 287$ J/kg K and $\gamma = 1.4$.

(i) When compressibility is considered, we get:

$$M_1 = \frac{V_1}{\sqrt{\gamma RT_1}} = \frac{200}{\sqrt{1.4\times287\times298.15}} = 0.578$$

$$\rho_1 = \frac{p_1}{RT_1} = \frac{215\times10^3}{287\times298.15} = 2.513 \text{ kg/m}^3$$

Since $$p_s = p_1 + \frac{\rho_1 V_1^2}{2}\left[1 + \frac{M_1^2}{4} + \frac{(2-\gamma)M_1^4}{24} + \ldots\right]$$

Thus $$p_s = 215 + \frac{2.513\times200^2}{2}\times10^{-3}\times\left[1 + \frac{0.578^2}{4} + \frac{(2-1.4)\times0.578^4}{24} + \ldots\right]$$

$$\therefore p_s = \mathbf{269.598 \text{ kPa}}$$

(ii) When compressibility is neglected, the stagnation pressure is given by,

$$p_s = p_1 + \frac{1}{2}\rho_1 V_1^2 = 215 + \frac{2.513\times200^2}{2}\times10^{-3} = \mathbf{265.26 \text{ kPa}}$$

17.8 ❐ AREA AND VELOCITY RELATIONSHIP FOR COMPRESSIBLE FLOW

The continuity equation for compressible flow is given by,

$$\frac{dV}{V} + \frac{dA}{A} + \frac{d\rho}{\rho} = 0 \quad \text{[Equation (17.2)]}$$

The Euler's equation for compressible flow is given by,

$$\frac{dp}{\rho} + gdz + VdV = 0$$

Assuming horizontal flow and neglecting the z term, we get:

$$\frac{dp}{\rho} + VdV = 0$$

Dividing and multiplying by $d\rho$, we get:

$$\frac{dp}{d\rho}\frac{d\rho}{\rho} + VdV = 0$$

But $$\frac{dp}{d\rho} = C^2 \quad \text{[From Equation (17.6)]}$$

Thus $$C^2\frac{d\rho}{\rho} + VdV = 0$$

$$\frac{d\rho}{\rho} = -\frac{VdV}{C^2}$$

Substituting this value of $(d\rho/\rho)$ in Equation (17.2), we get:

$$\frac{dV}{V} + \frac{dA}{A} - \frac{VdV}{C^2} = 0$$

$$\frac{dA}{A} = \frac{VdV}{C^2} - \frac{dV}{V} = \frac{dV}{V}\left[\frac{V^2}{C^2} - 1\right]$$

$$\therefore \frac{dA}{A} = \frac{dV}{V}\left[M^2 - 1\right] \tag{17.19}$$

The following conclusions can be drawn from Equation (17.19).

Case I: Accelerated flow This type of flow takes place in nozzles. Along the flow direction, pressure decreases and velocity increases. Based on the velocity of the fluid, two conditions arise for change in cross section.

(i) When $M < 1$ (i.e., $V < C$) and $(dA/A) = -ve$: For subsonic flow, the nozzle must be convergent nozzle and these nozzles are called subsonic nozzles. However, when $M = 1$, (i.e., $V = C$) and $(dA/A) = 0$, the throat of the nozzle is reached and the flow is sonic.

(ii) When $M > 1$ (i.e., $V > C$) and $(dA/A) = +ve$: For supersonic flow, the nozzle must be divergent nozzle. These nozzles are called supersonic nozzles or diverging nozzles.

Case II: Decelerated (retarded) flow This type of flow takes place in diffusers. The diffuser converts kinetic energy into pressure energy. Along the flow direction, pressure increases and velocity decreases. Based on the velocity of the fluid, again two conditions arise for change in cross section.

(i) When $M < 1$ (i.e., $V < C$) and $(dA/A) = +ve$: For subsonic flow, the diffuser must be divergent. However, when $M = 1$ (i.e., $V = C$) and $(dA / A) = 0$, the throat of the diffuser is reached and the flow is sonic.

(ii) When $M > 1$ (i.e., $V > C$) and $(dA/A) = -ve$: For supersonic flow, the diffuser must be convergent.

17.9 ❐ COMPRESSIBLE FLUID FLOW THROUGH A CONVERGENT NOZZLE

Figure 17.4 illustrates a large tank containing a compressible fluid fitted with a small convergent nozzle. Assuming that there is large pressure drop during the flow through the nozzle and the process is adiabatic.

Consider two points 1 and 2 inside the tank and exit of the nozzle, respectively. Let p_1, V_1, T_1 and ρ_1 be the pressure, velocity, temperature and density of fluid at point 1 and p_2, V_2, T_2 and ρ_2 be the corresponding values at point 2. Applying Bernoulli's equation for adiabatic flow at point 1 and 2, we get the below expression.

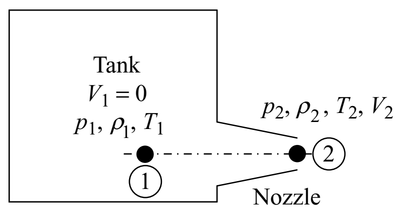


Figure 17.4 *Fluid flow through a convergent nozzle*

$$\left(\frac{\gamma}{\gamma - 1}\right)\frac{p_1}{\rho_1 g} + z_1 + \frac{V_1^2}{2g} = \left(\frac{\gamma}{\gamma - 1}\right)\frac{p_2}{\rho_2 g} + z_2 + \frac{V_2^2}{2g}$$

$$\left(\frac{\gamma}{\gamma - 1}\right)\frac{p_1}{\rho_1} = \left(\frac{\gamma}{\gamma - 1}\right)\frac{p_2}{\rho_2} + \frac{V_2^2}{2} \qquad [\because z_1 = z_2 \text{ and } V_1 = 0]$$

$$\frac{V_2^2}{2} = \left(\frac{\gamma}{\gamma - 1}\right)\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right)$$

Thus $$V_2 = \sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\left(\frac{p_1}{\rho_1}-\frac{p_2}{\rho_2}\right)} = \sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left(1-\frac{p_2}{\rho_2}\frac{\rho_1}{p_1}\right)} \quad (17.20)$$

Since $$\frac{\rho_1}{\rho_2} = \left(\frac{p_1}{p_2}\right)^{1/\gamma} \quad (i)$$

Thus $$V_2 = \sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-\frac{p_2}{p_1}\left(\frac{p_1}{p_2}\right)^{1/\gamma}\right]}$$

$$\therefore V_2 = \sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right]} \quad (17.21)$$

Let $(p_2/p_1) = p_r$ be the pressure ratio, then the above expression becomes,

$$V_2 = \sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-(p_r)^{\frac{\gamma-1}{\gamma}}\right]} \quad (17.21a)$$

The mass flow rate (m) of fluid is given by,

$$m = \rho_2 A_2 V_2 = \rho_2 A_2\sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-(p_r)^{\frac{\gamma-1}{\gamma}}\right]}$$

Thus $$m = A_2\sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}{\rho_2}^2\left[1-(p_r)^{\frac{\gamma-1}{\gamma}}\right]} \quad (ii)$$

From expression (i), we get:

$$\rho_2 = \rho_1\left(\frac{p_2}{p_1}\right)^{1/\gamma} = \rho_1\,(p_r)^{1/\gamma}$$

$${\rho_2}^2 = {\rho_1}^2\,(p_r)^{2/\gamma}$$

Substituting this value of ${\rho_2}^2$ in expression (ii), we get:

$$m = A_2\sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}{\rho_1}^2\,(p_r)^{\frac{2}{\gamma}}\left[1-(p_r)^{\frac{\gamma-1}{\gamma}}\right]}$$

$$m = A_2\sqrt{\left(\frac{2\gamma}{\gamma-1}\right)p_1\rho_1\left[(p_r)^{\frac{2}{\gamma}}-(p_r)^{\frac{\gamma+1}{\gamma}}\right]} \quad (17.22)$$

In Equation (17.22), all the other quantities except p_r are constant and the expression for maximum value of the mass flow rate (m_{max}) is given below.

$$\frac{dm}{dp_r} = 0$$

or
$$\frac{d}{dp_r}\left[(p_r)^{\frac{2}{\gamma}}-(p_r)^{\frac{\gamma+1}{\gamma}}\right]=0$$

$$\frac{2}{\gamma}(p_r)^{\frac{2}{\gamma}-1}-\frac{\gamma+1}{\gamma}(p_r)^{\frac{\gamma+1}{\gamma}-1}=0\Rightarrow(p_r)^{\frac{2-\gamma}{\gamma}}=\frac{\gamma+1}{2}(p_r)^{\frac{1}{\gamma}}$$

$$(p_r)^{2-\gamma}=\left(\frac{\gamma+1}{2}\right)^{\gamma}p_r\Rightarrow(p_r)^{2-\gamma-1}=\left(\frac{\gamma+1}{2}\right)^{\gamma}$$

$$(p_r)^{1-\gamma}=\left(\frac{\gamma+1}{2}\right)^{\gamma}\Rightarrow(p_r)^{\gamma-1}=\left(\frac{2}{\gamma+1}\right)^{\gamma}$$

$$\therefore p_r=\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \tag{17.23}$$

Thus, for obtaining $m_{\max}$, substituting Equation (17.23) in Equation (17.22), we get:

$$m_{\max}=A_2\sqrt{\left(\frac{2\gamma}{\gamma-1}\right)p_1\rho_1\left[\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}\times\frac{2}{\gamma}}-\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}\times\frac{\gamma+1}{\gamma}}\right]}$$

$$m_{\max}=A_2\sqrt{\left(\frac{2\gamma}{\gamma-1}\right)p_1\rho_1\left[\left(\frac{2}{\gamma+1}\right)^{\frac{2}{\gamma-1}}-\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\right]}$$

For air, $\gamma=1.4$ and thus, the value of $m_{\max}$ can be given by,

$$m_{\max}=A_2\sqrt{\left(\frac{2\times1.4}{1.4-1}\right)p_1\rho_1\left[\left(\frac{2}{1.4+1}\right)^{\frac{2}{1.4-1}}-\left(\frac{2}{1.4+1}\right)^{\frac{1.4+1}{1.4-1}}\right]}$$

$$\boxed{\therefore m_{\max}=0.685A_2\sqrt{p_1\rho_1}} \tag{17.24}$$

and
$$p_r=\frac{p_2}{p_1}=\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}=\left(\frac{2}{1.4+1}\right)^{\frac{1.4}{1.4-1}}=0.528 \tag{17.25}$$

Now for obtaining velocity at the outlet of nozzle for maximum flow rate of fluid, substituting Equation (17.23) in Equation (17.21(a)), we get the below expression.

$$V_2=\sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}\times\frac{\gamma-1}{\gamma}}\right]}=\sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-\frac{2}{\gamma+1}\right]}$$

$$=\sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left(\frac{\gamma-1}{\gamma+1}\right)}=\sqrt{\left(\frac{2\gamma}{\gamma+1}\right)\frac{p_1}{\rho_1}}$$

$$= \sqrt{\left(\frac{2\gamma}{\gamma+1}\right)\frac{p_1}{p_2}\frac{p_2}{\rho_2}\frac{\rho_2}{\rho_1}} = \sqrt{\left(\frac{2\gamma}{\gamma+1}\right)\frac{p_2}{\rho_2}\left(\frac{p_2}{p_1}\right)^{-1}\left(\frac{p_2}{p_1}\right)^{1/\gamma}}$$

$$= \sqrt{\left(\frac{2\gamma}{\gamma+1}\right)\frac{p_2}{\rho_2}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}-1}} = \sqrt{\left(\frac{2\gamma}{\gamma+1}\right)\frac{p_2}{\rho_2}(p_r)^{\frac{1-\gamma}{\gamma}}}$$

$$= \sqrt{\left(\frac{2\gamma}{\gamma+1}\right)\frac{p_2}{\rho_2}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}\times\frac{1-\gamma}{\gamma}}} \quad \text{[Substitute Equation (17.23)]}$$

$$\therefore V_2 = \sqrt{\left(\frac{2\gamma}{\gamma+1}\right)\frac{p_2}{\rho_2}\left(\frac{2}{\gamma+1}\right)^{-1}} = \sqrt{\gamma\frac{p_2}{\rho_2}} = \sqrt{\gamma R T_2} = C_2 \qquad (17.26)$$

The Equation (17.26) indicates that the velocity at the outlet of the nozzle for maximum flow rate is equal to the sound velocity.

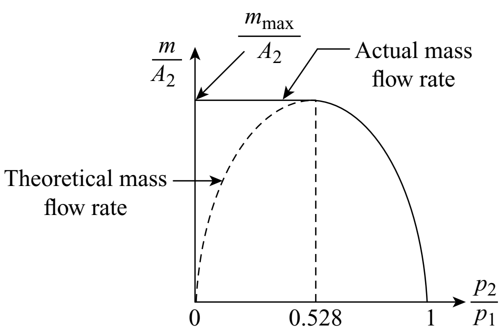


Figure 17.5 *Variation of mass flow rate versus pressure ratio*

A passage in which the sonic velocity has been reached and the mass flow rate is maximum is called choked passage or passage operating under choking conditions.

A plot of the mass flow rate per unit area (m/A_2) versus pressure ratio (p_2/p_1) for air is shown in Figure 17.5. The pressure ratio at which the mass rate of flow is maximum (m_{max}/A_2) is called the critical pressure ratio which is equal to 0.528. It can be seen that when the pressure ratio is more than 0.528, the mass flow rate decreases. It can also be seen from Figure 17.5 that the theoretical mass flow rate decreases for the values (p_2/p_1) < 0.528. However, in actual terms, it remains constant and would always be equal to the mass flow rate corresponding to pressure ratio equal to 0.528. This is because the critical conditions (for example, sonic velocity) are reached at throat and any further lowering of the exit pressure is not felt upstream, where it is being in the zone of silence.

Example 17.8 Determine the velocity of air flowing through a nozzle of diameter 10 mm at the outlet fitted to a large tank containing air at a pressure of 30 bar (abs) and at a temperature of 25°C. The pressure at the exit of the nozzle is 20 bar (abs). Also determine the maximum flow rate of air if the atmospheric pressure is 1 bar. Take $R = 287$ J/kg K and $\gamma = 1.4$.

Solution

Let $d = 10$ mm $= 0.01$ m, $p_1 = 30$ bar, $T_1 = 25°C = 25 + 273.15 = 298.15$ K, $p_2 = 20$ bar, $p_{atm} = 1$ bar, $R = 287$ J/kg K and $\gamma = 1.4$.

$$\rho_1 = \frac{p_1}{RT_1} = \frac{30\times10^5}{287\times298.15} = 35.06 \text{ kg/m}^3$$

Since

$$V_2 = \sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\times\left[1-\left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right]}$$

$$\therefore V_2 = \sqrt{\left(\frac{2\times1.4}{1.4-1}\right)\times\frac{30\times10^5}{35.06}\times\left[1-\left(\frac{20}{30}\right)^{(1.4-1)/1.4}\right]} = \mathbf{255.97\ m/s}$$

$$A_2 = \frac{\pi}{4} d_2^{\,2} = \frac{\pi}{4} \times 0.01^2 = 0.00007854 \text{ m}^2 \qquad [d_2 = d]$$

Since $$m_{max} = 0.685 A_2 \sqrt{p_1 \rho_1}$$

$$\therefore m_{max} = 0.685 \times 0.00007854 \times \sqrt{30 \times 10^5 \times 35.06} = \mathbf{0.55176\ kg/s}$$

Example 17.9 Air discharges through a convergent nozzle of tip diameter 25 mm fitted to a large vessel. The barometric pressure is 101 kPa. The pressure and temperature in the vessel is 700 kPa and 40°C, respectively. Determine the flow rate when the pressure outside the jet is 200 kPa. Also determine the pressure, temperature, velocity and sonic velocity at the nozzle tip for the given flow rate. Take $R = 287$ J/kg K and $\gamma = 1.4$.

Solution

Let $d = 25$ mm $= 0.025$ m, $p_{atm} = 101$ kPa, $p_1 = (700 + 101) = 801$ kPa, $T_1 = 40°C = 40 + 273.15 = 313.15$ K, $p_2 = (200 + 101) = 301$ kPa, $R = 287$ J/kg K and $\gamma = 1.4$.

$$A_2 = \frac{\pi}{4} d_2^{\,2} = \frac{\pi}{4} \times 0.025^2 = 0.000491 \text{ m}^2 \qquad [d_2 = d]$$

$$\rho_1 = \frac{p_1}{RT_1} = \frac{801 \times 10^3}{287 \times 313.15} = 8.9125 \text{ kg/m}^3$$

Since $$\frac{p_2}{p_1} = \frac{301}{801} = 0.376 < 0.528$$

Thus $$m_{max} = 0.685 A_2 \sqrt{p_1 \rho_1}$$

$$\therefore m_{max} = 0.685 \times 0.000491 \times \sqrt{801 \times 10^3 \times 8.9125} = \mathbf{0.8986\ kg/s}$$

$$p_2 = 0.528 \times p_1 = 0.528 \times 801 = \mathbf{422.928\ kPa}$$

Since $$V_2 = \sqrt{\left(\frac{2\gamma}{\gamma - 1}\right) \frac{p_1}{\rho_1} \times \left[1 - \left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right]}$$

$$\therefore V_2 = \sqrt{\left(\frac{2 \times 1.4}{1.4 - 1}\right) \times \frac{801 \times 10^3}{8.9125} \times \left[1 - \left(\frac{301}{801}\right)^{(1.4-1)/1.4}\right]} = \mathbf{391.753\ m/s}$$

$$T_2 = T_1 \left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma} = 313.15 \times \left(\frac{301}{801}\right)^{(1.4-1)/1.4} = \mathbf{236.76\ K}$$

17.10 ❐ COMPRESSIBLE FLUID FLOW THROUGH A VENTURIMETER

Consider an adiabatic flow of a compressible fluid through a horizontal venturimeter. Assume two points 1 and 2 at the inlet and throat of the venturimeter, respectively, as shown in Figure 17.6.

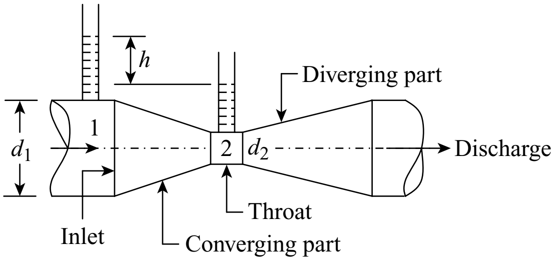


Figure 17.6 *Horizontal venturimeter carrying compressible fluid*

Let p_1, V_1, A_1 and ρ_1 be the pressure, velocity, area and density at point 1 and p_2, V_2, A_2 and ρ_2 be the corresponding values at point 2. Applying Bernoulli's equation for adiabatic flow at point 1 and 2, we get the below expression.

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1 g}+z_1+\frac{V_1^2}{2g}=\left(\frac{\gamma}{\gamma-1}\right)\frac{p_2}{\rho_2 g}+z_2+\frac{V_2^2}{2g}$$

$$\left(\frac{\gamma}{\gamma-1}\right)\left(\frac{p_1}{\rho_1}-\frac{p_2}{\rho_2}\right)=\frac{V_2^2}{2}-\frac{V_1^2}{2} \qquad [\because z_1=z_2]$$

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left(1-\frac{p_2}{\rho_2}\frac{\rho_1}{p_1}\right)=\frac{V_2^2}{2}-\frac{V_1^2}{2} \qquad \text{(i)}$$

Now $$\frac{\rho_1}{\rho_2}=\left(\frac{p_1}{p_2}\right)^{1/\gamma}=\left(\frac{p_2}{p_1}\right)^{-1/\gamma} \qquad [\because p/\rho^\gamma=k] \qquad \text{(ii)}$$

Substituting (ii) in (i), we get:

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-\frac{p_2}{p_1}\left(\frac{p_2}{p_1}\right)^{-1/\gamma}\right]=\frac{V_2^2}{2}-\frac{V_1^2}{2}$$

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{1-(1/\gamma)}\right]=\frac{V_2^2}{2}-\frac{V_1^2}{2}$$

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right]=\frac{V_2^2}{2}-\frac{V_1^2}{2} \qquad \text{(iii)}$$

Now $$V_1=\frac{\rho_2 A_2 V_2}{\rho_1 A_1} \qquad [\because \rho_1 A_1 V_1=\rho_2 A_2 V_2]$$

Thus $$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right]=\frac{V_2^2}{2}-\frac{1}{2}\left(\frac{\rho_2 A_2 V_2}{\rho_1 A_1}\right)^2$$

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right]=\frac{V_2^2}{2}\left[1-\frac{\rho_2^2}{\rho_1^2}\frac{A_2^2}{A_1^2}\right]$$

Again $$\frac{\rho_2}{\rho_1}=\left(\frac{p_2}{p_1}\right)^{1/\gamma} \text{ or } \frac{\rho_2^2}{\rho_1^2}=\left(\frac{p_2}{p_1}\right)^{2/\gamma} \qquad [\because p/\rho^\gamma=k]$$

Thus $$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right]=\frac{V_2^2}{2}\left[1-\left(\frac{p_2}{p_1}\right)^{2/\gamma}\frac{A_2^2}{A_1^2}\right]$$

$$V_2^2 = \left(\frac{2\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right]\Bigg/\left[1-\left(\frac{p_2}{p_1}\right)^{2/\gamma}\frac{A_2^2}{A_1^2}\right]$$

$$\therefore V_2 = \sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right]\Bigg/\left[1-\left(\frac{p_2}{p_1}\right)^{2/\gamma}\frac{A_2^2}{A_1^2}\right]}$$

Thus, the mass flow rate (m) through the venturimeter is given as $m = \rho_2 A_2 V_2$.

$$\therefore m = \rho_2 A_2\sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right]\Bigg/\left[1-\left(\frac{p_2}{p_1}\right)^{2/\gamma}\frac{A_2^2}{A_1^2}\right]} \tag{17.27}$$

In Equation (17.27), the value of pressure ratio (p_2/p_1) must be greater than the critical value (i.e., 0.528) for the flow.

Example 17.10 The inlet and throat diameters of a horizontal venturimeter are 0.32 m and 0.16 m, respectively. The pressure and temperature of air at the inlet section of the venturimeter are 140 kPa (abs) and 17°C, respectively. If pressure at the throat section is 130 kPa, then determine the mass flow rate of air through the venturimeter. Take $R = 287$ J/kg K and $\gamma = 1.4$.

Solution

Let $D_1 = 0.32$ m, $D_2 = 0.16$ m, $p_1 = 140$ kPa, $T_1 = 17°C = 290.15$ K, $p_2 = 130$ kPa, $R = 287$ J/kg K and $\gamma = 1.4$.

$$A_1 = \frac{\pi}{4}D_1^2 = \frac{\pi}{4}\times 0.32^2 = 0.080425 \text{ m}^2$$

$$A_2 = \frac{\pi}{4}D_2^2 = \frac{\pi}{4}\times 0.16^2 = 0.020106 \text{ m}^2$$

$$\rho_1 = \frac{p_1}{RT_1} = \frac{140\times 10^3}{287\times 290.15} = 1.6812 \text{ kg/m}^3$$

$$\rho_2 = \rho_1\left(\frac{p_2}{p_1}\right)^{1/\gamma} = 1.6812\times\left(\frac{130}{140}\right)^{1/1.4} = 1.5945 \text{ kg/m}^3$$

Since

$$m = \rho_2 A_2\sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right]\Bigg/\left[1-\left(\frac{p_2}{p_1}\right)^{2/\gamma}\frac{A_2^2}{A_1^2}\right]}$$

Thus

$$m = 1.5945\times 0.020106\times\sqrt{\frac{\left(\frac{2\times 1.4}{1.4-1}\right)\times\frac{140\times 10^3}{1.6812}\times\left[1-\left(\frac{130}{140}\right)^{(1.4-1)/1.4}\right]}{\left[1-\left(\frac{130}{140}\right)^{2/1.4}\times\left(\frac{0.020106}{0.080425}\right)^2\right]}}$$

$$\therefore m = \mathbf{3.6469 \ kg/s}$$

17.11 ❐ SHOCK WAVES

The abrupt transformation of supersonic flow into subsonic flow causes a pressure wave called the shock wave. It results in a sudden rise in pressure, density, temperature and entropy, but a drop in Mach number and velocity across a shock. The shock wave has a finite thickness of the order of 10^{-3} mm in the atmospheric pressure. Shock waves may occur in the diverging section of a convergent-divergent nozzle, in pipes and in front of a blunt-nosed body. This process of the formation of shock waves in a compressible fluid is analogous to the formation of hydraulic jump in open channels. With respect to the direction of flow, the shock waves are of two types, namely normal shock waves and oblique shock waves. As the name indicates, the normal shocks are almost perpendicular to the flow, whereas oblique shocks are inclined to the flow direction.

17.11.1 Normal Shock Wave

Consider a control surface (in a uniform duct) that includes a normal shock (Figure 17.7). The fluid is assumed to be in thermodynamic equilibrium upstream and downstream of the shock and its properties are designated by the subscripts 1 and 2, respectively. Let the duct has constant flow area, i.e., $A_1 = A_2 = A$ and it is horizontal, i.e., $z_1 = z_2$.

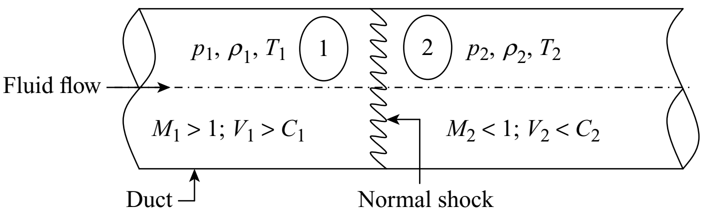


Figure 17.7 *Normal shock wave*

Let p_1 be the pressure, ρ_1 be the density, T_1 be the temperature, V_1 be the velocity and $M_1 > 1$ be the Mach number (supersonic flow) upstream of the shock and p_2, ρ_2, T_2, V_2 and $M_2 < 1$ (subsonic flow) be the corresponding values downstream of the shock. For the analysis of a normal shock wave continuity equation, momentum equation and energy equation have been considered.

For constant flow area, the continuity equation is written as follows.

$$\rho_1 V_1 = \rho_2 V_2 \tag{i}$$

Substitute $\rho = p/(RT)$ and $V = MC = M\sqrt{\gamma RT}$ in the above expression, we get:

$$\frac{p_1}{RT_1} M_1 \sqrt{\gamma RT_1} = \frac{p_2}{RT_2} M_2 \sqrt{\gamma RT_2}$$

Thus

$$\frac{p_1 M_1}{\sqrt{T_1}} = \frac{p_2 M_2}{\sqrt{T_2}} \tag{17.28}$$

When the effects of boundary friction are neglected, the momentum equation becomes,

$$(p_1 - p_2) = \rho_2 V_2^2 - \rho_1 V_1^2 \tag{ii}$$

$$p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2$$

$$p_1 + \frac{p_1}{RT_1} V_1^2 = p_2 + \frac{p_2}{RT_2} V_2^2 \Rightarrow p_1\left(1 + \frac{V_1^2}{RT_1}\right) = p_2\left(1 + \frac{V_2^2}{RT_2}\right)$$

Since $M = V/C$ and $C = \sqrt{\gamma RT}$, the above expression is written as follows.

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \tag{17.29}$$

Since for a shock $M_1 > 1$ and $M_2 < 1$ and from Equation (17.29), it can be observed that $p_2 > p_1$, i.e., static pressure increases across a shock wave.

When flow across the shock wave is considered adiabatic, the energy equation is written as follows.

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}+\frac{V_1^2}{2}=\left(\frac{\gamma}{\gamma-1}\right)\frac{p_2}{\rho_2}+\frac{V_2^2}{2} \qquad \text{(iii)}$$

By combining expressions (i) and (ii) and rearranging, we get:

$$p_1+\frac{(\rho_1 V_1)^2}{\rho_1}=p_2+\frac{(\rho_2 V_2)^2}{\rho_2} \qquad (17.30)$$

The Equation (17.30) is known as Rankine line equation.

By combining expressions (i) and (iii) and rearranging, we get:

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}+\frac{(\rho_1 V_1)^2}{2\rho_1^2}=\left(\frac{\gamma}{\gamma-1}\right)\frac{p_2}{\rho_2}+\frac{(\rho_2 V_2)^2}{2\rho_2^2} \qquad (17.31)$$

The Equation (17.31) is known as Fanno line equation.

By combining expressions (i), (ii) and (iii) and solving for (p_2/p_1) and (ρ_2/ρ_1), respectively, we get the following expressions.

$$\frac{p_2}{p_1}=\left[\left(\frac{\gamma+1}{\gamma-1}\right)\frac{\rho_2}{\rho_1}-1\right]\Bigg/\left[\left(\frac{\gamma+1}{\gamma-1}\right)-\frac{\rho_2}{\rho_1}\right] \qquad (17.32)$$

$$\frac{\rho_2}{\rho_1}=\frac{V_1}{V_2}=\left[1+\left(\frac{\gamma+1}{\gamma-1}\right)\frac{p_2}{p_1}\right]\Bigg/\left[\left(\frac{\gamma+1}{\gamma-1}\right)+\frac{p_2}{p_1}\right] \qquad (17.33)$$

The Equations (17.32) and (17.33) are called Rankine–Hugoniot equations.

In terms of Mach number, (T_2/T_1) can be expressed as follows.

$$\frac{T_2}{T_1}=\left[1+\frac{\gamma-1}{2}M_1^2\right]\Bigg/\left[1+\frac{\gamma-1}{2}M_2^2\right] \qquad (17.34)$$

The relationship for the Mach number upstream and downstream of a normal shock wave can be expressed as given below.

$$\boxed{M_2^2=\frac{(\gamma-1)M_1^2+2}{2\gamma M_1^2-(\gamma-1)}} \qquad (17.35)$$

The ratio of fluid properties in terms of Mach number upstream can be given by,

$$\frac{p_2}{p_1}=\frac{2\gamma M_1^2-(\gamma-1)}{\gamma+1} \qquad (17.36)$$

$$\frac{\rho_2}{\rho_1}=\frac{V_1}{V_2}=\frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2} \qquad (17.37)$$

$$\frac{T_2}{T_1}=\frac{[(\gamma-1)M_1^2+2][2\gamma M_1^2-(\gamma-1)]}{(\gamma+1)^2 M_1^2} \qquad (17.38)$$

The shock strength gives a measure of the strength of a shock and it can be defined as the ratio of the pressure rise across the shock to the upstream pressure. The mathematical expression for shock strength is given below.

$$\text{Shock strength} = \frac{p_2 - p_1}{p_1} = \frac{p_2}{p_1} - 1$$

Substitute Equation (17.36) in the above expression, we get:

$$\text{Shock strength} = \frac{2\gamma M_1^2 - (\gamma - 1)}{\gamma + 1} - 1 = \frac{2\gamma}{\gamma + 1}(M_1^2 - 1) \qquad (17.39)$$

17.11.2 Oblique Shock Wave

An oblique shock wave is one that is inclined with respect to the flow direction. Such a shock wave is formed when a supersonic flow is made to change direction near a sharp corner. The oblique shock wave in a two-dimensional flow over a concave corner is shown in Figure 17.8.

It can be seen that the streamlines after the shock remain parallel to each other but gets deflected by an angle α, i.e., angle made by the surface with original horizontal direction. The conical wave front (Mach cone) created by a supersonic body flowing through a stagnant fluid medium is called oblique shock wave. Since the losses for the case of oblique shock waves are much less than those of normal shock, a design is made sometimes in such a way that an oblique shock occurs instead of a normal shock. It is the main reason for making the nose angle of the fuselage of a supersonic aircraft small.

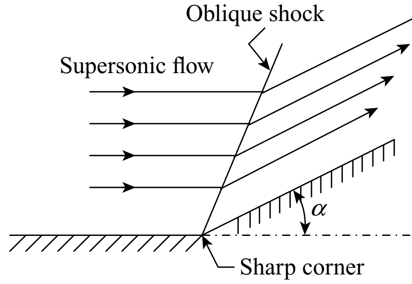


Figure 17.8 *Oblique shock wave over a concave corner*

Summary

1. In a compressible flow, the density of fluid does not remain constant.
2. **Differential form of the continuity equation in compressible flow:**
$$(dV/V) + (dA/A) + (d\rho/\rho) = 0$$
3. **Bernoulli's equation for isothermal process:**
$$\frac{p}{\rho g}\ln p + z + \frac{V^2}{2g} = C$$
4. **Bernoulli's equation for adiabatic process:**
$$\left(\frac{\gamma}{\gamma - 1}\right)\frac{p}{\rho g} + z + \frac{V^2}{2g} = C$$
5. **Velocity of sound wave in a fluid medium:**
   (i) $C = \sqrt{dp/d\rho} = \sqrt{K/\rho}$ (In terms of bulk modulus)
   (ii) $C = \sqrt{p/\rho} = \sqrt{RT}$ (For isothermal process)
   (iii) $C = \sqrt{\gamma RT}$ (For adiabatic process)
6. **Mach number (M):** $M = \dfrac{V}{C} = \dfrac{\text{Speed of flow}}{\text{Speed of sound}}$
   (i) For sonic flow: $M = 1$, (ii) for subsonic flow: $M < 1$, (iii) for supersonic flow: $M > 1$ and (iv) for hypersonic flow: $M >> 1$.
7. (i) When $M < 1$ (or $V < C$), the pressure waves travel ahead of the object, (ii) when $M = 1$ (or $V = C$), both the pressure wave and object reach at the same instant of time and (iii) when $M > 1$ (or $V > C$), the object moves faster than the pressure wave.
8. The point in a fluid stream where the velocity of flow becomes zero and the kinetic energy converts into pressure energy is called stagnation point. The stagnation pressure p_s, stagnation density ρ_s and stagnation temperature T_s are given below.
$$p_s = p_1\left[1 + \frac{(\gamma - 1)}{2}M_1^2\right]^{\frac{\gamma}{\gamma - 1}}, \ \rho_s = \rho_1\left[1 + \frac{(\gamma - 1)}{2}M_1^2\right]^{\frac{1}{\gamma - 1}},$$
$$T_s = T_1\left[1 + \frac{(\gamma - 1)}{2}M_1^2\right]$$

9. **Area velocity relationship for compressible flow:**

$$\frac{dA}{A} = \frac{dV}{V}\left[M^2 - 1\right]$$

10. **Velocity through a nozzle fitted to a large tank:**

$$V_2 = \sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right]}$$

$$= \sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-(p_r)^{(\gamma-1)/\gamma}\right]}$$

Here, p_1 and ρ_1 be the pressure and density inside the tank.

11. **Mass flow rate through the nozzle:**

$$m = A_2\sqrt{\left(\frac{2\gamma}{\gamma-1}\right)p_1\rho_1\left[(p_r)^{\frac{2}{\gamma}} - (p_r)^{\frac{\gamma+1}{\gamma}}\right]}$$

Here, A_2 be the area of the nozzle tip at its outlet.

12. **Maximum mass flow rate through the nozzle:**

$$m_{\max} = 0.685 A_2\sqrt{p_1\rho_1}$$

13. The velocity at the outlet of the nozzle for maximum flow rate is equal to the sound velocity.

14. **Mass flow rate (m) through the venturimeter:**

$$m = \rho_2 A_2\sqrt{\left(\frac{2\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right]\Big/\left[1-\left(\frac{p_2}{p_1}\right)^{2/\gamma}\frac{A_2^2}{A_1^2}\right]}$$

15. The abrupt change of supersonic flow into subsonic flow causes a pressure wave called the shock wave.

16. **Rankine line equation:** $p_1 + (\rho_1 V_1)^2/\rho_1 = p_2 + (\rho_2 V_2)^2/\rho_2$

17. **Fanno line equation:**

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1} + \frac{(\rho_1 V_1)^2}{2\rho_1^2} = \left(\frac{\gamma}{\gamma-1}\right)\frac{p_2}{\rho_2} + \frac{(\rho_2 V_2)^2}{2\rho_2^2}$$

18. **Rankine–Hugoniot equations:**

$$\frac{p_2}{p_1} = \left[\left(\frac{\gamma+1}{\gamma-1}\right)\frac{\rho_2}{\rho_1} - 1\right]\Big/\left[\left(\frac{\gamma+1}{\gamma-1}\right) - \frac{\rho_2}{\rho_1}\right] \text{ and}$$

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \left[1 + \left(\frac{\gamma+1}{\gamma-1}\right)\frac{p_2}{p_1}\right]\Big/\left[\left(\frac{\gamma+1}{\gamma-1}\right) + \frac{p_2}{p_1}\right]$$

19. Shock strength is the ratio of the pressure rise across the shock to the upstream pressure and it is equal to $2\gamma(M_1^2 - 1)/(\gamma+1)$.

Multiple-choice Questions

1. The compressibility effects can be considered negligible when Mach number is
   (a) 0.1.
   (b) 1.
   (c) 0.5.
   (d) 0.3.

2. The flow is said to be supersonic when Mach number is
   (a) Less than one.
   (b) Equal to one.
   (c) Greater than one.
   (d) None of the above.

3. The flow is said to be subsonic when Mach number is
   (a) Less than one.
   (b) Equal to one.
   (c) Greater than one.
   (d) None of the above.

4. The velocity of sound is large in
   (a) Steel.
   (b) Water.
   (c) Air.
   (d) Milk.

5. The speed of sound in air varies as
   (a) $\sqrt{p}$.
   (b) $\sqrt{\rho}$.
   (c) $\sqrt{T}$.
   (d) None of the above.

6. For accelerated flow, when $M < 1$, the nozzle is called
   (a) Sonic.
   (b) Subsonic.
   (c) Supersonic.
   (d) None of the above.

7. For accelerated flow when $M > 1$, the nozzle is called
   (a) Sonic.
   (b) Subsonic.
   (c) Supersonic.
   (d) None of the above.

8. The critical pressure ratio for adiabatic flow of air is equal to
   (a) 0.628.
   (b) 0.528.
   (c) 0.428.
   (d) 0.328.

Review Questions

1. What do you mean by compressible flow? When a liquid is treated as a compressible fluid? Also define Mach number.
2. Derive expressions for Bernoulli's equation for (i) isothermal process and (ii) adiabatic process.
3. Derive an expression for velocity of sound wave in a compressible fluid medium in terms of change of pressure and density.
4. Derive an expression for velocity of sound wave in a compressible fluid medium in terms of bulk modulus of elasticity and density.
5. Derive expressions for velocity of sound wave in a compressible fluid medium for (i) isothermal process and (ii) adiabatic process.
6. Explain the propagation of pressure wave in a compressible fluid with suitable diagrams when the Mach number is (i) less than one, (ii) equal to one and (iii) more than one.
7. Define stagnation pressure. Also derive an expression for the same in terms of approaching Mach number, pressure and specific heat ratio.
8. Derive an expression for stagnation density and stagnation temperature in terms of approaching Mach number, density and specific heat ratio.
9. Derive an expression for area velocity relationship for a compressible fluid in terms of area, velocity and Mach number.
10. Derive an expression for mass flow rate of a compressible fluid through a nozzle fitted to a large tank. Also give the conditions for maximum rate of flow.
11. Show that the velocity at the outlet of the nozzle for maximum flow rate is equal to the sound velocity.
12. Derive an expression for the maximum mass flow rate of a compressible fluid through a nozzle fitted to a large tank.
13. Derive an expression for the mass flow rate through a horizontal venturimeter.

Problems

1. A pipe of diameter 10 cm suddenly reduces to 5 cm . It carries air isothermally at 25°C. If the absolute pressures measured at the two sections just before and after the contraction are 4.75 bar and 3.8 bar, respectively, then determine the density and velocity at the two sections. Also determine the mass flow rate through the pipe. Take gas constant as 287 J/kg K.
[**Ans.** 5.551 kg/m^3, 4.441 kg/m^3, 39.89 m/s, 199.45 m/s, 1.74 kg/s]
2. Air with a velocity of 300 m/s flows through a horizontal pipe at a section where pressure is 78 kN/m^2 (abs) and temperature is 40°C. The pipe changes in its diameter and at this section, the pressure is 117 kN/m^2 (abs). Determine the velocity of air at this section if the flow of gas is adiabatic, the gas constant is 287 J/kg K and specific heat ratio is 1.4.
[**Ans.** 112.82 m/s]
3. The pressure, velocity and temperature at the upstream section of a horizontal pipe carrying air are 40 kN/m^2, 35 m/s and 147°C, respectively. Determine the pressure and the temperature at the downstream section if velocity at this section is 155 m/s and the process is adiabatic. Take the gas constant as 287 J/kg K and specific heat ratio as 1.4.
[**Ans.** 36.4 kN/m^2, 408.97 K]
4. Find the sonic velocity for the following given fluids, (i) crude oil of specific gravity 0.8 and bulk modulus 1.54 GN/m^2 and (ii) mercury having a bulk modulus of 27.2 GN/m^2.
[**Ans.** 1387.44 m/s, 1414.21 m/s]
5. An aeroplane is flying at a height of 15 km where temperature is −50°C. Find the speed of the plane corresponding to Mach number equal to 1.8. Take specific heat ratio as 1.4 and gas constant as 287 J/kg K.
[**Ans.** 1940.3 km/hr]
6. An aircraft is flying at a speed of 900 km/hr where air temperature is 7°C. Find the Mach number of the aircraft when specific heat ratio is 1.4 and gas constant is 287 J/kg K.
[**Ans.** 0.745]
7. A projectile travels in air having pressure and temperature as 1.011 bar and 10°C, respectively, at a speed of 1512 km/hr. Find the Mach number and Mach angle if $R = 287$ J/kg K and $\gamma = 1.4$.
[**Ans.** 1.245, 53.44°]
8. Find the velocity of a bullet fired in standard air if its Mach angle is 38° and temperature is 15°C. Take $R = 287$ J/kg K and $\gamma = 1.4$.
[**Ans.** 552.67 m/s]
9. Air has a velocity of 1000 km/hr at a pressure of 9.81 kPa (vacuum) and a temperature of 47°C. Determine its stagnation pressure, temperature, density and the local Mach number, if atmospheric pressure is 98.1 kPa, $R = 287$ J/kg K and $\gamma = 1.4$. Also determine the compressibility correction factor for a pitot-static tube to measure the velocity at a Mach number of 0.8.
[**Ans.** 131.26 kPa, 358.56 K, 1.2755 kg/m^3, 1.17]

10. An aeroplane flies at 849.6 km/hr through still air having a pressure of 78.5 kPa (abs) and temperature −8°C. Determine the stagnation properties if for air $R = 287$ J/kg K and $\gamma = 1.4$.

[**Ans.** 111.18 kPa, 292.87 K, 1.323 kg/m^3]

11. Air flows from a large vessel through a nozzle of diameter 25 mm fitted to the vessel. If the temperature of air is 30°C and its flow is adiabatic, then find the mass flow rate of air through the nozzle when pressure of air in the vessel is (i) 0.39 bar (gauge) and (ii) 3.35 bar (gauge). Take $R = 287$ J/kg K, $\gamma = 1.4$ and atmospheric pressure as 1.01 bar (abs).

[**Ans.** 0.1457 kg/s, 0.4969 kg/s]

12. Air flows from a large tank through a nozzle of diameter 20 mm fitted to it. If the pressure and temperature of air in the tank is 385 kPa (gauge) and 25°C, respectively, then find the maximum flow rate of air through the nozzle. Take $R = 287$ J/kg K, $\gamma = 1.4$ and atmospheric pressure as 100 kPa.

[**Ans.** 0.3568 kg/s]

13. A large tank containing air at a pressure of 2550 kPa (abs) and temperature of 22°C is fitted with a nozzle. Determine the velocity of air through the nozzle if the pressure of air at its exit is 1750 kPa, $R = 287$ J/kg K and $\gamma = 1.4$.

[**Ans.** 245.92 m/s]

14. A vessel containing air at a temperature of 22°C is fitted with a convergent nozzle of tip diameter 30 mm. Assuming adiabatic flow, determine the mass flow rate of air through the nozzle to the atmosphere when the pressure in the tank is (i) 150 kPa (abs) and (ii) 300 kPa. Take $R = 287$ J/kg K, $\gamma = 1.4$ and atmospheric pressure as 101.325 kPa.

[**Ans.** 0.2371 kg/s, 0.4991 kg/s]

15. The inlet and throat diameters of a horizontal venturimeter are 300 mm and 150 mm, respectively. The pressure and temperature of air at the inlet section of the venturimeter are 1.37 bar (abs) and 15°C, respectively. If pressure at the throat section is 1.27 bar (abs), then determine the mass flow rate of air through the venturimeter. Take $R = 287$ J/kg K and $\gamma = 1.4$.

[**Ans.** 3.179 kg/s]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (d)	**2.** (c)	**3.** (a)	**4.** (a)	**5.** (c)
6. (b)	**7.** (c)	**8.** (b)		

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Chapter **18**

Flow in Open Channels

18.1 ❐ INTRODUCTION

A channel may be considered as a passage through which water flows under atmospheric pressure. The flow of liquid (water) through a passage in which free liquid surface is open to atmosphere is called open channel flow. This type of flow is maintained due to gravity, i.e., flow takes place due to the downward slope of the channel bed. In case of flow of water through the pipe, there is no free surface as water flows under pressure. However, if the flow of water through a pipe is at atmospheric pressure or pipe does not run full, then the flow is considered similar to an open channel flow.

Generally, the cross section of pipes is circular and the flowing liquid fills the flow passages, whereas the open channels have a free surface open to atmosphere and may have any shape. Some of the examples of open channel flow are flow in canals, rivers and drainage channels. In this chapter, the basic concepts, which are used to describe and analyse flow in open channels pertaining to steady flow under uniform and non-uniform flow conditions are briefly discussed.

18.2 ❐ GEOMETRICAL PARAMETERS FOR OPEN CHANNELS

Some of the basic geometrical parameters of open channels are defined below (Figure 18.1).

1. **Depth of flow:** It is the vertical distance of the lowest point of a channel section from the free surface. It is denoted by y.
2. **Top width:** It is the width of the channel section at the free surface. It is denoted by T.
3. **Wetted area:** It is the cross-sectional area of the flow section of the channel. It is denoted by A.
4. **Wetted perimeter:** It is the length of the channel boundary in contact with the flowing water at any section. It is denoted by P.
5. **Hydraulic radius (or hydraulic mean depth):** It is the ratio of the wetted area to its wetted perimeter. It is denoted by m or R. Mathematically, it is given as $m = (A/P)$.
6. **Hydraulic depth:** It is the ratio of the wetted area to the top width. It is denoted by D. Mathematically, it is given as $D = (A/T)$.

18.3 ❐ TYPES OF FLOW IN OPEN CHANNELS

Depending upon the change in the depth of flow with respect to space and time, the flow in open channel can be classified as

(i) steady and unsteady flow,

(ii) uniform and non-uniform flow,

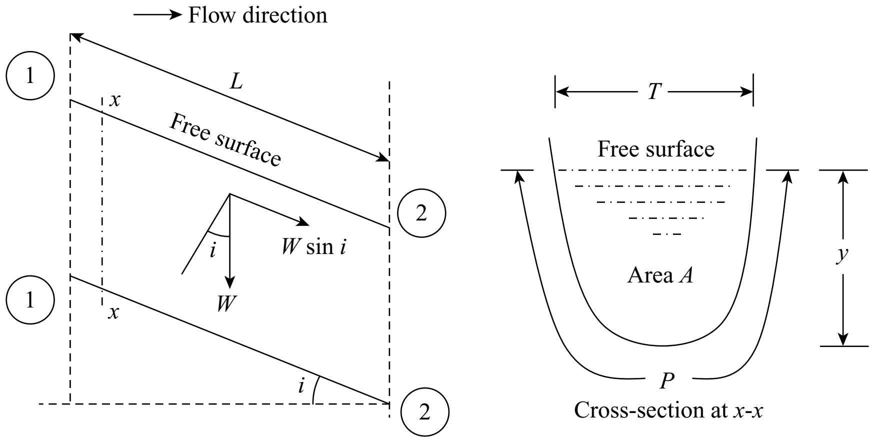

Figure 18.1 *Uniform flow in an open channel*

(iii) laminar and turbulent flow, and

(iv) sub-critical, critical and super critical flow. These various types of flows are discussed below.

1. **Steady and unsteady flow:** When the flow characteristics such as depth of flow (y), velocity of flow (V) and the rate of flow (Q) at any point in the flow do not change with respect to time (t), then the flow is called steady. The expression for steady and unsteady flow is mathematically given below.

$$\frac{\partial y}{\partial t} = 0, \frac{\partial V}{\partial t} = 0 \text{ and } \frac{\partial Q}{\partial t} = 0$$

However, when these flow parameters change with time, then the flow is called unsteady.

$$\frac{\partial y}{\partial t} \neq 0, \frac{\partial V}{\partial t} \neq 0 \text{ and } \frac{\partial Q}{\partial t} \neq 0$$

2. **Uniform and non-uniform flow:** When the depth (y), slope, cross section and the velocity of flow (V) for a given length of a channel (s or L) remain constant, then the flow is called uniform. The mathematical expression for uniform flow is given below.

$$\frac{\partial y}{\partial s} = 0 \text{ and } \frac{\partial V}{\partial s} = 0$$

However, when these parameters change along the length of the channel, the flow is said to be non-uniform. A non-uniform flow is also known as varied flow and its mathematical expression is given below.

$$\frac{\partial y}{\partial s} \neq 0 \text{ and } \frac{\partial V}{\partial s} \neq 0$$

Non-uniform flow can be classified as gradually varied flow and rapidly varied flow.

(a) **Gradually varied flow:** If the change in depth of flow is gradual over a long length of channel, then it is known as gradually varied flow.

(b) **Rapidly varied flow:** If the change in the depth of flow is abrupt in a small length of channel, then it is called rapidly varied flow.

3. **Laminar and turbulent flow:** Reynolds number (Re) is defined as the ratio of inertia force to viscous force. In an open channel, Reynolds number can be given as $Re = [(\rho Vm)/\mu]$, here ρ is the density of the liquid, V is the average or mean velocity of flow, μ is the viscosity of the liquid and m is the hydraulic radius (hydraulic mean depth).
   (a) **Laminar flow:** When Re < 500, the flow is called laminar.
   (b) **Turbulent flow:** When Re > 2000, the flow is called turbulent.
   (c) **Transitional flow:** When 500 < Re < 2000, the flow is called transitional.
4. **Sub-critical, critical and super critical flow:** The flow in a channel is caused due to gravitational force. Froude number (Fr) is an important parameter for analysing open channel flows. The ratio of inertia force to gravity force is called Froude number and it is given as $Fr = [V/\sqrt{gD}]$, here V is the mean velocity of flow, g is acceleration due to gravity and D is the hydraulic depth of channel section.
   (a) **Sub-critical flow:** When $Fr < 1$, the flow is called sub-critical (streaming or tranquil).
   (b) **Critical flow:** When $Fr = 1$, the flow is called critical.
   (c) **Supercritical flow:** When $Fr > 1$, the flow is called supercritical (rapid or shooting or torrential).

18.4 ❐ DISCHARGE THROUGH OPEN CHANNELS (CHEZY'S FORMULA)

Consider uniform flow through a longitudinal section of an open channel (Figure 18.1). Consider sections 1–1 and 2–2 in the direction of flow. Let L be the length of channel, A be the wetted cross-sectional area of channel, V be the mean velocity of flow, i be the slope of the bed, P be the wetted perimeter of the channel, f be the frictional resistance per unit velocity per unit area and w be the specific weight of water.

As the depths of water at the two sections are same, the pressure forces (F_1 and F_2) on the two sections are equal and acts in the opposite directions. Thus, the forces F_1 and F_2 cancel each other. As the flow is uniform, the velocity of flow is constant and therefore, no acceleration acts on the water and the resultant force acting in the direction of flow is zero.

The weight of water (W) between sections 1–1 and 2–2 is given by,

$$W = \text{Specific weight} \times \text{Volume} = w \times AL$$

$$\text{Component of } W \text{ in flow direction } = wAL\sin i$$

The frictional resistance (F) against the motion of water is given by,

$$F = f \times \text{Surface area} \times V^2 = fPLV^2$$

By resolving all forces in the direction of flow, we get:

$$wAL\sin i = fPLV^2$$

$$V^2 = \frac{wAL\sin i}{fPL} = \frac{w}{f} \times \frac{A}{P}\sin i$$

Thus
$$V = \sqrt{\frac{w}{f}}\sqrt{\frac{A}{P}\sin i}$$

Here, $C = \sqrt{w/f}$ is the Chezy's constant which depends upon the roughness of the channel surface and the Reynolds number and $m = (A/P)$ is the hydraulic radius.

For small values of i, we know that $\sin i \approx \tan i \approx i$ and thus, we get the below expression.

$$\boxed{V = C\sqrt{mi}} \quad (18.1)$$

The Equation (18.1) is known as Chezy's formula which was developed by French engineer Antoine Chezy in 1775.

Discharge (Q) through the channel is given by,

$$Q = AV = A \times C\sqrt{mi} = AC\sqrt{m} \times \sqrt{i} = K\sqrt{i} \tag{18.2}$$

Here, $K = (AC\sqrt{m})$ is called the conveyance of the channel section which measures the carrying capacity of the channel. For a channel of constant slope, the conveyance is directly proportional to discharge.

The Chezy's constant (C) is not a dimensionless coefficient and its value depends on the system of units. Its dimensions can be given below.

$$C = \frac{V}{\sqrt{mi}} = \frac{[LT^{-1}]}{[\sqrt{L}]} = [L^{1/2}T^{-1}]$$

The value of C can be determined by the following empirical formulae.

1. **Bazin formula:** According to this formula, the Chezy's constant expressed as C is given below.

$$\boxed{C = \frac{157.6}{1.81 + (k/\sqrt{m})}} \tag{18.3}$$

Here, k is the Bazin's constant whose value depends upon the surface roughness. Some of the typical values of k for various surface materials are given in Table 18.1.

Table 18.1 *Values of Bazin's constant (k) for different surface materials*

Surface of channel	Value of k
Very smooth cement or planed wood	0.11
Concrete or brick or unplanned wood	0.21
Ashlar, rubble masonry or poor brick work	0.83
Earth channel in very good condition	1.54
Earth channel in ordinary condition	2.36
Earth channel in rough condition	3.17

2. **Kutter's formula:** Two Swiss engineers Ganguillet and Kutter proposed the following formula in 1869 for the determination of Chezy's constant (C).

$$\boxed{C = \frac{23 + (0.00155/i) + (1/N)}{1 + [23 + (0.00155/i)] \times (N/\sqrt{m})}} \tag{18.4}$$

Here, N is the Kutter's constant whose value depends upon the channel surface and its condition. Some of the typical values of N for different surface materials are given in Table 18.2.

Table 18.2 *Values of Kutter's constant (N) for different surface materials*

Surface of channel	Value of N
Very smooth surface like plastic, glass, brass	0.010
Very smooth concrete and planned timber	0.011
Smooth concrete	0.012
Ordinary concrete lining, glazed brick work	0.013

(*Continued*)

Surface of channel	Value of N
Good wood	0.014
Earth channel in best condition	0.017
Straight unlined earth canals in good condition	0.020
Earth channel in ordinary condition	0.027
Earth channel with dense weed	0.035

3. **Manning's formula:** An Irish engineer Robert Manning proposed the following formula in 1889 for the determination of Chezy's constant (C).

$$\boxed{C = \frac{1}{N} m^{1/6}} \tag{18.5}$$

Here, N is the Manning's constant which has the same value as Kutter's constant given in Table 18.2.

Example 18.1 Determine the flow rate for a rectangular channel of width 6 m for uniform flow at a depth of 1 m if the bed slope is 1 vertical to 1000 horizontal. Also determine the conveyance for the given channel and comment on the state of flow. Take Chezy's constant as 60.

Solution

Let $b = 6$ m, $y = 1$ m, $i = (1/1000) = 0.001$ and $C = 60$.

$$A = by = 6 \times 1 = 6 \text{ m}^2$$

$$P = b + 2y = 6 + 2 \times 1 = 8 \text{ m}$$

$$m = \frac{A}{P} = \frac{6}{8} = 0.75$$

$$V = C\sqrt{mi} = 60 \times \sqrt{0.75 \times 0.001} = 1.6432 \text{ m/s}$$

$$Q = AV = 6 \times 1.6432 = \mathbf{9.8592 \text{ m}^3/\text{s}}$$

$$K = AC\sqrt{m} = 6 \times 60 \times \sqrt{0.75} = \mathbf{311.77}$$

$$Fr = \frac{V}{\sqrt{gy}} = \frac{1.6432}{\sqrt{9.81 \times 1}} = 0.525$$

Since $Fr < 1$, the flow in the channel is of tranquil nature.

Example 18.2 Using Kutter's formula determine the flow rate for a rectangular channel having depth of water 2 m, width 5 m, bed slope 1 in 2000 and Kutter's constant $N = 0.027$.

Solution

Let $y = 2$ m, $b = 5$ m, $i = (1/2000) = 0.0005$ and $N = 0.027$.

$$A = by = 5 \times 2 = 10 \text{ m}^2$$

$$P = b + 2y = 5 + 2 \times 2 = 9 \text{ m}$$

$$m = \frac{A}{P} = \frac{10}{9} = 1.11$$

Since $$C = \frac{23 + (0.00155/i) + (1/N)}{1 + [23 + (0.00155/i)] \times (N/\sqrt{m})}$$

$$\therefore C = \frac{23 + (0.00155/0.0005) + (1/0.027)}{1 + [23 + (0.00155/0.0005)] \times (0.027/\sqrt{1.11})} = 37.832$$

$$Q = AC\sqrt{mi} = 10 \times 37.832 \times \sqrt{1.11 \times 0.0005} = \mathbf{8.913\ m^3/s}$$

Example 18.3 A triangular gutter, whose sides include an angle of 60° conveys water at a uniform depth of 250 mm. If the discharge is 0.04 m^3/s, then determine the gradient of the trough (bed slope). Take Chezy's constant as 52.

Solution

Refer Figure 18.2. Let $\angle ADB = 60°$, $y = CD = 250$ mm $= 0.25$ m, $Q = 0.04\ m^3/s$ and $C = 52$.

$$AD = BD = \frac{CD}{\cos 30°} = \frac{0.25}{\cos 30°} = 0.2887 \text{ m}$$

$$AC = CD \tan 30° = 0.25 \tan 30° = 0.1443 \text{ m}$$

Thus $$AB = 2AC = 2 \times 0.1443 = 0.2886 \text{ m}$$

$$A = \frac{1}{2} AB \times CD = \frac{1}{2} \times 0.2886 \times 0.25 = 0.0361 \text{ m}^2$$

$$P = AD + BD = 0.2887 + 0.2887 = 0.5774 \text{ m}$$

$$m = \frac{A}{P} = \frac{0.0361}{0.5774} = 0.06252$$

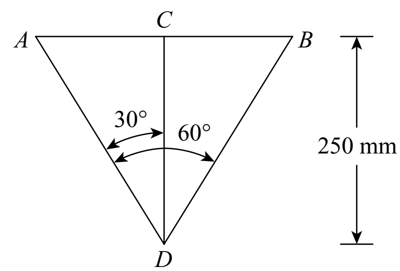

Figure 18.2

Since $$Q = AC\sqrt{mi}$$

Thus $$0.04 = 0.0361 \times 52 \times \sqrt{0.06252 \times i}$$

$$\therefore i = \left(\frac{0.04}{0.0361 \times 52}\right)^2 \times \frac{1}{0.06252} = \frac{1}{137.7}$$

Thus, the gradient of the trough is **1 in 137.7**.

Example 18.4 If an earthen channel of ordinary surface of width 3 m has depth of water 2 m and bed slope as 1 in 1500, then determine the discharge through the rectangular channel by taking suitable constants in (i) Bazin's formula and (ii) Manning's formula.

Solution

Let $b = 3$ m, $y = 2$ m and $i = (1/1500)$.

$$A = by = 3 \times 2 = 6 \text{ m}^2$$

$$P = b + 2y = 3 + 2 \times 2 = 7 \text{ m}$$

$$m = \frac{A}{P} = \frac{6}{7} = 0.857$$

(i) For an earthen channel of ordinary surface taking Bazin's constant as $k = 2.36$.

$$C = \frac{157.6}{1.81 + (k/\sqrt{m})} = \frac{157.6}{1.81 + (2.36/\sqrt{0.857})} = 36.1526$$

$$Q = AC\sqrt{mi} = 6 \times 36.1526 \times \sqrt{\frac{0.857}{1500}} = \mathbf{5.1848\ m^3/s}$$

(ii) For an earthen channel of ordinary surface taking Manning's constant as $N = 0.027$.

$$C = \frac{m^{1/6}}{N} = \frac{0.857^{1/6}}{0.027} = 36.097$$

$$Q = AC\sqrt{mi} = 6 \times 36.097 \times \sqrt{\frac{0.857}{1500}} = \mathbf{5.1769\ m^3/s}$$

Example 18.5 Determine the flow rate of water in a channel having semi-circular bottom of diameter 1.2 m and the two sides as vertical when the depth of flow is 1.2 m. Take Chezy's constant equal to 60 and slope of the bed of channel as 1 in 1000.

Solution

Refer Figure 18.3. Let $AB = CD = d = 1.2$ m, $y = 1.2$ m, $C = 60$ and $i = (1/1000) = 0.001$.

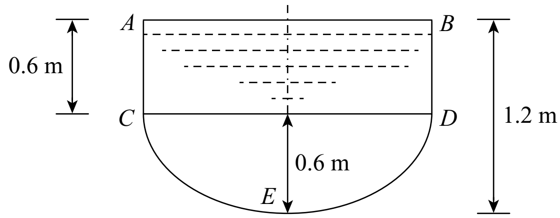

Figure 18.3

$$A = \text{Area}(ABDC) + \text{Area}(CED)$$

$$= 1.2 \times 0.6 + \frac{1}{2}\pi \times 0.6^2 = 1.2855\ \text{m}^2$$

$$P = AC + CED + DB = 0.6 + 0.6\pi + 0.6 = 3.085\ \text{m}$$

$$m = \frac{A}{P} = \frac{1.2855}{3.085} = 0.4167$$

$$Q = AC\sqrt{mi} = 1.2855 \times 60 \times \sqrt{0.4167 \times 0.001} = \mathbf{1.5745\ m^3/s}$$

Example 18.6 A circular sewer pipe is laid at a slope of 1 in 6000 and carries a discharge of 600 litres of water per second when flowing half full. Determine the diameter of the pipe if the Manning's constant is 0.017.

Solution

Let $i = (1/6000)$, $Q = 600\ l/\text{s} = 0.6\ \text{m}^3/\text{s}$, $d = (D/2)$ and $N = 0.017$.

Let D be the diameter of the pipe.

$$A = \frac{1}{2} \times \frac{\pi}{4} D^2 = \frac{\pi D^2}{8}$$

$$P = \frac{\pi D}{2}$$

$$m = \frac{A}{P} = \frac{(1/8)\pi D^2}{(1/2)\pi D} = \frac{D}{4}$$

$$Q = AC\sqrt{mi} = A \times \frac{m^{1/6}}{N} \times \sqrt{mi} = A \times \frac{m^{2/3}}{N} \times \sqrt{i}$$

Thus
$$0.6 = \frac{\pi D^2}{8} \times \frac{(D/4)^{2/3}}{0.017} \times \sqrt{\frac{1}{6000}} = 0.11835 D^{8/3}$$

$$\therefore D = \left(\frac{0.6}{0.11835}\right)^{3/8} = \mathbf{1.8381\ m}$$

Example 18.7 The base width of a trapezoidal channel section is 4 m and the side slopes are 1 : 2 (i.e., 1 vertical to 2 horizontal). The depth of water is 2 m. Determine the discharge through the channel when Chezy's constant is given 50 and the bed slope of the channel is 1 in 1000. Also find the shear stress at the channel boundary.

Solution

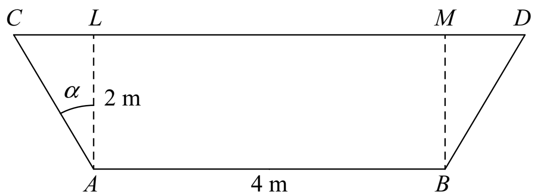

Figure 18.4

Refer Figure 18.4. Let $AB = 4$ m, side slopes = 1 : 2, $AL = BM = 2$ m, $C = 50$ and $i = (1/1000) = 0.001$.

Let F be the shear stress at the channel boundary.

$$\tan\alpha = \frac{CL}{AL} = \frac{2}{1}$$

Thus
$$CL = 2AL = 2 \times 2 = 4 \text{ m}$$

$$CD = AB + 2CL = 4 + 2 \times 4 = 12 \text{ m}$$

$$A = \frac{1}{2}(AB + CD) \times AL = \frac{1}{2}(4 + 12) \times 2 = 16 \text{ m}^2$$

$$AC = \sqrt{AL^2 + CL^2} = \sqrt{2^2 + 4^2} = 4.472 \text{ m}$$

$$P = AC + AB + BD = 4.472 + 4 + 4.472 = 12.944 \text{ m}$$

$$m = \frac{A}{P} = \frac{16}{12.944} = 1.2361$$

$$Q = AC\sqrt{mi} = 16 \times 50 \times \sqrt{1.2361 \times 0.001} = \mathbf{28.1266\ m^3/s}$$

Under equilibrium position, the frictional resistance to flow (FLP) equals the weight of liquid acting along the line of fluid motion ($wAL\sin\alpha$)and the expression is given below.

$$FLP = wAL\sin\alpha = wAL \times i \qquad [\because i = \sin\alpha]$$

$$\therefore F = w\left(\frac{A}{P}\right)i = w \times m \times i = 9810 \times 1.2361 \times 0.001 = \mathbf{12.126\ N/m^2}$$

18.5 ❐ MOST ECONOMICAL SECTION OF CHANNELS

The flow rate through a channel is given by Equation (18.2) as follows.

$$Q = AC\sqrt{mi} = AC\sqrt{\frac{A}{P} \times i}$$

It can be seen from the above equation that the discharge for a given channel slope, roughness and cross-sectional area would be maximum when the wetted perimeter is minimum. For the most economical shape of a channel the conditions are (i) maximum discharge, (ii) minimum excavation and lining, and (iii) least wetted perimeter. As mentioned above the discharge will be maximum when the wetted perimeter is minimum. Therefore, the channel cross section corresponding to the minimum perimeter for a given flow area is called the most economical section (or the most efficient or the best section).

In the following section, the conditions for the most economical section for the channels of rectangular, trapezoidal and circular sections have been discussed.

18.5.1 Most Economical Rectangular Channel Section

Consider a rectangular channel of width b and depth of flow y as shown in Figure 18.5. Let A be the area of flow and P be the wetted perimeter.

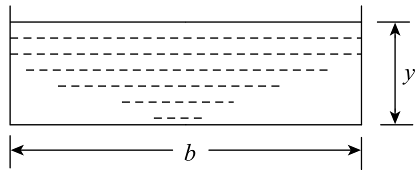

Figure 18.5 *Rectangular channel*

$$A = by$$

$$P = b + 2y$$

or $$P = \frac{A}{y} + 2y \qquad [\because b = A/y]$$

For the most economical section, the wetted perimeter must be minimum.

$$\frac{dP}{dy} = 0$$

$$\frac{d}{dy}\left[\frac{A}{y} + 2y\right] = 0 \Rightarrow -\frac{A}{y^2} + 2 = 0$$

$$A = 2y^2$$

or $$by = 2y^2 \qquad [\because A = by]$$

$$\boxed{y = (b/2) \text{ or } b = 2y} \tag{18.6}$$

Hydraulic radius (or hydraulic mean depth) is given by,

$$m = \frac{A}{P} = \frac{by}{b+2y} = \frac{2y \times y}{2y+2y} = \frac{2y^2}{4y} = \frac{y}{2} \quad [\because b = 2y] \tag{18.7}$$

Thus, the rectangular channel section will be most economical when the depth of flow is equal to half the base width or the hydraulic radius is equal to half the depth of flow.

Example 18.8 Find the most economical section of a rectangular channel having bed slope of 1 in 1000 and carrying water at a rate of 300 litres per second. Take Chezy's constant as 55.

Solution

Let $i = (1/1000)$, $Q = 300\ l/\text{s} = 0.3\ \text{m}^3/\text{s}$ and $C = 55$.

Let b be the width and y be the depth of flow. We know that for most economical rectangular section, $b = 2y$ and $m = y/2$.

$$A = by = 2y \times y = 2y^2$$

Since $$Q = AC\sqrt{mi}$$

Thus $$0.3 = 2y^2 \times 55 \times \sqrt{\frac{y}{2} \times \frac{1}{1000}} = 2.4597 y^{5/2}$$

$$\therefore y = \left(\frac{0.3}{2.4597}\right)^{2/5} = \mathbf{0.431\ m}$$

$$b = 2y = 2 \times 0.431 = \mathbf{0.862\ m}$$

Example 18.9 A rectangular channel of width 4 m has depth of water 1.5 m. The slope of the bed of the channel is 1 in 1000 and the value of Chezy's constant is 55. It is desired to increase the discharge to a maximum by changing the dimensions of the section for constant area of cross section, slope of the bed and roughness of the channel. Find the new dimensions of the channel and increase in discharge.

Solution

Let $b = 4$ m, $y = 1.5$ m, $i = (1/1000) = 0.001$ and $C = 55$.

$$A = by = 4 \times 1.5 = 6 \text{ m}^2$$

$$P = b + 2y = 4 + 2 \times 1.5 = 7 \text{ m}$$

$$m = \frac{A}{P} = \frac{6}{7} = 0.857$$

$$Q = AC\sqrt{mi} = 6 \times 55 \times \sqrt{0.857 \times 0.001} = 9.6606 \text{ m}^3/\text{s}$$

For determining maximum discharge (Q_1), for a given area of cross section, slope of the bed and roughness of the channel, let b_1 be the new width of the channel and y_1 be the new depth of flow. For maximum discharge through a rectangular channel, we know that $b_1 = 2y_1$.

$$A_1 = b_1 y_1 = 6 \text{ m}^2 \qquad [\because \text{Area} = \text{Constant}]$$

or $$2y_1 \times y_1 = 6$$

$$y_1^2 = 3$$

$$\therefore y_1 = \sqrt{3} = \mathbf{1.732\ m}$$

$$b_1 = 2y_1 = 2 \times 1.732 = \mathbf{3.464\ m}$$

$$P_1 = b_1 + 2y_1 = 3.464 + 2 \times 1.732 = 6.928 \text{ m}$$

$$m_1 = \frac{A_1}{P_1} = \frac{6}{6.928} = 0.866$$

$$Q_1 = AC\sqrt{m_1 i} = 6 \times 55 \times \sqrt{0.866 \times 0.001} = 9.7112 \text{ m}^3/\text{s}$$

Therefore, the increase in discharge (ΔQ) is given by,

$$\Delta Q = Q_1 - Q = 9.7112 - 9.6606 = \mathbf{0.0506\ m^3/s}$$

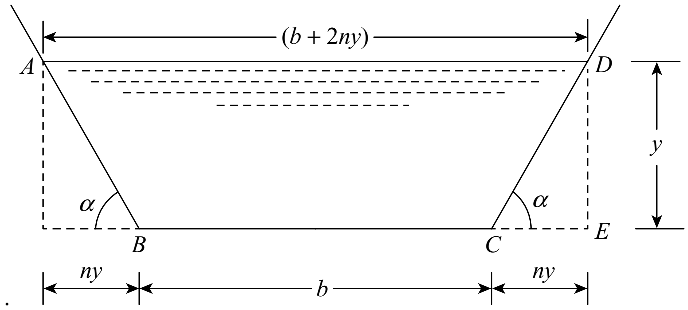

Figure 18.6 *Trapezoidal channel*

18.5.2 Most Economical Trapezoidal Channel Section

Consider a trapezoidal channel of base width b and depth of flow y as shown in Figure 18.6. Let α be the angle made by the sides with the horizontal and the side slope as one vertical to n horizontal as shown in Figure 18.6. Let A be the area of flow and P be the wetted perimeter.

$$A = \left(\frac{BC + AD}{2}\right)y = \frac{b + (b + 2ny)}{2}y = (b + ny)y \quad \text{(i)}$$

$$b = \frac{A}{y} - ny \quad \text{(ii)}$$

$$P = AB + BC + CD = BC + 2CD \quad [\because AB = CD]$$

But $$CD = \sqrt{CE^2 + DE^2} = \sqrt{n^2y^2 + y^2} = y\sqrt{n^2 + 1}$$

Thus $$P = b + 2y\sqrt{n^2 + 1} \quad \text{(iii)}$$

Now substituting Equation (ii) in Equation (iii), we get:

$$P = \frac{A}{y} - ny + 2y\sqrt{n^2 + 1} \quad \text{(iv)}$$

For the most economical section, the wetted perimeter must be minimum.

$$\frac{dP}{dy} = 0 \Rightarrow \frac{d}{dy}\left[\frac{A}{y} - ny + 2y\sqrt{n^2 + 1}\right] = 0$$

$$-\frac{A}{y^2} - n + 2\sqrt{n^2 + 1} = 0$$

$$-\frac{(b + ny)y}{y^2} - n + 2\sqrt{n^2 + 1} = 0 \qquad \text{[Substitute (i)]}$$

$$\frac{(b + ny) + ny}{y} = 2\sqrt{n^2 + 1}$$

$$\boxed{\frac{b + 2ny}{2} = y\sqrt{n^2 + 1}} \quad (18.8)$$

Thus, half of the top width is equal to one of the sloping sides.

From Equation (18.8), we get:

$$b + 2ny = 2y\sqrt{n^2 + 1} \tag{v}$$

The hydraulic radius (or hydraulic mean depth) is given by,

$$m = \frac{A}{P} = \frac{(b + ny)y}{b + 2y\sqrt{n^2 + 1}} = \frac{(b + ny)y}{b + b + 2ny} = \frac{(b + ny)y}{2(b + ny)} = \frac{y}{2} \tag{18.9}$$

Thus, for a most economical trapezoidal section, the hydraulic radius is half of the flow depth.

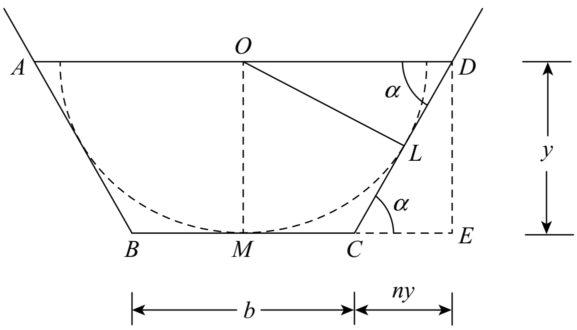

Figure 18.7 *Trapezoidal channel of most economical section*

Figure 18.7 illustrates a trapezoidal channel of most economical section in which a circle drawn with centre O and radius equal to the depth of the flow will be tangential to the three sides of it. Let OL be the perpendicular to the sloping side CD.

From triangle DEC, we get:

$$\sin\alpha = \frac{DE}{DC} = \frac{y}{\sqrt{y^2 + n^2 y^2}} = \frac{1}{\sqrt{n^2 + 1}}$$

From triangle OLD, we get:

$$OL = OD \sin\alpha$$

But $OD = y\sqrt{n^2 + 1}$ [From Equation (18.8)]

$$\therefore OL = y\sqrt{n^2 + 1} \times \frac{1}{\sqrt{n^2 + 1}} = y$$

Thus, if a circle is drawn with O as centre and radius equal to the depth of flow y, then the three sides of the most economical trapezoidal channel will be tangential to the circle.

Best side slope for most economical trapezoidal channel For minimum wetted perimeter, n can be determined from $(dP/dn) = 0$, taking A and y as constants.

Thus
$$\frac{d}{dn}\left[\frac{A}{y} - ny + 2y\sqrt{n^2 + 1}\right] = 0$$

$$-y + 2y \times \frac{1}{2}(n^2 + 1)^{-\frac{1}{2}} \times 2n = 0$$

$$-y + \frac{2ny}{\sqrt{(n^2 + 1)}} = 0 \Rightarrow 2n = \sqrt{(n^2 + 1)}$$

$$4n^2 = n^2 + 1 \Rightarrow 3n^2 = 1$$

$$\therefore \boxed{n = \frac{1}{\sqrt{3}}} \tag{18.10}$$

$$\tan\alpha = \frac{y}{ny} = \frac{1}{n} = \sqrt{3} = \tan 60^\circ$$

Thus $\alpha = 60°$

Thus, the best slope is at 60° to the horizontal.

For the most economical trapezoidal section, using Equation (18.8), we get:

$$\frac{b + 2ny}{2} = y\sqrt{n^2 + 1}$$

Substitute $n = 1/\sqrt{3}$ [Equation (18.10)] in the above equation, we get:

$$\frac{b + 2 \times (1/\sqrt{3})y}{2} = y\sqrt{(1/\sqrt{3})^2 + 1}$$

$$\sqrt{3}b + 2y = 2\sqrt{3}y \times \frac{2}{\sqrt{3}}$$

$$\sqrt{3}b + 2y = 4y$$

Thus
$$b = \frac{2y}{\sqrt{3}} \tag{18.11}$$

Now
$$P = b + 2y\sqrt{n^2 + 1}$$

Substituting Equations (18.10) and (18.11) in the above expression, we get:

$$P = \frac{2y}{\sqrt{3}} + 2y\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + 1} = \frac{2y}{\sqrt{3}} + 2y \times \frac{2}{\sqrt{3}} = \frac{6y}{\sqrt{3}} = 3\left(\frac{2y}{\sqrt{3}}\right) = 3b \tag{18.12}$$

Example 18.10 A trapezoidal channel has side slopes of 3 horizontal to 4 vertical and the slope of its bed is 1 in 2000. Determine the optimum dimensions of the channel, if it is to carry water at 0.6 m³/s. Take Chezy's constant as 75.

Solution

Let $n = \frac{\text{Horizontal}}{\text{Vertical}} = \frac{3}{4}$, $i = (1/2000)$, $Q = 0.6\ \text{m}^3/\text{s}$ and $C = 75$.

Since
$$\frac{b + 2ny}{2} = y\sqrt{n^2 + 1} \quad \text{[Most economical section]}$$

Thus
$$\frac{b + 2 \times (3/4)y}{2} = y\sqrt{\left(\frac{3}{4}\right)^2 + 1}$$

$$\frac{b + 1.5y}{2} = 1.25y \Rightarrow b + 1.5y = 2.5y$$

$$\therefore b = y$$

$$m = \frac{y}{2} \quad \text{[Most economical section]}$$

$$A = (b + ny)y = \left(y + \frac{3}{4}y\right)y = 1.75y^2$$

Since
$$Q = AC\sqrt{mi}$$

Thus
$$0.6 = 1.75y^2 \times 75 \times \sqrt{\frac{y}{2} \times \frac{1}{2000}} = 2.075245y^{5/2}$$

$$\therefore y = \left(\frac{0.6}{2.075245}\right)^{2/5} = 0.609 \text{ m}$$

$$\therefore b = y = \mathbf{0.609\ m}$$

Example 18.11 A trapezoidal channel having side slope equal to 60° with the horizontal and laid on a slope of 1 in 850 carries a discharge of 6 m^3/s. Determine the width at the base and depth of flow for most economical cross section. Take Chezy's constant as 55.

Solution

Let $\alpha = 60°$, $i = (1/850)$, $Q = 6 \text{ m}^3/\text{s}$ and $C = 55$.

Since
$$\tan\alpha = \tan 60° = \sqrt{3} = \frac{1}{n} \qquad \text{[Most economical section]}$$

Thus
$$n = \frac{1}{\sqrt{3}}$$

Since
$$\frac{b + 2ny}{2} = y\sqrt{n^2 + 1}$$

Thus
$$\frac{b + 2(1/\sqrt{3})y}{2} = y\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + 1}$$

$$\frac{\sqrt{3}b + 2y}{2\sqrt{3}} = \frac{2y}{\sqrt{3}} \Rightarrow \sqrt{3}b + 2y = 4y$$

$$\therefore b = \frac{2}{\sqrt{3}}y$$

$$A = (b + ny)y = \left(\frac{2}{\sqrt{3}}y + \frac{1}{\sqrt{3}}y\right)y = \sqrt{3}y^2$$

$$m = \frac{y}{2} \qquad \text{[Most economical section]}$$

Since
$$Q = AC\sqrt{mi}$$

Thus
$$6 = \sqrt{3}y^2 \times 55 \times \sqrt{\frac{y}{2} \times \frac{1}{850}} = 2.3105y^{5/2}$$

$$\therefore y = \left(\frac{6}{2.3105}\right)^{2/5} = \mathbf{1.465\ m}$$

$$b = \frac{2}{\sqrt{3}}y = \frac{2}{\sqrt{3}} \times 1.465 = \mathbf{1.692\ m}$$

Example 18.12 A power canal of trapezoidal section having side slope equal to 60° with the horizontal and laid on a slope of 1 in 2500 carries a discharge of 14 m^3/s. Determine the width at the base and depth of flow for most economical cross section. Take Manning's constant as 0.02.

Solution

Let $\alpha = 60°,\ i = (1/2500),\ Q = 14\ \text{m}^3/\text{s}$ and $N = 0.02$.

Since $$\tan\alpha = \tan 60° = \sqrt{3} = \frac{1}{n} \qquad \text{[Most economical section]}$$

Thus $$n = \frac{1}{\sqrt{3}}$$

Since $$\frac{b+2ny}{2} = y\sqrt{n^2+1}$$

Thus $$\frac{b+2(1/\sqrt{3})y}{2} = y\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2+1}$$

$$\frac{\sqrt{3}b+2y}{2\sqrt{3}} = \frac{2y}{\sqrt{3}} \Rightarrow \sqrt{3}b+2y = 4y$$

$$\therefore b = \frac{2}{\sqrt{3}}y$$

$$A = (b+ny)y = \left(\frac{2}{\sqrt{3}}y + \frac{1}{\sqrt{3}}y\right)y = \sqrt{3}y^2$$

$$m = \frac{y}{2} \qquad \text{[Most economical section]}$$

$$C = \frac{m^{1/6}}{N} = \frac{(y/2)^{1/6}}{0.02} = 44.545y^{1/6}$$

Since $$Q = AC\sqrt{mi}$$

Thus $$14 = \sqrt{3}y^2 \times 44.545y^{1/6} \times \sqrt{\frac{y}{2}\times\frac{1}{2500}}$$

$$14 = 1.09y^{8/3}$$

$$\therefore y = \left(\frac{14}{1.09}\right)^{3/8} = \mathbf{2.605\ m}$$

$$b = \frac{2}{\sqrt{3}}y = \frac{2}{\sqrt{3}}\times 2.605 = \mathbf{3.01\ m}$$

Example 18.13 A trapezoidal channel with side slopes of 1:1 has to be designed to convey 10 m^3/s of water at a velocity of 2 m/s so that the amount of concrete lining for the bed and sides is minimum. Determine the area of lining for one metre length of the canal.

Solution

Let $n = 1$, $Q = 10\ \text{m}^3/\text{s}$ and $V = 2\ \text{m/s}$.

$$A = \frac{Q}{V} = \frac{10}{2} = 5\ \text{m}^2$$

Since $$\frac{b + 2ny}{2} = y\sqrt{n^2 + 1} \qquad \text{[Most economical section]}$$

Thus $$\frac{b + 2\times 1\times y}{2} = y\sqrt{1^2 + 1}$$

$$b + 2y = 2\sqrt{2}y$$

$$\therefore b = (2\sqrt{2} - 2)y = 0.828y$$

$$A = (b + ny)y = (0.828y + y)y = 1.828y^2$$

Thus $$5 = 1.828y^2$$

$$\therefore y = \sqrt{\frac{5}{1.828}} = 1.654\ \text{m}$$

$$b = 0.828y = 0.828\times 1.654 = 1.3695\ \text{m}$$

Area of lining required for one metre length of canal is given by,

$$a = P\times 1 = (b + 2y\sqrt{n^2 + 1}) = (1.3695 + 2\times 1.654\times \sqrt{1^2 + 1}) = \mathbf{6.048\ m}$$

18.5.3 Most Economical Circular Channel Section

Figure 18.8 shows a circular channel through which water is flowing. Let R be the radius of the channel, D be its diameter, y be the depth of flow, 2α be the angle in radians subtended by water surface AB at the centre, P be the wetted perimeter, A be the wetted area and i be the slope of the bed.

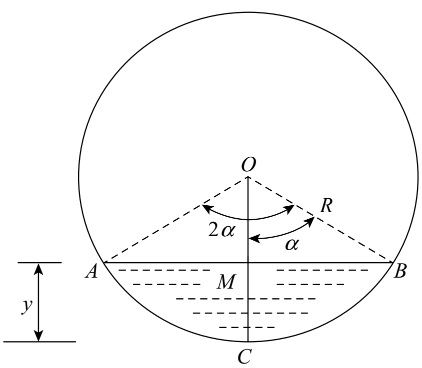

Figure 18.8 *Circular channel*

$$P = \text{Length of arc} = \frac{2\pi R}{2\pi}\times 2\alpha = 2R\alpha \qquad (18.13)$$

$$A = \text{Area } ACBA = \text{Area of sec } OACB - \text{Area of } \Delta OAB$$

$$A = \frac{\pi R^2}{2\pi}\times 2\alpha - \frac{1}{2}AB\times OM = R^2\alpha - \frac{1}{2}\times 2MB\times OM \qquad [\because AB = 2MB]$$

Now $MB = R\sin\alpha$ and $OM = R\cos\alpha$

Thus $$A = R^2\alpha - \frac{1}{2}\times 2R\sin\alpha\times R\cos\alpha = R^2\alpha - \frac{1}{2}R^2\times 2\sin\alpha\cos\alpha$$

$$A = R^2\alpha - \frac{1}{2}R^2\sin 2\alpha = R^2\left(\alpha - \frac{\sin 2\alpha}{2}\right) \qquad (18.14)$$

Hydraulic radius is given by,

$$m = \frac{A}{P} = \frac{R^2\left(\alpha - \dfrac{\sin 2\alpha}{2}\right)}{2R\alpha} = \frac{R}{2\alpha}\left(\alpha - \frac{\sin 2\alpha}{2}\right) \tag{18.15}$$

The discharge is given by,

$$Q = AC\sqrt{mi}$$

For a circular pipe, the shape of the flow area varies with the depth of flow. Thus, both the wetted area as well as the wetted perimeter varies with the depth of flow and the condition of area of flow being constant cannot be applied. Therefore, in the case of circular pipes two separate conditions, namely maximum velocity of flow and maximum discharge are to be derived.

(i) **Condition for maximum velocity:** The velocity of flow is given by Equation (18.1) as follows.

$$V = C\sqrt{mi} = C\sqrt{\frac{A}{P} \times i}$$

For a given value of C and i, the velocity will be maximum when (A/P) is maximum which varies with α.

Thus
$$\frac{d(A/P)}{d\alpha} = 0 \tag{i}$$

$$\frac{P\dfrac{dA}{d\alpha} - A\dfrac{dP}{d\alpha}}{P^2} = 0$$

or
$$P\frac{dA}{d\alpha} - A\frac{dP}{d\alpha} = 0 \tag{ii}$$

Differentiating Equation (18.14), we get:

$$\frac{dA}{d\alpha} = R^2(1 - \cos 2\alpha)$$

Differentiating Equation (18.13), we get:

$$\frac{dP}{d\alpha} = 2R$$

Substituting the values of P, $(dA/d\alpha)$, A and $(dP/d\alpha)$ in expression (ii), we get:

$$2R\alpha[R^2(1 - \cos 2\alpha)] - R^2\left(\alpha - \frac{\sin 2\alpha}{2}\right)2R = 0$$

$$2R^3\alpha(1 - \cos 2\alpha) - 2R^3\left(\alpha - \frac{\sin 2\alpha}{2}\right) = 0$$

$$\alpha(1 - \cos 2\alpha) - \left(\alpha - \frac{\sin 2\alpha}{2}\right) = 0$$

$$\alpha - \alpha\cos 2\alpha - \alpha + \frac{\sin 2\alpha}{2} = 0$$

$$\alpha\cos 2\alpha = \frac{\sin 2\alpha}{2} \Rightarrow \frac{\sin 2\alpha}{\cos 2\alpha} = 2\alpha$$

$$\tan 2\alpha = 2\alpha$$

By hit and trial method, the solution is given by,

$$2\alpha = 257.5° \Rightarrow \alpha = 128.75°$$

Now $$y = OC - OM = R - R\cos\alpha = R(1-\cos\alpha)$$

Thus $$\boxed{y = R(1-\cos 128.75°) \approx 1.62R \approx 0.81D} \quad (18.16)$$

In other words, maximum velocity will occur when the depth of flow is 0.81 times the diameter of the circular pipe.

$$\alpha = 128.75° = 128.75 \times \frac{\pi}{180} = 2.247 \text{ radians} \quad (18.17)$$

Hydraulic radius for maximum velocity is given by substituting the values of α in Equation (18.15) as follows.

$$m = \frac{R}{2\times 2.247} \times \left(2.247 - \frac{\sin 257.5°}{2}\right) \approx 0.6R \approx 0.3D \quad (18.18)$$

Thus, in a circular channel for maximum velocity, the hydraulic radius is equal to 0.3 times of its diameter.

(ii) **Condition for maximum discharge:** The discharge is given by Equation (18.2) as follows.

$$Q = AC\sqrt{mi} = C\sqrt{\frac{A^3}{P}\times i} \qquad [\because m = A/P]$$

For a given value of C and i, the discharge will be maximum when (A^3/P) is maximum which varies with α.

Thus $$\frac{d(A^3/P)}{d\alpha} = 0$$

$$\frac{P\times 3A^2\dfrac{dA}{d\alpha} - A^3\dfrac{dP}{d\alpha}}{P^2} = 0$$

or $$3PA^2\frac{dA}{d\alpha} - A^3\frac{dP}{d\alpha} = 0$$

$$3P\frac{dA}{d\alpha} - A\frac{dP}{d\alpha} = 0 \quad \text{(i)}$$

Differentiating Equation (18.14), we get:

$$\frac{dA}{d\alpha} = R^2(1-\cos 2\alpha)$$

Differentiating Equation (18.13), we get:

$$\frac{dP}{d\alpha} = 2R$$

Substituting values of P, $(dA/d\alpha)$, A and $(dP/d\alpha)$ in expression (i), we get:

$$3\times 2R\alpha[R^2(1-\cos 2\alpha)] - R^2\left(\alpha - \frac{\sin 2\alpha}{2}\right)2R = 0$$

$$6R^3\alpha(1-\cos 2\alpha) - 2R^3\left(\alpha - \frac{\sin 2\alpha}{2}\right) = 0$$

$$3\alpha(1-\cos 2\alpha) - \left(\alpha - \frac{\sin 2\alpha}{2}\right) = 0$$

$$3\alpha - 3\alpha\cos 2\alpha - \alpha + \frac{\sin 2\alpha}{2} = 0$$

$$2\alpha - 3\alpha\cos 2\alpha + \frac{\sin 2\alpha}{2} = 0$$

$$4\alpha - 6\alpha\cos 2\alpha + \sin 2\alpha = 0$$

By hit and trial method, the solution is given by,

$$2\alpha = 308^\circ \Rightarrow \alpha = 154^\circ$$

Now $\quad y = OC - OM = R - R\cos\alpha = R(1-\cos\alpha)$

Thus $\quad \boxed{y = R(1-\cos 154^\circ) \approx 1.9R \approx 0.95D} \quad (18.19)$

In other words, maximum discharge will occur when the depth of flow is 0.95 times the diameter of the circular pipe.

$$\alpha = 154^\circ = 154\times\frac{\pi}{180} = 2.6878 \text{ radians} \tag{18.20}$$

Hydraulic radius for maximum discharge is given by substituting the values of α in Equation (18.15) as follows.

$$m = \frac{R}{2\times 2.6878}\times\left(2.6878 - \frac{\sin 308^\circ}{2}\right) \approx 0.58R \approx 0.29D \tag{18.21}$$

Thus, in a circular channel for maximum discharge, the hydraulic radius is equal to 0.29 times of its diameter.

Example 18.14 A circular pipe of diameter 4 m is laid at a slope of 1 in 1200. Determine the discharge through the pipe if the depth of water in it is 1.5 m and Chezy's constant is 65.

Solution

Refer Figure 18.8. Let $D = 4$ m, $i = (1/1200)$, $MC = y = 1.5$ m and $C = 65$.

$$R = \frac{D}{2} = \frac{4}{2} = 2 \text{ m}$$

$$OM = OC - MC = R - 1.5 = 2 - 1.5 = 0.5 \text{ m}$$

$$\cos\alpha = \frac{OM}{OB} = \frac{0.5}{2} = 0.25$$

$$\therefore \alpha = \cos^{-1}(0.25) = 75.52° = 75.52 \times \frac{\pi}{180} = 1.3181 \text{ radians}$$

$$P = 2R\alpha = 2 \times 2 \times 1.3181 = 5.2724 \text{ m}$$

$$A = R^2\left(\alpha - \frac{\sin 2\alpha}{2}\right) = 2^2 \times \left[1.3181 - \frac{\sin(2\times 75.52°)}{2}\right] = 4.304 \text{ m}^2$$

$$m = \frac{A}{P} = \frac{4.304}{5.2724} = 0.8163$$

$$Q = AC\sqrt{mi} = 4.304 \times 65 \times \sqrt{\frac{0.8163}{1200}} = \mathbf{7.2966\ m^3/s}$$

Example 18.15 Determine the velocity and discharge for the conditions of maximum velocity and maximum discharge for a concrete lined circular channel of diameter 4 m having a bed slope of 1 in 750. Take Chezy's constant as 55.

Solution

Let $D = 4$ m, $i = (1/750)$ and $C = 55$.

$$R = \frac{D}{2} = \frac{4}{2} = 2 \text{ m}$$

(i) For maximum velocity, we get:

$$\alpha = 128.75° = 128.75 \times \frac{\pi}{180} = 2.2471 \text{ radians}$$

$$P = 2R\alpha = 2 \times 2 \times 2.2471 = 8.9884 \text{ m}$$

$$A = R^2\left(\alpha - \frac{\sin 2\alpha}{2}\right) = 2^2 \times \left[2.2471 - \frac{\sin(2\times 128.75°)}{2}\right] = 10.941 \text{ m}^2$$

$$m = \frac{A}{P} = \frac{10.941}{8.9884} = 1.2172$$

$$V = C\sqrt{mi} = 55 \times \sqrt{\frac{1.2172}{750}} = \mathbf{2.2157\ m/s}$$

$$Q = AV = 10.941 \times 2.2157 = \mathbf{24.242\ m^3/s}$$

(ii) For maximum discharge, we get:

$$\alpha = 154° = 154 \times \frac{\pi}{180} = 2.68781 \text{ radians}$$

$$A = R^2\left(\alpha - \frac{\sin 2\alpha}{2}\right) = 2^2 \times \left[2.68781 - \frac{\sin(2\times 154°)}{2}\right] = 12.3273 \text{ m}^2$$

$$P = 2R\alpha = 2 \times 2 \times 2.68781 = 10.75124 \text{ m}$$

$$m = \frac{A}{P} = \frac{12.3273}{10.75124} = 1.1466$$

$$V = C\sqrt{mi} = 55 \times \sqrt{\frac{1.1466}{750}} = \mathbf{2.1505\ m/s}$$

$$Q = AV = 12.3273 \times 2.1505 = \mathbf{26.51\ m^3/s}$$

18.6 ❐ NON-UNIFORM FLOW THROUGH OPEN CHANNELS

In non-uniform flow, the water surface in the open channels does not remain parallel to the bed and the velocity varies from section to section. A non-uniform flow is also known as the varied flow (or the flow of varying depth) and it may be rapidly varied flow (R.V.F.) or gradually varied flow (G.V.F.). A flow is called rapidly varied flow when the depth of flow changes abruptly over a small length of the channel, whereas a flow is said to be gradually varied flow when the depth of flow in a channel changes gradually over a long length of channel.

18.6.1 Specific Energy Curve

The curve showing the variation of specific energy with the depth of flow is called specific energy curve. The specific energy (e) is the total energy of flow per unit weight with the channel bed taken as the datum and it is given by the following expression.

$$e = \text{P.E. of flow} + \text{K.E. of flow} = e_p + e_k = y + \frac{V^2}{2g} \tag{18.22}$$

Here, y is the depth of flow and V is the average velocity of flow.

Consider a steady but non-uniform flow through a rectangular section of width b and depth of flow y.

The discharge (Q) through the channel is given by,

$$Q = AV = by \times V$$

The discharge per unit width is given by,

$$q = \frac{Q}{b} = \frac{byV}{b} = yV \tag{18.23}$$

$$V = \frac{Q}{A} = \frac{Q}{by} = \frac{q}{y}$$

Substituting the value of $V = (q/y)$ in Equation (18.22), we get:

$$e = y + \frac{(q/y)^2}{2g} = y + \frac{q^2}{2gy^2} \tag{18.24}$$

The Equation (18.24) gives the variation of specific energy with the depth of flow. This equation can also be represented graphically as a plot of specific energy versus the depth of flow as shown in Figure 18.9. Such a plot is called specific energy curve.

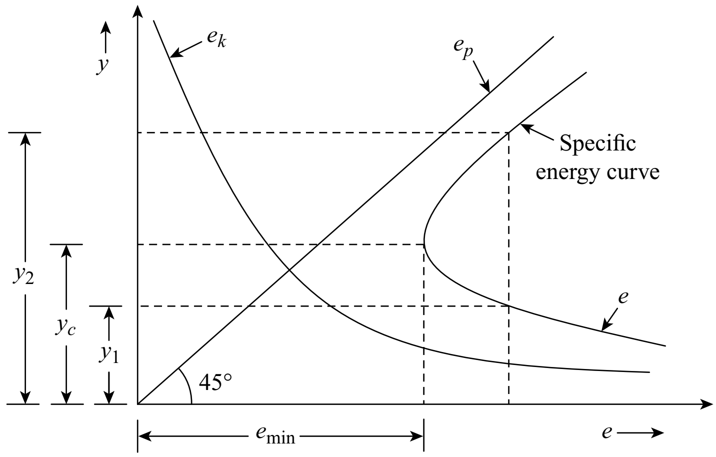

Figure 18.9 *Specific energy curve*

The specific energy curve may also be obtained by first drawing a curve for potential energy (e_p) which is a straight line passing through the origin at 45° with the x-axis and then drawing another curve for kinetic energy (e_k) which will be a parabola. By combining these two curves the resultant curve called specific energy curve can be obtained (Figure 18.9). Generally, for a particular value of specific energy, there are two values of the depth which are known as alternate or conjugate depths. There is a certain depth at which the specific energy is minimum which is known as critical depth (y_c) and the corresponding velocity is known as critical velocity (V_c).

18.6.2 Critical Depth

The critical depth (y_c) can be obtained from Equation (18.24) as given below.

$$\frac{d[e]}{dy} = \frac{d}{dy}\left[y + \frac{q^2}{2gy^2}\right] = 0$$

$$1 - \frac{q^2}{gy^3} = 0 \Rightarrow y^3 = \frac{q^2}{g}$$

$$y = \left(\frac{q^2}{g}\right)^{1/3}$$

Thus

$$\boxed{y_c = \left(\frac{q^2}{g}\right)^{1/3}} \qquad (18.25)$$

18.6.3 Critical Velocity

The critical velocity can be obtained as given below.

$$y_c^{\,3} = \frac{q^2}{g} \qquad (18.26)$$

$$gy_c^{\,3} = q^2$$

$$gy_c^{\,3} = (y_c V_c)^2 \qquad [\because q = y_c V_c]$$

$$V_c^{\,2} = g y_c$$

$$\therefore \boxed{V_c = \sqrt{g y_c}} \tag{18.27}$$

$$\frac{V_c}{\sqrt{g y_c}} = 1$$

Thus
$$Fr = \frac{V_c}{\sqrt{g y_c}} = 1 \tag{18.28}$$

Therefore, the Froude number for critical flow is unity.

18.6.4 Sub-Critical Flow

The flow is sub-critical (or streaming or tranquil) when the depth of flow in a channel is more than the critical depth. For this type of flow $Fr < 1$.

18.6.5 Super-Critical Flow

The flow is super-critical (or shooting or torrential) when the depth of flow in a channel is less than the critical depth. For this type of flow $Fr > 1$.

18.6.6 Minimum Specific Energy in Terms of Critical Depth

When specific energy is minimum, the depth of flow is critical and thus, the minimum specific energy ($e_{\min}$) can be expressed in terms of critical depth from Equation (18.24) as given below.

$$e_{\min} = y_c + \frac{q^2}{2 g y_c^{\,2}}$$

Substituting Equation (18.26) in the above expression, we get:

$$e_{\min} = y_c + \frac{y_c^{\,3}}{2 y_c^{\,2}} = y_c + \frac{y_c}{2} = 1.5 y_c \qquad [\because y_c^{\,3} = q^2 / g] \tag{18.29}$$

18.6.7 Condition for Maximum Discharge for a Given Value of Specific Energy

The specific energy is given by Equation (18.24) as follows.

$$e = y + \frac{q^2}{2 g y^2} = y + \left(\frac{Q}{b}\right)^2 \frac{1}{2 g y^2} = y + \frac{Q^2}{2 g b^2 y^2} \qquad [\because q = Q / b]$$

Thus
$$Q = \sqrt{(e - y) 2 g b^2 y^2} = b\sqrt{2g(e y^2 - y^3)}$$

For maximum discharge, the term $(ey^2 - y^3)$ in the above expression should be maximum and it is given below.

$$\frac{d}{dy}[e y^2 - y^3] = 0 \Rightarrow 2 e y - 3 y^2 = 0$$

$$2e - 3y = 0$$

$$\therefore \boxed{e = 1.5y} \quad (18.30)$$

Thus, the specific energy is 1.5 times of the depth of flow. However, from Equation (18.29), we can see that specific energy is minimum when it is equal to 1.5 times the value of depth of critical flow. Hence, the condition for maximum discharge for a given value of specific energy is that the depth of flow should be critical.

Example 18.16 The flow of water through a 6 m wide channel is 15 m^3/s. If the depth of water in the channel is 0.5 m, then determine (i) the specific energy of the flow, (ii) critical depth, (iii) critical velocity and (iv) minimum specific energy. (v) Also state whether the flow is sub-critical or super-critical.

Solution

Let $b = 6$ m, $Q = 15$ m^3/s and $y = 0.5$ m.

(i) $q = \frac{Q}{b} = \frac{15}{6} = 2.5 \text{ m}^3/\text{s per m}$

$$V = \frac{q}{y} = \frac{2.5}{0.5} = 5 \text{ m/s}$$

$$e = y + \frac{V^2}{2g} = 0.5 + \frac{5^2}{2 \times 9.81} = \mathbf{1.7742 \text{ m}}$$

(ii) $y_c = \left(\frac{q^2}{g}\right)^{1/3} = \left(\frac{2.5^2}{9.81}\right)^{1/3} = \mathbf{0.8605 \text{ m}}$

(iii) $V_c = \sqrt{gy_c} = \sqrt{9.81 \times 0.8605} = \mathbf{2.9054 \text{ m/s}}$

(iv) $e_{\min} = 1.5y_c = 1.5 \times 0.8605 = \mathbf{1.29075 \text{ m}}$

(v) $Fr = \frac{V}{\sqrt{gy}} = \frac{5}{\sqrt{9.81 \times 0.5}} = 2.258$

Since $Fr > 1$, thus flow is super-critical.

Example 18.17 Determine the maximum possible discharge if the specific energy for a 4 m wide channel is to be 4 Nm/N.

Solution

Let $b = 4$ m and $e = 4$ Nm/N.

$$y_c = \frac{2}{3}e = \frac{2}{3} \times 4 = 2.667 \text{ m}$$

$$V_c = \sqrt{gy_c} = \sqrt{9.81 \times 2.667} = 5.115 \text{ m/s}$$

$$Q_{\max} = by_cV_c = 4 \times 2.667 \times 5.115 = \mathbf{54.56682 \text{ m}^3/\text{s}}$$

Example 18.18 Water flows at a steady and uniform depth of 2 m in an open channel of rectangular cross section having base width equal to 5 m and laid at a slope of 1 in 1000. It is desired to obtain critical flow in the channel by providing a hump in the bed. Calculate the height of the hump. Consider the value of Manning's rugosity coefficient as 0.02 for the channel surface.

Solution

Let $y = 2$ m, $b = 5$ m, $i = (1/1000)$ and $N = 0.02$.

Let h be the height of hump.

$$A = by = 5 \times 2 = 10 \text{ m}^2$$

$$P = b + 2y = 5 + 2 \times 2 = 9 \text{ m}$$

$$m = \frac{A}{P} = \frac{10}{9} = 1.111 \text{ m}$$

Since
$$Q = AC\sqrt{mi} = A \times \frac{m^{1/6}}{N} \times \sqrt{mi} = A \times \frac{m^{2/3}}{N} \times \sqrt{i}$$

$$\therefore Q = 10 \times \frac{1.111^{2/3}}{0.02} \times \sqrt{\frac{1}{1000}} = 16.961 \text{ m}^3/\text{s}$$

$$V = \frac{Q}{A} = \frac{16.961}{10} = 1.6961 \text{ m/s}$$

$$q = \frac{Q}{b} = \frac{16.961}{5} = 3.3922 \text{ m}^3/\text{s per m}$$

$$\therefore y_c = \left(\frac{q^2}{g}\right)^{1/3} = \left(\frac{3.3922^2}{9.81}\right)^{1/3} = 1.0546 \text{ m}$$

$$e_{\min} = 1.5 y_c = 1.5 \times 1.0546 = 1.5819 \text{ m}$$

$$e = y + \frac{V^2}{2g} = 2 + \frac{1.6961^2}{2 \times 9.81} = 2.1466 \text{ m}$$

$$h = e - e_{\min} = 2.1466 - 1.5819 = \mathbf{0.5647 \text{ m}}$$

18.7 ❐ HYDRAULIC JUMP

It can be seen from the specific energy curve (Figure 18.9) that for a particular specific energy (e), there are two possible depths y_1 and y_2 such that $y_1 < y_c$ and $y_2 > y_c$, here y_c is the critical depth. The flow will be shooting flow when $y < y_c$, whereas it will be streaming flow when $y > y_c$. Such flows occur when the water flows over a dam as shown in Figure 18.10.

The shooting flow (supercritical flow) is unstable and always has a tendency to convert itself into stable streaming flow (sub-critical flow) by increasing its depth on the downstream side. This phenomenon of sudden increase in depth of water flow is termed as hydraulic jump. Due to its wave motion and stationary position, the hydraulic jump is also known as standing wave. In practice, hydraulic jump occurs at the toe of spillways or below a sluice gate.

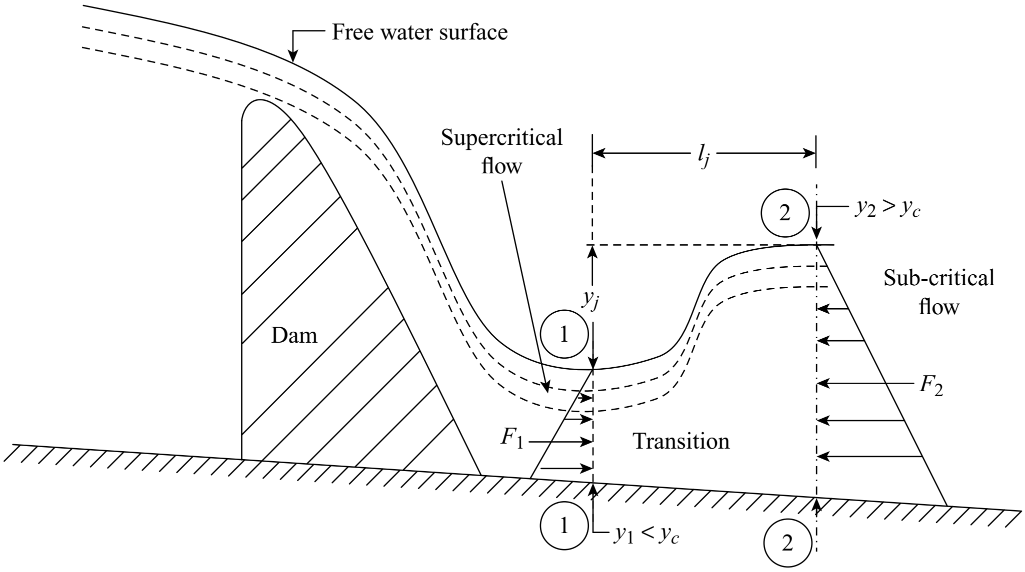

Figure 18.10 *Hydraulic jump*

18.7.1 Depth of Hydraulic Jump

The assumptions for the analysis of hydraulic jump are (i) the friction at the walls and channel bed is negligible, (ii) the slope of the channel bed is small and thus, the component of the weight of fluid in the flow direction is neglected and (iii) the flow is uniform and pressure distribution is hydrostatic before and after the jump.

Consider the sections 1–1 and 2–2 before and after the hydraulic jump, respectively, as shown in Figure 18.10. Let y_1 be the depth of flow, V_1 be the velocity of flow, A_1 be the cross-sectional area, F_1 be the pressure force and y_{G1} be the depth of centroid of area below free water surface at section 1–1 and y_2, V_2, A_2, F_2 and y_{G2} be the corresponding values at section 2–2.

Discharge per unit of width (q) is given by,

$$q = V_1 y_1 = V_2 y_2$$

Thus
$$V_1 = (q/y_1) \text{ and } V_2 = (q/y_2)$$

The pressure forces at sections 1–1 and 2–2 are respectively given by,

$$F_1 = \rho_w g A_1 y_{G1} = \rho_w g \times (y_1 \times 1) \times \frac{y_1}{2} = \frac{1}{2}\rho_w g {y_1}^2$$

$$F_2 = \rho_w g A_2 y_{G2} = \rho_w g \times (y_2 \times 1) \times \frac{y_2}{2} = \frac{1}{2}\rho_w g {y_2}^2$$

Thus, net force $F = (F_2 - F_1)$ acting on the mass of water between sections 1-1 and 2-2 is given below.

$$F = \frac{1}{2}\rho_w g {y_2}^2 - \frac{1}{2}\rho_w g {y_1}^2 = \frac{1}{2}\rho_w g({y_2}^2 - {y_1}^2) \quad \text{(i)}$$

Rate of change of momentum in the direction of force is given by,

$$= \text{Mass of water per second} \times \text{Change in velocity in direction of force}$$

$$= (\rho_w \times q \times 1) \times (V_1 - V_2) = \rho_w q (V_1 - V_2) = \rho_w q \left(\frac{q}{y_1} - \frac{q}{y_2} \right) \quad \text{(ii)}$$

According to impulse-momentum equation, the net force acting on the mass of water is equal to the rate of change of momentum in the direction of force. Thus, by equating the expressions (i) and (ii), we get the below expression.

$$\frac{1}{2} \rho_w g (y_2^2 - y_1^2) = \rho_w q \left(\frac{q}{y_1} - \frac{q}{y_2} \right)$$

$$\frac{1}{2} g (y_2 + y_1)(y_2 - y_1) = q^2 \left(\frac{y_2 - y_1}{y_1 y_2} \right)$$

$$\frac{g}{2} (y_2 + y_1) = \frac{q^2}{y_1 y_2}$$

$$(y_2 + y_1) = \frac{2q^2}{g y_1 y_2} \quad \text{(iii)}$$

$$y_2^2 + y_1 y_2 = [(2q^2)/(g y_1)] \qquad \text{[Multiplying both sides by } y_2\text{]}$$

$$y_2^2 + y_1 y_2 - [(2q^2)/(g y_1)] = 0$$

Thus $$y_2 = \frac{-y_1 \pm \sqrt{y_1^2 + 4 \times 1 \times [(2q^2)/(g y_1)]}}{2 \times 1}$$

Neglecting negative root, we get:

$$y_2 = \frac{-y_1}{2} + \sqrt{\frac{y_1^2}{4} + \frac{2q^2}{g y_1}} = \frac{-y_1}{2} + \sqrt{\frac{y_1^2}{4} + \frac{2(V_1 y_1)^2}{g y_1}}$$

$$\therefore y_2 = \frac{-y_1}{2} + \sqrt{\frac{y_1^2}{4} + \frac{2V_1^2 y_1}{g}} \quad (18.31)$$

Thus, the depth (height) of hydraulic jump (y_j) is given by,

$$y_j = y_2 - y_1 \quad (18.32)$$

The upstream Froude number (Fr_1) is given by,

$$Fr_1 = \frac{V_1}{\sqrt{g y_1}} \text{ or } (Fr_1)^2 = \frac{V_1^2}{g y_1}$$

Thus, Equation (18.31) can also be expressed in terms of upstream Froude number (Fr_1) as given below.

$$y_2 = \frac{-y_1}{2} + \sqrt{\frac{y_1^2}{4} + \frac{2V_1^2 y_1}{g}} = \frac{-y_1}{2} + \frac{y_1}{2}\sqrt{1 + \frac{8V_1^2}{g y_1}} = \frac{-y_1}{2} + \frac{y_1}{2}\sqrt{(1 + 8Fr_1^2)}$$

or
$$y_2 = \frac{y_1}{2}\left[\sqrt{(1+8Fr_1^2)} - 1\right] \quad (18.33)$$

Strength of the jump is given by,

$$y_s = \frac{y_2}{y_1} \quad (18.34)$$

18.7.2 Length of Hydraulic Jump

The distance between the sections over which the hydraulic jump takes place is known as length of the hydraulic jump (l_j) as shown in Figure 18.10. The length of the hydraulic jump for rectangular channels with horizontal floor varies from 5 to 7 times of the depth of the hydraulic jump, i.e., $l_j = 5$ to $7\ y_j$.

18.7.3 Loss of Energy Due to Hydraulic Jump

During hydraulic jump, eddying turbulence causes considerable head or energy losses (e_L) from the flowing water. This energy loss would be equal to the difference of specific energies at the upstream section (section 1–1) and downstream section (section 2–2) of the jump, i.e., $e_L = (e_1 - e_2)$. Thus, the expression for energy loss due to hydraulic jump is given below.

$$e_L = e_1 - e_2 = \left(y_1 + \frac{q^2}{2gy_1^2}\right) - \left(y_2 + \frac{q^2}{2gy_2^2}\right) = \frac{q^2}{2g}\left[\frac{1}{y_1^2} - \frac{1}{y_2^2}\right] - (y_2 - y_1)$$

Substitute $q^2 = (1/2)gy_1y_2(y_2 + y_1)$ obtained from (iii) in the above expression, we get:

$$e_L = \frac{(1/2)gy_1y_2(y_2+y_1)}{2g}\left[\frac{y_2^2 - y_1^2}{y_1^2 y_2^2}\right] - (y_2 - y_1)$$

$$e_L = \frac{(y_2+y_1)}{4} \times \frac{(y_2+y_1)(y_2-y_1)}{y_1y_2} - (y_2 - y_1) = (y_2 - y_1)\left[\frac{(y_2+y_1)^2}{4y_1y_2} - 1\right]$$

$$e_L = (y_2 - y_1)\left[\frac{y_2^2 + y_1^2 + 2y_2y_1 - 4y_1y_2}{4y_1y_2}\right] = (y_2 - y_1)\left[\frac{y_2^2 + y_1^2 - 2y_2y_1}{4y_1y_2}\right]$$

$$\therefore e_L = (y_2 - y_1)\frac{(y_2 - y_1)^2}{4y_1y_2} = \frac{(y_2 - y_1)^3}{4y_1y_2} \quad (18.35)$$

Power dissipated in hydraulic jump is given by,

$$P = \frac{\rho_w g Q e_L}{1000}\ kW \quad (18.36)$$

Example 18.19 A 2.5 m wide rectangular channel conveys 7.2 m^3/s of water. If the velocity of water before the jump is 4.8 m/s, then find (i) the condition for hydraulic jump to happen (ii) the height, length and strength of the jump and (iii) the loss of energy per kg of water.

Solution

Let $b = 2.5$ m, $Q = 7.2\ m^3/s$ and $V_1 = 4.8$ m/s.

(i) $q = \frac{Q}{b} = \frac{7.2}{2.5} = 2.88\ m^3/s$ per m

$$y_1 = \frac{q}{V_1} = \frac{2.88}{4.88} = 0.5902 \text{ m}$$

$$y_c = \left(\frac{q^2}{g}\right)^{1/3} = \left(\frac{2.88^2}{9.81}\right)^{1/3} = 0.9456 \text{ m}$$

Since $y_1 < y_c$, a jump would occur.

(ii) $Fr_1 = \frac{V_1}{\sqrt{gy_1}} = \frac{4.8}{\sqrt{9.81 \times 0.5902}} = 1.995$

$$y_2 = \frac{y_1}{2}[\sqrt{(1+8Fr_1^2)} - 1] = \frac{0.5902}{2} \times [\sqrt{(1+8\times1.995^2)} - 1] = 1.396 \text{ m}$$

$$\therefore y_j = y_2 - y_1 = 1.396 - 0.5902 = \mathbf{0.8058\ m}$$

$$\therefore l_j \approx 6y_j = 6 \times 0.8058 = \mathbf{4.8348\ m}$$

$$y_s = \frac{y_2}{y_1} = \frac{1.396}{0.5902} = \mathbf{2.3653}$$

(iii) $e_L = \frac{(y_2 - y_1)^3}{4y_1y_2} = \frac{(1.396 - 0.5902)^3}{4 \times 0.5902 \times 1.396} = \mathbf{0.159\ m}$

Example 18.20 A hydraulic jump takes place in a 0.6 m wide rectangular channel at a point where the depth of water flow is 0.16 m and the Froude number is 2.4. Find (i) the specific energy, (ii) critical and sequent depth, (iii) loss of head and (iv) power dissipated in jump.

Solution

Let $b = 0.6$ m, $y_1 = 0.16$ m and $Fr_1 = 2.4$.

Since
$$Fr_1 = \frac{V_1}{\sqrt{gy_1}}$$

$$\therefore V_1 = Fr_1 \times \sqrt{gy_1} = 2.4 \times \sqrt{9.81 \times 0.16} = 3.01 \text{ m/s}$$

$$q = V_1 y_1 = 3.01 \times 0.16 = 0.4816\ m^3/s \text{ per m}$$

(i) $e_1 = y_1 + \frac{V_1^2}{2g} = 0.16 + \frac{3.01^2}{2 \times 9.81} = \mathbf{0.6218\ m}$

(ii) $y_c = \left(\frac{q^2}{g}\right)^{1/3} = \left(\frac{0.4816^2}{9.81}\right)^{1/3} = \mathbf{0.287\ m}$

Sequent depth (y_2) is given as,

$$y_2 = \frac{y_1}{2}[\sqrt{(1+8Fr_1^2)} - 1] = \frac{0.16}{2} \times [\sqrt{(1+8\times 2.4^2)} - 1] = \mathbf{0.4689\ m}$$

(iii) $e_L = \frac{(y_2 - y_1)^3}{4y_1y_2} = \frac{(0.4689-0.16)^3}{4\times 0.16\times 0.4689} = \mathbf{0.0982\ m}$

(iv) $Q = A_1 \times V_1 = by_1 \times V_1 = 0.6\times 0.16\times 3.01 = 0.289\ m^3/s$

$$\therefore P = \frac{\rho_w g Q e_L}{1000} = \frac{1000\times 9.81\times 0.289\times 0.0982}{1000} = \mathbf{0.2784\ kW}$$

Summary

1. The flow of water through a passage in which free liquid surface is open to atmosphere is called open channel flow.
2. When the flow characteristics at any point in the flow do not change with respect to time, then the flow is called steady. Otherwise, it is called unsteady flow.
3. When the depth, slope, cross section, and the velocity of flow for a given length of a channel remain constant, then the flow is called uniform. Otherwise, it is called non-uniform flow.
4. Reynolds number (Re) can be given as $\text{Re} = [(\rho Vm)/\mu]$, here ρ is the density of the liquid, V is the average velocity of flow, μ is the viscosity of the liquid and m is the hydraulic radius that can be defined as the ratio of cross-sectional area of flow to the wetted perimeter, i.e., $m = (A/P)$.
5. In open channel flow, (i) for laminar flow: $\text{Re} < 500$, (ii) for turbulent flow: $\text{Re} > 2000$, (iii) for transitional flow: $500 < \text{Re} < 2000$.
6. Froude number can be given as $Fr = [V/\sqrt{gD}]$, here V is the mean velocity of flow, g is acceleration due to gravity and D is the hydraulic depth of channel section which is defined as the ratio of wetted area to the top width of the channel, i.e., $D = (A/T)$.
7. (i) For sub-critical flow: $Fr < 1$, (ii) for critical flow: $Fr = 1$, (iii) for supercritical flow: $Fr > 1$.
8. Discharge (Q) through the channel is given by Chezy's formula as $Q = AC\sqrt{mi}$, here A is the wetted cross-sectional area of channel, C is the Chezy's constant, $m = (A/P)$ is the hydraulic radius and i is the slope of the bed of the channel.
9. The value of C can be determined by the following empirical formulae:

   (i) $C = \frac{157.6}{1.81 + (k/\sqrt{m})}$ (Bazin formula)

   Here, k is the Bazin's constant.

   (ii) $C = \frac{23 + (0.00155/i) + (1/N)}{1 + [23 + (0.00155/i)] \times (N/\sqrt{m})}$ (Kutter's formula)

   Here, N is the Kutter's constant

   (iii) $C = m^{1/6}/N$ (Manning's formula)

   Here, N is the Manning's constant
10. The channel cross section corresponding to the minimum perimeter for a given flow area is called the most economical section.
11. The rectangular channel section will be most economical when the depth of flow is equal to half the base width or the hydraulic radius is equal to half the depth of flow.
12. The trapezoidal channel section will be most economical when half of the top width is equal to one of the sloping sides, i.e., $(b+2ny)/2 = y\sqrt{n^2+1}$ and the hydraulic radius is half of the flow depth, i.e., $m = y/2$.
13. The best side slope for trapezoidal channel section is at 60° to the horizontal.
14. Maximum velocity through a circular channel will occur when the depth of flow is 0.81 times the diameter of the circular pipe and $m = 0.3D$.

15. Maximum discharge through a circular channel will occur when the depth of flow is 0.95 times the diameter of the circular pipe and $m = 0.29D$.
16. **Specific energy curve:** Curve showing specific energy variation with depth of flow.
17. **Specific energy:** $e = y + q^2/(2gy^2)$, here q is the discharge per unit width.
18. **Critical depth:** $y_c = \left(q^2/g\right)^{1/3}$ and

    critical velocity: $V_c = \sqrt{gy_c}$
19. Minimum specific energy: $e_{\min} = 1.5y_c$
20. Condition for maximum discharge for a given value of specific energy: $e = 1.5y$ or the depth of flow should be critical
21. The phenomenon of sudden increase in depth of water flow due to conversion of shooting flow (supercritical flow) into stable streaming flow (sub-critical flow) is termed as hydraulic jump.
22. The depth (height) of hydraulic jump (y_j) is given as $y_j = y_2 - y_1$, here y_1 is the depth of flow upstream to the jump and y_2 is the depth of flow downstream to the jump and it given by the following expressions.

    $$y_2 = \frac{-y_1}{2} + \sqrt{\frac{y_1^2}{4} + \frac{2V_1^2 y_1}{g}} \text{ and } y_2 = \frac{y_1}{2}\left[\sqrt{\left(1 + 8Fr_1^2\right)} - 1\right]$$
23. Strength of the jump is given by $y_s = y_2/y_1$.
24. The distance between the sections over which the hydraulic jump takes place is known as length of the hydraulic jump (l_j). For rectangular channels with horizontal floor, it is given by $l_j = 5$ to $7\ y_j$.
25. Energy loss due to hydraulic jump: $e_L = (y_2 - y_1)^3/(4y_1y_2)$
26. Power dissipated in hydraulic jump:

    $P = [(\rho_w gQe_L)/1000]$ kW

Multiple-choice Questions

1. In an open channel, the conjugate depths of flow are the depths
   (a) At which total energy is same.
   (b) Of same specific force.
   (c) Which occur at the same specific energy.
   (d) None of the above.
2. The specific energy at the critical depth will be
   (a) Unity.
   (b) Minimum.
   (c) Maximum.
   (d) None of the above.
3. The channel flow is sub-critical when Froude number (Fr)
   (a) $Fr > 1$.
   (b) $Fr = 1$.
   (c) $Fr < 1$.
   (d) None of the above.
4. In most economical rectangular channel section, the depth is kept equal to
   (a) Hydraulic mean depth.
   (b) One third of the width.
   (c) One fourth of the width.
   (d) Half the width.
5. When Froude number is equal to unity, the flow in open channel is
   (a) Tranquil flow.
   (b) Streaming flow.
   (c) Shooting flow.
   (d) Critical flow.
6. The best side slope with the horizontal for the most economical trapezoidal channel is
   (a) 60°.
   (b) 45°.
   (c) 30°.
   (d) None of the above.
7. The discharge through a trapezoidal channel is maximum when
   (a) Half of top width is equal to sloping side.
   (b) Top width is equal to half of sloping side.
   (c) Top width is 1.5 times the sloping side.
   (d) None of the above.
8. The wetted perimeter for a circular channel in terms of radius (R) and angle (α) is equal to
   (a) $R\alpha$.
   (b) $2R\alpha$.
   (c) $R\alpha/2$.
   (d) $3R\alpha$.

Review Questions

1. What do you mean by flow in open channel? What are the types of flow in open channels? Why bed slope is provided in open channels?
2. Define the following terms, (i) depth of flow, (ii) top width, (iii) wetted area, (iv) wetted perimeter, (v) hydraulic radius and (vi) hydraulic depth.

3. Derive an expression for the Chezy's formula.
4. Define the term most economical section of a channel. How it is determined for rectangular channel section?
5. Show that the hydraulic mean depth of a trapezoidal channel having the best proportion is half of the minimum depth.
6. Prove that for the trapezoidal channel of most economical section half of top width is equal to length of one of the sloping side.
7. Derive the condition for the best side slope of the most economical trapezoidal channel section.
8. State and prove the conditions of maximum discharge and maximum velocity for circular channel section.
9. What do you mean by specific energy? Also explain a specific energy curve.
10. Derive expressions for critical depth and critical velocity.
11. Derive an expression for minimum specific energy in terms of critical depth.
12. Derive an expression for conditions of maximum discharge for a given value of specific energy.
13. Define hydraulic jump. Derive expressions for height of hydraulic jump and energy loss during it.

Problems

1. Calculate the rate of flow and conveyance for a rectangular channel 7 m wide and at a depth of 2 m for uniform flow. The bed slope of the channel is 1 in 1000 and the Chezy's constant is 55. Also state whether the flow is tranquil or rapid in nature.
[**Ans.** 27.468 m^3/s, 868.77, tranquil]
2. Determine the discharge through a rectangular channel 3 m wide if the depth of water is 2 m and the bed slope is 1 in 2000. Take the value of Bazin's constant as 2.36.
[**Ans.** 4.49 m^3/s]
3. Using Kutter's formula, determine the flow rate for a rectangular channel having depth of water 3 m, width 4 m, bed slope 1 in 2000 and Kutter's constant $N = 0.03$.
[**Ans.** 10.185 m^3/s]
4. Find the discharge through a rectangular channel if its width is 2 m, bed slope is 1 in 1500, depth of flow is 1.5 m and the Manning's constant is 0.012.
[**Ans.** 4.5924 m^3/s]
5. Calculate the bed slope of a trapezoidal channel 5 m wide at base and having 3 m depth of water if the side slope is 2 horizontal to 3 vertical and the discharge through the channel is 25 m^3/s. Use Manning's formula and take constant as 0.03.
[**Ans.** 1/1612.9]
6. A rectangular channel is to be dug in the rocky portion of a soil. Determine its most economical section if it is to carry 10000 litres of water per second with an average velocity of 4 m/s. Take Chezy's constant as 50.
[**Ans.** 2.236 m, 1.118 m, (1/87.34)]
7. A rectangular channel having a bed slope of 1 in 2000 is 3 m wide. Determine the maximum discharge through the channel if Chezy's constant is 50.
[**Ans.** 4.357 m^3/s]
8. Determine the most economical section of rectangular channel carrying water at a rate of 0.5 m^3/s when bed slope is 1 in 2000 and Chezy's constant is 50.
[**Ans.** 0.631 m, 1.262 m]
9. A trapezoidal channel has side slopes of 3 horizontal to 4 vertical and the slope of its bed is 1 in 2000. Determine the optimum dimensions of the channel, if it is to carry water at 0.5 m^3/s. Take Chezy's constant as 80.
[**Ans.** 0.55 m]
10. A trapezoidal channel having the side slope equal to 60° with the horizontal and laid on a slope of 1 in 750, carries a discharge of 10 m^3/s. Determine the width at the base and depth of flow for most economical cross section. Take Chezy's constant as 66.
[**Ans.** 1.63 m, 1.88 m]
11. An open channel of most economical section having the form of a half hexagon with horizontal bottom is required to give a maximum discharge of 8.5 m^3/s of water. The slope of the channel bottom is 1 in 2500. Determine the dimensions of the cross section taking Chezy's constant as 60.
[**Ans.** 2.02 m, 2.33 m]
12. A trapezoidal channel is designed to carry 2.4 m^3/s of water for minimum cross section. Determine the bottom width and depth if the bed slope is 1 in 1000, the side slopes are at 45° and the Chezy's coefficient is 56.
[**Ans.** 1.02 m, 0.844 m]
13. A trapezoidal channel has side slopes of 1 horizontal to 2 vertical and the slope of the bed is 1 in 1500. The area of the section is 40 m^2. Determine the dimensions of the section if it is most economical. Also find the discharge of the most economical section when the Chezy's constant is 54.
[**Ans.** 4.8 m, 5.93 m, 86.4 m^3/s]

14. A pipe of 2 m diameter is laid down with 5° inclination to the horizontal ground. Determine the discharge through the pipe when the depth of water in the pipe is 0.8 m. Take Chezy's constant as 70.

[**Ans.** 19.53 m^3/s]

15. Find the discharge through a pipe of diameter 3 m, if the depth of water in the pipe is 2.5 m and it is laid at a slope of 1 in 1000. Take Chezy's constant as 65.

[**Ans.** 8.973 m^3/s]

16. The discharge of water through a pipe of diameter 60 cm is 0.15 m^3/s. Determine the slope of the bed of the channel for maximum velocity when the Chezy's constant is 50.

[**Ans.** 1 in 1177]

17. Determine the maximum discharge of water through a circular channel of diameter 1.5 m when the bed slope of the channel is 1 in 1000. Take Chezy's constant as 55.

[**Ans.** 1.5674 m^3/s]

18. A concrete lined circular channel of diameter 3 m has a bed slope of 1 in 500. Determine the velocity and flow rate for the conditions maximum velocity and maximum discharge. Assume Chezy's constant as 50.

[**Ans.** 9.536 m^3/s, 11.406 m^3/s]

19. The specific energy for a 3 m wide channel is to be 3 Nm/N. What would be the maximum possible discharge?

[**Ans.** 26.58 m^3/s]

20. An open channel of rectangular cross section 4 m wide is laid at a slope of 1 in 800. The depth of water in the channel is 2 m. To obtain critical flow, a hump is proposed in its bed. Find the height of the hump if the value of Manning's rugosity coefficient is 0.02.

[**Ans.** 0.533 m]

21. A sluice gate discharges water into horizontal rectangular channel with a velocity of 12 m/s and depth of flow of 1.2 m. Calculate (i) the depth of flow of water after the jump and (ii) consequent loss in total head.

[**Ans.** 5.366 m, 2.807 m]

22. A sluice gate discharges water with a velocity of 6.2 m/s and depth of flow 0.42 m into a horizontal rectangular channel of width 8.2 m. Find the condition for hydraulic jump, its height, loss of energy per kg of water and the power lost in the jump.

[**Ans.** 1.196 m, 0.63 m-kg/kg of water, 131.967 kW]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (c) **2.** (b) **3.** (c) **4.** (d) **5.** (d)
6. (a) **7.** (a) **8.** (b)

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Chapter **19**

Dimensional Analysis and Model Similitude

19.1 ❒ INTRODUCTION

Dimensional analysis is a powerful mathematical tool for engineers and scientists. It combines the dimensional variables, non-dimensional variables and dimensional constants into non-dimensional parameters. Thus, it also reduces the number of necessary independent parameters in a problem and it systematically arranges them into dimensionless groups. Dimensional analysis finds applications in all fields of engineering. Especially, it is very useful when it is necessary to design and perform experiments. There are two commonly used dimensional analysis methods, namely Rayleigh method and Buckingham π method.

Generally, the model is a small scale replica of the actual machine or turbine. The actual machine is called a prototype. The study to determine the performance of machines (turbines) by conducting various tests on their models is called model analysis. The similarity between the model and its prototype is known as similitude. The results obtained from experiments in a model can be applied to its prototype only if a complete similarity exists between them. For existing complete similarity between the model and its prototype, three types of similarities are to be established, namely geometric similarity, kinematic similarity and dynamic similarity.

19.2 ❒ DIMENSIONS AND UNITS OF PHYSICAL QUANTITIES

All physical quantities are expressed by magnitudes and units. A dimension is a measure of physical quantity without numerical values (i.e., qualitative characteristics), while a unit is a method to assign a number to that dimension (i.e., quantitative characteristics). For example, length is a dimension, but metre is a unit.

Various physical quantities independent of each other, which is used to describe a phenomenon are called primary quantities (also called fundamental or basic quantities). According to SI system, there are seven primary quantities, namely mass, length, time, temperature, electric current, amount of substance and luminous intensity. Generally, the primary quantities, such as mass, length and time are used in fluid mechanics, and are assigned the dimensions of [M], [L] and [T], respectively.

All other physical quantities such as velocity, force, work and power can be expressed in terms of primary quantities and they are called secondary or derived quantities. For example, the dimensions of velocity are determined by the following expression.

$$\text{Velocity} = \frac{\text{Distance}}{\text{Time}} = \frac{L}{T} = [LT^{-1}]$$

The dimensions of various physical quantities in M-L-T system used in fluid mechanics are presented in Table 19.1.

Table 19.1 *Dimensions of various physical quantities*

Quantity	Dimensions	Quantity	Dimensions
Mass (m)	$[M]$	Momentum (M)	$[MLT^{-1}]$
Length (L)	$[L]$	Power (P)	$[ML^2T^{-3}]$
Time (t)	$[T]$	Frequency (n)	$[T^{-1}]$
Temperature (T)	$[\theta]$	Pressure (p), stress (τ)	$[ML^{-1}T^{-2}]$
Diameter, radius	$[L]$	Velocity potential	$[L^2T^{-1}]$
Area (A)	$[L^2]$	Surface tension (σ)	$[MT^{-2}]$
Volume (v)	$[L^3]$	Dynamic viscosity (μ)	$[ML^{-1}T^{-1}]$
Speed, velocity (V)	$[LT^{-1}]$	Kinematic viscosity (v)	$[L^2T^{-1}]$
Angular speed (ω)	$[T^{-1}]$	Moment of inertia (I)	$[L^4]$
Acceleration (a)	$[LT^{-2}]$	Discharge (Q)	$[L^3T^{-1}]$
Angular acceleration (α)	$[T^{-2}]$	Gravitational acceleration (g)	$[LT^{-2}]$
Density (ρ)	$[ML^{-3}]$	Specific weight (w)	$[ML^{-2}T^{-2}]$
Impulse	$[MLT^{-1}]$	Modulus of elasticity (E)	$[ML^{-1}T^{-2}]$
Force, thrust (F)	$[MLT^{-2}]$	Compressibility ($1/E$)	$[M^{-1}LT^2]$
Weight (W)	$[MLT^{-2}]$	Gas constant (R)	$[L^2T^{-2}\theta^{-1}]$
Angular momentum	$[ML^2T^{-1}]$	Vorticity (Ω)	$[T^{-1}]$
Moment, torque (T)	$[ML^2T^{-2}]$	Stream function (ψ)	$[L^2T^{-1}]$
Work, energy	$[ML^2T^{-2}]$	Circulation (Γ)	$[L^2T^{-1}]$
Enthalpy, quantity of heat	$[ML^2T^{-2}]$	Thermal conductivity (k)	$[MLT^{-3}\theta^{-1}]$
Specific heat	$[L^2T^{-2}\theta]$	Entropy (s)	$[ML^2T^{-2}\theta^{-1}]$

Example 19.1 Find the dimensions of the following physical quantities: (i) force (F), (ii) pressure (p), (iii) dynamic viscosity (μ), (iv) surface tension (σ), (v) work, (vi) momentum (M), (vii) discharge (Q), (viii) kinematic viscosity (v), (ix) power (P) and (x) modulus of elasticity (E).

Solution

(i) $F = ma = [M][LT^{-2}] = [MLT^{-2}]$

(ii) $p = \dfrac{\text{Force}}{\text{Area}} = \dfrac{[MLT^{-2}]}{[L^2]} = [ML^{-1}T^{-2}]$

(iii) $\mu = \dfrac{\tau}{(\partial u / \partial y)} = \dfrac{\text{Force/Area}}{(\partial u / \partial y)} = \dfrac{[MLT^{-2}]/[L^2]}{[LT^{-1}]/[L]} = [ML^{-1}T^{-1}]$

(iv) $\sigma = \dfrac{\text{Force}}{\text{Length}} = \dfrac{[MLT^{-2}]}{[L]} = [MT^{-2}]$

(v) $\text{Work} = \text{Force} \times \text{Distance} = [MLT^{-2}][L] = [ML^2T^{-2}]$

(vi) $M = \text{Mass} \times \text{Velocity} = [M][LT^{-1}] = [MLT^{-1}]$

(vii) $Q = \dfrac{\text{Volume}}{\text{Time}} = \dfrac{[L^3]}{[T]} = [L^3T^{-1}]$

(viii) $\nu = \dfrac{\mu}{\rho} = \dfrac{[ML^{-1}T^{-1}]}{[ML^{-3}]} = [L^2T^{-1}]$

(ix) $P = \dfrac{\text{Work}}{\text{Time}} = \dfrac{[ML^2T^{-2}]}{[T]} = [ML^2T^{-3}]$

(x) $E = \dfrac{\text{Stress}}{\text{Strain}} = \dfrac{\text{Force}}{\text{Area}} = \dfrac{[MLT^{-2}]}{[L^2]} = [ML^{-1}T^{-2}]$

19.3 ❐ DIMENSIONAL HOMOGENEITY

The law of dimensional homogeneity states that every additive term in an equation must have the same dimensions. In other words, two quantities may be added or subtracted only when they have the same dimensions. According to this law, any mathematical equation which correctly expresses a physical phenomenon should be dimensionally homogeneous. Therefore, the dimensions of the terms on the left hand side of the equation are identical as in its right hand side. A dimensional homogeneous equation is true and can be used for all systems of units without any modification.

Let us consider the equation for the time of swing of a simple pendulum as given below.

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Dimensions of the L.H.S. = $[T]$

On the R.H.S. of the equation, 2π is a constant and therefore, it has no dimension.

$$\text{Dimensions of the R.H.S.} = \left[\frac{L}{LT^{-2}}\right]^{1/2} = [T]$$

As the dimensions of the terms on both sides of the equation are same, it is dimensionally homogeneous and can be used in any system of units.

There are many equations which are dimensionally non-homogeneous and are applicable to a flow system. However, such equations will be valid only in a particular system of units.

Applications of principle of dimensional homogeneity The applications of dimensional homogeneity is used in the following ways.

(i) It helps in determining the dimensions of a physical quantity.
(ii) It helps in the conversion of units from one system to another.
(iii) It checks whether an equation is dimensionally homogeneous or not.

19.4 ❒ METHODS OF DIMENSIONAL ANALYSIS

Generally, there are two commonly used methods of dimensional analysis, namely Rayleigh method and Buckingham π-method.

19.4.1 Rayleigh Method

This method was proposed by Lord Rayleigh in 1899 for determining the effect of temperature on the viscosity of a gas. This method is also known as method of indices. It is used to determine the expression for a variable which depends upon maximum three or four variables only. It becomes difficult to obtain an expression for dependent variables when the numbers of independent variables are more than four. In this method, a functional relationship of some variables is expressed in the form of an exponential equation which must be dimensionally homogeneous.

Let x be a variable which depends upon x_1, x_2 and x_3 variables. The functional relation can be written as given below.

$$x = f(x_1, x_2, x_3)$$

The above equation can also be written as given below.

$$x = C(x_1^a\ x_2^b\ x_3^c)$$

In the above expression, C is a dimensionless constant which is either determined from physical characteristics of the problem or from experimental results and a, b and c are arbitrary powers.

The values of a, b and c are obtained by comparing the powers of the basic dimensions on both sides. Thus, the expression for the dependent variable can be obtained.

Consider the problem of the pendulum time period (T) which depends on the length of the pendulum (l) and acceleration due to gravity (g). The functional relationship for T can be written as follows.

$$T = f(l, g)$$

According to Rayleigh method, the above equation may be expressed in exponential form as follows.

$$T = C(l^a . g^b) \qquad \text{(i)}$$

Here, C is a dimensionless constant.

Substituting the dimensions for each variable on both sides, we get:

$$[T] = [1][L]^a\ [LT^{-2}]^b$$

The dimension of C being a constant is considered as unity.

For dimensional homogeneity, the powers of each dimension on both sides of the equation must be same.

For $$T: \quad 1 = -2b \qquad \text{(ii)}$$

For $$L: \quad 0 = a + b \qquad \text{(iii)}$$

From expressions (ii) and (iii), we get:

$$b = -(1/2),\ a = -b = (1/2)$$

Substituting the values of a and b in expression (i) and rearranging, we get:

$$T = C(L^{1/2}\ g^{-(1/2)})$$

Thus $$T = C\sqrt{\frac{l}{g}}$$

Here, $C = 2\pi$, which is determined experimentally.

Example 19.2 Find an expression for the drag force F on smooth sphere of diameter D, moving with a uniform velocity V in a fluid of density ρ and dynamic viscosity μ. Apply Rayleigh method of dimensional analysis.

Solution

The functional relationship for the drag force F can be given by,

$$F = f(D, V, \rho, \mu)$$

According to Rayleigh method, the above equation may be expressed in exponential form as given below.

$$F = C(D^a\ V^b\ \rho^c\ \mu^d) \tag{i}$$

Here, C is a dimensionless constant.

Substituting the dimensions for each variable on both sides, we get:

$$[MLT^{-2}] = [1]\ [L]^a\ [LT^{-1}]^b\ [ML^{-3}]^c\ [ML^{-1}T^{-1}]^d$$

The dimension of C being a constant is considered as unity.

For dimensional homogeneity, the powers of each dimension on both sides of the equation must be same.

For $$M:\quad 1 = c + d \tag{ii}$$

For $$L:\quad 1 = a + b - 3c - d \tag{iii}$$

For $$T:\quad -2 = -b - d \tag{iv}$$

There are four variables and we have three equations. Therefore, expressing a, b and c in terms of d, we get the following expressions.

$$c = 1 - d,\ b = 2 - d \text{ and}$$

$$a = 1 - b + 3c + d = 1 - (2 - d) + 3(1 - d) + d = 2 - d$$

Substituting these values in the expression (i) and rearranging, we get:

$$F = C(D^{2-d}\ V^{2-d}\ \rho^{1-d}\ \mu^d) = C\rho D^2 V^2 \left(\frac{\mu}{\rho VD}\right)^d$$

$$\therefore F = \boldsymbol{\rho D^2 V^2 \phi\left[\frac{\mu}{\rho VD}\right]}$$

Example 19.3 The efficiency η of a fan depends on the density ρ, the dynamic viscosity μ of the fluid, the angular velocity ω, diameter D of the rotor and the discharge Q. Express η in terms of dimensionless parameters using Rayleigh method of dimensional analysis.

Solution

The functional relationship for efficiency η is given by,

$$\eta = f(\rho, \mu, \omega, D, Q)$$

According to Rayleigh method, the above equation may be expressed in exponential form as given below.

$$\eta = C(\rho^a \ \mu^b \ \omega^c \ D^d \ Q^e) \tag{i}$$

Here, C is a dimensionless constant.

Substituting the dimensions for each variable on both sides, we get:

$$[M^0 L^0 T^0] = [1] \ [ML^{-3}]^a \ [ML^{-1}T^{-1}]^b \ [T^{-1}]^c \ [L]^d \ [L^3T^{-1}]^e$$

The dimension of C being a constant is considered as unity.

For dimensional homogeneity, the powers of each dimension on both sides of the equation must be same.

For $\quad M: \quad 0 = a + b \quad$ (ii)

For $\quad L: \quad 0 = -3a - b + d + 3e \quad$ (iii)

For $\quad T: \quad 0 = -b - c - e \quad$ (iv)

There are five variables and we have three equations. In the given problem, viscosity and discharge are more important. Therefore, expressing a, b and c in terms of b and e, we get the following expressions.

$$a = -b \,, \ c = -b - e \ \text{ and}$$

$$d = 3a + b - 3e = 3(-b) + b - 3e = -2b - 3e$$

Substituting these values in expression (i) and rearranging, we get:

$$\eta = C(\rho^{-b} \ \mu^b \ \omega^{-b-e} \ D^{-2b-3e} \ Q^e) = C\left[\left(\frac{\mu}{\rho\omega D^2}\right)^b \left(\frac{Q}{\omega D^3}\right)^e\right]$$

$$\therefore \eta = \phi\left[\left(\frac{\mu}{\rho\omega D^2}\right), \left(\frac{Q}{\omega D^3}\right)\right]$$

Example 19.4 The thrust F of a propeller depends on its diameter D, the flow velocity V, the fluid density ρ, the revolution per minute N and the viscosity of the fluid μ. Obtain an expression for F in terms of given parameters by using Rayleigh method of dimensional analysis.

Solution

The functional relationship for thrust F is given by,

$$F = f(D, V, \rho, N, \mu)$$

According to Rayleigh method, the above equation may be expressed in exponential form as given below.

$$F = C(D^a \ V^b \ \rho^c \ N^d \ \mu^e) \tag{i}$$

Here, C is a dimensionless constant.

Substituting the dimensions for each variable on both sides, we get:

$$[MLT^{-2}] = [1] \ [L]^a \ [LT^{-1}]^b \ [ML^{-3}]^c \ [T^{-1}]^d \ [ML^{-1}T^{-1}]^e$$

The dimension of C being a constant is considered as unity.

For dimensional homogeneity, the powers of each dimension on both sides of the equation must be same.

For $M: \quad 1 = c + e$ (ii)

For $L: \quad 1 = a + b - 3c - e$ (iii)

For $T: \quad -2 = -b - d - e$ (iv)

There are five variables and we have three equations. Therefore, expressing *a*, *b* and *c* in terms of *d* and *e*, we get the following expressions.

$$c = 1 - e\,,\ b = 2 - d - e \text{ and}$$

$$a = 1 - b + 3c + e = 1 - (2 - d - e) + 3(1 - e) + e = 2 + d - e$$

Substituting these values in the expression (i) and rearranging, we get:

$$F = C(D^{2+d-e}\ V^{2-d-e}\ \rho^{1-e}\ N^d\ \mu^e) = C\rho D^2 V^2 \left(\frac{\mu}{\rho VD}\right)^e \left(\frac{DN}{V}\right)^d$$

$$\therefore \frac{F}{\rho D^2 V^2} = \phi\left[\frac{\mu}{\boldsymbol{\rho VD}}, \frac{\boldsymbol{DN}}{\boldsymbol{V}}\right]$$

19.4.2 Buckingham π Method

When the number of variables in a phenomenon becomes considerably large, the Rayleigh method becomes tedious. This difficulty can be avoided by using Buckingham π method. This method was proposed in 1914 by E. Buckingham and is now known as the Buckingham Pi theorem. This theorem states that if there are *n* dimensional variables (dependent or independent) involved in a dimensional homogeneous equation which contains *m* fundamental quantities, then the variables can be grouped into $(n - m)$ dimensionless and independent terms. These independent dimensionless terms are called π terms.

Let x_1 be a variable which depends on independent variables $x_2, x_3, x_4 .....x_n$. The functional relation can be written as given below.

$$x_1 = f(x_2, x_3, x_4, ... x_n) \quad \text{(i)}$$

The above equation can also be written as follows.

$$f_1(x_1, x_2, x_3, x_4, ... x_n) = 0 \quad \text{(ii)}$$

It is a dimensionally homogenous equation containing *n* variables. If there are *m* fundamental quantities (such as *M*, *L*, *T*), then according to Buckingham π theorem, equation (ii) can be represented in $(n - m)$ number of π terms. Thus, equation (ii) is expressed as follows.

$$f_1(\pi_1, \pi_2, \pi_3, \pi_4, ... \pi_{n-m}) = 0 \quad \text{(iii)}$$

Each dimensionless π term is formed by combining *m* variables out of the total *n* variables with one of the remaining $(n - m)$ variables. It means each π term contains $(m + 1)$ variables. The *m* variables which is repeatedly used in π terms are called repeating variables. The repeating variables should be such that they together involve all the *m* fundamental quantities and they themselves do not form a dimensionless parameter. Let x_2, x_3 and x_4 be the repeating variables and $m\,(M, L, \text{and } T) = 3$. Thus, the different π terms can be expressed as given below.

$$\pi_1 = x_2^{a_1}\ x_3^{b_1} x_4^{c_1} x_1$$

$$\pi_2 = x_2^{a_2}\ x_3^{b_2} x_4^{c_2} x_5$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\pi_{n-m} = x_2^{a_{n-m}}\ x_3^{b_{n-m}} x_4^{c_{n-m}} x_n$$

The exponents a_1, b_1 and c_1, a_2, b_2 and c_2, ..., *etc.* in the above equations are determined by considering dimensional homogeneity for each equation so that each π term is dimensionless. The final general equation for the phenomenon is represented by expressing any one of the π term as a function of the others or any other required relationship may also be obtained.

$$\pi_1 = f_1(\pi_2, \pi_3, ...\pi_{n-m})$$

$$\pi_2 = f_1(\pi_1, \pi_3, ...\pi_{n-m})$$

The limitation of this method is that the exact functional relationship in Equation (iii) cannot be obtained from the analysis. Generally, the functional relationship can be obtained by experimental results.

Selection of repeating variables While selecting repeating variables, the following points should be considered.

1. The chosen repeating variables must represent all the fundamental dimensions involved in the problem and should not have the same dimensions.
2. Never select the dependent variable as repeating variables.
3. The repeating variables should be chosen in such a way that one variable contains geometric property (Example: *L*, *D*, *H*, etc.), other variable contains flow property (Example: *V*, *a*, etc.) and the third variable contains fluid property (Example: $\rho, \mu,$ etc.). A clever choice of the repeating variables for most of the problems may be (i) L, V, ρ (ii) D, V, ρ (iii) L, V, μ and (iv) D, V, μ.

Procedure for Buckingham Pi theorem Typically, the following five steps are involved in Buckingham Pi theorem.

1. List and count the *n* variables involved in the problem. If any variables are missing, then the dimensional analysis will fail. Express the variables in terms of primary dimensions.
2. Out of the *n* variables select *m* variables which will be used as repeating variables.
3. Write the general equations for different pi terms which may be expressed as the product of the repeating variables each raised to an unknown exponent and one of remaining variables, which is taken in turn and usually, has power as one.
4. Write the dimensional equations for the equations of the π terms. Evaluate the values of the unknown exponents by equating the exponents of the respective fundamental dimensions on both sides of each of the dimensional equation and obtain the different dimensionless groups.
5. Write the final general equation for the phenomenon in terms of π terms.

Suggestions for finding the final expression In order to obtain the final expression in the desired form, some of the useful suggestions are given below.

1. Any π term may be replaced by any power of that term. For example, π_1 may be replaced by $\pi_1^{-1}, \pi_1^{2}, \pi_1^{-1/2}$.
2. Any π term may be replaced by multiplying it with a numerical constant. For example, π_1 may be replaced by $2\pi_1$.
3. Any π term may be replaced by another π term obtained by adding or subtracting an absolute numeral from it.
4. Any π term may be replaced by multiplying or dividing it by another π term to get a single π term. For example, π_1 may be replaced by $(\pi_1 \times \pi_2)$ or (π_1 / π_2).
5. A dimensionless quantity is a π term.
6. The ratio of two quantities with same dimensions will be one of the π terms.

An illustration for the procedure of solving problems by Buckingham π theorem We shall illustrate the Buckingham pi theorem by considering the problem, where 'The resistance *R* of a partially submerged body moving in water to its motion depends on the density ρ_w, viscosity μ of water, length *L* of the body, velocity *V* of the body and the acceleration due to gravity *g*. Express the functional relationship between the given variables and the resistance *R*'.

Solution

The following steps may be adopted to solve it by Buckingham π method.

Step 1: The problem can be expressed as,

$$R = f(\rho_w, \mu, L, V, g) \tag{i}$$

or

$$f_1(R, \rho_w, \mu, L, V, g) = 0 \tag{ii}$$

Total number of variables: $n = 6$

Writing dimensions of each term, we get:

$$R = [MLT^{-2}];\ \rho_w = [ML^{-3}];\ \mu = [ML^{-1}T^{-1}];\ L = [L];\ V = [LT^{-1}];\ g = [LT^{-2}]$$

Thus, the fundamental dimensions in the problem are M, L, T and hence, $m = 3$

Therefore, number of π terms = $n - m = 6 - 3 = 3$

Thus, three π terms say π_1, π_2 and π_3 are formed.

Thus, Equation (i) may be written as,

$$f_1(\pi_1, \pi_2, \pi_3) = 0 \tag{iii}$$

Step 2: Since $m = 3$, we have to choose 3 repeating variables. R is a dependent variable and thus, it cannot be selected as a repeating variable. Out of the remaining five variables, the repeating variables should be chosen in such a way that they contain all the three fundamental dimensions and they themselves do not form a dimensionless parameter. Therefore, choosing length L, velocity V and density ρ_w as the repeating variables.

Step 3: Each π term contains $(m+1)$ variables and it is given by,

$$\pi_1 = L^{a_1} V^{b_1} \rho_w^{c_1} R \tag{a}$$

$$\pi_2 = L^{a_2} V^{b_2} \rho_w^{c_2} \mu \tag{b}$$

$$\pi_3 = L^{a_3} V^{b_3} \rho_w^{c_3} g \tag{c}$$

Step 4: Expressing each π term in terms of *M-L-T* system and solving it by the principle of dimensional homogeneity.

(i) **For π_1 term:**

$$\pi_1 = L^{a_1} V^{b_1} \rho_w^{c_1} R \tag{a}$$

$$M^0 L^0 T^0 = [L]^{a_1} [LT^{-1}]^{b_1} [ML^{-3}]^{c_1} [MLT^{-2}]$$

or

$$M^0 L^0 T^0 = M^{c_1+1} L^{a_1+b_1-3c_1+1} T^{-b_1-2}$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = c_1 + 1,\ 0 = a_1 + b_1 - 3c_1 + 1,\ 0 = -b_1 - 2$$

Solution: $c_1 = -1$, $b_1 = -2$ and $a_1 = -2$

Substituting the values of a_1, b_1 and c_1 in Equation (a) and rearranging, we get:

$$\pi_1 = L^{-2} V^{-2} \rho_w^{-1} R = \frac{R}{L^2 V^2 \rho_w}$$

(ii) **For π_2 term:**

$$\pi_2 = L^{a_2}\, V^{b_2}\, \rho_w^{\;c_2}\, \mu \tag{b}$$

$$M^0 L^0 T^0 = [L]^{a_2}\, [LT^{-1}]^{b_2}\, [ML^{-3}]^{c_2}\, [ML^{-1}T^{-1}]$$

Equating the exponents of *M*, *L* and *T*, respectively, we get:

$$0 = c_2 + 1,\ 0 = a_2 + b_2 - 3c_2 - 1,\ 0 = -b_2 - 1$$

Solution gives: $c_2 = -1$, $b_2 = -1$ and $a_2 = -1$

Substituting the values of a_2, b_2 and c_2 in Equation (b) and rearranging, we get:

$$\pi_2 = L^{-1}\, V^{-1}\, \rho_w^{\;-1}\, \mu = \frac{\mu}{L\,V\,\rho_w}$$

(iii) **For π_3 term:**

$$\pi_3 = L^{a_3}\, V^{b_3}\, \rho_w^{\;c_3}\, g \tag{c}$$

$$M^0 L^0 T^0 = [L]^{a_3}\, [LT^{-1}]^{b_3}\, [ML^{-3}]^{c_3}\, [LT^{-2}]$$

Equating the exponents of *M*, *L* and *T*, respectively, we get:

$$0 = c_3,\ 0 = a_3 + b_3 - 3c_3 + 1,\ 0 = -b_3 - 2$$

Solution: $c_3 = 0$, $b_3 = -2$ and $a_3 = 1$

Substituting the values of a_3, b_3 and c_3 in Equation (c) and rearranging, we get:

$$\pi_3 = L^1\, V^{-2}\, \rho_w^{\;0}\, g = \frac{Lg}{V^2}$$

Step 5: For obtaining the functional relationship, substituting the values of π_1, π_2 and π_3 in Equation (iii), we get the below expression.

$$f_1\left(\frac{R}{L^2\, V^2\, \rho_w}, \frac{\mu}{L\,V\,\rho_w}, \frac{Lg}{V^2}\right) = 0$$

Since expression is required for resistance *R*, we get:

$$\frac{R}{L^2 V^2 \rho_w} = \phi\left(\frac{\mu}{LV\rho_w}, \frac{Lg}{V^2}\right)$$

The reciprocal of π term and its square root is non-dimensional, we get:

$$\frac{R}{L^2 V^2 \rho_w} = \phi\left(\frac{\rho_w VL}{\mu}, \frac{V}{\sqrt{Lg}}\right)$$

$$\therefore R = \boldsymbol{L^2 V^2 \rho_w \phi\left(\frac{\rho_w VL}{\mu}, \frac{V}{\sqrt{Lg}}\right)}$$

19.4.3 Advantages and Limitations of Dimensional Analysis

Advantages

(i) It gives a relationship between the variables involved in a problem in terms of dimensionless parameters which helps in performing tests on the models.

(ii) It gives rapid analysis without going into deep mathematics required for the formation of fundamental equations and thus, considerably saves work and time.

(iii) It suggests small number of experiments and simultaneously, gives reliable results.

Limitations

(i) Before applying this method, all variables must be known. Dimensional analysis only suggests a relationship between parameters and does not provide information about the nature of phenomena.

(ii) It does not give the values of coefficients in the functional relationship which can be known only by conducting experiments.

(iii) If wrong variables are taken or any variable is missing, then the correct non-dimensional group will not be formed means the relationship will be erroneous.

Example 19.5 The velocity through a circular orifice depends on the head H causing the flow, diameter of the orifice D, coefficient of viscosity μ, mass density ρ and the acceleration due to gravity g. Using Buckingham pi theorem, obtain an expression for V.

Solution

The problem can be expressed as,

$$V = f(H, D, \mu, \rho, g) \qquad \text{(i)}$$

or

$$f_1(V, H, D, \mu, \rho, g) = 0 \qquad \text{(ii)}$$

Total number of variables: $n = 6$

Writing the dimensions of each term, we get:

$$V = [LT^{-1}],\ H = [L],\ D = [L],\ \mu = [ML^{-1}T^{-1}],\ \rho = [ML^{-3}],\ g = [LT^{-2}]$$

Thus, fundamental dimensions in the problem are M, L, T and hence, $m = 3$.

Therefore, the number of π terms = $n - m = 6 - 3 = 3$.

The three π terms say π_1, π_2 and π_3 are formed.

Equation (ii) may be written as,

$$f_1(\pi_1, \pi_2, \pi_3) = 0 \qquad \text{(iii)}$$

Since $m = 3$, choosing H, g and ρ as the repeating variables.

Each π term contains $(m+1)$ variables and can be written as,

$$\pi_1 = H^{a_1}\ g^{b_1}\ \rho^{c_1}\ V \qquad \text{(a)}$$

$$\pi_2 = H^{a_2}\ g^{b_2}\ \rho^{c_2}\ D \qquad \text{(b)}$$

$$\pi_3 = H^{a_3}\ g^{b_3}\ \rho^{c_3}\ \mu \qquad \text{(c)}$$

For π_1 term:

$$\pi_1 = H^{a_1}\ g^{b_1}\ \rho^{c_1}\ V \qquad \text{(a)}$$

$$M^0 L^0 T^0 = [L]^{a_1}\ [LT^{-2}]^{b_1}\ [ML^{-3}]^{c_1}\ [LT^{-1}]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = c_1,\ 0 = a_1 + b_1 - 3c_1 + 1,\ 0 = -2b_1 - 1$$

Solution: $c_1 = 0,\ b_1 = -(1/2)$ and $a_1 = -(1/2)$

Substituting the values of a_1, b_1 and c_1 in Equation (a) and rearranging, we get:

$$\pi_1 = H^{-(1/2)}\ g^{-(1/2)}\ \rho^0\ V = \frac{V}{\sqrt{gH}}$$

For π_2 term:

$$\pi_2 = H^{a_2}\ g^{b_2}\ \rho^{c_2}\ D \tag{b}$$

$$M^0 L^0 T^0 = [L]^{a_2}\ [LT^{-2}]^{b_2}\ [ML^{-3}]^{c_2}\ [L]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = c_2,\ 0 = a_2 + b_2 - 3c_2 + 1,\ 0 = -2b_2$$

Solution: $c_2 = 0$, $b_2 = 0$ and $a_2 = -1$

Substituting the values of a_2, b_2 and c_2 in Equation (b) and rearranging, we get:

$$\pi_2 = H^{-1}\ g^0\ \rho^0\ D = \frac{D}{H}$$

For π_3 term:

$$\pi_3 = H^{a_3}\ g^{b_3}\ \rho^{c_3}\ \mu \tag{c}$$

$$M^0 L^0 T^0 = [L]^{a_3}\ [LT^{-2}]^{b_3}\ [ML^{-3}]^{c_3}\ [ML^{-1}T^{-1}]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = c_3 + 1,\ 0 = a_3 + b_3 - 3c_3 - 1,\ 0 = -2b_3 - 1$$

Solution: $c_3 = -1$, $b_3 = -(1/2)$ and $a_3 = -(3/2)$

Substituting the values of a_3, b_3 and c_3 in Equation (c) and rearranging, multiplying and dividing by V, we get:

$$\pi_3 = H^{-(3/2)}\ g^{-(1/2)}\ \rho^{-1}\ \mu = \frac{\mu}{H^{3/2}\rho\sqrt{g}} = \frac{\mu}{H\rho V}\pi_1$$

For obtaining the functional relationship, substituting the values of π_1, π_2 and π_3 in Equation (iii), we get:

$$f_1\left(\frac{V}{\sqrt{gH}},\ \frac{D}{H},\ \frac{\mu}{H\rho V}\pi_1\right) = 0$$

Since expression is required for velocity V, we get:

$$\frac{V}{\sqrt{gH}} = \phi\left(\frac{D}{H},\ \frac{\mu}{H\rho V}\pi_1\right)$$

$$\therefore V = \boldsymbol{\sqrt{gH}\ \phi\left(\frac{D}{H},\ \frac{\mu}{H\rho V}\pi_1\right)}$$

Example 19.6 Derive on the basis of dimensional analysis suitable parameters to present the thrust developed by a propeller. Given that the thrust P depends upon the angular velocity ω, speed of advance V, diameter D, dynamic viscosity μ, mass density ρ, elasticity of the fluid medium which can be denoted by the speed of sound C in the medium.

Solution

The problem can be expressed as,

$$P = f(\omega, V, D, \mu, \rho, C) \tag{i}$$

or

$$f_1(P, \omega, V, D, \mu, \rho, C) = 0 \tag{ii}$$

Total number of variables: $n = 7$

Writing the dimensions of each term, we get:

$$P = [MLT^{-2}],\ \omega = [T^{-1}],\ V = [LT^{-1}],\ D = [L],\ \mu = [ML^{-1}T^{-1}],\ \rho = [ML^{-3}],\ C = [LT^{-1}]$$

Thus, the fundamental dimensions in the problem are M, L, T and hence, $m = 3$.

Therefore, the number of π terms $= n - m = 7 - 3 = 4$.

There are four π terms say π_1, π_2, π_3 and π_4 are formed.

Equation (ii) may be written as,

$$f_1(\pi_1, \pi_2, \pi_3, \pi_4) = 0 \tag{iii}$$

Since $m = 3$, choosing *D*, *V* and ρ as the repeating variables.

Each π term contains $(m+1)$ variables and can be written as,

$$\pi_1 = D^{a_1}\ V^{b_1}\ \rho^{c_1}\ P \tag{a}$$

$$\pi_2 = D^{a_2}\ V^{b_2}\ \rho^{c_2}\ \omega \tag{b}$$

$$\pi_3 = D^{a_3}\ V^{b_3}\ \rho^{c_3}\ \mu \tag{c}$$

$$\pi_4 = D^{a_4}\ V^{b_4}\ \rho^{c_4}\ C \tag{d}$$

For π_1 term:

$$\pi_1 = D^{a_1}\ V^{b_1}\ \rho^{c_1}\ P \tag{a}$$

$$M^0 L^0 T^0 = [L]^{a_1}\ [LT^{-1}]^{b_1}\ [ML^{-3}]^{c_1}\ [MLT^{-2}]$$

Equating the exponents of *M*, *L* and *T*, respectively, we get:

$$0 = c_1 + 1,\ 0 = a_1 + b_1 - 3c_1 + 1,\ 0 = -b_1 - 2$$

Solution: $c_1 = -1$, $b_1 = -2$ and $a_1 = -2$

Substituting the values of a_1, b_1 and c_1 in Equation (a) and rearranging, we get:

$$\pi_1 = D^{-2}\ V^{-2}\ \rho^{-1}\ P = \frac{P}{\rho D^2 V^2}$$

For π_2 term:

$$\pi_2 = D^{a_2}\ V^{b_2}\ \rho^{c_2}\ \omega \tag{b}$$

$$M^0 L^0 T^0 = [L]^{a_2}\ [LT^{-1}]^{b_2}\ [ML^{-3}]^{c_2}\ [T]^{-1}$$

Equating the exponents of *M*, *L* and *T*, respectively, we get:

$$0 = c_2,\ 0 = a_2 + b_2 - 3c_2,\ 0 = -b_2 - 1$$

Solution: $c_2 = 0$, $b_2 = -1$ and $a_2 = 1$

Substituting the values of a_2, b_2 and c_2 in Equation (b) and rearranging, we get:

$$\pi_2 = D^1\ V^{-1}\ \rho^0\ \omega = \frac{D\omega}{V}$$

For π_3 term:

$$\pi_3 = D^{a_3}\ V^{b_3}\ \rho^{c_3}\ \mu \tag{c}$$

$$M^0 L^0 T^0 = [L]^{a_3}\ [LT^{-1}]^{b_3}\ [ML^{-3}]^{c_3}\ [ML^{-1}T^{-1}]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = c_3 + 1, 0 = a_3 + b_3 - 3c_3 - 1, 0 = -b_3 - 1$$

Solution: $c_3 = -1$, $b_3 = -1$ and $a_3 = -1$

Substituting the values of a_3, b_3 and c_3 in Equation (c) and rearranging, we get:

$$\pi_3 = D^{-1} V^{-1} \rho^{-1} \mu = \frac{\mu}{DV\rho}$$

For π_4 term:

$$\pi_4 = D^{a_4} V^{b_4} \rho^{c_4} C \quad \text{(d)}$$

$$M^0 L^0 T^0 = [L]^{a_4} [LT^{-1}]^{b_4} [ML^{-3}]^{c_4} [LT^{-1}]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = c_4, 0 = a_4 + b_4 - 3c_4 + 1, 0 = -b_4 - 1$$

Solution: $c_4 = 0$, $b_4 = -1$ and $a_4 = 0$

Substituting the values of a_4, b_4 and c_4 in Equation (d) and rearranging, we get:

$$\pi_4 = D^0 V^{-1} \rho^0 C = \frac{C}{V}$$

For obtaining the functional relationship, substituting the values of π_1, π_2, π_3 and π_4 in Equation (iii), we get:

$$f_1\left(\frac{P}{\rho D^2 V^2}, \frac{D\omega}{V}, \frac{\mu}{DV\rho}, \frac{C}{V}\right) = 0$$

$$\frac{P}{\rho D^2 V^2} = \phi\left(\frac{D\omega}{V}, \frac{\mu}{DV\rho}, \frac{C}{V}\right)$$

$$\therefore P = \rho \boldsymbol{D^2 V^2} \phi\left(\frac{\boldsymbol{D\omega}}{\boldsymbol{V}}, \frac{\mu}{\boldsymbol{DV\rho}}, \frac{\boldsymbol{C}}{\boldsymbol{V}}\right)$$

Example 19.7 Derive a suitable form of the equation to represent the discharge Q through a sharp-edged triangular notch by using Buckingham pi theorem. Given that the discharge Q depends on the mass density ρ, head H, gravitational acceleration g, dynamic viscosity μ, surface tension σ of the fluid and the central angle α of the notch.

Solution

The problem can be expressed as,

$$Q = f(\rho, H, g, \mu, \sigma, \alpha) \quad \text{(i)}$$

or

$$f_1(Q, \rho, H, g, \mu, \sigma, \alpha) = 0 \quad \text{(ii)}$$

Total number of variables: $n = 7$

Writing dimensions of each term, we get:

$$Q = [L^3T^{-1}], \rho = [ML^{-3}], H = [L], g = [LT^{-2}],$$
$$\mu = [ML^{-1}T^{-1}], \sigma = [MT^{-2}], \alpha = [M^0 L^0 T^0]$$

Thus, the fundamental dimensions in the problem are M, L, T and hence, $m = 3$.

Therefore, the number of π terms $= n - m = 7 - 3 = 4$.

Thus, the four π terms say π_1, π_2, π_3 and π_4 are formed.

Equation (ii) may be written as,

$$f_1(\pi_1, \pi_2, \pi_3, \pi_4) = 0 \tag{iii}$$

Since $m = 3$, choosing ρ, g and H as the repeating variables.

Each π term contains $(m+1)$ variables and can be written as,

$$\pi_1 = \rho^{a_1}\ g^{b_1}\ H^{c_1}\ Q \tag{a}$$

$$\pi_2 = \rho^{a_2}\ g^{b_2}\ H^{c_2}\ \mu \tag{b}$$

$$\pi_3 = \rho^{a_3}\ g^{b_3}\ H^{c_3}\ \sigma \tag{c}$$

$$\pi_4 = \alpha \tag{d}$$

For π_1 term:

$$\pi_1 = \rho^{a_1}\ g^{b_1}\ H^{c_1}\ Q \tag{a}$$

$$M^0 L^0 T^0 = [ML^{-3}]^{a_1}\ [LT^{-2}]^{b_1}\ [L]^{c_1}\ [L^3T^{-1}]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = a_1,\ 0 = -3a_1 + b_1 + c_1 + 3,\ 0 = -2b_1 - 1$$

Solution: $a_1 = 0$, $b_1 = -(1/2)$ and $c_1 = -(5/2)$

Substituting the values of a_1, b_1 and c_1 in Equation (a) and rearranging, we get:

$$\pi_1 = \rho^0\ g^{-(1/2)}\ H^{-(5/2)}\ Q = \frac{Q}{g^{1/2} H^{5/2}}$$

For π_2 term:

$$\pi_2 = \rho^{a_2}\ g^{b_2}\ H^{c_2}\ \mu \tag{b}$$

$$M^0 L^0 T^0 = [ML^{-3}]^{a_2}\ [LT^{-2}]^{b_2}\ [L]^{c_2}\ [ML^{-1}T^{-1}]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = a_2 + 1,\ 0 = -3a_2 + b_2 + c_2 - 1,\ 0 = -2b_2 - 1$$

Solution: $a_2 = -1$, $b_2 = -(1/2)$ and $c_2 = (-3/2)$

Substituting the values of a_2, b_2 and c_2 in Equation (b) and rearranging, we get:

$$\pi_2 = \rho^{-1}\ g^{-(1/2)}\ H^{-(3/2)}\ \mu = \frac{\mu}{\rho g^{1/2} H^{3/2}}$$

For π_3 term:

$$\pi_3 = \rho^{a_3}\ g^{b_3}\ H^{c_3}\ \sigma \tag{c}$$

$$M^0 L^0 T^0 = [ML^{-3}]^{a_3}\ [LT^{-2}]^{b_3}\ [L]^{c_3}\ [MT^{-2}]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = a_3 + 1,\ 0 = -3a_3 + b_3 + c_3,\ 0 = -2b_3 - 2$$

Solution: $a_3 = -1$, $b_3 = -1$ and $c_3 = -2$

Substituting the values of a_3, b_3 and c_3 in Equation (c) and rearranging, we get:

$$\pi_3 = \rho^{-1}\ g^{-1}\ H^{-2}\ \sigma = \frac{\sigma}{\rho g H^2}$$

For π_4 term:

$$\pi_4 = \alpha \tag{d}$$

For obtaining the functional relationship, substituting the values of π_1, π_2, π_3 and π_4 in Equation (iii), we get:

$$f_1\left(\frac{Q}{g^{1/2} H^{5/2}}, \frac{\mu}{\rho g^{1/2} H^{3/2}}, \frac{\sigma}{\rho g H^2}, \alpha\right) = 0$$

$$\frac{Q}{g^{1/2} H^{5/2}} = \phi\left(\frac{\mu}{\rho g^{1/2} H^{3/2}}, \frac{\sigma}{\rho g H^2}, \alpha\right)$$

$$\therefore Q = \mathbf{g^{1/2} H^{5/2}}\, \phi\left(\frac{\boldsymbol{\mu}}{\boldsymbol{\rho g^{1/2} H^{3/2}}}, \frac{\boldsymbol{\sigma}}{\boldsymbol{\rho g H^2}}, \boldsymbol{\alpha}\right)$$

or

$$Q = CH^{5/2}$$

$$C = g^{1/2}\, \phi\left(\frac{\mu}{\rho g^{1/2} H^{3/2}}, \frac{\sigma}{\rho g H^2}, \alpha\right)$$

Example 19.8 The resisting force F during the flight of a supersonic plane depends upon the length of aircraft L, velocity V, dynamic viscosity μ, mass density ρ and bulk modulus of air K. Express the functional relationship between the variables and the resisting force using Buckingham pi theorem.

Solution

The problem can be expressed as,

$$F = f(L, V, \mu, \rho, K) \tag{i}$$

or

$$f_1(F, L, V, \mu, \rho, K) = 0 \tag{ii}$$

Total number of variables: $n = 6$
Writing the dimensions of each term, we get:

$$F = [MLT^{-2}],\ L = [L],\ V = [LT^{-1}],\ \mu = [ML^{-1}T^{-1}],\ \rho = [ML^{-3}],\ K = [ML^{-1}T^{-2}]$$

Thus, the fundamental dimensions in the problem are M, L, T and hence, $m = 3$.
Therefore, the number of π terms $= n - m = 6 - 3 = 3$.

The three π terms say π_1, π_2 and π_3 are formed.

Equation (ii) may be written as,

$$f_1(\pi_1, \pi_2, \pi_3) = 0 \tag{iii}$$

Since $m = 3$, choosing L, V and ρ as the repeating variables.

Each π term contains $(m+1)$ variables and can be written as,

$$\pi_1 = L^{a_1}\, V^{b_1}\, \rho^{c_1}\, F \tag{a}$$

$$\pi_2 = L^{a_2}\, V^{b_2}\, \rho^{c_2}\, \mu \tag{b}$$

$$\pi_3 = L^{a_3}\, V^{b_3}\, \rho^{c_3}\, K \tag{c}$$

For π_1 term:

$$\pi_1 = L^{a_1}\ V^{b_1}\ \rho^{c_1}\ F \tag{a}$$

$$M^0 L^0 T^0 = [L]^{a_1}\ [LT^{-1}]^{b_1}\ [ML^{-3}]^{c_1}\ [MLT^{-2}]$$

Equating exponents of M, L and T, respectively, we get:

$$0 = c_1 + 1,\ 0 = a_1 + b_1 - 3c_1 + 1,\ 0 = -b_1 - 2$$

Solution: $c_1 = -1,\ b_1 = -2$, and $a_1 = -2$

Substituting the values of a_1, b_1 and c_1 in Equation (a) and rearranging, we get:

$$\pi_1 = L^{-2}\ V^{-2}\ \rho^{-1}\ F = \frac{F}{L^2 V^2 \rho}$$

For π_2 term:

$$\pi_2 = L^{a_2}\ V^{b_2}\ \rho^{c_2}\ \mu \tag{b}$$

$$M^0 L^0 T^0 = [L]^{a_2}\ [LT^{-1}]^{b_2}\ [ML^{-3}]^{c_2}\ [ML^{-1}T^{-1}]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = c_2 + 1,\ 0 = a_2 + b_2 - 3c_2 - 1,\ 0 = -b_2 - 1$$

Solution: $c_2 = -1,\ \ b_2 = -1$ and $a_2 = -1$

Substituting the values of a_2, b_2 and c_2 in Equation (b) and rearranging, we get:

$$\pi_2 = L^{-1}\ V^{-1}\ \rho^{-1}\ \mu = \frac{\mu}{LV\rho}$$

For π_3 term:

$$\pi_3 = L^{a_3}\ V^{b_3}\ \rho^{c_3}\ K \tag{c}$$

$$M^0 L^0 T^0 = [L]^{a_3}\ [LT^{-1}]^{b_3}\ [ML^{-3}]^{c_3}\ [ML^{-1}T^{-2}]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = c_3 + 1,\ 0 = a_3 + b_3 - 3c_3 - 1,\ 0 = -b_3 - 2$$

Solution: $c_3 = -1,\ \ b_3 = -2$ and $a_3 = 0$

Substituting the values of a_3, b_3 and c_3 in Equation (c) and rearranging, we get:

$$\pi_3 = L^0\ V^{-2}\ \rho^{-1}\ K = \frac{K}{V^2 \rho}$$

For obtaining the functional relationship, substituting the values of π_1, π_2 and π_3 in Equation (iii), we get:

$$f_1\left(\frac{F}{L^2 V^2 \rho}, \frac{\mu}{LV\rho}, \frac{K}{V^2\rho}\right) = 0$$

$$\frac{F}{L^2 V^2 \rho} = \phi\left(\frac{\mu}{LV\rho}, \frac{K}{V^2\rho}\right)$$

$$\therefore F = \boldsymbol{L^2 V^2 \rho\, \phi\left(\frac{\mu}{LV\rho}, \frac{K}{V^2\rho}\right)}$$

Example 19.9 In a centrifugal pump, the rate of discharge Q is assumed to depend on the mass density ρ of fluid, speed of the pump N, the diameter of impeller D, acceleration due to gravity g, head H and dynamic viscosity μ of the fluid. Using Buckingham pi theorem show that $Q = ND^3\,\phi\left[\dfrac{gH}{N^2D^2}, \dfrac{v}{ND^2}\right]$, where v is the kinematic viscosity of the fluid.

Solution

The problem can be expressed as,

$$Q = f(\rho, N, D, g, H, \mu) \tag{i}$$

or

$$f_1(Q, \rho, N, D, g, H, \mu) = 0 \tag{ii}$$

Total number of variables: $n = 7$
Writing the dimensions of each term, we get:

$$Q = [L^3T^{-1}],\ \rho = [ML^{-3}],\ N = [T^{-1}],\ D = [L],\ g = [LT^{-2}],\ H = [L],\ \mu = [ML^{-1}T^{-1}]$$

Thus, the fundamental dimensions in the problem are M, L, T and hence, $m = 3$.
Therefore, the number of π terms $= n - m = 7 - 3 = 4$.
The four π terms say π_1, π_2, π_3 and π_4 are formed.
Equation (ii) may be written as,

$$f_1(\pi_1, \pi_2, \pi_3, \pi_4) = 0 \tag{iii}$$

Since $m = 3$, choosing D, N and ρ as the repeating variables.

Each π term contains $(m+1)$ variables and can be written as,

$$\pi_1 = D^{a_1}\ N^{b_1}\ \rho^{c_1}\ Q \tag{a}$$

$$\pi_2 = D^{a_2}\ N^{b_2}\ \rho^{c_2}\ g \tag{b}$$

$$\pi_3 = D^{a_3}\ N^{b_3}\ \rho^{c_3}\ H \tag{c}$$

$$\pi_4 = D^{a_4}\ N^{b_4}\ \rho^{c_4}\ \mu \tag{d}$$

For π_1 term:

$$\pi_1 = D^{a_1}\ N^{b_1}\ \rho^{c_1}\ Q \tag{a}$$

$$M^0L^0T^0 = [L]^{a_1}\ [T^{-1}]^{b_1}\ [ML^{-3}]^{c_1}\ [L^3T^{-1}]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = c_1,\ 0 = a_1 - 3c_1 + 3,\ 0 = -b_1 - 1$$

Solution: $c_1 = 0, b_1 = -1$ and $a_1 = -3$

Substituting the values of a_1, b_1 and c_1 in Equation (a) and rearranging, we get:

$$\pi_1 = D^{-3}\ N^{-1}\ \rho^0\ Q = \frac{Q}{ND^3}$$

For π_2 term:

$$\pi_2 = D^{a_2}\ N^{b_2}\ \rho^{c_2}\ g \tag{b}$$

$$M^0L^0T^0 = [L]^{a_2}\ [T^{-1}]^{b_2}\ [ML^{-3}]^{c_2}\ [LT^{-2}]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = c_2,\ 0 = a_2 - 3c_2 + 1,\ 0 = -b_2 - 2$$

Solution: $c_2 = 0,\ b_2 = -2$, and $a_2 = -1$

Substituting the values of a_2, b_2 and c_2 in Equation (b) and rearranging, we get:

$$\pi_2 = D^{-1}\ N^{-2}\ \rho^0\ g = \frac{g}{DN^2}$$

For π_3 term:

$$\pi_3 = D^{a_3}\ N^{b_3}\ \rho^{c_3}\ H \tag{c}$$

$$M^0 L^0 T^0 = [L]^{a_3}\ [T^{-1}]^{b_3}\ [ML^{-3}]^{c_3}\ [L]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = c_3,\ 0 = a_3 - 3c_3 + 1,\ 0 = -b_3$$

Solution: $c_3 = 0,\ b_3 = 0$ and $a_3 = -1$

Substituting the values of a_3, b_3 and c_3 in Equation (c) and rearranging, we get:

$$\pi_3 = D^{-1}\ N^0\ \rho^0\ H = \frac{H}{D}$$

For π_4 term:

$$\pi_4 = D^{a_4}\ N^{b_4}\ \rho^{c_4}\ \mu \tag{d}$$

$$M^0 L^0 T^0 = [L]^{a_4}\ [T^{-1}]^{b_4}\ [ML^{-3}]^{c_4}\ [ML^{-1}T^{-1}]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = c_4 + 1,\ 0 = a_4 - 3c_4 - 1,\ 0 = -b_4 - 1$$

Solution: $c_4 = -1,\ b_4 = -1$ and $a_4 = -2$

Substituting the values of a_4, b_4 and c_4 in Equation (d) and rearranging, we get:

$$\pi_4 = D^{-2}\ N^{-1}\ \rho^{-1}\ \mu = \frac{\mu}{D^2 N \rho}$$

For obtaining the functional relationship, substituting the values of π_1, π_2, π_3 and π_4 in Equation (iii), we get:

$$f_1\left(\frac{Q}{ND^3}, \frac{g}{DN^2}, \frac{H}{D}, \frac{\mu}{D^2 N \rho}\right) = 0$$

or

$$f_1\left(\frac{Q}{ND^3}, \frac{gH}{N^2 D^2}, \frac{\mu}{D^2 N \rho}\right) = 0 \qquad [\text{Multiply } \pi_2 \text{ and } \pi_3]$$

or

$$\frac{Q}{ND^3} = \phi\left(\frac{gH}{N^2 D^2}, \frac{\mu/\rho}{D^2 N}\right)$$

$$\therefore Q = ND^3 \phi\left(\frac{gH}{N^2 D^2}, \frac{\nu}{ND^2}\right) \qquad [\because (\mu/\rho) = \nu]$$

Hence proved.

Example 19.10 The pressure drop Δp in a pipe of diameter D and length L depends on the velocity of flow V, dynamic viscosity μ, mass density ρ and roughness k. Obtain an expression for Δp using Buckingham pi theorem.

Solution

The problem can be expressed as,

$$\Delta p = f(D, L, V, \mu, \rho, k) \quad \text{(i)}$$

or

$$f_1(\Delta p, D, L, V, \mu, \rho, k) = 0 \quad \text{(ii)}$$

Total number of variables: $n = 7$

Writing the dimensions of each term, we get:

$$\Delta p = [ML^{-1}T^{-2}],\ D = [L],\ L = [L],\ V = [LT^{-1}],\ \mu = [ML^{-1}T^{-1}],\ \rho = [ML^{-3}],\ k = [L]$$

Thus, the fundamental dimensions in the problem are M, L, T and hence, $m = 3$.

Therefore, the number of π terms $= n - m = 7 - 3 = 4$.

The four π terms say π_1, π_2, π_3 *and* π_4 are formed.

Equation (ii) may be written as,

$$f_1(\pi_1, \pi_2, \pi_3, \pi_4) = 0 \quad \text{(iii)}$$

Since $m = 3$, choosing D, V and ρ as the repeating variables.

Each π term contains $(m+1)$ variables and can be written as,

$$\pi_1 = D^{a_1}\, V^{b_1}\, \rho^{c_1}\, \Delta p \quad \text{(a)}$$

$$\pi_2 = D^{a_2}\, V^{b_2}\, \rho^{c_2}\, L \quad \text{(b)}$$

$$\pi_3 = D^{a_3}\, V^{b_3}\, \rho^{c_3}\, \mu \quad \text{(c)}$$

$$\pi_4 = D^{a_4}\, V^{b_4}\, \rho^{c_4}\, k \quad \text{(d)}$$

For π_1 term:

$$\pi_1 = D^{a_1}\, V^{b_1}\, \rho^{c_1}\, \Delta p \quad \text{(a)}$$

$$M^0L^0T^0 = [L]^{a_1}\, [LT^{-1}]^{b_1}\, [ML^{-3}]^{c_1}\, [ML^{-1}T^{-2}]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = c_1 + 1,\ 0 = a_1 + b_1 - 3c_1 - 1,\ 0 = -b_1 - 2$$

Solution: $c_1 = -1,\ b_1 = -2$ and $a_1 = 0$

Substituting the values of a_1, b_1 and c_1 in Equation (a) and rearranging, we get:

$$\pi_1 = D^0\, V^{-2}\, \rho^{-1}\, \Delta p = \frac{\Delta p}{\rho V^2}$$

For π_2 term:

$$\pi_2 = D^{a_2}\, V^{b_2}\, \rho^{c_2}\, L \quad \text{(b)}$$

$$M^0L^0T^0 = [L]^{a_2}\, [LT^{-1}]^{b_2}\, [ML^{-3}]^{c_2}\, [L]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = c_2,\ 0 = a_2 + b_2 - 3c_2 + 1,\ 0 = -b_2$$

Solution: $c_2 = 0,\ b_2 = 0$ and $a_2 = -1$

Substituting the values of a_2, b_2 and c_2 in Equation (b) and rearranging, we get:

$$\pi_2 = D^{-1} V^0 \rho^0 L = \frac{L}{D}$$

For π_3 term:

$$\pi_3 = D^{a_3} V^{b_3} \rho^{c_3} \mu \quad \text{(c)}$$

$$M^0 L^0 T^0 = [L]^{a_3} [LT^{-1}]^{b_3} [ML^{-3}]^{c_3} [ML^{-1}T^{-1}]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = c_3 + 1, 0 = a_3 + b_3 - 3c_3 - 1, 0 = -b_3 - 1$$

Solution: $c_3 = -1$, $b_3 = -1$ and $a_3 = -1$

Substituting the values of a_3, b_3 and c_3 in Equation (c) and rearranging, we get:

$$\pi_3 = D^{-1} V^{-1} \rho^{-1} \mu = \frac{\mu}{DV\rho}$$

For π_4 term:

$$\pi_4 = D^{a_4} V^{b_4} \rho^{c_4} k \quad \text{(d)}$$

$$M^0 L^0 T^0 = [L]^{a_4} [LT^{-1}]^{b_4} [ML^{-3}]^{c_4} [L]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = c_4, 0 = a_4 + b_4 - 3c_4 + 1, 0 = -b_4$$

Solution: $c_4 = 0$, $b_4 = 0$ and $a_4 = -1$

Substituting the values of a_4, b_4 and c_4 in Equation (d) and rearranging, we get:

$$\pi_4 = D^{-1} V^0 \rho^0 k = \frac{k}{D}$$

For obtaining the functional relationship, substituting the values of π_1, π_2, π_3 and π_4 in Equation (iii), we get:

$$f_1\left(\frac{\Delta p}{\rho V^2}, \frac{L}{D}, \frac{\mu}{DV\rho}, \frac{k}{D}\right) = 0$$

$$\therefore \frac{\Delta p}{\rho V^2} = \phi\left(\boldsymbol{\frac{L}{D}}, \boldsymbol{\frac{\mu}{DV\rho}}, \boldsymbol{\frac{k}{D}}\right)$$

Experiments indicate that the drop in pressure Δp is a function of (L/D)

$$\frac{\Delta p}{\rho V^2} = \frac{L}{D}\phi\left(\frac{\mu}{DV\rho}, \frac{k}{D}\right)$$

$$\frac{\Delta p}{\rho g} = \frac{V^2}{g}\frac{L}{D}\phi\left(\frac{\mu}{DV\rho}, \frac{k}{D}\right) \quad \text{[Divide both sides by g]}$$

$$\frac{\Delta p}{\rho g} = \frac{4V^2}{2g}\frac{L}{D} f \quad \left[\text{where } f = \phi\left(\frac{\mu}{DV\rho}, \frac{k}{D}\right)\right]$$

(Multiplication or division by any constant does not change the π term)

$$\therefore \frac{\Delta p}{\rho g} = h_f = \frac{4 fLV^2}{2gD}$$

Example 19.11 Show by the use of Buckingham pi theorem that the power P developed in the hydraulic turbine is given by, $P = \rho_w N^3 D^5 \phi\left(\frac{D}{B}, \frac{\rho_w D^2 N}{\mu}, \frac{ND}{\sqrt{gH}}\right)$ where ρ_w is the mass density of water, N is the speed, D is the diameter, B is the width of the runner, μ is dynamic viscosity, H is the head and g is the gravitational acceleration.

Solution

The problem can be expressed as,

$$P = f(\rho_w, N, D, B, \mu, H, g) \quad \text{(i)}$$

or

$$f_1(P, \rho_w, N, D, B, \mu, H, g) = 0 \quad \text{(ii)}$$

Total number of variables: $n = 8$

Writing the dimensions of each term, we get:

$$P = [ML^2T^{-3}],\ \rho_w = [ML^{-3}],\ N = [T^{-1}],\ D = [L],$$
$$B = [L],\ \mu = [ML^{-1}T^{-1}],\ H = [L],\ g = [LT^{-2}]$$

Thus, the fundamental dimensions in the problem are M, L, T and hence, $m = 3$.

Therefore, the number of π terms = $n - m = 8 - 3 = 5$.

The five π terms say $\pi_1, \pi_2, \pi_3, \pi_4$ and π_5 are formed.

Equation (ii) may be written as,

$$f_1(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5) = 0 \quad \text{(iii)}$$

Since $m = 3$, choosing D, N and ρ_w as the repeating variables.

Each π term contains $(m+1)$ variables and can be written as,

$$\pi_1 = D^{a_1}\ N^{b_1}\ \rho_w^{c_1}\ P \quad \text{(a)}$$

$$\pi_2 = D^{a_2}\ N^{b_2}\ \rho_w^{c_2}\ B \quad \text{(b)}$$

$$\pi_3 = D^{a_3}\ N^{b_3}\ \rho_w^{c_3}\ \mu \quad \text{(c)}$$

$$\pi_4 = D^{a_4}\ N^{b_4}\ \rho_w^{c_4}\ H \quad \text{(d)}$$

$$\pi_5 = D^{a_5}\ N^{b_5}\ \rho_w^{c_5}\ g \quad \text{(e)}$$

For π_1 term:

$$\pi_1 = D^{a_1}\ N^{b_1}\ \rho_w^{c_1}\ P \quad \text{(a)}$$

$$M^0L^0T^0 = [L]^{a_1}\ [T^{-1}]^{b_1}\ [ML^{-3}]^{c_1}\ [ML^2T^{-3}]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = c_1 + 1,\ 0 = a_1 - 3c_1 + 2,\ 0 = -b_1 - 3$$

Solution: $c_1 = -1$, $b_1 = -3$ and $a_1 = -5$

Substituting the values of a_1, b_1 and c_1 in Equation (a) and rearranging, we get:

$$\pi_1 = D^{-5}\ N^{-3}\ \rho_w^{-1}\ P = \frac{P}{\rho_w N^3 D^5}$$

For π_2 term:

$$\pi_2 = D^{a_2}\ N^{b_2}\ \rho_w^{\ c_2}\ B \tag{b}$$

$$M^0 L^0 T^0 = [L]^{a_2}\ [T^{-1}]^{b_2}\ [ML^{-3}]^{c_2}\ [L]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = c_2,\ 0 = a_2 - 3c_2 + 1,\ 0 = -b_2$$

Solution: $c_2 = 0,\ b_2 = 0$, and $a_2 = -1$

Substituting the values of a_2, b_2 and c_2 in Equation (b) and rearranging, we get:

$$\pi_2 = D^{-1}\ N^0\ \rho_w^{\ 0}\ B = \frac{B}{D}$$

For π_3 term:

$$\pi_3 = D^{a_3}\ N^{b_3}\ \rho_w^{\ c_3}\ \mu \tag{c}$$

$$M^0 L^0 T^0 = [L]^{a_3}\ [T^{-1}]^{b_3}\ [ML^{-3}]^{c_3}\ [ML^{-1}T^{-1}]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = c_3 + 1,\ 0 = a_3 - 3c_3 - 1,\ 0 = -b_3 - 1$$

Solution: $c_3 = -1,\ b_3 = -1$ and $a_3 = -2$

Substituting the values of a_3, b_3 and c_3 in Equation (c) and rearranging, we get:

$$\pi_3 = D^{-2}\ N^{-1}\ \rho_w^{\ -1}\ \mu = \frac{\mu}{\rho_w D^2 N}$$

For π_4 term:

$$\pi_4 = D^{a_4}\ N^{b_4}\ \rho_w^{\ c_4}\ H \tag{d}$$

$$M^0 L^0 T^0 = [L]^{a_4}\ [T^{-1}]^{b_4}\ [ML^{-3}]^{c_4}\ [L]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = c_4,\ 0 = a_4 - 3c_4 + 1,\ 0 = -b_4$$

Solution: $c_4 = 0,\ b_4 = 0$ and $a_4 = -1$

Substituting the values of a_4, b_4 and c_4 in Equation (d) and rearranging, we get:

$$\pi_4 = D^{-1}\ N^0\ \rho_w^{\ 0}\ H = \frac{H}{D}$$

For π_5 term:

$$\pi_5 = D^{a_5}\ N^{b_5}\ \rho_w^{\ c_5}\ g \tag{e}$$

$$M^0 L^0 T^0 = [L]^{a_5}\ [T^{-1}]^{b_5}\ [ML^{-3}]^{c_5}\ [LT^{-2}]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = c_5,\ 0 = a_5 - 3c_5 + 1,\ 0 = -b_5 - 2$$

Solution: $c_5 = 0,\ b_5 = -2$ and $a_5 = -1$

Substituting the values of a_5, b_5 and c_5 in Equation (e) and rearranging, we get:

$$\pi_5 = D^{-1} N^{-2} \rho_w{}^0 g = \frac{g}{DN^2}$$

For obtaining the functional relationship, substituting the values of $\pi_1, \pi_2, \pi_3, \pi_4$ and π_5 in Equation (iii), we get:

$$f_1\left(\frac{P}{\rho_w N^3 D^5}, \frac{B}{D}, \frac{\mu}{\rho_w D^2 N}, \frac{H}{D}, \frac{g}{DN^2}\right) = 0$$

or $$f_1\left(\frac{P}{\rho_w N^3 D^5}, \frac{B}{D}, \frac{\mu}{\rho_w D^2 N}, \frac{gH}{N^2 D^2}\right) = 0 \qquad \text{[Multiply } \pi_4 \text{ and } \pi_5\text{]}$$

or $$f_1\left(\frac{P}{\rho_w N^3 D^5}, \frac{B}{D}, \frac{\mu}{\rho_w D^2 N}, \frac{\sqrt{gH}}{ND}\right) = 0$$

or $$f_1\left(\frac{P}{\rho_w N^3 D^5}, \frac{B}{D}, \frac{\rho_w D^2 N}{\mu}, \frac{ND}{\sqrt{gH}}\right) = 0$$

$$\frac{P}{\rho_w N^3 D^5} = \phi\left(\frac{D}{B}, \frac{\rho_w D^2 N}{\mu}, \frac{ND}{\sqrt{gH}}\right)$$

$$\therefore P = \rho_w N^3 D^5 \phi\left(\frac{D}{B}, \frac{\rho_w D^2 N}{\mu}, \frac{ND}{\sqrt{gH}}\right)$$

Hence proved.

Example 19.12 The discharge Q in a device is a function of diameter D, speed N, mass density ρ, dynamic viscosity μ, surface tension σ and specific weight w. Obtain an expression for Q using Buckingham pi theorem.

Solution

The problem can be expressed as,

$$Q = f(D, N, \rho, \mu, \sigma, w) \qquad \text{(i)}$$

or $$f_1(Q, D, N, \rho, \mu, \sigma, w) = 0 \qquad \text{(ii)}$$

Total number of variables: $n = 7$

Writing the dimensions of each term, we get:

$$Q = [L^3T^{-1}], D = [L], N = [T^{-1}], \rho = [ML^{-3}],$$
$$\mu = [ML^{-1}T^{-1}], \ \sigma = [MT^{-2}], w = [ML^{-2}T^{-2}]$$

Thus, the fundamental dimensions in the problem are M, L, T and hence, $m = 3$.

Therefore, the number of π terms $= n - m = 7 - 3 = 4$.

The four π terms say π_1, π_2, π_3 and π_4 are formed.

Equation (ii) may be written as,

$$f_1(\pi_1, \pi_2, \pi_3, \pi_4) = 0 \qquad \text{(iii)}$$

Since $m = 3$, choosing D, N and ρ as the repeating variables.

Each π term contains $(m+1)$ variables and can be written as,

$$\pi_1 = D^{a_1}\ N^{b_1}\ \rho^{c_1}\ Q \tag{a}$$

$$\pi_2 = D^{a_2}\ N^{b_2}\ \rho^{c_2}\ \mu \tag{b}$$

$$\pi_3 = D^{a_3}\ N^{b_3}\ \rho^{c_3}\ \sigma \tag{c}$$

$$\pi_4 = D^{a_4}\ N^{b_4}\ \rho^{c_4}\ w \tag{d}$$

For π_1 term:

$$\pi_1 = D^{a_1}\ N^{b_1}\ \rho^{c_1}\ Q \tag{a}$$

$$M^0 L^0 T^0 = [L]^{a_1}\ [T^{-1}]^{b_1}\ [ML^{-3}]^{c_1}\ [L^3T^{-1}]$$

Equating the exponents of *M*, *L* and *T*, respectively, we get:

$$0 = c_1,\ 0 = a_1 - 3c_1 + 3,\ 0 = -b_1 - 1$$

Solution: $c_1 = 0$, $b_1 = -1$, and $a_1 = -3$

Substituting the values of a_1, b_1 and c_1 in Equation (a) and rearranging, we get:

$$\pi_1 = D^{-3}\ N^{-1}\ \rho^0\ Q = \frac{Q}{D^3 N}$$

For π_2 term:

$$\pi_2 = D^{a_2}\ N^{b_2}\ \rho^{c_2}\ \mu \tag{b}$$

$$M^0 L^0 T^0 = [L]^{a_2}\ [T^{-1}]^{b_2}\ [ML^{-3}]^{c_2}\ [ML^{-1}T^{-1}]$$

Equating the exponents of *M*, *L* and *T*, respectively, we get:

$$0 = c_2 + 1,\ 0 = a_2 - 3c_2 - 1,\ 0 = -b_2 - 1$$

Solution: $c_2 = -1$, $b_2 = -1$, and $a_2 = -2$

Substituting the values of a_2, b_2 and c_2 in Equation (b) and rearranging, we get:

$$\pi_2 = D^{-2}\ N^{-1}\ \rho^{-1}\ \mu = \frac{\mu}{\rho N D^2}$$

For π_3 term:

$$\pi_3 = D^{a_3}\ N^{b_3}\ \rho^{c_3}\ \sigma \tag{c}$$

$$M^0 L^0 T^0 = [L]^{a_3}\ [T^{-1}]^{b_3}\ [ML^{-3}]^{c_3}\ [MT^{-2}]$$

Equating the exponents of *M*, *L* and *T*, respectively, we get:

$$0 = c_3 + 1,\ 0 = a_3 - 3c_3,\ 0 = -b_3 - 2$$

Solution: $c_3 = -1$, $b_3 = -2$, and $a_3 = -3$

Substituting the values of a_3, b_3 and c_3 in Equation (c) and rearranging, we get:

$$\pi_3 = D^{-3}\ N^{-2}\ \rho^{-1}\ \sigma = \frac{\sigma}{D^3 N^2 \rho}$$

For π_4 term:

$$\pi_4 = D^{a_4}\ N^{b_4}\ \rho^{c_4}\ w \tag{d}$$

$$M^0 L^0 T^0 = [L]^{a_4}\ [T^{-1}]^{b_4}\ [ML^{-3}]^{c_4}\ [ML^{-2}T^{-2}]$$

Equating the exponents of M, L and T, respectively, we get:

$$0 = c_4 + 1, 0 = a_4 - 3c_4 - 2, 0 = -b_4 - 2$$

Solution: $c_4 = -1,\ b_4 = -2$ and $a_4 = -1$

Substituting the values of a_4, b_4 and c_4 in Equation (d) and rearranging, we get:

$$\pi_4 = D^{-1}\, N^{-2}\, \rho^{-1}\, w = \frac{w}{DN^2\rho}$$

For obtaining the functional relationship, substituting the values of π_1, π_2, π_3 and π_4 in Equation (iii), we get:

$$f_1\left(\frac{Q}{D^3N}, \frac{\mu}{\rho ND^2}, \frac{\sigma}{D^3N^2\rho}, \frac{w}{DN^2\rho}\right) = 0$$

$$\frac{Q}{D^3N} = \phi\left(\frac{\mu}{\rho ND^2}, \frac{\sigma}{D^3N^2\rho}, \frac{w}{DN^2\rho}\right)$$

$$\therefore Q = \boldsymbol{D^3 N\phi\left(\frac{\mu}{\rho ND^2}, \frac{\sigma}{D^3N^2\rho}, \frac{w}{DN^2\rho}\right)}$$

19.5 ❐ MODEL STUDIES

Before the actual construction of the hydraulic structures (dams, spillways, etc.) or hydraulic machines (turbines, pumps, etc.) their models are made and tested to obtain the desired information. Such experimental investigation is also required when the problems cannot be solved by theoretical analysis. The model is a small scale replica of the actual structure or the machine while the actual structure or machine is called the prototype. In many cases, the models are much smaller than its prototypes, whereas in some cases the models may be larger than the prototypes. For example, investigation of a carburetor and a wrist watch is carried out in a large scale model.

Applications of Model Studies

Model studies find applications in various branches of engineering. Some of the important applications in different fields are given below.

(i) To determine the full size of civil engineering structures, such as dams, spillways, etc.
(ii) To predict the performance of mechanical engineering devices, such as turbines, pumps, compressors, etc.
(iii) To predict the behaviour of naval engineering devices, such as ships, submarines, etc.
(iv) To predict the stability characteristics and wind loads of tall buildings in architectural engineering.
(v) To predict the performance of aviation engineering equipment and devices, such as aeroplanes, rockets and missiles.

Importance of Model Studies

The advantages of model studies are given below.

(i) The model tests are quite economical and convenient because without incurring much expenditure, the design of the model may be changed until the most suitable design is obtained.
(ii) Model testing can also be used to incorporate required modifications in an existing prototype.
(iii) Based on the final results obtained from the model test, the performance and behaviour of the prototype can be easily predicted in advance. In order to achieve this there should be a complete similarity between the model and prototypes.

19.6 ❐ SIMILITUDE-TYPES OF SIMILARITIES

Similitude means the complete similarity between the model and its prototype. The results obtained from experiments on models can be applied to the prototype only if a complete similarity exists between them. For establishing a complete similarity between the model and its prototype, three type of similarities are to be established, namely (i) geometric similarity, (ii) kinematic similarity and (iii) dynamic similarity.

19.6.1 Geometric Similarity

Geometric similarity exists between the model and its prototype when the ratios of their corresponding linear dimensions are equal. The ratio is known as scale ratio. Thus, for geometric similarity, the model must be of the same shape as the prototype, but it may be scaled by some constant scale factor. In other words, two geometric similar systems may differ in size but they will be identical in shape.

Let L_m, B_m, D_m, H_m, A_m and v_m be the length, breadth, diameter, height, area and volume of the model, respectively and L_p, B_p, D_p, H_p, A_p and v_p be the corresponding length, breadth, diameter, height, area and volume of the prototype, respectively. The length scale ratio (L_r), area scale ratio (A_r) and volume scale ratio (v_r) are respectively given as follows.

$$L_r = \frac{L_m}{L_p} = \frac{B_m}{B_p} = \frac{D_m}{D_p} = \frac{H_m}{H_p}$$

$$A_r = \frac{A_m}{A_p} = \left(\frac{L_m B_m}{L_p B_p}\right) = L_r^{\,2}$$

$$v_r = \frac{v_m}{v_p} = \left(\frac{L_m B_m H_m}{L_p B_p H_p}\right) = L_r^{\,3}$$

Thus, the geometric similarity implies that the corresponding areas are related by the square of the scale factor and the corresponding volumes by the cube of the scale factor. It means if the model and the prototype are geometrically similar, then they can be superimposed by merely changing the scale.

19.6.2 Kinematic Similarity

Kinematic similarity is the similarity of motion between the model and the prototype. Kinematic similarity exists when the velocities at the points in the model have a constant ratio to the velocities at the corresponding points in the prototype and their relative directions are also same. Geometric similarity is a prerequisite for kinematic similarity. Thus, kinematic similarity implies geometric similarity and in addition, similarity in the flow. The velocity triangles, speed ratio and the flow ratio for the model and the prototype will be equal. The kinematic similarity may be considered as time scale equivalence.

Let T_m, L_m, V_m, a_m and Q_m be the time, length, velocity, acceleration and discharge, respectively, for the model at any point and $T_p, L_p, V_p,\ a_p$ and Q_p be the corresponding time, length, velocity, acceleration and discharge, respectively, for the prototype at the corresponding point. The time scale ratio (T_r), velocity scale ratio (V_r), acceleration scale ratio (a_r) and discharge scale ratio (Q_r) are respectively given as follows.

$$T_r = \frac{T_m}{T_p}$$

$$V_r = \frac{V_m}{V_p} = \left(\frac{L_m / T_m}{L_p / T_p}\right) = \frac{L_r}{T_r}$$

$$a_r = \frac{a_m}{a_p} = \left(\frac{L_m / T_m^{\ 2}}{L_p / T_p^{\ 2}} \right) = \frac{L_r}{T_r^{\ 2}}$$

$$Q_r = \frac{Q_m}{Q_p} = \left(\frac{L_m^{\ 3} / T_m}{L_p^{\ 3} / T_p} \right) = \frac{L_r^{\ 3}}{T_r}$$

19.6.3 Dynamic Similarity

Dynamic similarity is the similarity of forces between the model and the prototype. Dynamic similarity exists when all forces at the points in the model have a constant ratio to the corresponding forces at the corresponding points in the prototype. The directions of the corresponding forces at the corresponding points should also be same. Thus, dynamic similarity implies similarity in the magnitude and directions of forces acting on the model and the prototype at all points. Both geometric and kinematic similarities are a prerequisite for dynamic similarity.

The forces acting on a fluid particle may be any one or a combination of the following forces.

(i) Gravitational force (F_g) = Mass × Acceleration due to gravity of the flowing fluid
(ii) Friction or viscous force (F_v) = Shear stress due to viscosity × Area of the flow
(iii) Pressure force (F_p)= Pressure intensity × Cross-sectional area of the flowing fluid.
(iv) Elastic force (F_e)= Elastic stress × Area of the flowing fluid
(v) Surface tension force (F_s)= Surface tension × Length of surface of the flowing fluid.
(vi) Inertia force (F_i) = Mass × Acceleration of the flowing fluid.

Inertia force is the resistive force which is due to the mass of the fluid particles and its direction is opposite to that of the fluid particles. The magnitude of inertia force is equal to the product of the mass of the particles (m) and acceleration (a) and it is given below.

$$F_i = ma = F_g + F_v + F_p + F_e + F_s$$

Let the subscripts m and p denotes the model and prototype, respectively.

The dynamic similarity force ratio (F_r) is given by,

$$F_r = \frac{(F_i)_m}{(F_i)_p} = \frac{(ma)_m}{(ma)_p} = \frac{(F_g + F_v + F_p + F_e + F_s)_m}{(F_g + F_v + F_p + F_e + F_s)_p} = \frac{(\Sigma F)_m}{(\Sigma F)_p}$$

For complete dynamic similarity, the ratios of the individual component forces must also be equal to the ratio of inertia forces.

$$F_r = \frac{(F_i)_m}{(F_i)_p} = \frac{(F_g)_m}{(F_g)_p} = \frac{(F_v)_m}{(F_v)_p} = \frac{(F_p)_m}{(F_p)_p} = \frac{(F_e)_m}{(F_e)_p} = \frac{(F_s)_m}{(F_s)_p}$$

Thus

$$\left(\frac{F_i}{F_g}\right)_m = \left(\frac{F_i}{F_g}\right)_p ; \left(\frac{F_i}{F_v}\right)_m = \left(\frac{F_i}{F_v}\right)_p ; \left(\frac{F_i}{F_p}\right)_m = \left(\frac{F_i}{F_p}\right)_p ;$$

$$\left(\frac{F_i}{F_e}\right)_m = \left(\frac{F_i}{F_e}\right)_p \text{ and } \left(\frac{F_i}{F_s}\right)_m = \left(\frac{F_i}{F_s}\right)_p$$

These force ratios are dimensionless numbers which appear in the fluid flow analysis.

19.7 ❒ DIMENSIONLESS NUMBERS AND THEIR SIGNIFICANCE

In a fluid flow phenomenon, due to the motion of fluid particles, inertia force always exists. The dimensionless numbers are obtained by dividing the inertia force by any of the remaining forces, namely gravitational force, viscous force, pressure force, elastic force and surface tension force. These force ratios are dimensionless and are also termed as non-dimensional parameters. The important dimensionless numbers are given below.

19.7.1 Reynolds Number

It is defined as the ratio of inertia force (F_i) to the viscous force (F_v). The expression for Reynolds number (*Re*) is given below.

$$F_i = \rho \times Q \times V = \rho \times AV \times V = \rho L^2 V^2$$

Here, L is the characteristic length.

$$F_v = \tau \times A = \left(\mu \frac{du}{dy}\right) \times A = \left(\mu \frac{V}{L}\right) \times L^2 = \mu VL$$

Thus $$\text{Re} = \frac{F_i}{F_v} = \frac{\rho L^2 V^2}{\mu VL} = \frac{\rho VL}{\mu} = \frac{VL}{\nu} \qquad [\because \nu = \mu / \rho]$$

In case of pipe flows, L is replaced by pipe diameter D, we get:

$$\boxed{\text{Re} = \frac{\rho VD}{\mu} = \frac{VD}{\nu}} \tag{19.1}$$

Reynolds number is the key parameter to determine the flow regime in pipes. It is named in the honour of Osborne Reynolds (1842-1912), a British physicist. This number signifies the relative predominance of the inertia force to the viscous force. At large Reynolds number, the inertia forces are large relative to the viscous forces which cause random and rapid fluctuations and thus, the flow is turbulent. At small Reynolds number, the viscous forces are large which keep the fluid in-line and thus, the flow is laminar.

This number is taken as a criterion of dynamic similarity in the flow situations where viscous forces predominate. For examples, flow through pipes, orificemeter, venturimeter, flow through low speed turbo machines and flow over submerged bodies.

19.7.2 Froude Number

It is defined as the square root of the ratio of the inertia force (F_i) to the gravity force (F_g). The expression for Froude number (F_r) is given below.

$$F_i = \rho L^2 V^2$$

$$F_g = \rho \times \text{Volume} \times g = \rho \times L^3 \times g$$

Thus $$\boxed{Fr = \sqrt{\frac{F_i}{F_g}} = \sqrt{\frac{\rho L^2 V^2}{\rho L^3 g}} = \frac{V}{\sqrt{Lg}}} \tag{19.2}$$

The Froude number is significant in free surface flows only and it governs the dynamic similarity of the flow where gravitational forces are predominant. Some of the examples are flow over spillways, weirs and notches and flow through open channels.

19.7.3 Euler Number

It is defined as the square root of the ratio of the inertia force (F_i) to the pressure force (F_p). The expression for Euler number (E_u) is given below.

$$F_i = \rho L^2 V^2$$

$$F_p = \text{pressure} \times \text{area} = p \times A = p \times L^2$$

Thus
$$\boxed{Eu = \sqrt{\frac{F_i}{F_p}} = \sqrt{\frac{\rho L^2 V^2}{pL^2}} = \frac{V}{\sqrt{p/\rho}}} \tag{19.3}$$

The Euler number may be considered in a fluid when its pressure drops low enough to cause vapour formation. This number becomes important when pressure changes in fluid flows are predominant and the other forces, such as viscous forces, gravity forces and surface tension forces are absent. Some examples of such flow may be pressure rise due to sudden closure of a valve, water hammer in pipes and discharge through orifice.

19.7.4 Weber Number

It is defined as the square root of the ratio of the inertia force (F_i) to the force of surface tension (F_s). The expression for Weber number (W_e) is given below.

$$F_i = \rho L^2 V^2$$

$$F_s = \text{Surface tension } \times \text{Length} = \sigma \times L$$

Thus
$$\boxed{We = \sqrt{\frac{F_i}{F_s}} = \sqrt{\frac{\rho L^2 V^2}{\sigma L}} = \frac{V}{\sqrt{\sigma/(\rho L)}}} \tag{19.4}$$

The Weber number is significant only when it has a smaller value in the order of unity or less which shows the predominance of surface tension force. It assumes importance in flow situations, such as droplets, capillary flows, blood flows in veins and arteries, and ripple waves.

19.7.5 Mach Number

It is defined as the square root of the ratio of the inertia force (F_i) to the elastic force (F_e). The expression for Mach number (M) is given below.

$$F_i = \rho L^2 V^2$$

$$F_e = \text{Bulk modulus of elasticity} \times \text{Area} = K \times A = K \times L^2$$

Thus
$$\boxed{M = \sqrt{\frac{F_i}{F_e}} = \sqrt{\frac{\rho L^2 V^2}{KL^2}} = \sqrt{\frac{V^2}{K/\rho}} = \frac{V}{\sqrt{K/\rho}} = \frac{V}{C} = \frac{\text{Speed of flow}}{\text{Speed of sound}}} \tag{19.5}$$

Here, $C = \sqrt{K/\rho}$ represents the velocity of sound in that fluid medium whose properties K and ρ are being taken. A flow is called sonic when $M = 1$, subsonic when $M < 1$, supersonic when $M > 1$ and hypersonic when $M >> 1$. A higher Mach number signifies the predominance of the effect of compressibility of the fluid. At a smaller value of Mach number (less than 0.3 for gas flows), the compressibility effect is neglected.

The Mach number is significant when there is substantial change in the density of a fluid due to pressure changes. This number is used in the analysis of systems involving high speed flows, spacecraft, rockets, water hammer problems, aerodynamic testing and compressor testing.

19.8. ❐ SIMILARITY LAWS OR MODEL LAWS

The ratios of forces given in Section 19.6.3 are non-dimensional numbers. For dynamic similarity, these ratios for model and prototype should be same. However, it is not possible to satisfy all the non-dimensional numbers. Thus, the model is generally designed on the basis of predominant force only. The various model laws are developed on the basis of each dimensionless number as described below.

19.8.1 Reynolds Model Law

This law states that for dynamic similarity, the Reynolds number of the model should be equal to the Reynolds number of the prototype. It is applicable in the flow problems where viscous force is predominant in addition to inertia force. According to this law, we get the below expression.

$$(\text{Re})_m = (\text{Re})_p$$

$$\boxed{\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p}} \tag{19.6}$$

$$\frac{\rho_r V_r L_r}{\mu_r} = 1 \tag{19.7}$$

Here, $\rho_r = (\rho_m / \rho_p)$, $V_r = (V_m / V_p)$, $L_r = (L_m / L_p)$ and $\mu_r = (\mu_m / \mu_p)$ are the scale ratios of density, velocity, length and viscosity, respectively.

The expressions for time scale ratio (T_r), acceleration scale ratio (a_r), discharge scale ratio (Q_r) and force scale ratio (F_r) for this model are given below.

$$T_r = \frac{L_r}{V_r}$$

$$a_r = \frac{V_r}{T_r}$$

$$Q_r = (\rho A V)_r = \rho_r L_r^2 V_r$$

$$F_r = m_r a_r = \rho_r L_r^3 a_r, \text{ where } m_r = \rho_r L_r^3$$

This law is used as similarity criterion in the phenomena like (i) incompressible fluid flow in pipe, (ii) motion of air planes, (iii) flow around completely immersed bodies under moving fluid and (iv) motion of submarines completely under water.

Example 19.13 A liquid of specific gravity 0.925 and viscosity 0.032 poise is to be transported through a pipe of diameter 1 m at the rate of 2250 litres per second. Tests were performed on a 10 cm diameter pipe using water at ambient conditions. If the viscosity and density of water at the given conditions is 0.01 poise and 1000 kg/m^3, respectively, then determine the velocity and rate of flow in the model.

Solution

Let $S_p = 0.925$, $\mu_p = 0.032$ poise, $D_p = 1$ m, $Q_p = 2250$ l/s = 2.25 m^3/s, $D_m = 10$ cm = 0.1 m, $\mu_m = 0.01$ poise and $\rho_m = 1000$ kg/m^3.

Let V_m and Q_m be the velocity and rate of flow in the model, respectively.

$$\rho_p = S_p \rho_w = 0.925 \times 1000 = 925 \text{ kg/m}^3$$

$$V_p = \frac{Q_p}{A_p} = \frac{Q_p}{(\pi/4) D_p^2} = \frac{2.25}{(\pi/4)\times 1^2} = 2.865 \text{ m/s}$$

For dynamic similarity of the pipe flow: $(\text{Re})_m = (\text{Re})_p$

$$\frac{\rho_m V_m D_m}{\mu_m} = \frac{\rho_p V_p D_p}{\mu_p} \Rightarrow V_m = \frac{\rho_p}{\rho_m} \times \frac{\mu_m}{\mu_p} \times \frac{D_p}{D_m} \times V_p$$

$$\therefore V_m = \frac{925}{1000} \times \frac{0.01}{0.032} \times \frac{1}{0.1} \times 2.865 = \mathbf{8.282\ m/s}$$

$$Q_m = A_m V_m = \frac{\pi}{4} D_m{}^2 \times V_m = \frac{\pi}{4} \times 0.1^2 \times 8.282 = \mathbf{0.06505\ m^3/s}$$

Example 19.14 A wind tunnel is used to test 5 : 1 scale model of a car. The velocity with prototype is 60 km/hr and for the dynamic similar conditions, the model drag is 240 N. If air is used with model as well as the prototype, then determine the drag and the power required for the prototype.

Solution

Let $L_p / L_m = 5:1$, $V_p = 60$ km/hr and $F_m = 240$ N.

Let F_p and P_p be the drag and the power required for the prototype.

For dynamic similarity, the Reynolds model is used, i.e., $(\text{Re})_m = (\text{Re})_p$.

Thus
$$\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p}$$

As air is used for both model and prototype, $\rho_m = \rho_p$ and $\mu_m = \mu_p$.

Thus
$$\frac{V_m}{V_p} = \frac{L_p}{L_m} = 5$$

$$\therefore V_m = 5V_p = 5 \times 60 = \mathbf{300\ km/hr}$$

Since
$$F = m \times a = \rho v \times a = \rho \times L^3 \times \frac{V}{T} = \rho \times L^2 \times \frac{L}{T} \times V = \rho L^2 V^2$$

Thus
$$\frac{F_p}{F_m} = \frac{\rho_p}{\rho_m} \times \left(\frac{L_p}{L_m}\right)^2 \times \left(\frac{V_p}{V_m}\right)^2$$

$$\therefore F_p = F_m \times \frac{\rho_p}{\rho_m} \times \left(\frac{L_p}{L_m}\right)^2 \times \left(\frac{V_p}{V_m}\right)^2 = 240 \times 1 \times \left(\frac{5}{1}\right)^2 \times \left(\frac{60}{300}\right)^2 = 240\ \text{N}$$

$$\therefore P_p = F_p \times V_p = \frac{240 \times 60 \times 1000}{3600} = \mathbf{4000\ Watts}$$

Example 19.15 A ship of length 300 m moves in sea water whose density is 1030 kg/m^3. A 1 : 100 model of this ship is to be tested in a wind tunnel. The velocity of air in the wind tunnel around the model is 30 m/s and the resistance of the model is 60 N. Determine the velocity of ship and its resistance in sea water. The density of air is given as 1.24 kg/m^3. Take the kinematic viscosities of sea water and air as 0.012 stokes and 0.018 stokes, respectively.

Solution

Let $L_p = 300$ m, $\rho_p = 1030$ kg/m^3, $L_m / L_p = 1/100$, $V_m = 30$ m/s, $F_m = 60$ N, $\rho_m = 1.24$ kg/m^3, $\nu_p = 0.012$ stokes and $\nu_m = 0.018$ stokes. Let V_p and F_p be the velocity and resistance of the ship, respectively.

For dynamic similarity, the Reynolds model is used, i.e., $(\text{Re})_m = (\text{Re})_p$.

Thus $$\frac{V_m L_m}{\nu_m} = \frac{V_p L_p}{\nu_p}$$

$$\therefore V_p = V_m \times \frac{\nu_p}{\nu_m} \times \frac{L_m}{L_p} = 30 \times \frac{0.012}{0.018} \times \frac{1}{100} = \mathbf{0.2\ m/s}$$

Since $$F_p = F_m \times \frac{\rho_p}{\rho_m} \times \left(\frac{L_p}{L_m}\right)^2 \times \left(\frac{V_p}{V_m}\right)^2$$

$$\therefore F_p = 60 \times \frac{1030}{1.24} \times \left(\frac{100}{1}\right)^2 \times \left(\frac{0.2}{30}\right)^2 = \mathbf{22150.54\ N}$$

19.8.2 Froude Model Law

According to this law, for the similarity of the flow, the Froude number of the model should be equal to the Froude number of the prototype. It finds applications in the flow problems where gravitational force is predominant which controls the motion in addition to inertia force. According to this law, we get the following expression.

$$(Fr)_m = (Fr)_p$$

$$\boxed{\frac{V_m}{\sqrt{L_m g_m}} = \frac{V_p}{\sqrt{L_p g_p}}} \tag{19.8}$$

$$\frac{V_r}{\sqrt{L_r g_r}} = 1 \tag{19.9}$$

Practically, the two sites of model and prototype testing have equal values of acceleration due to gravity. Thus, $g_r = 1$ and Equation (19.9) is simplified as follows.

$$\frac{V_r}{\sqrt{L_r}} = 1 \text{ or } V_r = \sqrt{L_r} \tag{19.10}$$

Here, $V_r = (V_m / V_p)$ and $L_r = (L_m / L_p)$ are the scale ratios of velocity and length, respectively. The Equation (19.10) may be used to obtain other scale ratios.

For this model, the time scale ratio (T_r), acceleration scale ratio (a_r), discharge scale ratio (Q_r), force scale ratio (F_r), pressure scale ratio (p_r), work (energy) scale ratio (w_r) and power scale ratio (P_r) in terms of the length scale ratio is given below.

$$T_r = \frac{L_r}{V_r} = \frac{L_r}{\sqrt{L_r}} = \sqrt{L_r}$$

$$a_r = \frac{V_r}{T_r} = \frac{\sqrt{L_r}}{\sqrt{L_r}} = 1$$

$$Q_r = \rho_r A_r V_r = L_r^2 \sqrt{L_r} = L_r^{2.5}$$

When the same fluid is used in model and prototype, ρ_r is equal to one.

$$F_r = \rho_r L_r^3 a_r = 1 \times L_r^3 \times 1 = L_r^3$$

$$p_r = \frac{F_r}{A_r} = \frac{L_r^3}{L_r^2} = L_r$$

$$w_r = F_r L_r = L_r^{\,3} L_r = L_r^{\,4}$$

$$P_r = \frac{F_r L_r}{T_r} = \frac{L_r^{\,3} L_r}{\sqrt{L_r}} = L_r^{\,3.5}$$

The similitude based on this law finds applications in (i) free surface flows, such as flow over spillways and sluices, (ii) flow of jet from a nozzle or an orifice, (iii) flow problems in which waves are likely to be formed on the surface and (iv) flow problems in which fluids of different mass densities flow over one another.

Example 19.16 In 1 in 30 model of a spillway, the velocity and discharge are 1.5 m/s and 2 m^3/s, respectively. Determine the corresponding velocity and discharge in the prototype.

Solution

$$\text{Let } L_r = L_m / L_p = 1/30\,,\ V_m = 1.5 \text{ m/s and } Q_m = 2 \text{ m}^3/\text{s}\,.$$

Let V_p and Q_p be the velocity and discharge, respectively, in the prototype.

Since $$\frac{V_m}{\sqrt{L_m}} = \frac{V_p}{\sqrt{L_p}} \qquad \text{[Froude model law]}$$

$$\therefore V_p = V_m \times \sqrt{\frac{L_p}{L_m}} = \sqrt{30} \times 1.5 = \mathbf{8.216\ m/s}$$

Since $$Q_r = \frac{Q_m}{Q_p} = L_r^{\,2.5}$$

$$\therefore Q_p = \frac{Q_m}{L_r^{\,2.5}} = \frac{2}{(1/30)^{2.5}} = \mathbf{9859.01\ m^3/s}$$

Example 19.17 A 5 m ship model was tested in water having density as 1000 kg/m^3. The measurements showed a resistance of 60 N when the model moved at 2.5 m/s. Determine the velocity of 80 m prototype and the force required to drive the prototype at this speed through sea water having density as 1025 kg/m^3.

Solution

Let $L_m = 5$ m, $\rho_m = 1000$ kg/m^3, $F_m = 60$ N, $V_m = 2.5$ m/s, $L_p = 80$ m and $\rho_p = 1025$ kg/m^3.

$$L_r = \frac{L_m}{L_p} = \frac{5}{80} = \frac{1}{16}$$

Since $$\frac{V_m}{\sqrt{L_m}} = \frac{V_p}{\sqrt{L_p}} \qquad \text{[Froude model law]}$$

$$\therefore V_p = V_m \times \sqrt{\frac{L_p}{L_m}} = \frac{V_m}{\sqrt{L_r}} = \frac{2.5}{\sqrt{1/16}} = \mathbf{10\ m/s}$$

Since $$\frac{F_m}{F_p} = \frac{\rho_m}{\rho_p} \times \frac{L_m^{\,3}}{L_p^{\,3}} \times \frac{a_m}{a_p} = \frac{\rho_m}{\rho_p} \times L_r^{\,3} \qquad [\because (a_m / a_p) = 1]$$

$$\therefore F_p = \frac{F_m}{L_r^{\,3}} \times \frac{\rho_p}{\rho_m} = \frac{60}{(1/16)^3} \times \frac{1025}{1000} = \mathbf{251904\ N}$$

Example 19.18 A 7.2 m high and 15 m long spillway discharges 94 m^3/s discharge under a head of 2 m. If 1 : 9 scale model of this spillway is to be constructed, then determine the model dimensions, head over spillway model and the model discharge. If the model experiences a force of 7500 N, then determine the force on the prototype.

Solution

Let $h_p = 7.2$ m, $L_p = 15$ m, $Q_p = 94$ m^3/s, $H_p = 2$ m, $L_r = L_m / L_p = 1/9$, and $F_m = 7500$ N.

$$L_r = \frac{h_m}{h_p} = \frac{L_m}{L_p} = \frac{1}{9}$$

Height of the model is given by,

$$h_m = h_p \times L_r = 7.2 \times \frac{1}{9} = \mathbf{0.8\ m}$$

Length of the model is given by,

$$L_m = L_p \times L_r = 15 \times \frac{1}{9} = \mathbf{1.67\ m}$$

Head over the model is given by,

$$H_m = H_p \times L_r = 2 \times \frac{1}{9} = \mathbf{0.222\ m}$$

Discharge through the model is given by,

$$Q_m = Q_p \times L_r^{2.5} = 94 \times \left(\frac{1}{9}\right)^{2.5} = \mathbf{0.387\ m^3/s}$$

Force on the prototype is given by,

$$F_p = \frac{F_m}{L_r^3} = \frac{7500}{(1/9)^3} = \mathbf{5467500\ N}$$

19.8.3 Euler Model Law

According to this law, for the similarity of the flow, the Euler number of the model should be equal to the Euler number of the prototype. It finds applications in the flow problems where pressure force is predominant in addition to inertia force. According to this law, we get the following expression.

$$(Eu)_m = (Eu)_p$$

$$\boxed{\frac{V_m}{\sqrt{p_m/\rho_m}} = \frac{V_p}{\sqrt{p_p/\rho_p}}} \tag{19.11}$$

If the same fluid is used in model and prototype, then $\rho_m = \rho_p$ and we get the below expressions.

$$\boxed{\frac{V_m}{\sqrt{p_m}} = \frac{V_p}{\sqrt{p_p}}} \tag{19.12}$$

or

$$\frac{V_r}{\sqrt{p_r}} = 1 \tag{19.13}$$

Euler model law is used as similarity criterion for the flow in an enclosed fluid system in which turbulence is fully developed, so that the viscous forces are insignificant and the other forces, such as gravity and surface tension are absent. It is also applicable in the flow problems where cavitation occurs.

19.8.4 Weber Model Law

For the similarity of the flow, the Weber number of the model should be equal to the Weber number of the prototype. It finds applications in the flow problems where surface tension effects predominate in addition to inertia force. According to this law, we get the following expressions.

$$(We)_m = (We)_p$$

$$\boxed{\frac{V_m}{\sqrt{\sigma_m/(\rho_m L_m)}} = \frac{V_p}{\sqrt{\sigma_p/(\rho_p L_p)}}} \tag{19.14}$$

If the same fluid is used in model and prototype, then $\rho_m = \rho_p$ and we get the below expressions.

$$\boxed{\frac{V_m}{\sqrt{\sigma_m/L_m}} = \frac{V_p}{\sqrt{\sigma_p/L_p}}} \tag{19.15}$$

or

$$\frac{V_r}{\sqrt{\sigma_r / L_r}} = 1 \tag{19.16}$$

Weber model law is used as similarity criterion for the flow where surface tension forces dominate. This law finds application in problems like (i) flow over weirs for very low heads, (ii) very thin sheets of liquid flowing over a surface, (iii) capillary waves in channels and (iv) capillary rise in narrow passages.

19.8.5 Mach Model Law

According to this law, for the similarity of the flow, the Mach number of the model should be equal to the Mach number of the prototype. It finds applications in the flow problems where forces from elastic compression are significant in addition to inertia force. According to this law, we get the following expressions.

$$(M)_m = (M)_p$$

$$\boxed{\frac{V_m}{\sqrt{K_m/\rho_m}} = \frac{V_p}{\sqrt{K_p/\rho_p}}} \tag{19.17}$$

$$\frac{V_r}{\sqrt{K_r / \rho_r}} = 1 \tag{19.18}$$

For dynamic similarity, the Mach model law finds application in (i) aerodynamic testing, (ii) in the flow phenomena involving velocities exceeding the speed of sound, (iii) hydraulic model testing for water flow problems and (iv) water testing of torpedoes.

Example 19.19 In an aeroplane model of size 1/20 of its prototype, the pressure drop is 5 kN/m^2. Find the corresponding pressure drop in the prototype if the model is tested in water. Assume the densities of air and water as 1.24 kg/m^3 and 1000 kg/m^3, respectively and the viscosities of air and water as 1.8×10^{-4} poise and 1.0×10^{-2} poise, respectively.

Solution

Let $(L_m / L_p) = (1/20)$, $p_m = 5$ kN/m^2, $\rho_p = 1.24$ kg/m^3, $\rho_m = 1000$ kg/m^3, $\mu_p = 1.8\times10^{-4}$ poise and $\mu_m = 1.0\times10^{-2}$ poise.

Let p_p be the corresponding pressure drop in the prototype.

It is known that this problem involves pressure and viscous forces and therefore, Euler's and Reynolds numbers are to be considered for dynamic similarity.

Since $$\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p} \qquad \text{[Reynolds model law]}$$

$$\therefore \frac{V_p}{V_m} = \frac{\rho_m}{\rho_p} \times \frac{L_m}{L_p} \times \frac{\mu_p}{\mu_m} = \frac{1000}{1.24} \times \frac{1}{20} \times \frac{1.8 \times 10^{-4}}{1 \times 10^{-2}} = 0.7258$$

Since $$\frac{V_m}{\sqrt{p_m / \rho_m}} = \frac{V_p}{\sqrt{p_p / \rho_p}} \qquad \text{[Euler model law]}$$

$$\therefore p_p = \left(\frac{V_p}{V_m}\right)^2 \times \frac{\rho_p}{\rho_m} \times p_m = 0.7258^2 \times \frac{1.24}{1000} \times 5 \times 10^3 = \mathbf{3.266\ N/m^2}$$

Example 19.20 A 1 : 30 model of a sea water flying boat when moved through fresh water at 2 m/s has a drag of 8 N. For calculating the skin resistances, use the relation $F_f = fAV^2$, where f is the skin drag coefficient having values for model and prototype as 0.025 and 0.0018, respectively. The wetted surface area of the model is 20 m^2. Find the total drag on the prototype and the power required to drive it. Assume sea water and fresh water densities as 1025 kg/m^3 and 1000 kg/m^3, respectively.

Solution

Let $L_r = L_m / L_p = 1/30$, $V_m = 2$ m/s, $(F_w)_m = 8$ N, $F_f = fAV^2$, $f_m = 0.025$, $f_p = 0.0018$, $A_m = 20$ m^2, $\rho_p = 1025$ kg/m^3 and $\rho_m = 1000$ kg/m^3.

According to Reynolds model law, we get:

$$\frac{(F_w)_m}{(F_w)_p} = \frac{\rho_m}{\rho_p} \times \left(\frac{L_m}{L_p}\right)^2 \times \left(\frac{V_m}{V_p}\right)^2 = \frac{\rho_m}{\rho_p} \times L_r^2 \times \left(\sqrt{L_r}\right)^2 = \frac{\rho_m}{\rho_p} \times L_r^3$$

$$\therefore (F_w)_p = (F_w)_m \times \frac{\rho_p}{\rho_m} \times \frac{1}{L_r^3} = 8 \times \frac{1025}{1000} \times \frac{1}{(1/30)^3} = 221400 \text{ N}$$

According to Froude model law, we get:

$$\frac{V_m}{\sqrt{L_m}} = \frac{V_p}{\sqrt{L_p}}$$

$$\therefore V_p = V_m \times \sqrt{\frac{L_p}{L_m}} = \frac{V_m}{\sqrt{L_r}} = \frac{2}{\sqrt{1/30}} = 10.954 \text{ m/s}$$

Since $$A_r = \frac{A_m}{A_p} = L_r^2$$

Thus $$A_p = \frac{A_m}{L_r^2} = \frac{20}{(1/30)^2} = 18000 \text{ m}^2$$

Skin drag on prototype is given by,

$$(F_f)_p = f_p A_p V_p^2 = 0.0018 \times 18000 \times 10.954^2 = 3887.68 \text{ N}$$

Total drag on prototype is given by,

$$F_p = (F_w)_p + (F_f)_p = 221400 + 3887.68 = \mathbf{225287.68\ N}$$

Power required to drive the prototype is given by,

$$P_p = \frac{F_p V_p}{1000} = \frac{225287.68 \times 10.954}{1000} = \mathbf{2467.8\ kW}$$

19.9 ❐ TYPES OF MODELS

The models used for testing can be classified into two broad categories, namely undistorted models and distorted models. These models are described below.

1. **Undistorted Models:** These models are geometrically similar to their prototypes. Therefore, scale ratios for corresponding linear dimensions of the model and its prototype are same. Due to perfect similitude, the behaviour of the prototype can be easily and accurately predicted from the results obtained by testing the undistorted model.
2. **Distorted Models:** A model which is not similar to its prototype is called a distorted model. In these models, one or more terms of the model are not identical with their counterparts in the prototype. In distorted models, different scale ratios are adopted for the linear dimensions. Some of the examples for which distorted models are to be prepared are rivers, dams across very wide rivers, estuaries and harbours. A distorted model may have the following distortions.
   (a) **Geometrical distortion:** This distortion can be either of varying dimensions or that of configuration. It results due to the adoption of different scales for vertical and horizontal dimensions. Distortion of dimensions is frequently used in river models where a different scale ratio for depth is taken. The distortion of configuration results when the configuration of the model does not resemble to its prototype. In such distortion, the model is geometrically similar but its bed slope is increased by keeping it in tilted position when compared to the position of the prototype.
   (b) **Material distortion:** This distortion results when the materials in both the model and prototype do not satisfy the similitude conditions.
   (c) **Hydraulic distortion:** Such distortion occurs due to change in some uncontrollable hydraulic quantities, such as time, velocity and discharge.

Reasons for adopting distorted models Some of the reasons for adopting distorted models are (i) to maintain accuracy in vertical measurements, (ii) to maintain turbulent flow, (iii) to obtain suitable roughness condition, (iii) to obtain suitable bed material and its adequate movement and (iv) to accommodate the available facilities, for example, money, space, time and water supply.

Merits of distorted models Some of the merits of distorted models are (i) accurate measurements can be made easily due to increase in the depth of fluid or magnification of wave heights in models, (ii) model size can be sufficiently reduced and thereby, the cost is considerably lowered and its operation is simplified, (iii) considerably increased Reynolds number is possible and the surface tension can be lowered and thereby, better results can be obtained and (iv) sufficient tractive force can be developed to produce adequate bed movement.

Demerits of distorted models Some of the demerits of distorted models are (i) pressure and velocity may not be correctly reproduced in magnitude and direction, (ii) slopes, cuts, and bends in a river cannot be truly produced in sand or other erodible material, (iii) a model wave may differ from that of the prototype and (iv) results obtained from a distorted model are difficult to extrapolate and interpolate.

Even though when distorted models have many drawbacks, if sufficient allowances are made judiciously in the interpretation of their results, then very useful information can be obtained.

19.10 ❒ SCALE EFFECTS IN MODELS

Always some discrepancy or deviations remains between the results obtained from the prototype and its model when complete similarity does not exist between them. This discrepancy is called scale effect. Some of the factors which result scale effects are discussed below.

In a certain fluid flow problem, several forces exist. The complete similarity can be obtained only if all the pertinent model laws are simultaneously satisfied. However, it is very difficult to satisfy all the model laws involved in a phenomenon and thus, complete similarity cannot be achieved. Under such circumstances, the forces which have secondary influence on the fluid flow phenomenon are neglected, so that the number of model laws to be satisfied is reduced. By neglecting these forces, some discrepancy or scale effect would be developed between the results obtained from the model tests and those of prototype.

The scale effect may also be developed in cases where the forces which have practically no effect on the behaviour of the prototype significantly affects the behaviour of its model. It may also not be possible to correctly simulate all the conditions in the model as that of the prototype which may also result scale effect.

In order to know the scale effects, the models with different scales should be tested. The observations collected from different scales models can be used to develop an empirical relation between scale effects and the size of model. This factor may be utilized to correct the results of the model tests.

Summary

1. The law of dimensional homogeneity states that every additive term in an equation must have the same dimensions.
2. Rayleigh method is used for determining the expression for a variable which depends upon maximum three or four variables only.
3. Buckingham π theorem states that if there are n dimensional variables involved in a dimensional homogeneous equation which contains m fundamental quantities, then the variables can be grouped into (n-m) dimensionless terms called π terms.
4. The model is a small scale replica of the actual structure or the machine while the actual structure or machine is called prototype.
5. Similitude means the complete similarity between the model and its prototype.
6. **Geometric similarity:** Ratios of corresponding linear dimensions of model and its prototype are equal.
7. **Kinematic similarity:** Similarity of motion between the model and the prototype.
8. **Dynamic similarity:** Similarity of forces between the model and the prototype.
9. **Reynolds number:** $\text{Re} = \dfrac{\text{Inertia force}}{\text{Viscous force}} = \dfrac{\rho VL}{\mu} = \dfrac{VL}{\nu}$, replace L by D for a pipe.
10. Froude number is defined as the square root of the ratio of the inertia force to the gravity force and is given by $Fr = V/\sqrt{Lg}$.
11. Euler number is defined as the square root of the ratio of the inertia force to the pressure force and is given by $Eu = V/\sqrt{p/\rho}$.
12. Weber number is defined as the square root of the ratio of the inertia force to the force of surface tension and is given by $We = V/\sqrt{\sigma/(\rho L)}$.
13. Mach number is defined as the square root of the ratio of the inertia force to the elastic force and is given by $M = \dfrac{V}{\sqrt{K/\rho}} = \dfrac{V}{C}$.
14. Scale effect: Discrepancy between results obtained from prototype and its model.

Multiple-choice Questions

1. Flow has Froude number less than one if
   (a) Normal depth is less than critical depth.
   (b) Normal depth is greater than critical depth.
   (c) Normal depth is equal to critical depth.
   (d) None of the above.
2. The force scale ratio for Reynolds model law using the same fluid both in the model and prototype is equal to
   (a) L_r (b) L_r^2
   (c) $L_r^{3/2}$ (d) 1

3. The discharge scale ratio for Froude model law is given by
(a) $L_r^{1/2}$ (b) $L_r^{3/2}$
(c) L_r^2 (d) $L_r^{5/2}$

4. In the study of forces acting on aeroplane flying with supersonic velocity, which of the following number plays significant role?
(a) Weber number. (b) Reynolds number.
(c) Mach number . (d) None of the above.

5. The time scale ratio for a model based on Froude model law in terms of length scale ratio $L_r = (L_m/L_p)$ is
(a) $\sqrt{L_r}$ (b) $1/\sqrt{L_r}$
(c) $L_r^{3/2}$ (d) None of the above

6. The acceleration ratio for a model based on Froude model law is equal to
(a) $\sqrt{L_r}$ (b) $1/\sqrt{L_r}$
(c) L_r^2 (d) 1

7. Model analysis of free surface flow is based on
(a) Euler number. (b) Reynolds number.
(c) Mach number. (d) Froude number.

8. The scale effect in models can be
(a) Negative only . (b) Positive only.
(c) Both negative and positive. (d) None of the these.

9. Distorted models are used for
(a) Rivers.
(b) Harbours.
(c) Dams across very wide rivers.
(d) All the above.

10. Principle of similitude forms the basis of
(a) Comparing two identical equipments.
(b) Designing models so that the results can be converted to prototype.
(c) Comparing similarity between design and actual equipment.
(d) Hydraulic designs.

11. An orificemeter to carry water is calibrated with air in a geometrically similar model at 1/5 prototype scale. If the ratio of kinematic viscosity of air to water is 12.5, then dynamic similar flow will be obtained when the discharge ratio (air to water) is
(a) 0.4. (b) 2.5.
(c) 62.5. (d) None of the above.

Review Questions

1. What do you understand by dimensional analysis? What are its uses?
2. Explain the principle of dimensional homogeneity with suitable examples.
3. Explain the methods of dimensional analysis along with advantages and limitations.
4. Describe Rayleigh's method for dimensional analysis.
5. Describe Buckingham pi theorem, why it is preferred over Rayleigh's method?
6. Define repeating variables. How these are selected for dimensional analysis?
7. What is a model study? Also give its applications and importance.
8. What do you understand by similitude? Describe its types.
9. What is meant by geometric, kinematic and dynamic similarities? Are these similarities truly attainable? Explain.
10. Give the physical significance of Reynolds number, Froude number, Euler number, Weber number and Mach number for fluid flow problems.
11. Discuss different laws on which models are designed for dynamic similarity.
12. Define distorted and undistorted models. Also give their merits and demerits.
13. Discuss scale effect in model testing. How is it detected?

Problems

1. Due to viscous flow the pressure difference δp in a pipe of diameter D and length l depends upon velocity V, density ρ and viscosity μ of the fluid. Express the functional relationship between these variables and δp by using Rayleigh method of dimensional analysis.

[**Ans.** $\delta p = \{(\mu V)/D\}\phi\left[(l/D), (\rho VD)/\mu\right]$]

2. The power P developed by a hydraulic pump depends on the head H, the discharge Q and specific weight w of the fluid. Derive an expression for the power developed and the given variables by Rayleigh method of dimensional analysis.

[**Ans.** $P = C(HQw)$]

3. Using Buckingham pi theorem, show that the velocity through an orifice is given by $V = \sqrt{2gH}\ \phi\left[(D/H), \mu/(\rho VH), \sigma/(\rho V^2 H)\right]$ where H is the head causing flow, ρ is the mass density, σ is the surface tension, μ is the coefficient of viscosity, D is the diameter of the orifice and g is the gravitational acceleration.

4. The lift F_L on airfoil depends on the mass density ρ, velocity V, characteristic depth d, angle of incidence α and coefficient of viscosity μ of the fluid. By using Buckingham pi theorem, obtain an expression for F_L in terms of other given parameters.
[**Ans**. $F_L = \rho V^2 d^2\ \phi\,[\mu/(\rho VD), \alpha]$]

5. Using Buckingham pi theorem, show that the pressure difference Δp in a pipe is given by $\Delta p = \{(\mu V)/\mathrm{D}\}(L/D)\ \phi\,[(\rho DV)/\mu]$, where D is the diameter of pipe, L is the length of pipe, ρ is the mass density, V is the velocity and μ is the viscosity.

6. In film lubricated journal bearings, the frictional torque T is found to depend on the speed of rotation N, viscosity of the oil μ, the load on the projected area P, and the diameter D. Find the dimensionless parameters for application to such bearings in general.
[**Ans**. $T = N\mu D^3 \phi\,[P/(N\mu)]$]

7. Obtain an expression for the efficiency, $\eta = \phi[\mu/(\rho D^2\omega), Q/(D^2\omega)]$ of a fan in terms of dimensionless parameters when it depends on the discharge Q, runner diameter D, angular velocity ω, mass density ρ and dynamic viscosity μ.

8. In a flow through a sudden contraction in a circular duct, the head loss H is found to depend on the inlet velocity V, diameters D and d and the fluid properties density ρ, viscosity μ and acceleration due to gravity g. Obtain an expression for the ratio H/D.
[**Ans**. $(H/D) = \phi\,[(d/D), (\rho DV)/\mu, (gD)/V^2]$]

9. Using Buckingham pi theorem, show that the frictional torque T of a disc is given by $T = D^5 N^2 \rho\phi[\mu/(\rho ND^2)]$, where D is the diameter of disc, N is the speed, ρ is the density and μ is the viscosity of the fluid.

10. The discharge Q over a rectangular weir depends on the head H, acceleration due to gravity g, length of the weir crest L, height h and the kinematic viscosity v. Using the method of dimensional analysis obtain an expression for Q as $Q = \sqrt{g}\,H^{-5/2}\phi[v/(g^{1/2}H^{3/2}), (L/H), (h/H)]$.

11. The capillary rise h is observed to be influenced by the tube diameter D, density ρ, acceleration due to gravity g and surface tension σ. Find the dimensionless parameters for the correlation of experimental results.
[**Ans.** $(h/D) = \phi[\sigma/(D^2\rho g)]$]

12. Determine the form of equation for the discharge Q through a sharp edged triangular notch that depends upon the notch head H, velocity of approach V, central angle α and gravitational acceleration g. Using Buckingham pi theorem, show that $Q = \sqrt{g}\,H^{5/2}\phi[V/(g^{1/2}H^{1/2}), \alpha]$.

13. The thermal boundary layer thickness δ_t is influenced by the parameters, namely viscosity μ, thermal conductivity k, density ρ, specific heat of the fluid c, velocity of flow V and distance from leading edge x. Using Buckingham pi theorem, show that $(\delta_t/x) = \phi[(\rho xV)/\mu, (c\mu)/k]$.

14. The water is flowing through a pipe of diameter 24 cm at a velocity of 4 m/s. Determine the velocity of oil flowing in another pipe of diameter 8 cm, if the condition of dynamic similarity is existing between the two pipes. The viscosity of water and oil is given as 1×10^{-3} Ns/m^2 and 2.5×10^{-3} Ns/m^2. The specific gravity of oil is given as 0.8.
[**Ans.** 37.5 m/s]

15. A wind tunnel is used to test 5 : 1 scale model of a car. The velocity with prototype is 75 km/hr and for dynamic similar conditions, the model drag is 300 N. If air is used with model as well as the prototype, then determine the drag and the power required for the prototype.
[**Ans.** 300 N, 6.25 kW]

16. A model of submarine is scaled down to 1/10 of the prototype and is to be tested in a wind tunnel where free stream pressure is 2×10^3 kN/m^2 and absolute temperature is 50°C. The speed of the prototype is 7.72 m/s. Find the free stream velocity of air and the ratio of the drags between model and prototype. Take kinematic viscosity and density of sea water as 1.4×10^{-6} m^2/s and 1025 kg/m^3 and viscosity of air as 0.0184 centipoise.
[**Ans.** 47.04 m/s, 7.81×10^{-3}]

17. A pipeline of 4 m diameter is to be designed to carry oil at the rate 5 m^3/s having specific gravity as 0.92 and viscosity as 0.04 poise. Tests were conducted using a pipe of 40 cm diameter and water as a liquid. Determine the velocity and rate of flow required for the model pipe. Take the viscosity of water as 0.01 poise.
[**Ans.** 0.398 m/s, 0.115 m^3/s]

18. The water having kinematic viscosity of 1.2×10^{-6} m^2/s and a mass density of 1000 kg/m^3 flows at a mean speed of 1.6 m/s through a 50 mm diameter pipeline. What corresponding volumetric rate (measured at atmospheric pressure) of air flow through this pipeline would give rise to essentially similar dynamical flow conditions and what would this be so? The air has kinematic viscosity of 14.7×10^{-6} m^2/s and a mass density of 1.23 kg/m^3. Determine for each fluid, the pressure drop which would occur in 5 m length of this pipeline. Take Darcy's coefficient of friction as $f = 0.005$ for both fluids.
[**Ans.** 2560.41 N/m^2, 472.52 N/m^2]

19. The characteristics of a propeller of 1.8 m diameter and rotational speed 60 rpm are examined by means of a geometrically similar model of 22.5 cm diameter. When the model is run at 240 rpm by a torque of 15 Nm, the thrust developed is 150 N and the speed of advance is 1.6 m/s. Find the following

for the full scale propeller, such as (i) speed of advance, (ii) thrust, (iii) efficiency of the full scale propeller and (iv) torque when its efficiency is equal to that of the model.

[**Ans.** 3.2 m/s, 38.4 kN, 63.66%, 30.721 kNm]

20. A ship model 1/49 is towed through sea water at a speed of 2 m/s and a force of 4 N is required to tow the model. Find the speed of ship and the force on the ship when the prototype is subjected to wave resistance only.

[**Ans.** 14 m/s, 470.596 kN]

21. In the model test of a spillway, the velocity and discharge are 1.5 m/s and 2.5 m^3/s, respectively. Determine the corresponding velocity and discharge over the prototype which is 36 times the model size.

(**Ans.** 9 m/s, 19440 m^3/s)

22. In a geometrically similar model of spillway, the discharge per metre length is 0.25 m^3/s. If the scale of the model is 1/36, then find the discharge per metre run of the prototype.

[**Ans.** 54 m^3/s]

23. A spillway model is to be built to a geometrically similar scale of 1/50 across a flume of 600 mm width. The prototype is 15 m high and maximum head on it is expected to be 1.5 m. (i) What height of the model and what head on the model should be used? (ii) If the flow over the model at a particular head is 12 litres per second, what flow per metre length of the prototype is expected? (iii) If the negative pressure in the model is 200 mm, what is the negative pressure in prototype? Is it practicable?

[**Ans.** 0.3 m, 0.03 m, 7071.07 *l*/s, –10 m, not practicable]

24. A model of a torpedo is tested in a towing tank at a velocity of 36 m/s whilst the prototype is to run at 6 m/s. (i) What model scale has been used? For water, take kinematic viscosity as 1.13×10^{-6} m^2/s. (ii) What would be the model speed, if tested in wind tunnel under a pressure of 1950 kPa and a constant temperature of 25 ºC. The absolute viscosity of air under these conditions is 1.85×10^{-4} poise and gas constant is 0.287 kJ/kg K.

[**Ans.** 1 : 6, 25.85 m/s]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (b)	**2.** (d)	**3.** (d)	**4.** (c)	**5.** (a)
6. (d)	**7.** (d)	**8.** (c)	**9.** (d)	**10.** (c)
11. (b).				

Chapter 20

Impact of Free Jets and Basics of Fluid Machines

20.1 ❒ INTRODUCTION

The jet of water discharging from a nozzle in atmosphere is called a free jet, i.e., a jet having constant pressure throughout. The jet coming out of the nozzle has certain amount of kinetic energy. When this jet strikes a vane (flat or curved plate), it exerts a force on it. This impressed force is known as impact of the jet which is designated as hydrodynamic force. This force is due to fluid motion which always involves change in momentum. It can be obtained from impulse-momentum principle or from Newton's second law of motion.

Turbine is the most important and satisfactory prime mover that produces mechanical power (i.e., shaft power). The working fluid in a turbine may be water, steam, gas, wind and refrigerants. The power consuming devices, such as compressors, pumps, fans, blowers, etc., raise the pressure or velocity of working fluid. The turbines, compressors and pumps are used in electric power generation, aircraft propulsion, ship propulsion, and a wide variety of medium and heavy industries.

In this chapter, different cases of force exerted by free water jet on stationary and moving vanes of different shapes are discussed. The propulsion of ship by the reaction of jet and basics of fluid machines are also discussed.

20.2 ❒ IMPULSE-MOMENTUM PRINCIPLE

Impulse-momentum principle is a modified form of Newton's second law of motion which states that the resultant external force acting on anybody in any direction is equal to the rate of change of momentum of the body in that direction.

Let m be the mass of fluid, V be the velocity of fluid and F be the force.

According to Newton's second law of motion, we get:

$$F = \frac{d(mV)}{dt} = \frac{\text{Mass}}{\text{Time}} \times (\text{Initial velocity} - \text{Final velocity}) \tag{20.1}$$

Equation (20.1) is known as impulse-momentum principle which can also be written as follows.

$$F \cdot dt = d\,(mV) \tag{20.2}$$

Equation (20.2) is known as impulse-momentum equation in which $F.\ dt$ is impulse and $d(mV)$ is the resulting change in momentum in the direction of force.

20.3 ❒ FORCE EXERTED BY A JET ON A STATIONARY VERTICAL FLAT PLATE

Consider a jet of water coming out from the nozzle and strikes a flat vertical plate as shown in Figure 20.1. Assume that the plate is smooth and the loss of energy due to impact of the jet is zero.

Let ρ_w be the mass density of water, V be the velocity of the jet, d be the diameter of the jet and $A=(\pi/4)d^2$ be the area of the jet.

The quantity of water striking the plate per second is given by,

$$Q = A \times V$$

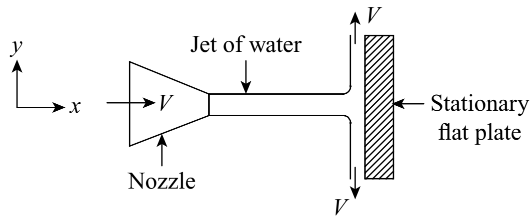


Figure 20.1 *Jet striking a stationary vertical flat plate*

Mass of water striking the flat plate per second $= \rho_w \times Q = \rho_w AV$

Applying impulse-momentum principle, we get:

$$F_x = \frac{\text{Mass}}{\text{Time}} \times (\text{Initial velocity in jet direction} - \text{Final velocity in jet direction})$$

Since the jet gets deflected through 90° after striking the plate, the component of the velocity of the jet leaving the plate in the direction of the jet will be zero.

$$\boxed{\therefore F_x = \rho_w AV \times (V - 0) = \rho_w AV^2} \quad (20.3)$$

Since the plate is stationary, i.e., $u = 0$, the work done per second by the jet on the plate is given below.

$$w = \text{Force} \times \text{Velocity} = F_x \times u = F_x \times 0 = 0$$

Example 20.1 A jet of water of diameter 50 mm having a velocity of 35 m/s strikes a flat smooth plate normally. Determine the force exerted by the jet on the plate if the plate is stationary and also find the work done.

Solution

Let $d = 50$ mm $= 0.05$ m, $V = 35$ m/s and $u = 0$.

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4} \times 0.05^2 = 0.0019635 \text{ m}^2$$

$$F_x = \rho_w AV^2 = 1000 \times 0.0019635 \times 35^2 = \mathbf{2405.29\ N}$$

$$w = F_x \times u = 2405.29 \times 0 = \mathbf{0}$$

20.4 ❒ FORCE EXERTED BY A JET ON A MOVING VERTICAL FLAT PLATE

Consider a jet of water coming out from the nozzle with a velocity V and it strikes a flat vertical plate which is moving with a uniform velocity u away from the jet as shown in Figure 20.2. Assume that the plate is smooth and loss of energy due to the impact of jet is zero. The jet strikes the plate with a relative velocity which is equal to $(V-u)$. Let ρ_w be the mass density of water, d be the diameter of the jet and $A=(\pi/4)d^2$ be the area of the jet.

The quantity of water striking the plate per second is given by,

$$Q = A(V-u)$$

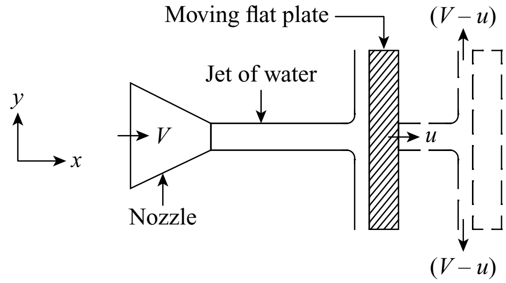


Figure 20.2 *Jet striking a vertical flat moving plate*

$$\text{Mass of water striking the plate per second} = \rho_w \times Q = \rho_w A(V-u)$$

Applying impulse-momentum principle, we get:

$$F_x = \frac{\text{Mass}}{\text{Time}} \times (\text{Initial velocity in jet direction} - \text{Final velocity in jet direction})$$

$$\boxed{\therefore F_x = \rho_w A(V-u) \times [(V-u)-0] = \rho_w A(V-u)^2} \quad (20.4)$$

Work done per second by the jet on the moving plate is given by,

$$\boxed{w = F_x \times u = \rho_w A(V-u)^2 \times u} \quad (20.5)$$

Efficiency of the jet is the ratio of output to the input, i.e., the ratio of work done per second by the jet (w) to the kinetic energy of the jet per second (K.E.).

Thus $$\eta = \frac{w}{\text{K.E.}} = \frac{\rho_w A(V-u)^2 \times u}{(1/2)(\rho_w A V)V^2} = \frac{2(V^2+u^2-2uV)u}{V^3} = \frac{2(V^2u+u^3-2Vu^2)}{V^3}$$

For a given jet velocity, the efficiency will be maximum when $(d\eta/du) = 0$.

$$\frac{d}{du}\left[\frac{2}{V^3}(V^2u+u^3-2Vu^2)\right] = 0$$

$$\frac{2}{V^3}(V^2+3u^2-4Vu) = 0$$

$$(V^2+3u^2-4Vu) = 0 \qquad [\because (2/V^3) \neq 0]$$

$$(V^2-3Vu-Vu+3u^2) = 0$$

$$V(V-3u)-u(V-3u) = 0$$

$$(V-u)(V-3u) = 0$$

$$\therefore V = u \text{ or } V = 3u$$

If $V = u$, then w from Equation (20.5) becomes,

$$w = \rho_w A(u-u)^2 \times u = 0$$

For maximum efficiency, we get:

$$\boxed{V = 3u \text{ or } u = \frac{V}{3}}$$

Maximum work done per second can be obtained by substituting $u = (V/3)$ in Equation (20.5).

$$\therefore w_{max} = \rho_w A\left(V - \frac{V}{3}\right)^2 \times \frac{V}{3} = \frac{4}{27}\rho_w A V^3$$

Maximum efficiency is given by,

$$\boxed{\eta_{max} = \frac{w_{max}}{\text{K.E.}} = \frac{(4/27)\rho_w A V^3}{(1/2)\times(\rho_w A V)\times V^2} = \frac{8}{27} \text{ or } 29.63\%}$$

Example 20.2 The diameter of the nozzle fitted at the end of pipe is 75 mm through which water is flowing and the head of water at the centre of nozzle is 200 m. The jet strikes the plate perpendicular to it. Determine the force exerted by the jet of water on the plate if the plate is moving away from the jet with a velocity of 10 m/s. Also find the work done per second on the plate and the efficiency of the jet. The coefficient of velocity is given as 0.95.

Solution

Let $d = 75\text{ mm} = 0.075\text{ m},\ H = 200\text{ m},\ u = 10\text{ m/s}$ and $C_v = 0.95$.

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 0.075^2 = 0.004418\text{ m}^2$$

$$V = C_v\sqrt{2gH} = 0.95\times\sqrt{2\times 9.81\times 200} = 59.51\text{ m/s}$$

$$F_x = \rho_w A(V-u)^2 = 1000\times 0.004418\times(59.51-10)^2 = \mathbf{10829.58\ N}$$

$$w = F_x\times u = 10829.58\times 10 = \mathbf{108295.8\ Nm/s}$$

$$\eta = \frac{w}{(1/2)\rho_w AV^3} = \frac{108295.8}{(1/2)\times 1000\times 0.004418\times 59.51^3}\times 100 = \mathbf{23.26\%}$$

20.5 ❐ FORCE EXERTED BY JET ON A STATIONARY INCLINED FLAT PLATE

Consider a flat stationary plate inclined at an angle α to the direction of flow of water jet as shown in Figure 20.3. Assume that the plate is smooth and loss of energy due to the impact of jet is zero.

Let V be the absolute velocity of the jet, d be the diameter of the jet, $A = (\pi/4)d^2$ be the area of the jet, $Q = AV$ be the quantity of water striking the plate per second and ρ_w be the mass density of the water.

$$\text{Mass of water striking the plate per second} = \rho_w\times Q = \rho_w AV$$

Applying impulse-momentum principle, we get:

$$F = \frac{\text{Mass}}{\text{Time}}(\text{Initial jet velocity in normal direction} - \text{Final jet velocity in normal direction})$$

$$\boxed{\therefore F = \rho_w AV\times[V\cos(90^\circ-\alpha)-0] = \rho_w AV^2\sin\alpha} \quad (20.6)$$

This force F can be resolved into two components, one in the direction of the jet (i.e., along x-axis) denoted by F_x and the other component perpendicular to the direction of flow (i.e., along y-axis) denoted by F_y.

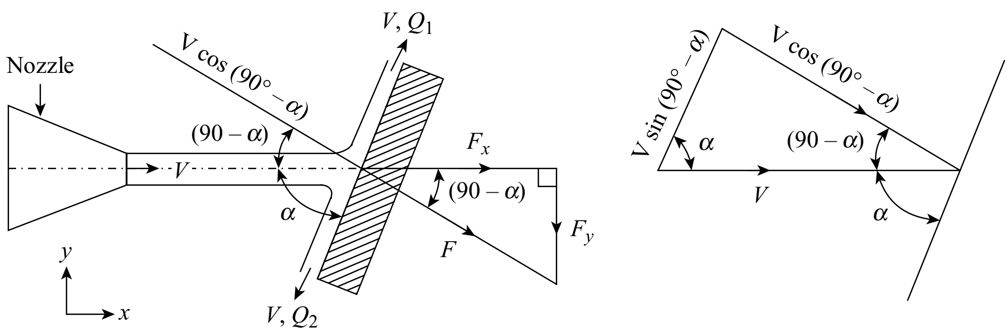


Figure 20.3 *Jet striking an inclined stationary flat plate*

$$\boxed{F_x = F\cos(90^\circ - \alpha) = (\rho_w AV^2 \sin\alpha)\times\sin\alpha = \rho_w AV^2 \sin^2\alpha} \tag{20.7}$$

$$\boxed{F_y = F\sin(90^\circ - \alpha) = (\rho_w AV^2 \sin\alpha)\times\cos\alpha = \rho_w AV^2 \sin\alpha\cos\alpha} \tag{20.8}$$

Since the plate is stationary, work done per second by the jet on the plate will be zero.

Let the total discharge (Q) of the jet divides into two portions Q_1 and Q_2 parallel to the plate surface. This division of discharge may be calculated by applying the condition that there is no resultant force acting in the direction parallel to the plate because there is no pressure change and no frictional resistance. According to momentum equation, the direction of discharge Q_1 is given below.

Final momentum rate − Initial momentum rate = Force

or
$$(\rho_w Q_1 V - \rho_w Q_2 V) - \rho_w QV\cos\alpha = 0$$

$$Q_1 - Q_2 = Q\cos\alpha \tag{i}$$

Since
$$Q = Q_1 + Q_2 \tag{ii}$$

By solving the expressions (i) and (ii), the values of Q_1 and Q_2 are obtained as follows.

$$Q_1 = \frac{Q}{2}(1+\cos\alpha) \tag{20.9}$$

$$Q_2 = \frac{Q}{2}(1-\cos\alpha) \tag{20.10}$$

Ratio of discharges is given by,

$$\boxed{\frac{Q_1}{Q_2} = \frac{1+\cos\alpha}{1-\cos\alpha}} \tag{20.11}$$

Example 20.3 A 25 mm diameter water jet exerts a force of 883 N in the direction of flow on a flat plate which is held inclined at an angle of 30° with the axis of stream. Find the rate of flow of water.

Solution

Let $d = 25$ mm $= 0.025$ m, $F_x = 883$ N and $\alpha = 30^\circ$.

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 0.025^2 = 0.000491\ \text{m}^2$$

Since
$$F_x = \rho_w AV^2 \sin^2\alpha$$

$$883 = 1000\times 0.000491\times V^2 \sin^2 30^\circ$$

$$\therefore V = \sqrt{\frac{883}{1000\times 0.000491\sin^2 30^\circ}} = 84.814\ \text{m/s}$$

$$Q = AV = 0.000491\times 84.814 = \mathbf{0.041644\ m^3/s}$$

20.6 ❐ FORCE EXERTED BY A JET ON A MOVING INCLINED FLAT PLATE

Consider a smooth flat plate inclined at an angle of α to the direction of jet flow and is moving with a uniform velocity in the direction of jet as shown in Figure 20.4.

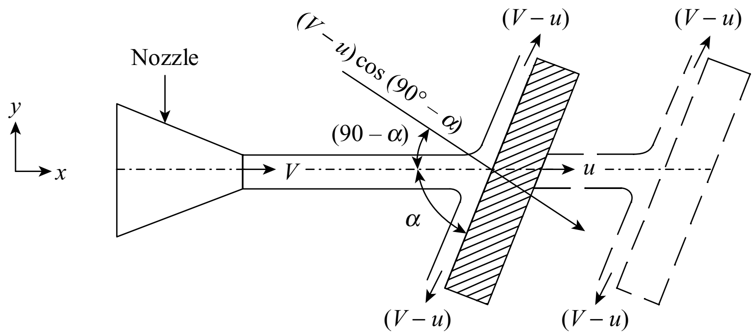


Figure 20.4 *Jet striking to an inclined moving flat plate*

Let V be the absolute velocity of the jet, d be the diameter of the jet, $A = (\pi/4)d^2$ be the area of the jet, u be the velocity of the inclined flat plate and ρ_w be the mass density of the water. The jet strikes the plate with a relative velocity which is equal to $(V - u)$.

The quantity of water striking the plate per second is given by,

$$Q = A(V - u)$$

$$\text{Mass of water striking the plate per second} = \rho_w \times Q = \rho_w A(V - u)$$

The force exerted by the jet on the plate in the direction normal to the plate is given by the impulse-momentum principle as follows.

$$F = \frac{\text{Mass}}{\text{Time}}(\text{Initial jet velocity in normal direction} - \text{Final jet velocity in normal direction})$$

$$\boxed{\therefore F = \rho_w A(V - u) \times [(V - u)\cos(90° - \alpha) - 0] = \rho_w A(V - u)^2 \sin\alpha} \tag{20.12}$$

This force F can be resolved into two components, one in the direction of the jet (i.e., along x-axis) denoted by F_x and the other component perpendicular to the direction of flow (i.e., along y-axis) denoted by F_y.

$$\boxed{F_x = F\sin\alpha = [\rho_w A(V - u)^2 \sin\alpha] \times \sin\alpha = \rho_w A(V - u)^2 \sin^2\alpha} \tag{20.13}$$

$$\boxed{F_y = F\cos\alpha = [\rho_w A(V - u)^2 \sin\alpha] \times \cos\alpha = \rho_w A(V - u)^2 \sin\alpha\cos\alpha} \tag{20.14}$$

Work done per second by the jet on the plate is given by,

$$\boxed{w = F_x \times u = \rho_w A(V - u)^2 \sin^2\alpha \times u} \tag{20.15}$$

Efficiency of the jet is the ratio of work done per second by the jet (w) to the kinetic energy of the jet per second (K.E.).

$$\therefore \eta = \frac{w}{\text{K.E.}} = \frac{\rho_w A(V - u)^2 \sin^2\alpha \times u}{(1/2) \times (\rho_w AV) \times V^2} = \frac{2u(V - u)^2 \sin^2\alpha}{V^3} \tag{20.16}$$

Example 20.4 A 75 mm diameter water jet having a velocity of 25 m/s strikes a flat plate, the normal of which is inclined at 45° to the axis of the jet. Find the normal force exerted on the plate (i) when plate is stationary and (ii) when plate is moving with a velocity of 15 m/s in the direction of jet away from the jet. Also determine the power and efficiency of the jet when the plate is moving.

Solution

Let $d = 75\text{ mm} = 0.075\text{ m}$, $V = 25\text{ m/s}$, angle $= 45°$ and $u = 15\text{ m/s}$.

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 0.075^2 = 0.004418 \text{ m}^2$$

Angle between the jet and the plate: $\alpha = 90° - 45° = 45°$

(i) The normal force on an inclined stationary flat plate is given by,

$$F = \rho_w AV^2 \sin\alpha = 1000\times 0.004418\times 25^2 \sin 45° = \mathbf{1952.5\ N}$$

(ii) When the plate is moving in the direction of jet, the normal force is given by,

$$F = \rho_w A(V-u)^2 \sin\alpha = 1000\times 0.004418\times (25-15)^2 \sin 45° = \mathbf{312.4\ N}$$

Force in the direction of jet is given by,

$$F_x = \rho_w A(V-u)^2 \sin^2\alpha = 1000\times 0.004418\times (25-15)^2 \sin^2 45° = \mathbf{220.9\ N}$$

$$P = \frac{F_x \times u}{1000} = \frac{220.9\times 15}{1000} = \mathbf{3.3135\ kW}$$

$$\eta = \frac{w}{(1/2)\rho_w AV^3} = \frac{3313.5}{(1/2)\times 1000\times 0.004418\times 25^3}\times 100 = \mathbf{9.6\%}$$

Example 20.5 Derive an expression for the normal force exerted by the water jet on the inclined flat plate when the plate is moving with a uniform velocity parallel to itself and in direction of the normal to its surface. A 75 mm diameter jet of water having a velocity of 30 m/s strikes a flat plate, the normal of which is inclined at 30° to the axis of the jet. Calculate the normal force exerted on the plate when the plate is moving with a velocity of 5 m/s parallel to itself and in the direction of the normal to its surface. Also calculate the work done, power and efficiency of the jet.

Solution

Refer Figure 20.5. Let $d = 75$ mm $= 0.075$ m, $V = 30$ m/s, $\alpha = 30°$ and $u = 5$ m/s.

Let V be the absolute velocity of the jet, d be the diameter of the jet, $A = (\pi/4)d^2$ be the area of the jet, u be the velocity of the flat plate, ρ_w be the mass density of the water and α be the angle of inclination of the normal to the axis of the jet.

Mass of the water issued by the jet per second = $\rho_w AV$, plate velocity in the jet direction = $u/\cos\alpha$ and jet striking with a relative velocity to the plate in the jet direction $= V - (u/\cos\alpha)$.

Quantity of water striking the plate per second is given by,

$$Q = \text{Area}\times\text{Velocity} = A\times\left(V - \frac{u}{\cos\alpha}\right)$$

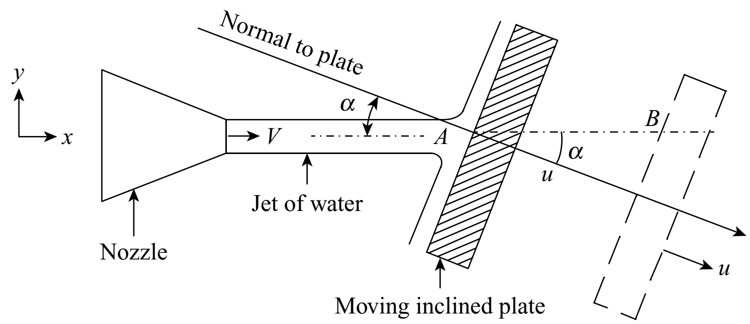


Figure 20.5 *Jet striking to moving inclined flat plate parallel to itself*

Mass of water actually striking the plate per second is given by,

$$\text{Mass/Second} = \rho_w \times Q = \rho_w \times A\left(V - \frac{u}{\cos\alpha}\right) = \rho_w A\left(\frac{V\cos\alpha - u}{\cos\alpha}\right)$$

Initial component of velocity normal to the plate = $V\cos\alpha$, final velocity normal to the plate = u and change of velocity normal to the plate = $(V\cos\alpha - u)$.

Force normal to the plate is given by,

$$F = (\text{Mass/Second}) \times \text{Change in velocity}$$

$$\therefore F = \rho_w A\left(\frac{V\cos\alpha - u}{\cos\alpha}\right) \times (V\cos\alpha - u) = \frac{\boldsymbol{\rho_w A(V\cos\alpha - u)^2}}{\boldsymbol{\cos\alpha}} \qquad (20.17)$$

Equation (20.17) is the required expression for normal force exerted by the water jet on the inclined plate when the plate is moving with a uniform velocity parallel to itself and in direction of the normal to its surface.

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4} \times 0.075^2 = 0.004418 \text{ m}^2$$

$$\alpha = 30°$$

Substituting the above values in Equation (20.17), we get:

$$F = \frac{1000 \times 0.004418 \times (30\cos 30° - 5)^2}{\cos 30°} = \mathbf{2245.63\ N}$$

The work done is given by,

$$w = F \times u = 2245.63 \times 5 = \mathbf{11228.15\ Nm/s}$$

$$P = \frac{w}{1000} = \frac{11228.15}{1000} = \mathbf{11.22815\ kW}$$

$$\eta = \frac{w}{(1/2)\rho_w A V^3} = \frac{11228.15}{(1/2) \times 1000 \times 0.004418 \times 30^3} \times 100 = \mathbf{18.82\%}$$

20.7 ❒ FORCE EXERTED BY A JET ON A SERIES OF FLAT PLATES

In actual engineering applications, a large number of plates are mounted on the circumference of a wheel at a fixed distance apart as shown in Figure 20.6. The water jet strikes a plate and exerts force on the plate which causes the wheel to rotate. As the wheel rotates, the second plate mounted on the wheel appears before the jet which again exerts force on the second plate. Thus, each plate appears successively before the jet and it exerts force on each plate. This force causes to move the plates with uniform velocity.

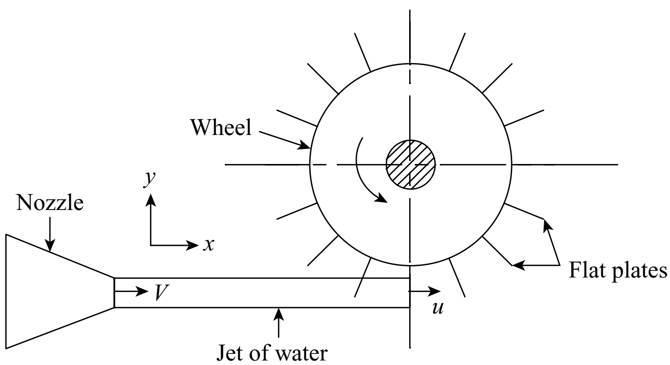


Figure 20.6 *Jet striking a series of flat plates*

Let V be the absolute velocity of the jet, d be the diameter of the jet, $A = (\pi/4)d^2$ be the area of the jet, u be the velocity of the flat plate and ρ_w be the mass density of the water. When all plates are considered, the plate will continuously intercept the jet and hence, in this case the plates will always be in contact with the jet. Therefore, the mass of water per second striking the series of plates is $(\text{mass/sec}) = \rho_w AV$.

Applying impulse-momentum principle, we get:

$$F_x = \frac{\text{Mass}}{\text{Time}}(\text{Initial velocity in jet direction} - \text{Final velocity in jet direction})$$

$$\boxed{\therefore F_x = \rho_w AV \times [(V-u) - 0] = \rho_w AV(V-u)} \quad (20.18)$$

Work done by the jet on the series of plates per second is given by,

$$\boxed{w = F_x \times u = \rho_w AV(V-u) \times u} \quad (20.19)$$

Efficiency of the jet is the ratio of output to the input, i.e., the ratio of work done per second by the jet (w) to the kinetic energy of the jet per second (K.E.).

$$\eta = \frac{\text{output}}{\text{input}} = \frac{w}{\text{K.E.}} = \frac{\rho_w AV(V-u)\times u}{(1/2)\times(\rho_w AV)\times V^2} = \frac{2u(V-u)}{V^2} \quad (20.20)$$

For a given jet velocity, the efficiency will be maximum when $(d\eta / du) = 0$ and it is expressed below.

$$\frac{d}{du}\left[\frac{2u(V-u)}{V^2}\right] = 0$$

$$\frac{2V - 4u}{V^2} = 0$$

Thus $\qquad 2V - 4u = 0 \qquad [\because (1/V^2) \neq 0]$

$$\boxed{\therefore u = \frac{V}{2}} \quad (20.21)$$

Maximum efficiency can be obtained by substituting $u = (V/2)$ in Equation (20.20) and we get the below expression.

$$\boxed{\eta_{\max} = \frac{2(V/2)[V-(V/2)]}{V^2} = \frac{V(V/2)}{V^2} = \frac{1}{2} \text{ or } 50\%} \quad (20.22)$$

Example 20.6 A jet of water 50 mm in diameter with a velocity of 25 m/s impinges on a series of plates. The plates are so arranged that each plate appears successively before the jet in the same direction and always moves with a velocity of 8 m/s. Find the force on the plate, work done per second, power and efficiency of the system.

Solution

Let $d = 50 \text{ mm} = 0.05 \text{ m}$, $V = 25 \text{ m/s}$ and $u = 8 \text{ m/s}$.

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 0.05^2 = 0.0019635 \text{ m}^2$$

$$F_x = \rho_w AV(V-u) = 1000\times 0.0019635\times 25\times(25-8) = \mathbf{834.49 \text{ N}}$$

$$w = F_x \times u = 834.49\times 8 = \mathbf{6675.92 \text{ Nm/s}}$$

$$P = \frac{w}{1000} = \frac{6675.92}{1000} = \mathbf{6.676 \text{ kW}}$$

$$\eta = \frac{w}{(1/2)\rho_w AV^3} = \frac{6675.92}{(1/2)\times 1000\times 0.0019635\times 25^3}\times 100 = \mathbf{43.52\%}$$

20.8 ❐ FORCE EXERTED BY A JET ON STATIONARY CURVED VANE

Three cases of the stationary curved vane are considered, such as (i) jet strikes the symmetrical curved vane at the centre, (ii) jet strikes the symmetrical curved vane at one end tangentially and (iii) jet strikes the unsymmetrical curved vane at one end tangentially.

20.8.1 Force Exerted on a Stationary Symmetrical Curved Vane When the Jet Strikes at the Centre of Vane

Consider a jet of water coming out from the nozzle striking a symmetrical curved vane at its centre on the concave side as shown in Figure 20.7. Assume that the vane is smooth and loss of energy due to the impact of jet is zero. Hence, the velocity of the leaving jet will be same. Let the curved vane be symmetrical about x-axis, α be the angle between leaving jet and x-axis at the outer tip, V be the absolute velocity of the jet, d be the diameter of the jet, $A = (\pi/4)d^2$ be the area of the jet, ρ_w be the mass density of the water, $Q = AV$ be the quantity of water striking per second to the vane and the mass of water striking the vane per second $= \rho_w \times Q = \rho_w AV$.

The velocity at the outlet of the plate can be resolved into two components, one in the direction of the jet and the other perpendicular to the direction of jet. The velocity at the outlet is in opposite direction to the jet of water coming out from the nozzle. Hence, the component of velocity in the direction of jet will be negative.

Component of velocity in the direction of jet $= -V\cos\alpha$

Component of velocity perpendicular to the jet $= V\sin\alpha$

Force exerted by the jet in the direction of the jet can be given by impulse-momentum principle as follows.

$$F_x = \frac{\text{Mass}}{\text{Time}}(\text{Initial velocity in jet direction} - \text{Final velocity in jet direction})$$

$$\boxed{\therefore F_x = \rho_w AV \times [V - (-V\cos\alpha)] = \rho_w AV^2(1+\cos\alpha)} \tag{20.23}$$

If there is any loss of energy either due to impact of the jet or due to frictional resistance on the vane, then relative velocity at the outlet tip will be kV, where k is the vane coefficient having value less than unity. Thus, Equation (20.23) is written as follows.

$$\boxed{F_x = \rho_w AV^2(1+k\cos\alpha)} \tag{20.23a}$$

Force exerted by the jet in the direction perpendicular to the jet is given by,

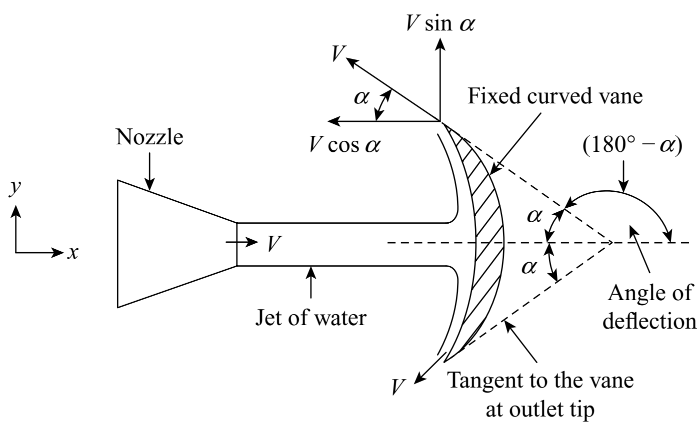


Figure 20.7 *Jet striking a fixed symmetrical curved vane at its centre*

$$F_y = \frac{\text{Mass}}{\text{Time}}(\text{Initial velocity in } y\text{-direction} - \text{Final velocity in } y\text{-direction})$$

$$\boxed{\therefore F_y = \rho_w AV \times (0 - V\sin\alpha) = -\rho_w AV^2 \sin\alpha} \quad (20.24)$$

Negative sign indicates that the force is acting in downward direction.

Work done per second by the jet on the stationary curved vane will be zero.

Limiting cases

Case I: When $\alpha = 90°$, the vane becomes a flat plate and Equation (20.23) is written as follows.

$$F_x = \rho_w AV^2(1+\cos 90°) = \rho_w AV^2 \quad (20.25)$$

Thus, it is same as that for a flat plate held normal to the jet direction.

Since $[\rho_w AV^2(1+\cos\alpha)] > \rho_w AV^2$, the force exerted on a curved vane is always greater than that on a flat plate.

Case II: When $\alpha = 0°$, the vane becomes semicircular and Equation (20.23) can be written as follows.

$$\boxed{F_x = \rho_w AV^2(1+\cos 0°) = 2\rho_w AV^2} \quad (20.26)$$

Equation (20.26) shows that the force exerted on the semicircular vane will be twice than that exerted on the flat plate.

Example 20.7 A jet of water 75 mm diameter having velocity 35 m/s strikes a curved fixed symmetrical vane at the centre. If the jet is deflected through an angle of 165° at the outlet of the curved vane, then find the force exerted by the jet of water in the direction of the jet when (i) vane is smooth and (ii) when coefficient of friction is 0.85.

Solution

Let $d = 75 \text{ mm} = 0.075 \text{ m}$, $V = 35 \text{ m/s}$, $(180° - \alpha) = 165°$ and $k = 0.85$.

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4} \times 0.075^2 = 0.004418 \text{ m}^2$$

$$180° - \alpha = 165° \Rightarrow \alpha = 15°$$

(i) Force exerted by jet on the smooth curved vane in the direction of the jet is given by,

$$F_x = \rho_w AV^2(1+\cos\alpha)$$

$$\therefore F_x = 1000 \times 0.004418 \times 35^2 \times (1+\cos 15°) = \mathbf{10639.69 \text{ N}}$$

(ii) Force exerted by the jet on the curved vane when coefficient of friction k is given by,

$$F_x = \rho_w AV^2(1+k\cos\alpha)$$

$$\therefore F_x = 1000 \times 0.004418 \times 35^2 \times (1+0.85\cos 15°) = \mathbf{9855.543 \text{ N}}$$

20.8.2 Force Exerted on a Stationary Curved Vane When the Jet Strikes the Symmetrical Curved Vane at One End Tangentially

The jet of water striking the curved vane at one end tangentially is shown in Figure 20.8. The curved vane is symmetrical about x-axis. Thus, the angle at the inlet tip and the outlet tip of the vane will be equal. Assume that the vane is smooth and loss of energy due to the impact of jet is zero. Hence, the velocity at inlet and outlet of vane will be same. Let α be the angle made by the jet with the x-axis at the inlet of the curved vane, V be the absolute velocity of the jet, d be the diameter of the jet, $A = (\pi/4)d^2$ be the area of the jet, ρ_w be the mass density of the water and mass of water striking the vane per second $= \rho_w AV$.

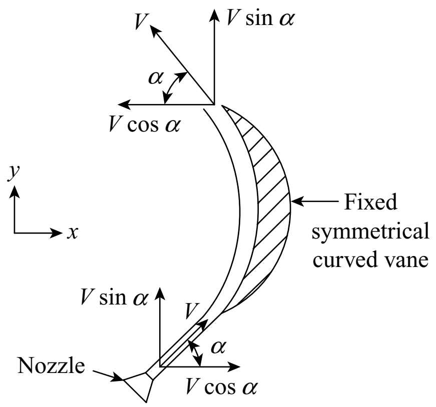


Figure 20.8

Force exerted by the jet in x-direction is given by impulse-momentum principle as follows.

$$F_x = \frac{\text{Mass}}{\text{Time}}(\text{Initial velocity in } x\text{-direction} - \text{Final velocity in } x\text{-direction})$$

$$\boxed{\therefore F_x = \rho_w AV \times [V\cos\alpha - (-V\cos\alpha)] = 2\rho_w AV^2 \cos\alpha} \qquad (20.27)$$

Force exerted by the jet in y-direction is given by impulse-momentum principle as follows.

$$F_y = \frac{\text{Mass}}{\text{Time}}(\text{Initial velocity in } y\text{-direction} - \text{Final velocity in } y\text{-direction})$$

$$\boxed{\therefore F_y = \rho_w AV \times (V\sin\alpha - V\sin\alpha) = 0} \qquad (20.28)$$

As the vane is stationary, work done per second by the jet on the curved vane is zero.

20.8.3 Force Exerted on a Stationary Curved Vane When the Jet Strikes the Unsymmetrical Curved Vane at One End Tangentially

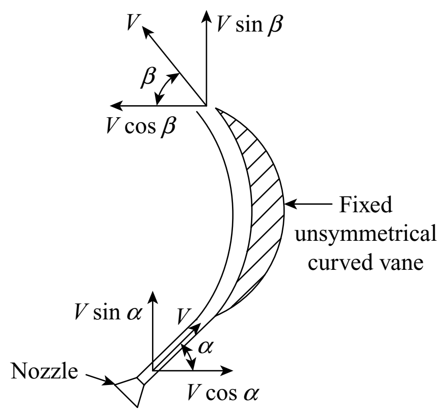


Figure 20.9

Consider that water jet strikes the unsymmetrical curved vane at one end tangentially as shown in Figure 20.9. The curved vane is unsymmetrical about x-axis. Thus, the angle at the inlet tip and outlet tip of the vane will not be equal. Assume that the vane is smooth and loss of energy due to the impact of jet is zero. Hence, velocity at the inlet and outlet of vane will be equal. Let α be the angle made by the jet with x-axis at the inlet of the curved vane, β be the angle made by the tangent at the outlet tip with x-axis, V be the absolute velocity of the jet, d be the diameter of the jet, $A = (\pi/4)d^2$ be the area of the jet, ρ_w be the mass density of the water and mass of water striking the vane per second $= \rho_w AV$.

Force exerted by the jet in x-direction is given by impulse-momentum principle as follows.

$$F_x = \frac{\text{Mass}}{\text{Time}}(\text{Initial velocity in } x\text{-direction} - \text{Final velocity in } x\text{-direction})$$

$$\boxed{\therefore F_x = \rho_w AV \times [V\cos\alpha - (-V\cos\beta)] = \rho_w AV^2(\cos\alpha + \cos\beta)} \qquad (20.29)$$

Force exerted by the jet in y-direction is given by impulse-momentum principle as follows.

$$F_y = \frac{\text{Mass}}{\text{Time}}(\text{Initial velocity in } y\text{-direction} - \text{Final velocity in } y\text{-direction})$$

$$\boxed{\therefore F_y = \rho_w AV \times (V\sin\alpha - V\sin\beta) = \rho_w AV^2(\sin\alpha - \sin\beta)} \qquad (20.30)$$

Work done by the jet on stationary unsymmetrical curved vane will be zero.

Example 20.8 A 25 mm diameter jet of water strikes a symmetrical stationary curved vane tangentially at one end and leaves at the other end with a velocity of 15 m/s. Determine the force exerted by the jet on the plate in the horizontal and vertical directions if the jet gets deflected through 150° by the vane.

Solution

Let $d = 25$ mm $= 0.025$ m, $V = 15$ m/s and $(180° - \alpha) = 150°$.

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 0.025^2 = 0.000491\ \text{m}^2$$

$$180^\circ - \alpha = 150^\circ \Rightarrow \alpha = 30^\circ$$

The force exerted by the jet of water in x-direction is given by,

$$F_x = 2\rho_w AV^2 \cos\alpha = 2\times 1000\times 0.000491\times 15^2 \cos 30^\circ = \mathbf{191.35\ N}$$

The force exerted by the jet of water in y-direction is given by,

$$F_y = \mathbf{0}$$

Example 20.9 A jet of water with diameter 50 mm moving with a velocity of 35 m/s strikes a curved fixed vane tangentially at one end at an angle of 30° to the horizontal. The jet leaves the vane at an angle of 20° to the horizontal. Find the force exerted by the jet on the plate in the horizontal and vertical directions.

Solution

Let $d = 50$ mm $= 0.05$ m, $V = 35$ m/s, $\alpha = 30^\circ$ and $\beta = 20^\circ$.

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 0.05^2 = 0.0019635\ \text{m}^2$$

The force exerted by the jet of water in x-direction is given by,

$$F_x = \rho_w AV^2(\cos\alpha + \cos\beta)$$

$$\therefore F_x = 1000\times 0.0019635\times 35^2\times(\cos 30^\circ + \cos 20^\circ) = \mathbf{4343.271\ N}$$

The force exerted by the jet of water in y-direction is given by,

$$F_y = \rho_w AV^2(\sin\alpha - \sin\beta)$$

$$\therefore F_y = 1000\times 0.0019635\times 35^2\times(\sin 30^\circ - \sin 20^\circ) = \mathbf{379.987\ N}$$

20.9 ❐ FORCE EXERTED BY JET ON MOVING CURVED VANE

Four cases of the moving curved vane are considered, such as (i) jet striking a single symmetrical moving curved vane at the centre, (ii) jet striking a series of symmetrical moving curved vanes at the centre, (iii) jet striking an unsymmetrical moving curved vane tangentially at the tip and (iv) jet striking a series of radial curved vanes.

20.9.1 Force Exerted on a Single Symmetrical Moving Curved Vane When the Jet Strikes at the Centre of Vane

Let a jet of water strikes a curved vane at the centre which is moving with a uniform velocity in the direction of the jet as shown in Figure 20.10. The vane is symmetrical about x-axis. Assume that the vane is smooth and loss of energy due to the impact of jet is zero. Let α be the angle made by the jet with x-axis at the inlet of the curved vane, V be the absolute velocity of the jet, u be the uniform velocity of vane in the direction of jet, d be the diameter of the jet, $A = (\pi/4)d^2$ be the area of the jet and ρ_w be the mass density of the water.

Here, the jet strikes the vane with a relative velocity which is equal to $(V - u)$.

This velocity can be resolved into two components, one in the direction of the jet along x-axis and the other perpendicular to the direction of the jet along y-axis.

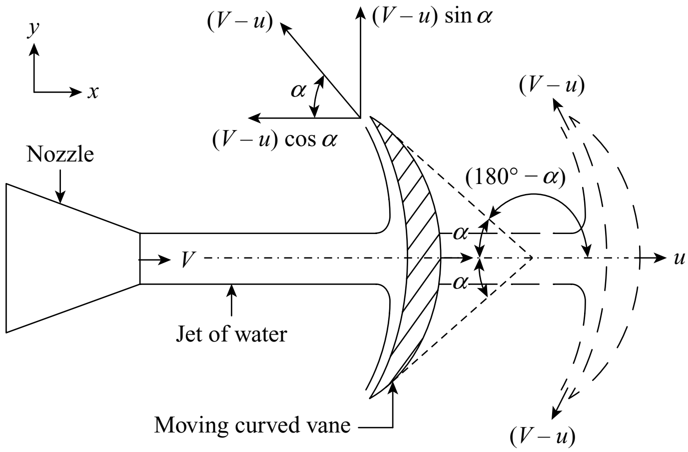


Figure 20.10 *Jet striking a symmetrical curved moving vane at its centre*

The quantity of water striking the vane per second is given by,

$$Q = A(V-u)$$

Mass of the water striking the vane per second $= \rho_w Q = \rho_w A(V-u)$

Force exerted by the jet in the direction of the jet is given by,

$$F_x = \frac{\text{Mass}}{\text{Time}}(\text{Initial velocity in jet direction} - \text{Final velocity in jet direction})$$

Thus $$F_x = \rho_w A(V-u)\times[(V-u)-\{-(V-u)\cos\alpha\}]$$

$$\boxed{\therefore F_x = \rho_w A(V-u)^2(1+\cos\alpha)} \quad (20.31)$$

If there is any loss of energy either due to the impact of jet or due to frictional resistance on the vane, then relative velocity at the outlet tip will be $k(V-u)$, where k is vane coefficient having a value less than unity. Thus, Eq. (20.31) can be written as follows.

$$\boxed{F_x = \rho_w A(V-u)^2(1+k\cos\alpha)} \quad (20.32)$$

Force exerted by the jet in y-direction is given by,

$$\boxed{F_y = \rho_w A(V-u)\times[0-(V-u)\sin\alpha] = -\rho_w A(V-u)^2\sin\alpha} \quad (20.33)$$

The negative sign indicates that the force is acting in downward direction.

Work done by the jet on the vane per second is given by,

$$\boxed{w = F_x\times u = \rho_w A(V-u)^2(1+\cos\alpha)\times u} \quad (20.34)$$

Efficiency of the jet is the ratio of work done per second by the jet (w) to the kinetic energy of the jet per second (K.E.).

$$\eta = \frac{w}{\text{K.E.}} = \frac{\rho_w A(V-u)^2(1+\cos\alpha)\times u}{(1/2)\times(\rho_w AV)\times V^2} = \frac{2u(V-u)^2(1+\cos\alpha)}{V^3} \quad (20.35)$$

Since V and α are constants, the efficiency will be maximum when $(d\eta/du) = 0$.

$$\frac{d}{du}\left[\frac{2u\,(V-u)^2(1+\cos\alpha)}{V^3}\right] = 0$$

$$\frac{d}{du}\left[\frac{2u\,(V^2+u^2-2Vu)\,(1+\cos\alpha)}{V^3}\right] = 0$$

$$\frac{d}{du}\left[\frac{2}{V^3}(V^2u+u^3-2Vu^2)(1+\cos\alpha)\right] = 0$$

$$\frac{2}{V^3}(1+\cos\alpha)(V^2+3u^2-4Vu) = 0$$

Thus $$V^2+3u^2-4Vu = 0 \qquad [\because (2/V^3)(1+\cos\alpha) \neq 0]$$

$$V^2-3Vu-Vu+3u^2 = 0$$

$$V(V-3u)-u\,(V-3u) = 0$$

$$(V-u)\,(V-3u) = 0$$

$$\therefore V = u \text{ or } V = 3u$$

If $V = u$ then from Equation (20.34), we get the below expression.

$$w = \rho_w A(V-V)^2(1+\cos\alpha)\times V = 0$$

Hence, for maximum efficiency (η_{max}), $V = 3u$, then from Equation (20.35), we get the below expression.

$$\boxed{\eta_{max} = \frac{2u\,(3u-u)^2(1+\cos\alpha)}{(3u)^3} = \frac{8}{27}(1+\cos\alpha)} \qquad (20.36)$$

or $$\eta_{max} = \frac{8}{27}\left(2\cos^2\frac{\alpha}{2}\right) = \frac{16}{27}\cos^2\frac{\alpha}{2} \qquad (20.36a)$$

When $\alpha = 0°$, the curved vanes become semicircular, then η_{max} from Equation (20.36) is given below.

$$\boxed{\eta_{max} = \frac{8}{27}(1+\cos 0°) = \frac{8}{27}(1+1) = \frac{16}{27} = 0.5926 \text{ or } 59.26\%}$$

Example 20.10 A jet of water with diameter 100 mm strikes a curved vane at its centre with a velocity of 25 m/s. The curved vane is moving with a velocity of 5 m/s in the direction of the jet. The jet is deflected through an angle of 160°. Assume that the plate is smooth. Calculate (i) the force exerted on the vane in the direction of jet, (ii) work done per second by the jet on the vane, (iii) power of the jet, (iv) efficiency of the jet and (v) maximum efficiency of the jet. (vi) Also calculate the force exerted on the vane in the direction of jet whose coefficient of friction is 0.9.

Solution

Let $d = 100$ mm $= 0.1$ m, $V = 25$ m/s, $u = 5$ m/s, $(180° - \alpha) = 160°$ and $k = 0.9$.

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 0.1^2 = 0.007854 \text{ m}^2$$

$$180° - \alpha = 160° \Rightarrow \alpha = 20°$$

(i) $F_x = \rho_w A(V-u)^2(1+\cos\alpha)$

$\therefore F_x = 1000\times 0.007854\times(25-5)^2\times(1+\cos 20°) = \mathbf{6093.74\ N}$

(ii) $w = F_x \times u = 6093.74\times 5 = \mathbf{30468.7\ Nm/s}$

(iii) $P = \dfrac{w}{1000} = \dfrac{30468.7}{1000} = \mathbf{30.4687\ kW}$

(iv) $\eta = \dfrac{2u(V-u)^2(1+\cos\alpha)}{V^3} = \dfrac{2\times 5\times(25-5)^2\times(1+\cos 20°)}{25^3}\times 100 = \mathbf{49.66\%}$

(v) $\eta_{\max} = \dfrac{8}{27}(1+\cos\alpha) = \left[\dfrac{8}{27}\times(1+\cos 20°)\right]\times 100 = \mathbf{57.47\%}$

(vi) $F_x = \rho_w A(V-u)^2(1+k\cos\alpha)$

$\therefore F_x = 1000\times 0.007854\times(25-5)^2\times(1+0.9\cos 20°) = \mathbf{5798.52\ N}$

20.9.2 Force On a Series of Symmetrical Moving Curved Vanes When the Jet Strikes at the Centre of Vanes

The single vane system is practically not feasible because the distance between the vane and the nozzle issuing the jet will be constantly increasing as it requires continuous lengthening of the jet. Therefore, for actual engineering applications, a series of curved vanes is mounted on the circumference of a wheel at a fixed distance apart. The water jet strikes the vane and it exerts force on the vane, and this force causes the wheel to rotate. When the wheel rotates, each vane appears successively before the jet and thereby, force is exerted on each vane which causes to move the vane with a uniform velocity.

Let α be the angle between the leaving jet and *x*-axis at the outer tip, *V* be the absolute velocity of the jet, *u* be the uniform velocity of vane in the direction of jet, *d* be the diameter of the jet, $A = (\pi/4)d^2$ be the area of the jet and ρ_w be the mass density of the water.

When all the vanes are considered, the vanes will always be in contact with the jet. Hence, the entire fluid coming out of nozzle will be utilized. Thus, the mass of water striking the vane per second is $\rho_w AV$.

The force exerted by the jet in the direction of motion of the jet is given by,

$$F_x = \frac{\text{Mass}}{\text{Time}}(\text{Initial velocity in jet direction} - \text{Final velocity in jet direction})$$

Thus $$F_x = \rho_w AV\times[(V-u)-(-(V-u)\cos\alpha)]$$

$$\boxed{\therefore F_x = \rho_w AV(V-u)(1+\cos\alpha)} \quad (20.37)$$

When coefficient of friction for vanes (*k*) is given, then Equation (20.37) is written as follows.

$$F_x = \rho_w AV(V-u)(1+k\cos\alpha) \quad (20.38)$$

Work done per second on the wheel is given by,

$$\boxed{w = F_x\times u = \rho_w AV(V-u)(1+\cos\alpha)\times u} \quad (20.39)$$

When there is any energy loss, then work done is given by,

$$w = \rho_w AV(V-u)(1+k\cos\alpha)\times u \quad (20.\ 40)$$

Efficiency of the jet is the ratio of work done per second by the jet (*w*) to the kinetic energy of the jet per second (K.E.).

Thus $$\eta = \frac{w}{\text{K.E.}} = \frac{\rho_w AV(V-u)(1+\cos\alpha)\times u}{(1/2)\times(\rho_w AV)\times V^2} = \frac{2u(V-u)(1+\cos\alpha)}{V^2} \quad (20.41)$$

The efficiency will be maximum when $(d\eta/du) = 0$ and we get the below expression.

$$\frac{d}{du}\left[\frac{2u(V-u)(1+\cos\alpha)}{V^2}\right] = 0$$

$$\frac{d}{du}\left[\frac{2}{V^2}(1+\cos\alpha)\,(Vu-u^2)\right] = 0$$

$$\frac{2}{V^2}(1+\cos\alpha)\,(V-2u) = 0$$

Thus $$V - 2u = 0 \qquad [\because (2/V^2)(1+\cos\alpha) \neq 0]$$

$$\therefore V = 2u$$

Hence, maximum efficiency can be obtained by substituting $V = 2u$ in Equation (20.41) as given below.

$$\boxed{\eta_{\max} = \frac{2u(2u-u)\,(1+\cos\alpha)}{(2u)^2} = \frac{1+\cos\alpha}{2}} \quad (20.42)$$

Case I: When $\alpha = 0°$, the curved vanes will become semicircular. Thus, maximum efficiency from Equation (20.42) is obtained as follows.

$$\eta_{\max} = \frac{1+\cos 0°}{2} = 1 \text{ or } 100\%$$

This is the theoretical value of maximum efficiency for a wheel provided with semicircular vanes mounted on its periphery. This concept is used in the design of buckets for Pelton turbine.

Case II: When $\alpha = 90°$, the curved vanes reduce to flat plates mounted on the periphery of a wheel, then maximum efficiency from Equation (20.42) is obtained as follows.

$$\eta_{\max} = \frac{1+\cos 90°}{2} = \frac{1}{2} \text{ or } 50\%$$

This is same as derived previously in Section 20.7.

Example 20.11 A jet of water with diameter 0.1 m strikes on a series of symmetrical hemispherical curved vanes at the centre attached to the circumference of a wheel with a velocity of 15 m/s. The linear velocity of the vane is 5 m/s in the direction of the jet. Assuming that the vane is smooth, find (i) the force exerted on the vane in the direction of the jet, (ii) work done per second and (iii) efficiency of the jet.

Solution

Let $d = 0.1$ m, $\alpha = 0°$ (hemispherical vanes), $V = 15$ m/s and $u = 5$ m/s.

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 0.1^2 = 0.007854 \text{ m}^2$$

(i) $F_x = \rho_w AV(V-u)\,(1+\cos\alpha)$

$\therefore F_x = 1000\times 0.007854\times 15\times(15-5)\times(1+\cos 0°) = \mathbf{2356.2\ N}$

(ii) $w = F_x \times u = 2356.2\times 5 = \mathbf{11781\ Nm/s}$

(iii) $\eta = \frac{2u(V-u)(1+\cos\alpha)}{V^2} = \frac{2\times 5\times(15-5)\times(1+\cos 0°)}{15^2}\times 100 = \mathbf{88.9\%}$

20.9.3 Force Exerted by a Jet on an Unsymmetrical Moving Curved Vane When the Jet Strikes Tangentially at One of the Tips

Consider a jet of water striking a moving curved vane tangentially at one of its tips as shown in Figure 20.11. The vane is moving in x-direction. Since the jet is striking tangentially, the loss of energy due to the impact of jet will be zero. Since the vane is moving, the effective velocity of the jet entering the vane will be equal to relative velocity of the jet with respect to the vane and it is given by the vector difference of the jet velocity and vane velocity at the inlet.

Let V_i and V_o be the absolute velocities of the jet at the inlet and outlet, respectively,

u_i and u_o be the velocities of the vane at the inlet and outlet, respectively,

V_{ri} and V_{ro} be the relative velocities of the jet and vane at the inlet and outlet, respectively,

V_{wi} and V_{wo} be the velocities of whirl at the inlet and outlet, respectively, (i.e., components of velocities V_i and V_o respectively, in the direction of motion of vane),

V_{fi} and V_{fo} be the velocities of flow at the inlet and outlet, respectively, (i.e., components of velocities of V_i and V_o respectively, perpendicular to the direction of motion of vane),

α and β be the angles made by absolute velocities with the direction of the vane at the inlet and outlet, respectively. The angle α is also known as guide blade angle.

θ and ϕ be the angles made by the relative velocities with the direction of motion of the vane at the inlet and outlet, respectively. The angles θ and ϕ are also known as vane angle at the inlet and outlet, respectively.

The triangles *ABD* and *EFH* are called the velocity triangles at the inlet and outlet, respectively.

Inlet velocity triangle Referring to Figure 20.11, draw *AD* to represent the velocity V_i in magnitude and direction to some convenient scale. Draw *DC* to represent the velocity u_i in magnitude and direction to the same scale. Here, *CA* represents the relative velocity V_{ri} between the jet and the vane in magnitude and direction to the same scale. In order that the jet enters the vane smoothly without any shock, *CA* must be parallel to the tangent to the vane at its inlet tip. From *A* draw a perpendicular which meets *DC* at *B* when produced. Thus, *AD* represents V_i, *AC* represents V_{ri}, *DC* represents u_i, *BD* represents V_{wi}, *AB* represents V_{fi}, $\angle ADB$ represents α and $\angle ACB$ represents θ.

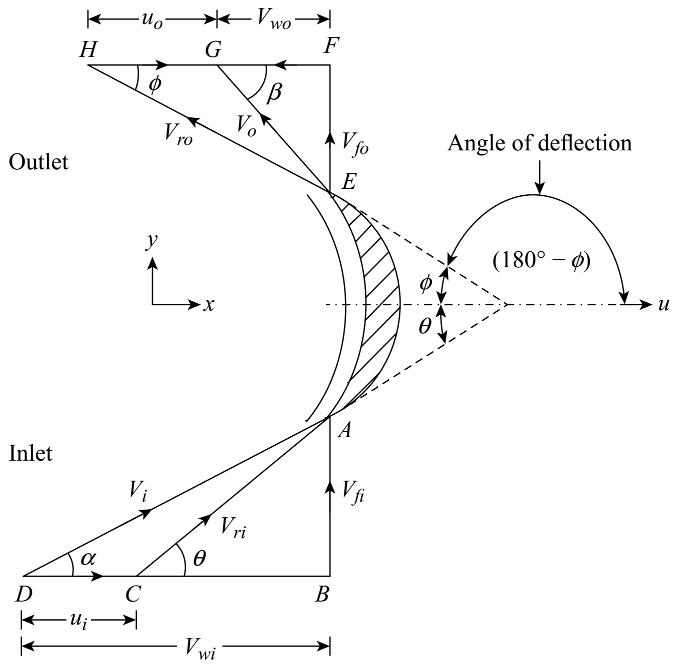


Figure 20.11 *Jet strikes tangentially at the tip of moving unsymmetrical curved vane*

Outlet velocity triangle Referring to Figure 20.11, draw *EH* to represent the relative velocity V_{ro} in magnitude and direction to some convenient scale. If the jet is to leave the vane without shock, then V_{ro} must be parallel to the tangent to the vane at outlet tip. Draw *HG* to represent the velocity of vane u_o to the same scale in magnitude and direction. Draw a perpendicular *EF* to *HG* which meets at F when produced. Thus, *EH* represents V_{ro}, *HG* represents u_o, *EG* represents V_o, *EF* represents V_{fo}, *FG* represents V_{wo}, $\angle EHG$ represents ϕ and $\angle EGF$ represents β.

Consider that the vane is smooth and the loss of energy due to friction is zero. Therefore, the velocity in the direction of motion at inlet and outlet are equal, i.e., $u_i = u_o = u$ and also relative velocity of the jet at inlet and outlet are equal, i.e., $V_{ri} = V_{ro}$. Let ρ_w be the mass density of water, d be the diameter of the jet, $A = (\pi/4)d^2$ be the area of the jet and mass of the water striking the vane per second $= \rho_w A V_{ri}$.

The force exerted by the jet in the direction of motion of the jet is given by,

$$F_x = \frac{\text{Mass}}{\text{Time}}(\text{Initial velocity in jet direction} - \text{Final velocity in jet direction})$$

Initial velocity of the jet with which it strikes the vane in the direction of motion = component of velocity V_{ri} in the direction of motion $= BC = V_{ri}\cos\theta = (V_{wi} - u_i)$.

Final velocity in the direction of motion equals the component of velocity V_{ro} in the direction of motion = FH = $-V_{ro}\cos\phi = -(u_o + V_{wo})$.

(The negative sign has been taken because the direction of this component is opposite to that of the motion of the vane.)

Thus $$F_x = \rho_w A V_{ri} \times \left[(V_{wi} - u_i) - \{-(u_o + V_{wo})\}\right]$$

Since $$u_i = u_o = u$$

$$\boxed{\therefore F_x = \rho_w A V_{ri}(V_{wi} + V_{wo})} \tag{20.43}$$

Equation (20.43) gives the force exerted on the vane in the direction of motion of vane for $\beta < 90°$ (i.e., β is an acute angle). When β is a right angle (i.e., $\beta = 90°$), $V_{wo} = 0$, then Equation (20.43) is written as follows.

$$\boxed{F_x = \rho_w A V_{ri} V_{wi}} \tag{20.43a}$$

When β is an obtuse angle (i.e., $\beta > 90°$), then Equation (20.43) becomes,

$$\boxed{F_x = \rho_w A V_{ri}(V_{wi} - V_{wo})} \tag{20.43b}$$

The general expression for F_x is given by,

$$\boxed{F_x = \rho_w A V_{ri}(V_{wi} \pm V_{wo})} \tag{20.44}$$

Work done per second on the vane by jet is given by,

$$w = F_x \times u = \rho_w A V_{ri}(V_{wi} \pm V_{wo}) \times u \tag{20.45}$$

Work done per second per unit weight of water striking per second is given by,

$$\boxed{w = \frac{\rho_w A V_{ri}(V_{wi} \pm V_{wo}) \times u}{(\rho_w A V_{ri})g} = \frac{(V_{wi} \pm V_{wo})u}{g} \text{ Nm/N}} \tag{20.46}$$

Work done per second per unit mass of water striking per second is given by,

$$\boxed{w = \frac{\rho_w A V_{ri}(V_{wi} \pm V_{wo}) \times u}{\rho_w A V_{ri}} = (V_{wi} \pm V_{wo})u \text{ Nm/kg}} \tag{20.47}$$

Efficiency of the jet is the ratio of work done per second by the jet (w) to the kinetic energy of the jet per second (K.E.).

Thus $$\eta = \frac{w}{\text{K.E.}} = \frac{\rho_w A V_{ri}(V_{wi} \pm V_{wo}) \times u}{(1/2) \times (\rho_w A V_i) \times V_i^2} = \frac{2\rho_w A V_{ri}(V_{wi} \pm V_{wo})u}{V_i^3} \tag{20.48}$$

If there is no friction at the vane surface, then work done equals the change in kinetic energy. Thus, work done is given below.

$$w = \frac{1}{2} \times (\text{mass/sec}) \times (V_i^2 - V_o^2)$$

Efficiency of the jet is given by,

$$\eta = \frac{w}{\text{K.E.}} = \frac{(1/2)(\text{mass/sec})(V_i^2 - V_o^2)}{(1/2)(\text{mass/sec})V_i^2} = \frac{V_i^2 - V_o^2}{V_i^2} = 1 - \left(\frac{V_o}{V_i}\right)^2 \quad (20.48a)$$

Example 20.12 A jet of water having a velocity of 25 m/s strikes a smooth curved vane which is moving with a velocity of 5 m/s. The jet makes an angle of 15° with the direction of motion of vane at inlet and leaves at an angle of 120° to the direction of motion of the vane at outlet. Find (i) the vane angle, so that water enters and leaves the vane without shock, (ii) work done per second per unit weight of water striking the vane per second and (iii) work done per second per unit of mass of water striking the vane per second and (iv) efficiency.

Solution

Refer Figure 20.12. Let V_i = 25 m/s, since vane is smooth, $u_i = u_o = u = 5$ m/s and $V_{ri} = V_{ro}$, $\alpha = 15°$ and $(180° - \beta) = 120°$.

$$180° - \beta = 120° \Rightarrow \beta = 60°$$

(i) From inlet velocity triangle, we get:

$$V_{fi} = V_i \sin\alpha = 25 \sin 15° = 6.47 \text{ m/s}$$

$$V_{wi} = V_i \cos\alpha = 25 \cos 15° = 24.15 \text{ m/s}$$

$$\theta = \tan^{-1}\left(\frac{V_{fi}}{V_{wi} - u_i}\right) = \tan^{-1}\left(\frac{6.47}{24.15 - 5}\right) = \mathbf{18.67°}$$

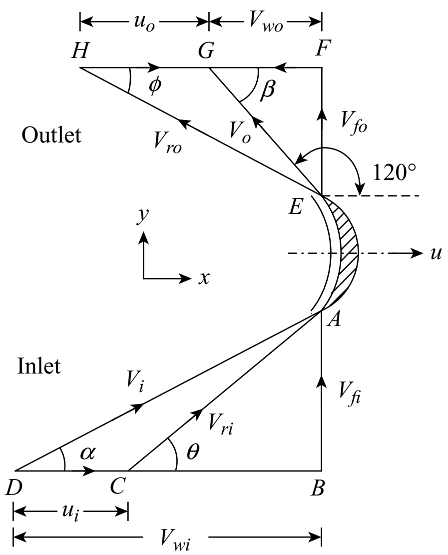


Figure 20.12

$$V_{ri} = \frac{V_{fi}}{\sin\theta} = \frac{6.47}{\sin 18.67°} = 20.21 \text{ m/s}$$

$$\therefore V_{ri} = V_{ro} = 20.21 \text{ m/s}$$

Applying sine rule to ΔEGH, we get:

$$\frac{V_{ro}}{\sin(180° - \beta)} = \frac{u_o}{\sin(\beta - \phi)} \Rightarrow \frac{20.21}{\sin(180° - 60°)} = \frac{5}{\sin(60° - \phi)}$$

$$\sin(60° - \phi) = \frac{5\sin 120°}{20.21} = 0.2143$$

$$60° - \phi = \sin^{-1}(0.2143) = 12.37°$$

$$\therefore \phi = 60° - 12.37° = \mathbf{47.63°}$$

(ii) From outlet velocity triangle, we get:

$$V_{wo} = V_{ro}\cos\phi - u_o = 20.21\cos 47.63° - 5 = 8.62 \text{ m/s}$$

Work done per unit weight per second is given by,

$$w = \frac{(V_{wi} + V_{wo})u}{g} = \frac{(24.15 + 8.62)\times 5}{9.81} = \mathbf{16.702\ Nm/N}$$

(iii) Work done per second per unit mass of water striking the vane per second is given by,

$$w = (V_{wi} + V_{wo})u = (24.15 + 8.62)\times 5 = \mathbf{163.85\ Nm/kg}$$

Example 20.13 A jet of water having a velocity of 30 m/s strikes a smooth curved vane which is moving with a velocity of 10 m/s. The jet makes an angle of 30° with the direction of motion of vane at the inlet and leaves at an angle of 90° to the direction of motion of vane at the outlet. Draw the velocity triangles at the inlet and outlet. Also determine (i) the vane angles at inlet and outlet so that water enters and leaves the vane without shock and (ii) work done per second per unit weight of water striking the vane per second.

Solution

Refer Figure 20.13. Let $V_i = 30$ m/s, since vane is smooth, $u_i = u_o = u = 10$ m/s and $V_{ri} = V_{ro}$, $\alpha = 30°$, $(180° - \beta) = 90°$.

$$180^o - \beta = 90° \Rightarrow \beta = 90°$$

(i) From inlet velocity triangle, we get:

$$V_{fi} = V_i \sin\alpha = 30\sin 30° = 15 \text{ m/s}$$

$$V_{wi} = V_i \cos\alpha = 30\cos 30° = 25.981 \text{ m/s}$$

$$\theta = \tan^{-1}\left(\frac{V_{fi}}{V_{wi} - u_i}\right) = \tan^{-1}\left(\frac{15}{25.981 - 10}\right) = \mathbf{43.19°}$$

$$V_{ri} = \frac{V_{fi}}{\sin\theta} = \frac{15}{\sin 43.19°} = 21.92$$

$$\therefore V_{ro} = V_{ri} = 21.92 \text{ m/s}$$

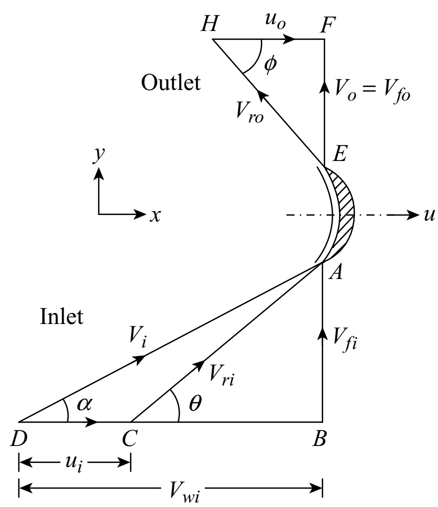


Figure 20.13

$$\phi = \cos^{-1}\left(\frac{u_o}{V_{ro}}\right) = \cos^{-1}\left(\frac{10}{21.92}\right) = \mathbf{62.86^\circ}$$

(ii) Work done per second per unit weight of water is given by,

$$w = \frac{V_{wi}u}{g} = \frac{25.981 \times 10}{9.81} = \mathbf{26.4842\ Nm/N}$$

Example 20.14 A jet of water with diameter 40 mm having a velocity of 15 m/s strikes a smooth curved vane which is moving with a velocity of 7.5 m/s in the direction of jet. The jet leaves the vane at an angle of 60° to the direction of motion of vane at the outlet. Determine (i) the force exerted by the jet on the vane in the direction of motion, (ii) work done per second by the jet and (iii) efficiency of the jet.

Solution

Refer Figure 20.14. Let $d = 40$ mm $= 0.04$ m, $V_i = 15$ m/s, since vane is smooth $u_i = u_o = u = 7.5$ m/s and $V_{ri} = V_{ro}$ and $(180^\circ - \beta) = 60^\circ$.

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4} \times 0.04^2 = 0.001257\ \text{m}^2$$

Since jet and vane move in the same direction, $\alpha = 0^\circ$ and $\theta = 0^\circ$.

$$180^\circ - \beta = 60^\circ \Rightarrow \beta = 120^\circ$$

From the inlet velocity triangle, which is a straight line, we get the following values.

$$V_{ri} = V_i - u_i = 15 - 7.5 = 7.5\ \text{m/s},$$

$$V_{wi} = V_i = 15\ \text{m/s}$$

and $$V_{ro} = V_{ri} = 7.5\ \text{m/s}$$

From the outlet triangle *EGH*, we get:

$$\angle GEH = 180^\circ - (\phi + 60^\circ) = (120^\circ - \phi)$$

Applying sine rule, we get:

$$\frac{EH}{\sin 60^\circ} = \frac{GH}{\sin(120^\circ - \phi)} \Rightarrow \frac{7.5}{\sin 60^\circ} = \frac{7.5}{\sin(120^\circ - \phi)}$$

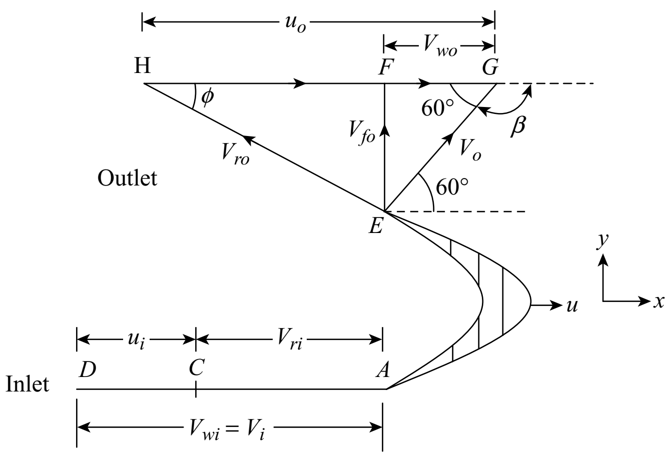


Figure 20.14

$$\sin(120° - \phi) = \sin 60° \Rightarrow 120° - \phi = 60° \Rightarrow \phi = 60°$$

$$V_{wo} = u_o - V_{ro}\cos\phi = 7.5 - 7.5\cos 60° = 3.75 \text{ m/s}$$

(i) Force exerted on the vane in the direction of motion of vane for $\beta > 90°$ is given by,

$$F_x = \rho_w A V_{ri}(V_{wi} - V_{wo}) = 1000 \times 0.001257 \times 7.5 \times (15 - 3.75) = \mathbf{106.06\ N}$$

(ii) $w = F_x \times u = 106.06 \times 7.5 = \mathbf{795.45\ Nm/s}$

(iii) $\eta = \dfrac{w}{(1/2)\rho_w A V_i^3} = \dfrac{795.45}{(1/2)\times 1000 \times 0.001257 \times 15^3} \times 100 = \mathbf{37.5\%}$

Example 20.15 A jet of water having a velocity of 25 m/s strikes a smooth curved vane which is moving with a velocity of 6 m/s in the direction as that of the jet at inlet. The vane is so shaped that the jet is deflected through 140°. The diameter of the jet is 75 mm. Assume the vane to be smooth, then find (i) the force exerted by the jet on the vane in the direction of motion, (ii) work done per second, (iii) power of the jet and (iv) efficiency.

Solution

Refer Figure 20.15. Let $V_i = 25$ m/s, since vane is smooth, $u_i = u_o = u = 6$ m/s and $V_{ri} = V_{ro}$, $(180° - \phi) = 140°$ and $d = 75 \text{ mm} = 0.075 \text{ m}$.

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4} \times 0.075^2 = 0.00442 \text{ m}^2$$

Since jet and vane move in the same direction, $\alpha = 0°$ and $\theta = 0°$.

$$180° - \phi = 140° \Rightarrow \phi = 40°$$

From the inlet velocity triangle, which is a straight line, we get the below values.

$$V_{ri} = V_i - u_i = 25 - 6 = 19 \text{ m/s}$$

and $$V_{wi} = V_i = 25 \text{ m/s}$$

From the outlet velocity triangle EFH, we get:

$$V_{ri} = V_{ro} = 19 \text{ m/s}$$

and $$V_{wo} = V_{ro}\cos\phi - u_o = 19\cos 40° - 6 = 8.555 \text{ m/s}$$

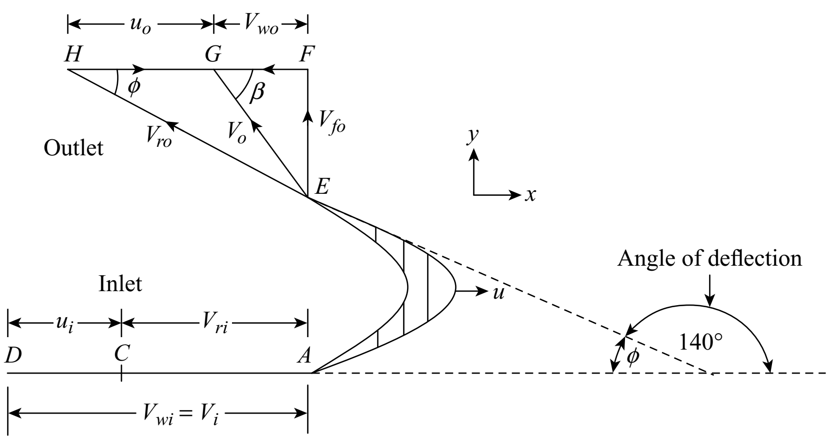


Figure 20.15

(i) $F_x = \rho_w A V_{ri}(V_{wi} + V_{wo}) = 1000 \times 0.00442 \times 19 \times (25 + 8.555) = \mathbf{2817.95\ N}$

(ii) $w = F_x \times u = 2817.95 \times 6 = \mathbf{16907.7\ Nm/s}$

(iii) $P = \dfrac{w}{1000} = \dfrac{16907.7}{1000} = \mathbf{16.9077\ kW}$

(iv) $\eta = \dfrac{w}{(1/2)\rho_w A V_i^3} = \dfrac{16907.7}{(1/2) \times 1000 \times 0.00442 \times 25^3} \times 100 = \mathbf{48.96\%}$

Example 20.16 A jet of water moving at 15 m/s impinges on a symmetrical smooth curved vane shaped to deflect the jet through 140°. If the vane is moving at 6 m/s, then find the angle of the jet so that there is no shock at the inlet. Also determine the absolute velocity of exit in magnitude and direction and the work done per unit weight of water. Assume the vane to be smooth.

Solution

Refer Figure 20.16. Let $V_i = 15$ m/s, $140° = 180° - (\theta + \phi)$, since vane is smooth, $u_i = u_o = u = 6$ m/s and $V_{ri} = V_{ro}$. Since vane is symmetrical, $\theta = \phi$

Since $$140° = 180° - (\theta + \phi) \Rightarrow \theta + \phi = 40° \Rightarrow \theta = \phi = 20° \qquad [\because \theta = \phi]$$

Applying sine rule to ΔADC, we get:

$$\frac{AD}{\sin(180° - \theta)} = \frac{DC}{\sin(\theta - \alpha)} \Rightarrow \frac{15}{\sin(180° - 20°)} = \frac{6}{\sin(20° - \alpha)}$$

$$\sin(20° - \alpha) = \frac{6}{15} \times \sin 160° = 0.13681$$

$$20° - \alpha = \sin^{-1}(0.13681) = 7.86°$$

$$\therefore \alpha = 20° - 7.86° = 12.14°$$

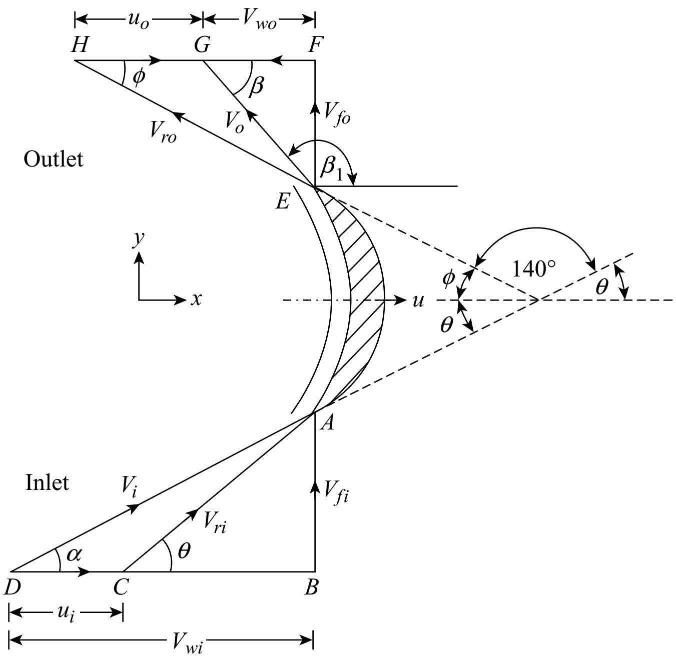


Figure 20.16

Again applying sine rule to ΔADC, we get:

$$\frac{AD}{\sin(180^\circ-\theta)}=\frac{AC}{\sin\alpha}\Rightarrow\frac{15}{\sin(180^\circ-20^\circ)}=\frac{V_{ri}}{\sin 12.14^\circ}$$

$$\therefore V_{ri}=\frac{15\sin 12.14^\circ}{\sin 160^\circ}=9.223\text{ m/s}$$

$$\therefore V_{ro}=V_{ri}=9.223\text{ m/s}$$

From the outlet velocity ΔEHF, we get:

$$V_{wo}=V_{ro}\cos\phi-u_o=9.223\cos 20^\circ-6=2.67\text{ m/s}$$

$$V_{fo}=V_{ro}\sin\phi=9.223\sin 20^\circ=3.154\text{ m/s}$$

In right angled ΔEFG, we get:

$$V_o=\sqrt{V_{fo}^2+V_{wo}^2}=\sqrt{3.154^2+2.67^2}=\mathbf{4.132\text{ m/s}}$$

$$\beta=\tan^{-1}\left(\frac{V_{fo}}{V_{wo}}\right)=\tan^{-1}\left(\frac{3.154}{2.67}\right)=\mathbf{49.75^\circ}$$

Angle made by absolute velocity at the outlet with the direction of motion is given by,

$$\beta_1=180^\circ-\beta=180^\circ-49.75^\circ=130.25^\circ$$

$$V_{wi}=V_i\cos\alpha=15\cos 12.14^\circ=\mathbf{14.664\text{ m/s}}$$

Work done per unit weight of water striking the vane per second is given by,

$$w=\frac{(V_{wi}+V_{wo})u}{g}=\frac{(14.664+2.67)\times 6}{9.81}=\mathbf{10.602\text{ Nm/N}}$$

20.9.4 Force Exerted by a Jet on a Series of Radial Curved Vanes

Consider a series of vanes mounted on a wheel as shown in Figure 20.17.

The jet of water strikes the vanes and the wheel starts rotating at a constant angular speed. All the notations used in previous Section 20.9.3 remains same. Let N be the speed of wheel in rpm, $\omega=(2\pi N)/60$ be the angular speed of the wheel, R_i and R_o be the radii of the wheel at the inlet and outlet of the vane, respectively, u_i and u_o be the velocity of vane at the inlet and outlet, respectively.

As the blades are situated radially around the wheel, velocity of the blades at the inlet and outlet tips of the vane will be different, i.e., $u_i=\omega R_i$ and $u_o=\omega R_o$.

The mass of water striking per second for a series of vanes is equal to the mass of water coming out from the nozzle per second, i.e., $(\text{Mass/Second})=\rho_w A V_i$.

Momentum of water striking in the tangential direction per second at the inlet

$$=(\text{Mass/Second})\times\text{Component of } V_i \text{ in tangential direction}$$

$$=\rho_w A V_i\times(V_i\cos\alpha)=\rho_w A V_i V_{wi}\qquad[\because V_{wi}=V_i\cos\alpha]$$

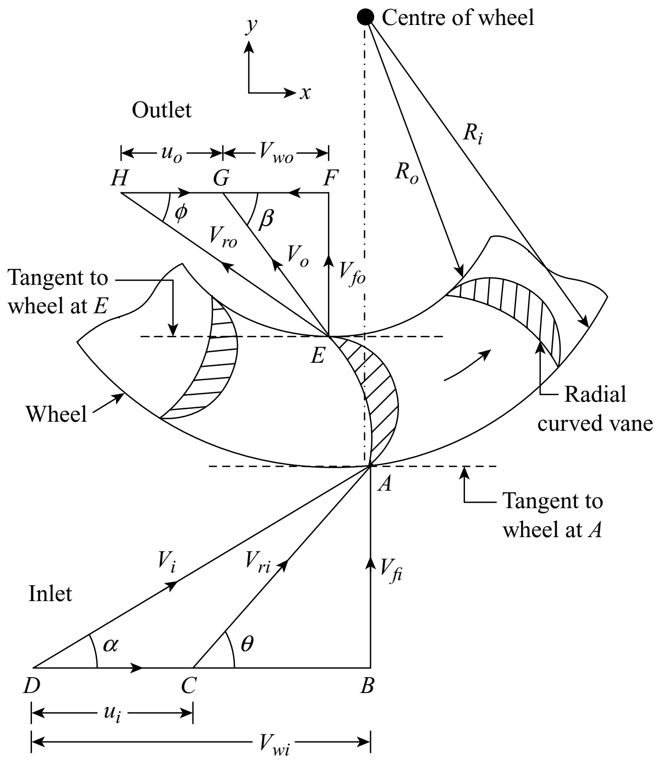


Figure 20.17 *Series of curved vanes mounted radially on a wheel*

Momentum of water at the outlet per second

$$= (\text{Mass/Second}) \times \text{Component of } V_o \text{ in tangential direction}$$

$$= \rho_w A V_i \times (-V_o \cos\beta) = -\rho_w A V_i V_{wo} \qquad [\because V_{wo} = V_o \cos\beta]$$

Negative sign is taken because V_o at the outlet is in opposite direction.

Angular momentum per second at the inlet = Momentum at inlet × Radius at inlet

$$= \rho_w A V_i V_{wi} \times R_i$$

Angular momentum per second at outlet = Momentum at outlet × Radius at outlet

$$= -\rho_w A V_i V_{wo} \times R_o$$

Hence, torque exerted by the water on the wheel is given below.

$$T = \text{Rate of change of angular momentum}$$

$$= (\text{Initial angular momentum per second} - \text{Final angular momentum per second})$$

$$= [\rho_w A V_i V_{wi} R_i - (-\rho_w A V_i V_{wo} R_o)] = (\rho_w A V_i V_{wi} R_i + \rho_w A V_i V_{wo} R_o)$$

Thus $\quad T = \rho_w A V_i (V_{wi} R_i + V_{wo} R_o)$

Work done per second on the wheel is given by,

$$w = \text{Torque} \times \text{Angular speed} = \rho_w A V_i (V_{wi} R_i + V_{wo} R_o) \times \omega$$

$$\therefore w = \rho_w A V_i (V_{wi} R_i \omega + V_{wo} R_o \omega) = \rho_w A V_i (V_{wi} u_i + V_{wo} u_o) \qquad (20.49)$$

If angle β is obtuse, then work done per second is given below.

$$w = \rho_w A V_i (V_{wi} u_i - V_{wo} u_o) \tag{20.50}$$

Thus, by combining Equations (20.49) and (20.50), the general expression for the work done on the wheel per second is given below.

$$\boxed{w = \rho_w A V_i (V_{wi} u_i \pm V_{wo} u_o)} \tag{20.51}$$

Equation (20.51) is known as Euler's momentum equation.

For radial discharge at the outlet tip of the vane $\beta = 90°$ and $V_{wo} = 0$, then from Equation (20.51), work done on wheel per second is given below.

$$\boxed{w = \rho_w A V_i (V_{wi} u_i)} \tag{20.52}$$

Work done on the wheel per second per unit weight of the water is given by,

$$\boxed{w = \frac{\rho_w A V_i (V_{wi} u_i \pm V_{wo} u_o)}{\rho_w A V_i g} = \frac{V_{wi} u_i \pm V_{wo} u_o}{g}} \tag{20.53}$$

Work done per second per unit mass of the water is given by,

$$\boxed{w = \frac{\rho_w A V_i (V_{wi} u_i \pm V_{wo} u_o)}{\rho_w A V_i} = (V_{wi} u_i \pm V_{wo} u_o)} \tag{20.54}$$

Efficiency of the radial curved vanes is the ratio of work done per second by the jet (w) to the initial kinetic energy per second of the jet (K.E.).

Thus
$$\eta = \frac{w}{\text{K.E.}} = \frac{\rho_w A V_i (V_{wi} u_i \pm V_{wo} u_o)}{(1/2) \times (\rho_w A V_i) \times V_i^2} = \frac{2(V_{wi} u_i \pm V_{wo} u_o)}{V_i^2} \tag{20.55}$$

If there is no loss of energy, then work done on the wheel per second is also equal to the change in kinetic energy of the jet per second. Hence, work done per second is given below.

$$w = \frac{1}{2}(\text{mass/sec})(V_i^2 - V_o^2) = \frac{1}{2} \times (\rho_w A V_i) \times (V_i^2 - V_o^2)$$

$$\eta = \frac{w}{K.E.} = \frac{(1/2)(\rho_w A V_i)(V_i^2 - V_o^2)}{(1/2) \times (\rho_w A V_i) \times V_i^2} = \frac{V_i^2 - V_o^2}{V_i^2} = 1 - \left(\frac{V_o}{V_i}\right)^2 \tag{20.56}$$

It can be seen from Equation (20.56) that for a given value of the initial jet velocity V_i, the efficiency will be maximum when V_o has minimum value. However, V_o cannot be zero as in that case there will be no flow. But V_o can be reduced by keeping the angle of vane at the outlet (ϕ) minimum. Equation (20.55) shows that the efficiency will be maximum when β is an acute angle, so that there will be positive sign between V_{wi} and V_{wo}. For maximum efficiency, V_{wo} should also be maximum. This is possible only if $\beta = 0$, then $V_{wo} = V_o$ and $\phi = 0$. But in actual practice, ϕ has some value and it cannot be zero. Thus, it may be concluded that smaller the value of ϕ, higher is the efficiency.

Example 20.17 A jet of water with diameter 100 mm having a velocity of 25 m/s strikes a series of curved vanes mounted on a wheel which is rotating at 200 rpm. The jet makes an angle of 25° with the tangent to wheel at the inlet and leaves the wheel with a velocity of 5 m/s at an angle of 120° to the tangent to the wheel at outlet. Water is flowing from outward in a radial direction. The outer and inner radii of the wheel are 0.5 m and 0.25 m, respectively. Determine (i) the vane angle at inlet and outlet, (ii) work done per second, (iii) work done per second per kg of water and (iv) efficiency of the wheel.

Solution

Refer Figure 20.18. Let $d = 100\text{ mm} = 0.1\text{ m}$, $V_i = 25\text{ m/s}$, $N = 200\text{ rpm}$, $\alpha = 25°$, $V_o = 5\text{ m/s}$, $(180° - \beta) = 120°$, $R_i = 0.5\text{ m}$ and $R_o = 0.25\text{ m}$.

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4} \times 0.1^2 = 0.007854\text{ m}^2$$

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 200}{60} = 20.944\text{ rad/s}$$

$$u_i = \omega R_i = 20.944 \times 0.5 = 10.472\text{ m/s}$$

$$u_o = \omega R_o = 20.944 \times 0.25 = 5.236\text{ m/s}$$

$$180° - \beta = 120° \Rightarrow \beta = 60°$$

(i) From inlet velocity triangle, we get:

$$V_{wi} = V_i \cos\alpha = 25\cos 25° = 22.66\text{ m/s}$$

$$V_{fi} = V_i \sin\alpha = 25\sin 25° = 10.565\text{ m/s}$$

$$\theta = \tan^{-1}\left(\frac{V_{fi}}{V_{wi} - u_i}\right) = \tan^{-1}\left(\frac{10.565}{22.66 - 10.472}\right) = \mathbf{40.92°}$$

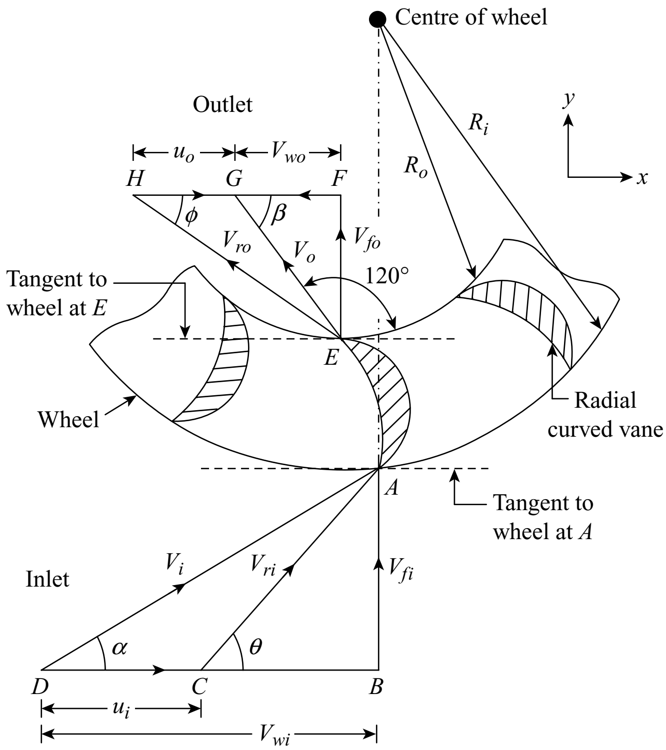


Figure 20.18

From outlet velocity triangle, we get:

$$V_{wo} = V_o \cos\beta = 5\cos 60° = 2.5 \text{ m/s}$$

$$V_{fo} = V_o \sin\beta = 5\sin 60° = 4.33 \text{ m/s}$$

$$\phi = \tan^{-1}\left(\frac{V_{fo}}{V_{wo}+u_o}\right) = \tan^{-1}\left(\frac{4.33}{2.5+5.236}\right) = \mathbf{29.24°}$$

(ii) Work done per second on the wheel is given by,

$$w = \rho_w A V_i (V_{wi}u_i + V_{wo}u_o)$$

$$\therefore w = 1000\times 0.007854\times 25\times(22.66\times 10.472+2.5\times 5.236) = \mathbf{49163.197\ Nm/s}$$

(iii) Work done per second per kg of water is given by,

$$w = (V_{wi}u_i + V_{wo}u_o) = (22.66\times 10.472+2.5\times 5.236) = \mathbf{250.385\ Nm/kg}$$

(iv) $\eta = \dfrac{2(V_{wi}u_i + V_{wo}u_o)}{V_i^2} = \dfrac{2(22.66\times 10.472+2.5\times 5.236)}{25^2}\times 100 = \mathbf{80.12\%}$

20.10 ❐ FORCE EXERTED BY A JET ON A HINGED PLATE

Consider a jet of water striking a vertical flat plate of uniform thickness at the centre which is hinged at point A as shown in Figure 20.19.

Due to the force exerted by the jet on the plate, the plate will swing freely through an angle about the hinged point *A*. Let ρ_w be the mass density of water, *A* be the area of the jet, *V* be the velocity of the jet, *l* be the distance of the centre of jet from hinged point *A*, α be the angle of swing about hinge and *W* be the weight of plate acting at the centre of gravity of the plate.

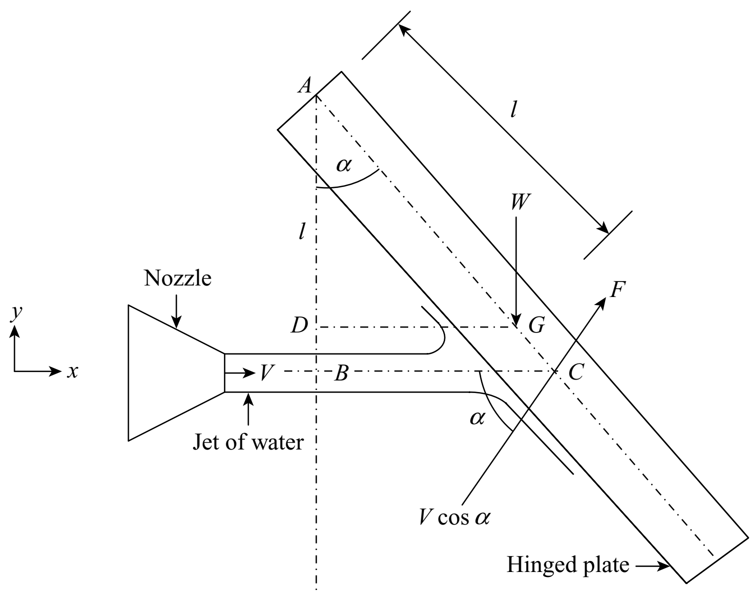


Figure 20.19 *Force on a hinged plate*

When the jet strikes the plate, the point B on the plate shifts to point G. Thus, the distance $AB = AG = l$.

The following two forces are acting on the plate:

(i) Force due to jet of water normal to the plate is equal to the rate of change of momentum in normal direction as given below.

$$F = \frac{\text{Mass}}{\text{Time}}(\text{Initial velocity in normal direction} - \text{Final velocity in normal direction})$$

$$\boxed{\therefore F = \rho_w AV \times (V\cos\alpha - 0) = \rho_w AV^2 \cos\alpha} \tag{20.57}$$

(ii) Weight of the plate W is acting vertically downward at the centre of gravity of the plate.

Under equilibrium conditions, the moments of these forces about the hinge point of the plate must be zero. Thus, moment of force F about hinge point A is equal to the moment of weight W about hinge point A.

$$\therefore F \times \text{Length } AC = W \times \text{Length } GD$$

Since $$AC = \frac{AB}{\cos\alpha} = \frac{l}{\cos\alpha} \text{ and } GD = AG\sin\alpha = l\sin\alpha$$

Thus $$\rho_w AV^2 \cos\alpha \times \frac{l}{\cos\alpha} = W \times l\sin\alpha$$

$$\rho_w AV^2 = W\sin\alpha$$

$$\boxed{\therefore \sin\alpha = \frac{\rho_w AV^2}{W}} \tag{20.58}$$

Example 20.18 A jet of water with 30 mm diameter moving with a velocity of 20 m/s strikes a hinged square plate of weight 420 N at the centre of the plate. The plate is of uniform thickness. Find the angle through which the plate will swing.

Solution

Let $d = 30 \text{ mm} = 0.03 \text{ m}$, $V = 20 \text{ m/s}$ and $W = 420 \text{ N}$.

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4} \times 0.03^2 = 0.000707 \text{ m}^2$$

Since $$\alpha = \sin^{-1}\left(\frac{\rho_w AV^2}{W}\right) \quad \text{[From Equation (20.58)]}$$

$$\therefore \alpha = \sin^{-1}\left(\frac{1000 \times 0.000707 \times 20^2}{420}\right) = \mathbf{43.32°}$$

Example 20.19 A metal plate of 12 mm thickness and 220 mm square is hung so that it can swing freely about the upper horizontal edge. A horizontal jet of water of 25 mm diameter impinges with its axis perpendicular and 55 mm below the edge of the hinge, and keeps steadily inclined at 30° to the vertical. Find the velocity of the jet if the specific weight of the metal is 80 kN/m^3.

Solution

Refer Figure 20.20. Let $t = 12 \text{ mm} = 0.012 \text{ m}$, $b = 220 \text{ mm} = 0.22 \text{ m}$, $d = 25 \text{ mm} = 0.025 \text{ m}$, $AB = 55 \text{ mm} = 0.055 \text{ m}$, $\alpha = 30°$ and $w = 80 \text{ kN/m}^3$.

Let W be the weight of the plate and v be its volume.

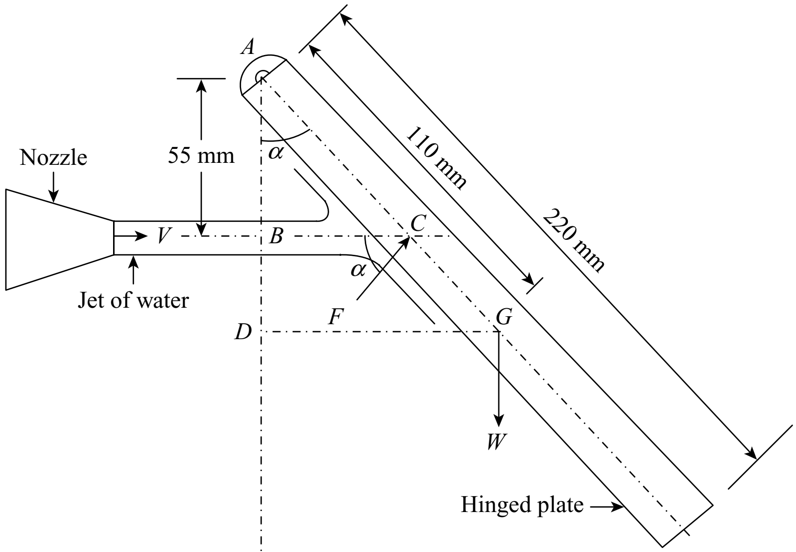


Figure 20.20

$$A = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 0.025^2 = 0.000491 \text{ m}^2$$

$$v = b \times b \times t = 0.22 \times 0.22 \times 0.012 = 5.808 \times 10^{-4} \text{ m}^3$$

$$W = v \times w = 5.808 \times 10^{-4} \times 80 \times 10^3 = 46.464 \text{ N}$$

$$F = \rho_w A V^2 \cos\alpha = 1000 \times 0.000491 \times V^2 \cos 30° = 0.4252 \text{ V}^2$$

The moments of force F about hinge point A is given by,

$$= F \times AC = F \times \frac{AB}{\cos\alpha} = 0.4252\, V^2 \times \frac{0.055}{\cos 30°} = 0.027 \text{ V}^2$$

The moment of weight W about hinge point A is given by,

$$\Rightarrow W \times GD = W \times AG \sin\alpha = 46.464 \times 0.110 \sin 30° = 2.55552$$

Since Moment of F about A = Moment of W about A [Under equilibrium]

Thus $0.027V^2 = 2.55552$

$$\therefore V = \sqrt{\frac{2.55552}{0.027}} = \mathbf{9.73\ m/s}$$

Example 20.20 A rectangular plate weighing 60 N is suspended vertically by a hinge on the top horizontal edge. The centre of gravity of the plate is 10 cm from the hinge. A horizontal jet of water of 2.5 cm diameter, whose axis is 15 cm below the hinge, impinges normally to the plate with a velocity of 5 m/s. Find the horizontal force applied at the centre of gravity to maintain the plate in vertical position. Find the change in velocity of the jet if the plate is deflected by 30° and the same horizontal force continues to act at the centre of gravity of the plate.

Solution

Refer Figure 20.21. Let $W = 60$ N, $AG = 10$ cm $= 0.1$ m, $d = 2.5$ cm $= 0.025$ m, $AB = 15$ cm $= 0.15$ m and $\alpha = 30°$.

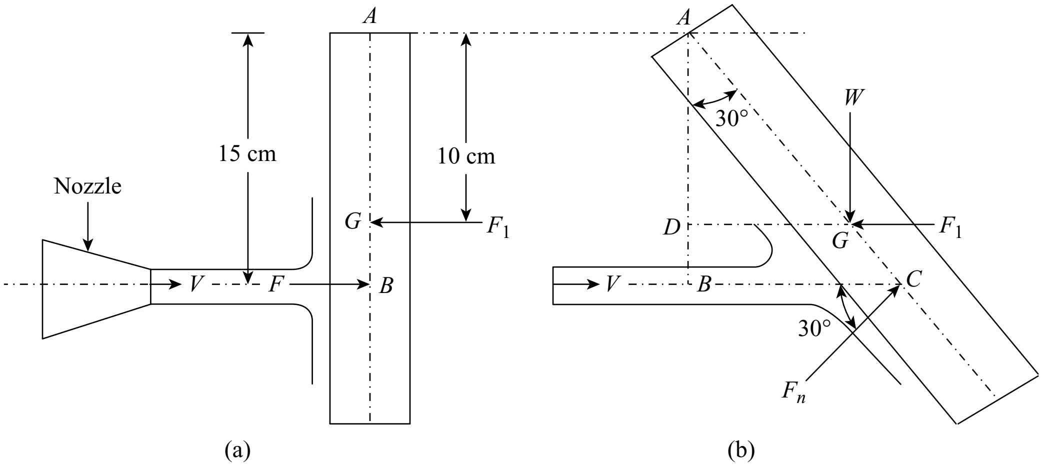


Figure 20.21

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4} \times 0.025^2 = 0.000491 \text{ m}^2$$

(i) Let the force applied at the centre of gravity of the plate to keep the plate in vertical position be F_1 as shown in Fig. 20.21(a).

$$F = \rho_w A V^2 = 1000 \times 0.000491 \times 5^2 = 12.275 \text{ N}$$

Let F_1 be the force applied at the centre of gravity to maintain the plate in vertical position. Taking moments about the hinge point 'A', we get the following expression.

$$F \times AB = F_1 \times AG$$

$$12.275 \times 0.15 = F_1 \times 0.1$$

$$\therefore F_1 = \frac{12.275 \times 0.15}{0.1} = 18.4125 \text{ N}$$

(ii) The plate is deflected through an angle of 30° as shown in Figure 20.21(b), the plate is in equilibrium under the action of three forces, such as (a) weight of the plate, W acting at G at a distance 10 cm from A, (b) horizontal force acting at G, $F_1 = 14.4125\ N$ and (c) normal force, F_n on the plate due to jet of water.

Since $\quad F_n = \rho_w A V(V\cos\alpha - 0) = \rho_w A V^2 \cos\alpha$

Thus $\quad F_n = 1000 \times 0.000491 \times V^2 \cos 30° = 0.42522\, V^2$

Taking moments of all forces about hinge A, we get:

$$F_n \times AC = F_1 \times AD + W \times DG$$

$$F_n \times \frac{AB}{\cos 30°} = F_1 \times AG\cos 30° + W \times AG\sin 30°$$

$$0.42522\, V^2 \times \frac{0.15}{\cos 30°} = 14.4125 \times 0.10\cos 30° + 60 \times 0.10\sin 30°$$

$$0.07365\, V^2 = 4.24816$$

$$\therefore V = \sqrt{\frac{4.24816}{0.07365}} = 7.595 \text{ m/s}$$

$\therefore$ Change in velocity = 7.595 − 5 = **2.595 m/s** **(Increase)**

20.11 ❐ JET PROPULSION OF SHIPS

Jet propulsion of ship means the propulsion or movement of the ship. It is one of the applications of the impulse-momentum principle wherein the reaction of a high velocity jet coming out from a nozzle is used to move the ship. This principle is also used to propel the aircrafts and missiles. Based on the position of the inlet orifices to the direction of the motion of the ship, two cases are considered, such as (i) inlet orifices at right angle to the motion of the ship and (ii) inlet orifices facing the direction of the ship.

20.11.1 Inlet Orifices at Right Angle to the Motion of the Ship

The Figure 20.22 illustrates the propulsion of ship in which the inlet orifices are provided at right angles to the direction of motion of the ship. The orifices are provided in the middle of the ship (amidships). The ship carries a centrifugal pump which draws water from the surrounding sea and discharges it through a nozzle at the rear of the ship (also called stern).

Let u be the velocity of the ship, A be the area of orifice, V be the absolute velocity of the jet coming out from the stern and ρ_w be the mass density of water. The velocity V and u are in opposite direction and therefore, relative velocity of the jet with respect to ship is $V_r = V - (-u) = (V + u)$.

Mass of the water coming out from the orifice per second at the back of ship is given by,

$$\text{Mass/Second} = \rho_w A V_r = \rho_w A(V + u)$$

Force exerted on the ship = Mass flow rate of water × Change in velocity

Here, change in velocity is the difference between the relative velocity of the jet with respect to ship (V_r) and velocity of ship (u).

$$\boxed{\therefore F = \rho_w A(V+u) \times [(V+u) - u] = \rho_w A(V+u)V} \quad (20.59)$$

Work done on the ship by jet per second is given by,

$$\boxed{w = F \times u = \rho_w A(V+u)V \times u} \quad (20.60)$$

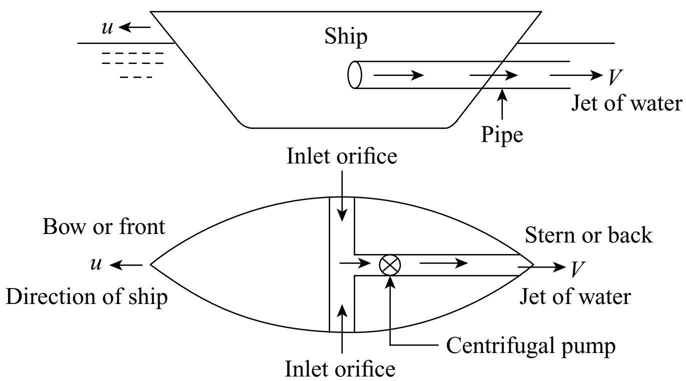


Figure 20.22 *Inlet orifices at right angle to the motion of the ship*

Efficiency of the propulsion is the ratio of work done per second by the jet (w) to the initial kinetic energy per second of the jet (K.E.).

Thus $$\eta = \frac{w}{\text{K.E.}} = \frac{\rho_w A(V+u)Vu}{(1/2)\times \rho_w A(V+u)\times (V+u)^2} = \frac{2Vu}{(V+u)^2} \tag{20.61}$$

For maximum efficiency, $(d\eta/du) = 0$ and we get the following result.

$$\frac{d}{du}\left[\frac{2Vu}{(V+u)^2}\right] = 0$$

$$\frac{(V+u)^2 \times 2V - 2Vu \times 2(V+u)}{(V+u)^4} = 0$$

$$(V+u)^2 \times 2V - 2Vu \times 2(V+u) = 0$$

$$(V+u)[(V+u)\times 2V - 4Vu] = 0$$

$$2V[(V+u) - 2u] = 0 \qquad [\because V \neq -u]$$

$$(V+u) - 2u = 0 \qquad [\because 2V \neq 0]$$

$$\boxed{\therefore u = V} \tag{20.62}$$

For obtaining maximum efficiency, substituting $V = u$ in Equation (20.61), we get:

$$\boxed{\eta_{\max} = \frac{2(u)u}{(u+u)^2} = \frac{2u^2}{4u^2} = 0.5 \text{ or } 50\%} \tag{20.63}$$

20.11.2 Inlet Orifices Face the Direction of Motion of the Ship

The Figure 20.23 shows an arrangement in which the inlet orifices face the direction of the motion of the ship. In this case, water is drawn in the pipe from the bow (front) of the ship and gets discharged at the stern.

The expressions for propelling force and work done per second are same as for the previous case and are respectively given below.

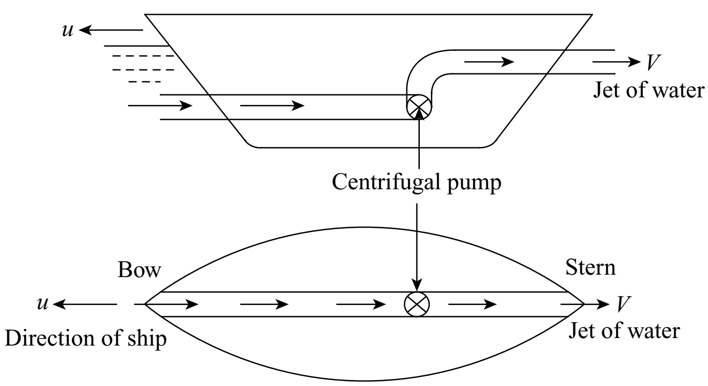


Figure 20.23 *Inlet orifices face the direction of motion of the ship*

$$F = \rho_w A(V+u)V$$

$$w = \rho_w A(V+u)V \times u$$

In this case, water enters with a velocity equal to the velocity of the ship. Thus, the energy supplied by the jet per second (K.E.) is different and it is given below.

$$\text{K.E.} = \frac{1}{2}\rho_w A(V+u)[(V+u)^2 - u^2]$$

or
$$\text{K.E.} = \frac{1}{2}\rho_w A(V+u)(V^2+u^2+2Vu-u^2) = \frac{1}{2}\rho_w A(V+u)(V^2+2Vu)$$

Efficiency of propulsion is given by,

$$\eta = \frac{w}{\text{K.E.}} = \frac{\rho_w A(V+u)Vu}{(1/2)\rho_w A(V+u)(V^2+2Vu)} = \frac{2Vu}{V(V+2u)} = \frac{2u}{V+2u} \tag{20.64}$$

Normally, $u < V$ and therefore, the limiting value of u is equal to V. For this case, it is not practically feasible and it is a theoretical consideration only. Thus, the maximum possible value of propulsive efficiency can be obtained by substituting $u = V$ in Equation (20.64) as given below.

$$\boxed{\eta = \frac{2V}{V+2V} = \frac{2}{3} = 66.67\%}$$

Example 20.21 A jet propelled boat draws water amidship and discharges it through a jet of cross-sectional area 0.024 m^2 at the back with an absolute velocity of 25 m/s. If the boat moves with a speed of 36 km/hr, then determine (i) the force exerted on the boat, (ii) work done on the boat, (iii) power of the motor required to work the pump and (iv) efficiency of propulsion.

Solution

Let $A = 0.024$ m^2, $V = 25$ m/s and $u = 36$ km/hr.

(i) $u = \dfrac{36 \times 1000}{60 \times 60} = 10$ m/s

$F = \rho_w A(V+u)V = 1000 \times 0.024 \times (25+10) \times 25 = \mathbf{21000\ N}$

(ii) $w = F \times u = 21000 \times 10 = \mathbf{210000\ Nm/s}$

(iii) $P = \dfrac{w}{1000} = \dfrac{210000}{1000} = \mathbf{210\ kW}$

(iv) $\eta = \dfrac{2Vu}{(V+u)^2} = \dfrac{2 \times 25 \times 10}{(25+10)^2} \times 100 = \mathbf{40.82\%}$

Example 20.22 A ship propelled by reaction jets and discharging astern has a resistance to its motion of 3000 N when moving with a speed of 25 km/hr. The velocity of jet relative to the ship is 15 m/s and the area of each jet is 0.0067 m^2. Determine (i) the number of jets and (ii) power required to drive the pump. Determine the efficiency of the jet propulsion for both the arrangement of inlet orifices.

Solution

Let $F = 3000$ N, $u = 25$ km/hr, $V_r = V + u = 15$ m/s and $a = 0.0067$ m^2. Let n be the number of jets.

(i) $u = \dfrac{25 \times 1000}{60 \times 60} = 6.944$ m/s

$V = V_r - u = 15 - 6.944 = 8.056$ m/s

Since $$F = \rho_w A(V + u)V$$

Thus $$3000 = 1000 \times A \times 15 \times 8.056$$

$$\therefore A = \frac{3000}{1000 \times 15 \times 8.056} = 0.02483 \text{ m}^2$$

$$n = \frac{\text{total area of jets}}{\text{area of each jet}} = \frac{A}{a} = \frac{0.02483}{0.0067} = 3.706 \approx \mathbf{4}$$

(ii) $P = \dfrac{F \times u}{1000} = \dfrac{3000 \times 6.944}{1000} = \mathbf{20.832\ kW}$

This will remain same for both the arrangements of inlet orifices.

(iii) When the inlet orifices are at right angles to the motion of the ship, then efficiency is obtained as given below.

$$\eta = \frac{2Vu}{(V+u)^2} = \frac{2 \times 8.056 \times 6.944}{15^2} \times 100 = \mathbf{49.72\%}$$

(iv) When the inlet orifices face the direction of motion of the ship, then efficiency is obtained as given below.

$$\eta = \frac{2u}{V + 2u} = \frac{2 \times 6.944}{8.056 + 2 \times 6.944} \times 100 = \mathbf{63.29\%}$$

Example 20.23 A jet propelled ship discharges water through a jet of area 0.02 m^2. The water is drawn from inlet orifices facing the direction of motion of the ship. The total drag is estimated to be $20\,u^2$ Nm, where u is the speed of the ship in m/s. If the ship moves with a speed of 60 km/hr, then determine (i) the relative velocity of the jet, (ii) energy supplied by the jet, (iii) power of the motor required to work the pump and (iv) jet propulsion efficiency. Assume the efficiency of pump and the density of water as 0.75 and 1020 kg/m^3, respectively.

Solution

Let $A = 0.02 \text{ m}^2$, $F = 20\,u^2$, $u = 60$ km/hr, $\eta_p = 0.75$ and $\rho_w = 1020 \text{ kg/m}^3$.

(i) $u = \dfrac{60 \times 1000}{60 \times 60} = 16.667$ m/s

Since $F = \rho_w A V_r (V_r - u) = 20\,u^2$

Thus $1020 \times 0.02 \times V_r \times (V_r - 16.667) = 20 \times 16.667^2$

$$V_r^2 - 16.667 V_r - 272.342 = 0$$

$$\therefore V_r = \frac{16.667 \pm \sqrt{16.667^2 + 4 \times 272.342}}{2 \times 1} = \mathbf{26.821\ m/s}$$

(ii) Kinetic energy supplied by the jet per second is given by,

$$\text{K.E.} = \frac{1}{2}(\rho_w A V_r)(V_r^2 - u^2)$$

$$\therefore \text{K.E.} = \frac{1020 \times 0.02 \times 26.821 \times (26.821^2 - 16.667^2)}{2} = \mathbf{120804.12\ Nm/s}$$

(iii) Power of the motor is given by,

$$P = \frac{\text{K.E.}}{1000\eta_p} = \frac{120804.12}{1000 \times 0.75} = \mathbf{161.0722\ kW}$$

(iv) $\eta = \frac{2u}{V + 2u} = \frac{2u}{V_r + u} = \frac{2 \times 16.667}{26.821 + 16.667} \times 100 = \mathbf{76.65\%}$

20.12 ❐ FLUID MACHINES

Fluid machines include all machines or devices which handle fluids (i.e., liquids and gases) and convert either fluid power into shaft power or shaft power into fluid power. Fluid machines which convert fluid power into shaft power are called turbines while fluid machines which convert shaft power into fluid power are called pumps, fans, blowers or compressors. A general classification of fluid machines is given below.

1. **Turbomachines:** A turbomachine is a power or head generating machine which employs the dynamic action of a rotary element called the rotor. The action of the rotor changes the energy levels of the continuously flowing fluid through the turbomachines. Turbines, pumps, compressors, blowers and fans are some examples of turbomachines and they are also known as rotodynamic machines. The turbomachines may be classified on the following basis.

   (i) **Based on quantity of fluid**

   (a) Machines which influence an indefinite quantity of fluid are known as open turbomachines. Examples: Windmills, unshrouded fans, electric fans, etc.

   (b) Machines in which a finite quantity of fluid is affected are called closed turbomachines. Examples: Turbines, compressors, etc.

   (ii) **Based on power**

   (a) Machines which absorb power to increase the energy level of the working fluid. Examples: Pump, compressor, etc.

   (b) Machines which produces power by decreasing the energy level of the working fluid. Examples: Hydraulic turbine, steam turbine and gas turbine.

   (iii) **Based on the type of fluid handled**

   (a) Machines which handle water, such as hydraulic turbines and pumps.

   (b) Machines which handle steam, such as steam turbines and steam engines.

   (c) Machines which handle gas, such as compressors, gas turbines and air turbines.

   (iv) **Based on the nature of flow path in moving over blade rows**

   (a) When the working fluid flows through the runner along the direction parallel to the axis of rotation of the runner, then the turbomachines are known as axial flow turbomachines.

   (b) When the working fluid flows in the radial direction through the runner, then the turbomachines are known as radial flow turbomachines. The radial machines may be inward radial flow machines or outward radial flow machines.

   (c) When the working fluid flows through the runner in the radial direction but leaves in the direction parallel to the axis of rotation of the runner, then the turbomachines are known as mixed flow turbomachines.

   (v) **Based on pressure changes**

   (a) If there is any pressure change when the moving fluid passes over moving blades, then the machine is called a reaction turbomachine.

   (b) If there is no pressure change, then the machine is called an impulse turbomachine.

(vi) **Based on the type of flow within it**

(a) If the density of the fluid does not change appreciably through the machine, then it is called an incompressible flow machine. For example, all types of hydraulic machines, such as turbines and pumps are incompressible flow machines.

(b) If the density of the fluid changes appreciably through the machine, then it is called a compressible flow machine.

Mach number (M) is defined as the square root of the ratio of the inertia force to the elastic force. Based on the value of Mach number, compressible machines are classified as subsonic flow machines ($M < 1$), transonic flow machines ($M \approx 1$), supersonic flow machines ($M > 1$) and hypersonic flow machines ($M >> 1$).

2. **Reciprocating machines** The reciprocating machines are also known as positive displacement machines. Examples of these machines are reciprocating pumps and compressors.
3. **Various water lifting devices** Some of the water lifting devices are jet pump, air lift pump and hydraulic ram.
4. **Pumps transmitting oils** These devices use oil under pressure to operate and control systems, for example, gear pumps, constant delivery pumps, variable delivery pumps, various appliances and accessories relating to fluid systems.

20.13 ❐ HYDRAULIC MACHINES AND ITS MAIN PARTS

In this book, only closed type hydraulic machines, such as hydraulic turbines and pumps have been discussed. The main parts of these machines are discussed in this section.

1. **Shaft:** Power generating hydraulic machines have only output shaft, for example, hydraulic turbines. However, power absorbing hydraulic machines have only input shaft, for example, hydraulic pumps. Power transmitting hydraulic machines have both input and output shafts, for example, hydraulic coupling and hydraulic torque converter.
2. **Runner or impeller or rotor:** The rotating element having blades or vanes on its periphery fitted on the shaft is called runner or impeller or rotor. In case of radial flow hydraulic turbines and pumps, it is called runner, impeller in the case of centrifugal pumps and rotor in the case of axial flow gas and steam turbines.
3. **Guide blade or nozzle:** It is not a compulsory part of every hydraulic machine. Guide blades are provided in radial flow and in axial flow reaction turbines while nozzle is provided in impulse turbines.
4. **Casing:** It is also not a compulsory part of every hydraulic machine. A hydraulic machine with casing is called a closed machine. The volute casing is used in hydraulic turbines (radial flow and axial flow reaction turbines) to increase the velocity of fluid before it enters the runner. The volute casing is also provided in centrifugal pumps to increase the pressure of the water flowing through it.
5. **Draft tube:** A tube of gradually increasing area (also called diffuser) is used for discharging water from the exit of the hydraulic reaction turbines to the tail race.
6. **Penstock:** A penstock is a pipe of large diameter which carries water under pressure from the storage reservoir to the hydraulic turbines.
7. **Cylinder and piston arrangement:** This arrangement is required in reciprocating pumps.
8. **Suction and delivery pipes:** These pipes are fitted in reciprocating as well as centrifugal pumps. One end of the suction pipe is connected to the inlet of the pump and the other end dips into water in the sump, whereas one end of the delivery pipe is connected to the outlet of the pump and the other end delivers water at the required height.

Summary

1. The jet of water discharging from a nozzle in atmosphere is called free jet and the force exerted by it on a vane is called impact of jet.
2. $F \cdot dt = d(mV)$ is called impulse-momentum equation in which $F \cdot dt$ is impulse and $d(mV)$ is the resulting change in momentum in the direction of force.
3. The force exerted by jet of water in a stationary flat vertical plate: $F_x = \rho_w A V^2$

4.	**Inclined stationary flat plate**	**Inclined moving flat plate**
Force exerted by jet (F)	$\rho_w A V^2 \sin\alpha$	$\rho_w A(V-u)^2 \sin\alpha$
Force in direction of jet (F_x)	$\rho_w A V^2 \sin^2\alpha$	$\rho_w A(V-u)^2 \sin^2\alpha$
Force $\perp$ to flow direction (F_y)	$\rho_w A V^2 \sin\alpha\cos\alpha$	$\rho_w A(V-u)^2 \sin\alpha\cos\alpha$
Work done per second (w)	0	$[\rho_w A(V-u)^2 \sin^2\alpha]u$

5.	**Moving flat vertical plate**	**Series of flat plates**
Force exerted by jet (F_x)	$\rho_w A(V-u)^2$	$\rho_w AV(V-u)$
Work done per second by jet (w)	$\rho_w A(V-u)^2 u$	$\rho_w AV(V-u)u$
Efficiency of the jet (η)	$(2/V^3)(V^2u + u^3 - 2Vu^2)$	$[2u(V-u)]/V^2$
Condition for η_{max}	$V = 3u$ or $u = V/3$	$u = V/2$
Maximum efficiency (η_{max})	8/27 or 29.63%	1/2 or 50%

6. Stationary curved vanes			
	Jet strikes at the centre	**Jet strikes symmetrical vane tangentially**	**Jet strikes unsymmetrical vane tangentially**
Force exerted in jet direction (F_x)	$\rho_w AV^2(1+\cos\alpha)$	$2\rho_w AV^2\cos\alpha$	$\rho_w AV^2(\cos\alpha + \cos\beta)$
Force exerted in $\perp$ direction (F_y)	$-\rho_w AV^2 \sin\alpha$	0	$-\rho_w AV^2(\sin\alpha - \sin\beta)$

7. Moving symmetrical curved vanes		
	Jet strikes at the centre of single vane	**Jet strikes at the centre of a series of vanes**
Force exerted by the jet (F_x)	$\rho_w A(V-u)^2(1+\cos\alpha)$	$\rho_w AV(V-u)(1+\cos\alpha)$
Work done by the jet (w)	$\rho_w A(V-u)^2(1+\cos\alpha)u$	$\rho_w AV(V-u)(1+\cos\alpha)u$

Efficiency (η)	$[2u(V-u)^2(1+\cos\alpha)]/V^3$	$[2u\,(V-u)\,(1+\cos\alpha)]/V^2$
Condition for η_{max}	$u = V/3$	$u = V/2$
Maximum efficiency (η_{max})	$(8/27)(1+\cos\alpha)$	$(1/2)(1+\cos\alpha)$

8.	Jet strikes an unsymmetrical moving curved vane tangentially	Jet strikes a series of radial curved vanes
Force (F_x) or torque (T) exerted	$F_x = \rho_w A V_{ri}(V_{wi} \pm V_{wo})$ (+ve for $\beta < 90°$; −ve for $\beta > 90°$)	$T = \rho_w A V_i(V_{wi}R_i \pm V_{wo}R_o)$
Work done per second (w)	$\rho_w A V_{ri}(V_{wi} \pm V_{wo})u$ (+ve for $\beta < 90°$; −ve for $\beta > 90°$)	$\rho_w A V_i(V_{wi}u_i \pm V_{wo}u_o)$
Work done per second per unit weight (w)	$[(V_{wi} \pm V_{wo})u]/g$	$[(V_{wi}u_i \pm V_{wo}u_o)]/g$
Work done per second per unit mass (w)	$(V_{wi} \pm V_{wo})\,u$	$(V_{wi}u_i \pm V_{wo}u_o)$
Efficiency (η)	$[\rho_w A V_{ri}(V_{wi} \pm V_{wo})u]/[(1/2)\rho_w A V_i^3]$	$2[V_{wi}u_i \pm V_{wo}u_o]/V_i^2$

9. Propulsion of ship means the movement of ship with the help of jet

	Inlet orifices at right angle to the motion of the ship	Inlet orifices face the direction of motion of the ship
Force exerted on ship (F)	$\rho_w a(V+u)V$	$\rho_w a(V+u)V$
Work done on the ship (w)	$\rho_w a(V+u)Vu$	$\rho_w a(V+u)Vu$
Efficiency η	$(2Vu)/(V+u)^2$	$2u/(V+2u)$
Condition for η_{max}	$u = V$	$u = V$
Maximum efficiency (η_{max})	50%	66.67%

10. Angle of swing of the hinged plate: $\sin\alpha = (\rho_w A V^2)/W$

11. Fluid machines include turbomachines, reciprocating machines, various water lifting devices and pumps transmitting oils.

12. Turbomachine is a power or head generating machine by employing a rotor.

13. The main parts of hydraulic machines are shaft, runner, guide blades, casing, draft tube, cylinder and piston arrangement and suction and delivery pipes.

Multiple-choice Questions

1. Force exerted by a water jet on a fixed vertical plate in the jet direction in terms of mass density of water (ρ_w), area of jet (A) and velocity of jet (V) is equal to

(a) $\rho_w AV^3$. (b) $\rho_w AV$.
(c) $(\rho_w A)/V^2$. (d) $\rho_w AV^2$.

2. Force exerted by a water jet on a fixed inclined plate in the jet direction in terms of mass density of water (ρ_w), area of jet (A), velocity of jet (V) and angle of inclination of the plate (α) is equal to

(a) $\rho_w AV^2 \sin^2 \alpha$. (b) $\rho_w AV(1+\sin\alpha)$.
(c) $\rho_w AV^2(1+\cos^2 \alpha)$. (d) $\rho_w AV^2(1+\sin\alpha)$.

3. Maximum efficiency in the case of a flat moving plate is given by
(a) (8/25). (b) (4/27).
(c) (16/27). (d) (8/27).

4. Maximum possible efficiency for a jet striking a series of flat plate is
(a) 1. (b) 0. (c) 0.75. (d) 0.5.

5. Force exerted on a stationary semicircular vane when the jet strikes at its centre is
(a) Twice than that on a flat plate.
(b) Half than that on a flat plate.
(c) Equal to than that on a flat plate.
(d) None of the above.

6. Maximum efficiency in the case of a single symmetrical moving curved vane when the jet strikes at the centre of vane is equal to

(a) $(8/27)(1-\cos\alpha)$. (b) $(16/27)(1+\cos\alpha)$.
(c) $(8/27)(1+\cos\alpha)$. (d) $(8/27)$.

7. Maximum possible efficiency in the case of a single semicircular vane when the jet strikes at the centre of vane will be
(a) 49% (b) 49.2%
(c) 59% (d) 59.2%

8. Maximum efficiency $(\eta_{\max})$ when the jet strikes at the centre of a series of symmetrical moving curved vanes is equal to

(a) $(1-\cos\alpha)/2$. (b) $(1+\cos\alpha)/2$.
(c) $\cos\alpha/2$. (d) $(1+\sin\alpha)/2$.

9. For maximum efficiency, when the jet strikes at the centre of a series of symmetrical moving curved vanes, the absolute velocity of the jet V relative to the velocity of vane u will be
(a) Half. (b) Twice.
(c) One third. (d) Equal.

10. In actual practice, for obtaining maximum efficiency, vane angle at the outlet ϕ for the series of radial curved vanes should be
(a) Maximum. (b) Minimum.
(c) 0°. (d) None of the above.

Review Questions

1. What do you mean by fluid machines? How will you classify them?

2. Discuss the main parts of hydraulic machines. Also derive Euler's equation applied to fluid machines.

3. Define free jet, impact of jet, jet propulsion and impulse-momentum equation.

4. Derive an expression for the force exerted by a water jet on a fixed vertical plate in the direction of jet.

5. Derive an expression for the force exerted by a water jet on a moving flat plate. Also derive an expression for its maximum efficiency.

6. Derive an expression for the distribution of flow in the two directions parallel to the plate when a jet of water impinges on a plate at an angle α. Assume that friction between the fluid and plate is negligible.

7. Derive an expression for the force F exerted by a jet of area A which strikes a flat plate at an angle α to the normal of the plate with velocity V. The plate itself is moving with velocity u in the direction of normal to the plate surface.

8. Show that the force exerted by a jet of water in an inclined fixed plate in the direction of the jet is given by $F_x = \rho_w AV^2 \sin^2\alpha$, where ρ_w is the mass density of water, A is the area of the jet, V is the velocity of the jet and α is the inclination of the plate with the jet.

9. Derive an expression for the efficiency when a water jet strikes a series of flat plates. Also show that the efficiency can never exceed 50%.

10. Show that when a jet of water impinges on a series of curved vanes, maximum efficiency is obtained when the vane is semicircular in section and the velocity of vane is half that of jet.

11. Derive expressions for the force exerted by a jet on stationary curved plate with conditions, such as (i) when the jet strikes the curved plate at the centre and (ii) when the jet strikes the curved plate at one end tangentially under symmetrical and unsymmetrical condition of plate.

12. Show that the force exerted by a jet of water striking at the centre on a fixed semi-circular plate in the direction of the jet is twice the force exerted by the jet on a fixed vertical plate.

13. Derive the given expression for the efficiency when a fluid jet strikes a series of radial curved vanes, $\eta = \{2(V_{wi}u_i \pm V_{wo}u_o)\}/V_i^2$.

14. Prove that the maximum efficiency when a jet strikes the series of symmetrical vanes at their centre is given by $\eta_{max} = (1+\cos\alpha)/2$, where α is the angle made by the jet with x-axis at the inlet of the curved vane.

15. A jet having a velocity V strikes a single curved vane moving in the jet direction with velocity u so that velocity of jet relative to vane is $(V-u)$. The vane causes the jet to be reversed in direction. Show that maximum efficiency is obtained when $V = 3u$. Also find the resultant maximum efficiency.

16. Derive expressions for efficiency of propulsion of ship when (i) the inlet orifices at right angle to the motion of ship and (ii) inlet orifices face the direction of ship motion.

17. Show that the force exerted by a jet of fluid on a moving inclined plate in the direction of the jet is given by $F_x = \rho A(V-u)^2\sin^2\alpha$, where A is the area of the jet, V is the velocity of the jet and α is the inclination of the plate with the jet.

18. Establish the ratio of force exerted by a water jet when it strikes (i) a stationary flat plate, (ii) a flat plate moving in direction of jet at one-third of velocity of jet, and (iii) a series of flat plates mounted on a wheel and moving at one-third the velocity of jet.

19. Derive an expression for the angle of swing of a vertical hinged plate.

20. Show that the maximum efficiency is slightly less than 60% when a jet strikes tangentially a smooth curved vane moving in the same direction as the jet and jet gets reversed in the direction.

Problems

1. Find the force exerted by a jet of water of diameter 50 mm on a stationary flat plate, when the jet strikes the plate normally with a velocity of 10 m/s. Also find the work done.
[**Ans.** 196 N, 0]

2. A jet of water cross-sectional area 40 cm^2 functions with a velocity of 25 m/s and strikes a stationary flat plate held at 30° to the axis of jet. Find the force exerted by the jet on the plate (i) in the direction normal to the plate, (ii) in the direction of the jet and (iii) in the direction perpendicular to the flow. (iv) Find how the discharge gets distributed after striking the plate.
[**Ans.** 1250 N, 625 N, 1082.53 N, 13.93]

3. A jet of water of diameter 75 mm having a velocity of 20 m/s strikes normally a flat smooth plate. Determine the thrust on the plate (i) if the plate is at rest and (ii) if the plate is moving in the same direction as the jet with a velocity of 5 m/s. Also determine the work done per second on the plate in each case and the efficiency of the jet when the plate is moving.
[**Ans.** 1767.14 N, 994.02 N, 0, 4970.1 Nm, 28.12%]

4. A square plate weighing 110 N and of uniform thickness and 30 cm edge is hung so that horizontal jet 20 mm diameter and having a velocity of 10 m/s impinges on the plate. The centre line of the jet is 150 mm below the upper edge of the plate and when the plate is vertical, the jet strikes the plate normally at its centre. Find what force must be applied at the lower edge of the plate in order to keep the plate vertical. If the plate is allowed to swing freely, then find the inclination to the vertical which the plate will assume under the action of jet.
[**Ans.** 15.7 N, 16.59°]

5. A jet of water of diameter 75 mm having a velocity of 25 m/s strikes a flat smooth plate normally. Determine the force exerted by the jet on the plate (i) if the plate is stationary and (ii) if the plate is moving in the direction of the jet with a velocity of 5 m/s. Also find the work done per second on the plate in each case and the efficiency of the jet when the plate is moving.
[**Ans.** 2761.16 N, 0, 1767.14 N, 8835.7 Nm, 25.6%]

6. A jet of water having a velocity of 12 m/s strikes a curved vane which is moving with a velocity of 4 m/s. The vane is symmetrical and it is so shaped that the jet is deflected through 120°. Find the angle of the jet at inlet so that there is no shock. What is the absolute velocity of the jet at outlet in magnitude and direction and the work done per unit weight of water? Assume the vane to be smooth.
[**Ans.** 20.41°, 5.29 m/s, 52.87°, 127.87°, 5.91 Nm/N]

7. A jet of water having a velocity of 30 m/s strikes a series of curved vanes which are moving with a velocity of 10 m/s in the same direction as that of the jet at inlet. The vanes are so shaped that the jet is deflected through 130°. The diameter of the jet is 100 mm. Assume that the vanes are smooth, find (i) the force exerted by the jet on the vane in the direction of motion, (ii) work done, (iii) power exerted on the vane and (iv) the efficiency of the vane.
[**Ans.** 7740.12 N, 77401.2 Nm/s, 77.4 kW, 73%]

8. A water jet with diameter 50 mm having a velocity of 35 m/s strikes a flat plate, the normal of which is inclined at 30° to the axis of the jet. Calculate the normal force exerted on the plate, (i) when the plate is stationary and (ii) when the plate is moving with a velocity of 10 m/s in the direction of jet. Also calculate the work done, power and efficiency of the jet when the plate is moving.
[**Ans.** 1202.64 N, 613.59 N, 3067.9 Nm/s, 3.07 kW, 7.29%]

9. A jet of water moving at 12 m/s impinges on a vane shaped to deflect the jet through 120° when stationary. If the vane is moving at 5 m/s, then find the angle of the jet so that there is no shock at the inlet. What is the absolute velocity of the jet at

exit in magnitude and direction and the work done per second per unit weight of water striking per second? Assume that the vane is smooth and symmetrical.

[**Ans.** 30°, 17.98°, 110.93°, 3.97 m/s, 6.54 Nm/N]

10. A water jet of diameter 5 cm strikes a curved plate at its centre with a velocity of 25 m/s. The curved plate is moving with a velocity of 10 m/s in the direction of the jet. The jet is deflected through an angle of 165°. Assume that the plate is smooth, find (i) the force exerted on the plate in the direction of jet, (ii) power of the jet and (iii) efficiency of the jet.

[**Ans.** 868.52 N, 8.67 kW, 56.62%]

11. A water jet of diameter 5 cm impinges on a curved vane and is deflected through an angle of 175°. The vane moves in the same direction as that of the jet with a velocity of 35 m/s. If the rate of flow is 170 litres per second, then determine the component of force on the vane in the direction of motion. How much would be the power developed by the vane and what would be the vane efficiency? Neglect friction. How these parameters would change if instead of one vane there is a series of vanes fixed to a wheel and moving in the direction of jet with a velocity of 35 m/s?

[**Ans.** 10432.9 N, 365.15 kW, 57.31%, 17510 N, 612.85 kW, 96.13%]

12. A 6 cm diameter jet having a velocity of 25 m/s strikes a flat plate, the normal of which is inclined at 45° to the axis of the jet. Find the normal pressure on the plate (i) when the plate is stationary and (ii) when the plate is moving with a velocity of 10 m/s in the direction of jet. Also determine the power and efficiency of system when the plate is moving.

[**Ans.** 1249.56 N, 450.25 N, 3.184 kW, 14.4%]

13. A circular water jet having a cross-sectional area of 25 cm^2 moves with a velocity of 35 m/s and strike a curved symmetrical plate at its centre. The angle of curvature of plate at the outlet with x-axis direction is 120°. Find the force exerted by the jet on the plate in x-direction, (i) when the plate is stationary and (ii) when the plate is moving in the direction of jet with a velocity of 15 m/s.

[**Ans.** 4593.75 N, 1500 N]

14. A water jet of diameter 2 cm strikes a 20 cm × 20 cm square plate of uniform thickness with a velocity of 10 m/s at the centre of the plate which is suspended vertically by a hinge on its top horizontal edge. The weight of the plate is 49 N. The jet strikes normal to the plate. Determine (i) what force must be applied at the lower edge of the plate so that the plate is kept vertical and (ii) if the plate is allowed to deflect freely and then what will be the inclination of the plate with vertical due to the force exerted by the jet of water?

[**Ans.** 15.7 N, 39.85°]

15. A stationary vane having an inlet angle of zero degree and an outlet angle of 25° receives water at a velocity of 50 m/s. Determine the components of force acting on it in the direction of jet velocity and normal to it. Also find the resultant force in magnitude and direction per kg of flow per second. If the vane stated above is moving with a velocity of 20 m/s in the direction of the jet, then determine the force components in the direction of the vane velocity and across it, also the resultant force in magnitude and direction. Calculate the work done and power developed per kg of flow.

[**Ans.** 95.315 N, –21.13 N, 97.63 N, 12.5°, 57.19 N, –12.67 N, 58.57 N, 12.5°, 1143.8 Nm/s, 1.144 kW]

16. A jet propelled ship discharges water through a jet area of 0.02 m^2. The water is drawn from inlet orifices facing the direction of motion of the ship. The total drag is estimated to be $17u^2$ Nm, where u is the speed of the ship in m/s. If the ship moves with a speed of 54 km/hr, then determine (i) the relative velocity of the jet, (ii) energy supplied by the jet, (iii) power of the motor required to work the pump and (iv) the jet propulsion efficiency. Assume the efficiency of pump and the density of water as 0.75 and 1020 kg/m^3, respectively.

[**Ans.** 23.11 m/s, 72855.09 Nm/s, 97.14 kW, 78.72%]

17. A jet propelled boat draws water amidship and discharges it through a jet of cross-sectional area 0.02 m^2 at the back with an absolute velocity of 10 m/s. If the boat moves with a speed of 15 km/hr, then determine (i) the force exerted on the boat, (ii) power of the motor required to work the pump and (iii) efficiency of propulsion.

[**Ans.** 2834 N,11.82 kW, 41.54%]

18. A ship propelled by reaction jets and discharging astern has a resistance to its motion of 3500 N when moving with a speed of 8.33 m/s. The velocity of jet relative to the ship is 18 m/s and the area of each jet is 100 cm^2. Determine the number of jets and the power required to drive the pump. Also determine the efficiency of the jet propulsion for both the arrangement of inlet orifices.

[**Ans.** 2, 29.16 kW, 50%, 63.3%]

19. A jet of water having a velocity of 25 m/s strikes a series of radial curved vanes mounted on a wheel which is rotating at 200 rpm. The jet makes an angle of 20° with the tangent to the wheel at the inlet and leaves the wheel with a velocity of 5 m/s at an angle of 130° to the tangent to the wheel at outlet. Water is flowing from outward in a radial direction. The outer and inner radii of the wheel are 0.5 m and 0.25 m, respectively. Determine (i) the vane angles at the inlet and outlet, (ii) work done per unit weight of water and (iii) efficiency of the wheel.

[**Ans.** 33.29°, 24.38°, 26.78 Nm/N, 84.08%]

20. A jet of water from a nozzle is deflected through 60° from its original direction by a curved plate which it enters tangentially without shock with a velocity of 30 m/s and leaves with a mean velocity of 25 m/s. If the discharge from nozzle is 1 kg/s, then calculate the magnitude and direction of the resultant force on the stationary vane.

[**Ans.** 27.84 N, 51.05°]

21. A water jet of diameter 50 mm strikes a smooth curved vane at its centre with a velocity of 15 m/s. The curved vane is moving with a velocity of 5 m/s in the direction of the jet. The jet is deflected through an angle of 165°. Find thrust on the plate in the direction of jet, power and efficiency of the jet.
[**Ans.** 385.32 N, 1.927 kW, 58.25%]

22. A jet of water having a velocity of 45 m/s impinges without shock on a series of vanes moving at 15 m/s. The direction of motion of the vanes is inclined at 20° to that of jet, the relative velocity at outlet is 0.9 times that of at inlet and absolute velocity of water at exit is to be normal to the vanes. Determine the vane angles, work done on vanes per unit weight of water supplied and efficiency of the jet.
[**Ans.** 29.4°, 57.9°, 64.7 Nm/N, 62.6%]

23. A jet of water having a velocity of 40 m/s strikes a curved vane, which is moving with a velocity of 20 m/s. The jet makes an angle of 30° with the direction of vane at the inlet and leaves at an angle of 90° to the direction of motion of vane at outlet. Determine the vane angles so that water enters and leaves the vane without shock.
[**Ans.** 53.8°, 36.2°]

24. A jet of diameter 50 mm impinges on a curved vane and is deflected through an angle of 175°. The vane moves in the same direction as that of jet with a velocity of 35 m/s. Determine the power developed and the efficiency of the jet if the rate of flow is 170 litres per second.
[**Ans.** 365.15 kW, 57.3%]

25. A water jet of diameter 50 mm strikes a curved vane at its centre with a velocity of 30 m/s. The curved vane is moving with a velocity of 10 m/s in the direction of jet. The jet is deflected through an angle of 150°. Assuming that the vane is smooth, find (i) the force exerted on the vane in the direction of jet, (ii) work done per second, (iii) power of the jet and (iv) efficiency. Also calculate the force exerted on the vane in the direction of jet whose coefficient of friction is 0.9.
[**Ans.** 1465.57 N, 14655.73 Nm/s, 14.656 kW, 55.29%, 1397.55 N]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (d)	**2.** (a)	**3.** (d)	**4.** (d)	**5.** (a)
6. (c)	**7.** (d)	**8.** (b)	**9.** (b)	**10.** (b)

Chapter 21

Pelton Turbine (Impulse Turbine)

21.1 ❒ INTRODUCTION

Hydraulic energy means the energy possessed by water in the form of potential energy, kinetic energy and intermolecular energy. The power transmitted by a rotating shaft is usually known as mechanical energy. The hydraulic machines are energy conversion devices in which hydraulic energy is either converted into mechanical energy or mechanical energy is converted into hydraulic energy. The hydraulic machines which convert hydraulic energy into mechanical energy are known as turbines, whereas the hydraulic machines which convert mechanical energy into hydraulic energy are known as pumps.

A hydraulic turbine consists of a wheel called runner (or rotor) having a number of evenly spaced vanes (blades or buckets) on its periphery. When water falls from certain height, its potential energy is converted into kinetic energy, which is further converted into mechanical energy by allowing the water to flow through the turbine runner. The mechanical energy so developed is utilized to run an electric generator which is coupled to the turbine shaft and thus, electric energy is generated. The electric power which has been obtained from hydraulic energy is known as hydroelectric power.

In impulse turbines, the entire pressure energy of water is converted into kinetic energy by passing it through a nozzle. Thus, the energy available at the inlet of an impulse turbine is only kinetic energy. The Pelton turbine is the only impulse hydraulic turbine which is now commonly used. Some important Pelton turbine installations in India are Mandi hydroelectric project (Himachal Pradesh), Mahatma Gandhi hydroelectric project (Karnataka), Koyna hydroelectric project (Maharashtra), Pykara hydroelectric scheme (Tamil Nadu), Kundah hydroelectric project (Chennai) and Pallivasal power station (Kerala). In this chapter, the characteristics and properties of Pelton turbine is briefly explained.

21.2 ❒ CLASSIFICATION OF HYDRAULIC TURBINES

The hydraulic turbines are classified as follows:

1. According to the type of energy available at the inlet.
   - (i) **Impulse turbine:** At the inlet of the turbine, water has only kinetic energy. For example: Pelton turbine, Banki turbine and Jonval turbine.
   - (ii) **Reaction turbine:** At the inlet of the turbine, water has both kinetic energy and pressure energy. For example: Francis turbine, propeller and Kaplan turbines.

2. According to the direction of flow through the runner.
   (i) **Tangential flow turbine:** Water flows along the tangent to the axis of rotation of the runner. For example: Pelton wheel.
   (ii) **Radial flow turbine:** Water flows in the radial direction through the runner. The radial flow turbine is either inward radial flow type or outward radial flow type.
      (a) **Inward radial flow turbine:** Water enters at the outer circumference and flows inwards radially towards the centre of the runner. For example: Old Francis turbine and Thomson turbine.
      (b) **Outward radial flow turbine:** Water enters at the centre and flows radially towards the outer periphery of the runner. For example: Fourneyron turbine.
   (iii) **Axial flow turbine:** Water flows through the runner along the direction parallel to the axis of rotation of the runner. For example: Propeller turbine, Kaplan turbine and Jonval turbine.
   (iv) **Mixed flow turbine:** Water enters the runner at the outer periphery in the radial direction and leaves it at the centre in the direction parallel to the axis of rotation of the runner. For example: Modern Francis turbine.
3. According to the head available at the inlet of turbine.
   (i) **High head turbine:** High head turbines are capable of working under very high heads usually more than 250 m. For example: Pelton turbine and it requires relatively less quantity of water only.
   (ii) **Medium head turbine:** These turbines are capable of working under medium heads ranging from 60 m to 250 m. For example: Modern Francis turbine and it requires relatively large quantity of water.
   (iii) **Low head turbine:** These turbines are capable of working under heads less than 60 m. For example: Kaplan and propeller turbines and it requires large quantity of water.
4. According to the specific speed of the turbine.

   Specific speed is the speed of a geometrically similar turbine which would develop unit power when working under unit head. It is denoted by N_s and it is given by the following expression.

$$\boxed{N_s = \frac{N\sqrt{P}}{H^{5/4}}} \tag{21.1}$$

   Here, N is the normal working speed in rpm, P is the power output of the turbine in kW and H is the net head in metres. Based on the specific speed of the turbines, the following classification is made.
   (i) **Low specific speed turbine:** Specific speed of these turbines varies from 8.5 to 50. If the specific speed of turbines varies from 8.5 to 30, then it will be Pelton wheel with single jet and if it varies from 30 to 50, then it will be Pelton wheel with double jet.
   (ii) **Medium specific speed turbine:** Specific speed of these turbines varies from 50 to 255. For example: Francis turbine.
   (iii) **High specific speed turbine:** Specific speed of these turbines varies from 255 to 860. For example: Kaplan and propeller turbines.
5. According to the name of the originator.
   (i) **Pelton turbine:** It is named after Lester A. Pelton, an American engineer. It is the only impulse type of turbine which is commonly used for high head and low discharge.
   (ii) **Francis turbine:** It is named in the honour of James B. Francis, an American engineer, who developed an inward radial flow turbine but later on it was modified. The modern Francis turbine is a mixed flow reaction turbine which is used for medium head and medium discharge.
   (iii) **Kaplan turbine:** It is named after the Austrian engineer V. Kaplan. It is an axial flow reaction turbine that is used for low heads. It requires large quantity of water to produce large amount of power.
6. According to the disposition of the turbine shaft.

   The turbines may be disposed with either vertical or horizontal shafts. Commonly, the vertical disposition of shafts is used for turbines. Pelton wheel is an example of horizontal shaft, whereas Kaplan turbine is a vertical shaft turbine.

21.3 ❐ IMPULSE TURBINE OPERATION PRINCIPLE

The operation principle of impulse turbines is schematically shown in Figure 21.1. In impulse turbines, the available hydraulic energy is converted into kinetic energy by passing through a nozzle fitted at the end of the penstock. High velocity jet of water coming out from the nozzle strikes a series of suitably shaped vanes mounted on the periphery of the runner (or wheel). The vanes change the direction of the jet without changing its pressure. The resulting change in momentum causes the rotation of the vanes and the runner. The runner revolves freely in air and thus, mechanical energy is obtained at the turbine shaft. The water coming out of the nozzle operates under atmospheric pressure throughout its action on the runner and its subsequent flow to the tail race. Thus, these turbines are also termed as free jet turbines.

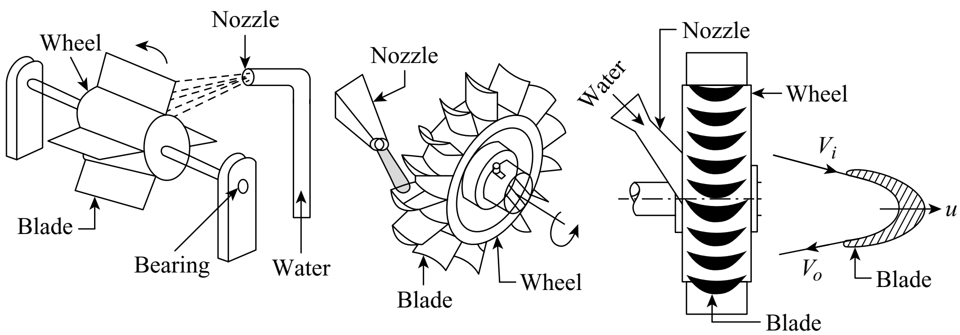


Figure 21.1 *Principle of impulse turbine*

Some of the important impulse turbines are Pelton turbine (or Pelton wheel), Turgo impulse wheel, Girard turbine, Banki turbine and Jonval turbine. Pelton turbine is the only impulse turbine type which is now commonly used.

21.4 ❐ GENERAL LAYOUT OF A HYDROELECTRIC POWER PLANT

Figure 21.2 shows a general layout of a hydroelectric power plant. A hydroelectric power plant consists of the following components.

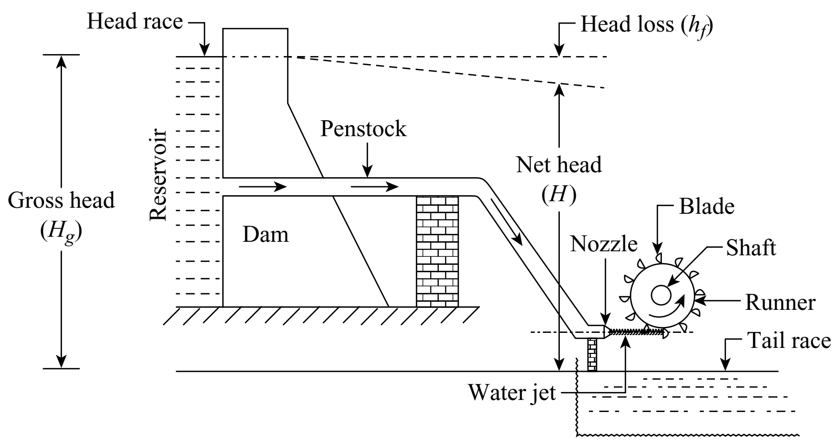


Figure 21.2 *General layout of a hydroelectric power plant*

1. **Dam:** It is a structure which is built across a river at an appropriate site for impounding river water for its storage and to create the head. The height through which water is stored behind the dam above the turbine level is called head of water. The water surface of the stored water in a reservoir is called head race level or simply head race.
2. **Penstocks:** These are pipes of large diameters which carry water under pressure from the storage reservoir to the turbine. Penstocks are usually made of steel, wood or reinforced concrete.
3. **Turbines:** Turbines are machines which converts hydraulic energy into mechanical energy. These are also termed as hydraulic motors or prime movers. The turbines have two main components, namely stator and rotor. Stator means a nozzle or a row of guide or fixed blades which convert potential energy of water into kinetic energy. The rotor or runner is a wheel which is free to rotate about an axis and a number of blades are mounted on its periphery. It rotates under the dynamic action of water flowing over its blades.
4. **Tail race:** It is a passage for discharging water after it has passed through the turbines into the river. The water surface in the tail race channel is called tail race level or simply tail race.
5. **Forebay:** It is a smaller reservoir which temporarily stores water at the head of the penstocks and supplies water when required. It may be constructed at a short distance downstream of the main reservoir. No forebay is required when the power house is located at the base of the dam. The forebays are constructed in installations where the power house is located at a far distance from the storage reservoir. The use of forebay is common in high head hydroelectric power plants (i.e., head about >250 m).
6. **Surge tank:** It is a reservoir which is fitted at some opening before the turbine to receive the rejected flow when the pipeline is suddenly closed by a valve at its steep end. The rapid velocity fluctuation due to sudden closure and opening sets up large magnitude pressure transients. These excessive pressures may lead to bursting of the pipe and this phenomenon is called water hammer. The sudden surge of water in penstock is taken by the surge tank when the water requirement reduces suddenly. The surge tank also supplies additional water required by the turbine due to the sudden increase in demand, before the water comes from the reservoir.

21.5 ❐ HEADS AND EFFICIENCIES OF A HYDRAULIC TURBINE

The heads and important hydraulic turbine efficiencies are given below.

1. **Gross head:** Gross head is the difference between the head race level and the tail race level when no water is flowing (Figure 21.2). It is denoted by H_g.
2. **Net head:** It is the head available at the inlet of the turbine and it is also known as effective head. When water flows from the head race to the turbine inlet, certain head loss occurs mainly due to friction between water and the penstocks. If h_f is the head loss due to friction then the net head is given by the following expression.

$$H = H_g - h_f = H_g - \frac{4fLV^2}{2gD} = H_g - \frac{f_f LV^2}{2gD} \tag{21.2}$$

   Here, f = coefficient of friction, $4f = f_f$ = friction factor, L = length of penstock, V = velocity of flow in penstock, D = diameter of penstock and g = gravitational acceleration.
3. **Hydraulic efficiency (η_h):** It is defined as the ratio of power developed by the turbine runner (R.P.) to the power supplied by the water jet at the inlet of the turbine (W.P.). It represents the effectiveness with which energy is transferred from the water to the runner. The hydraulic losses are taken into account by the hydraulic efficiency of a turbine and it is given by the following relation.

$$\boxed{\eta_h = \frac{\text{Runner power}}{\text{Water power}} = \frac{\text{R.P.}}{\text{W.P.}}} \tag{21.3}$$

   The hydraulic losses mainly occur due to blade friction, eddy formation and change of direction of flow when water flows from inlet to the exit of the turbine.

The water power (W.P.) can be evaluated by the product of weight of water striking the vanes of the turbine per second ($\rho_w Qg$) and the net head (H) as expressed below.

$$\text{W.P.} = \frac{\rho_w Qg \times H}{1000} \text{ kW, where } Q \text{ is the volume of water per second.}$$

4. **Mechanical efficiency (η_m):** Due to mechanical losses, the power available at the shaft for use is always less than the power produced by the runner. Thus, mechanical efficiency is defined as the ratio of the power available at the turbine shaft (S.P.) to the power developed by the runner (R.P.). The mechanical losses (e.g., bearing friction) are taken into account by the mechanical efficiency. The mathematical expression for mechanical efficiency is represented below.

$$\boxed{\eta_m = \frac{\text{Shaft power}}{\text{Runner power}} = \frac{\text{S.P.}}{\text{R.P.}}} \tag{21.4}$$

Depending upon the size and capacity of the Pelton turbine, the values of mechanical efficiency generally varies from 97% to 99%.

5. **Volumetric efficiency (η_v):** When water enters the turbine, there is a possibility that some amount of water may go to the tail race without striking the runner blades. Thus, volumetric efficiency is defined as the ratio of the volume of the water actually striking the runner to the volume of water supplied to the turbine by the jet. The leakage losses are taken into account by volumetric efficiency. The volumetric efficiency is mathematically represented as shown below.

$$\eta_v = \frac{\text{Volume of water actually striking the runner}}{\text{Volume of water supplied to the jet}} \tag{21.5}$$

For Pelton turbine, the volumetric efficiency generally varies from 97% to 99%.

6. **Overall efficiency (η_o):** It is defined as the ratio of power available at the turbine shaft (S.P.) to the power available from the water jet at the turbine inlet (W.P.). The judgment of the performance of a hydraulic turbine is made by its overall efficiency. Generally, shaft power (P) is taken in kW and water power is given by [($\rho_w gQH$)/1000] kW. Thus, the overall efficiency is mathematically represented as given below.

$$\eta_o = \frac{\text{Shaft power}}{\text{Water power}} = \frac{\text{S.P.}}{\text{W.P.}} = \frac{1000P}{\rho_w gQH} \tag{21.6}$$

$$Q = \frac{1000P}{\rho_w g H \eta_o} \quad \text{[From Equation (21.6)]}$$

Multiply and divide Equation (21.6) by R.P., we get:

$$\eta_o = \frac{S.P}{R.P} \times \frac{R.P}{W.P} = \eta_m \times \eta_h \tag{21.7}$$

When volumetric efficiency (η_v) is also considered, the overall efficiency is expressed as given below.

$$\boxed{\eta_o = \eta_m \times \eta_h \times \eta_v} \tag{21.8}$$

If η_g is the efficiency of the generator, then plant efficiency (η_p) and power output of hydro unit (P_o) is given by the following expression.

$$\eta_p = \eta_g \times \eta_o \tag{21.9}$$

$$P_o = \frac{\eta_p \rho_w gQH}{1000} = \frac{\eta_g \eta_o \rho_w gQH}{1000} \tag{21.10}$$

21.6 ❒ WATERWHEEL

Waterwheel is the oldest form of water turbine. It converts the energy of free-flowing or falling water into useful form of power. In olden days, waterwheels were used in a watermill and nowadays, it is not commonly used. The waterwheel consists of large wooden or metal wheel which is provided with blades or buckets around its periphery. Generally, the wheel is mounted vertically on a horizontal axle. The water is delivered to the wheel at some point on its circumference striking one or more buckets at a time. Waterwheels can be classified on the basis of water applied to the wheel relative to the wheel axle. The three main types of waterwheels are shown in Figure 21.3.

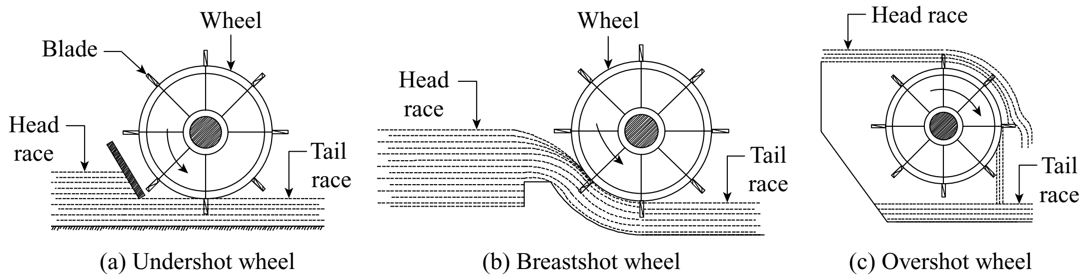


Figure 21.3 *Types of water wheel*

1. **Undershot wheel:** An undershot wheel is rotated due to striking of the water to the blades at the bottom of the wheel as shown in Figure 21.3(a). It is the least efficient and oldest type of wheel. Its efficiency varies from about 35% to 45%. The subtypes of this waterwheel, namely Poncelet wheel and Sagebien wheel provide greater efficiency than it.
2. **Breastshot wheel:** A breastshot wheel is rotated by falling water striking the buckets near the centre of the wheel's edge or just above it as shown in Figure 21.3(b). These wheels are more efficient than the undershot wheels. Its efficiency varies from about 50% to 60%. The breastshot and undershot wheels can be used in rivers and in places where high volume of water flows in a large reservoir.
3. **Overshot wheel:** An overshot wheel is rotated by falling water striking the buckets near the top of the wheel as shown in Figure 21.3(c). These are more efficient than breastshot and undershot wheels. Its efficiency varies from about 65% to 85%. The overshot wheels are suitable for small reservoir and where a small stream with a height difference more than 2 m is available.

21.7 ❒ PELTON TURBINE (PELTON WHEEL)

The Pelton turbine (or Pelton wheel) is a tangential flow impulse turbine. It is the only impulse type turbine which is now commonly used and it is named after L. A. Pelton, an American Engineer. This turbine is used for high heads and is generally used for heads in excess of 250 m. So far, Pelton turbine has been used under a head of 1770 m developing 23080 kW power when running at 750 rpm and it is installed at Reisseck (Austria). The largest Pelton turbine of the world is in Italy which produces 110.25 MW power under a head of 711 m when running at 300 rpm. The main parts of the Pelton turbine are shown in Figure 21.4.

1. **Penstock:** It is a large sized pipe through which water flows from the reservoir to the turbine. On the basis of available water head, a penstock may consist of wood, concrete or steel. It is equipped with screens called trash racks at the inlet which prevents the debris from entering into it. For regulating the flow from the reservoir to the turbine, the penstocks are fitted with control valves.
2. **Spear and nozzle arrangement:** The nozzle fitted at the end of the penstock is provided with a spear which controls the quantity of water striking the buckets of runner as shown in Figure 21.4. The spear is a conical needle having streamlined head. The axial movement of spear in case of smaller units is controlled by a wheel while in the case of

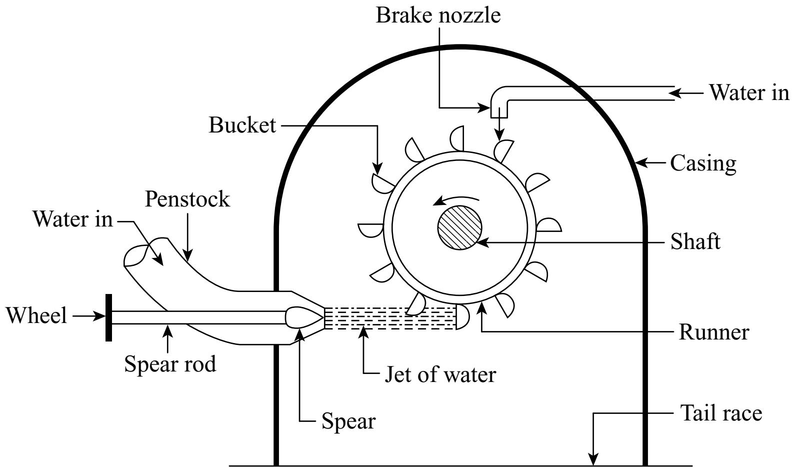


Figure 21.4 *Pelton turbine*

larger units it is controlled automatically by a governor. When the spear moves in forward direction, the flow area decreases and thus, the amount of water striking the runner is reduced. On the other hand, if the spear moves in backward direction, then the flow area increases, as a result, the amount of water striking the runner increases.

Usually, the shaft of a Pelton turbine is horizontal. When the Pelton turbine shaft is horizontal, then no more than two jets are used. However, for vertical shaft turbine it is limited to six numbers of jets. The number of nozzles depends upon the specific speed and these are spaced that a jet, after striking a bucket does not interfere with another jet.

3. **Runner with buckets:** The runner consists of a large circular disc in which a number of buckets (always ≥ 15) are evenly spaced round its periphery as shown in Figure 21.4. The buckets take a shape of double hemispherical cup or bowl as shown in Figure 21.5.

   Each bucket is divided into two symmetrical parts by a sharp edged ridge called a splitter. The jet of water strikes on the splitter, which divides the jet into two equal parts, each of which after flowing round the inner smooth bucket surface leaves at its outer edge as shown in Figure 21.5. The advantage of double shaped buckets is that the axial thrust produced by the water in each half neutralizes each other. Thus, the bearings supporting the wheel shaft are not subjected to any axial or end thrust. The buckets are so shaped that the jet of water gets deflected through 160° to 170°.

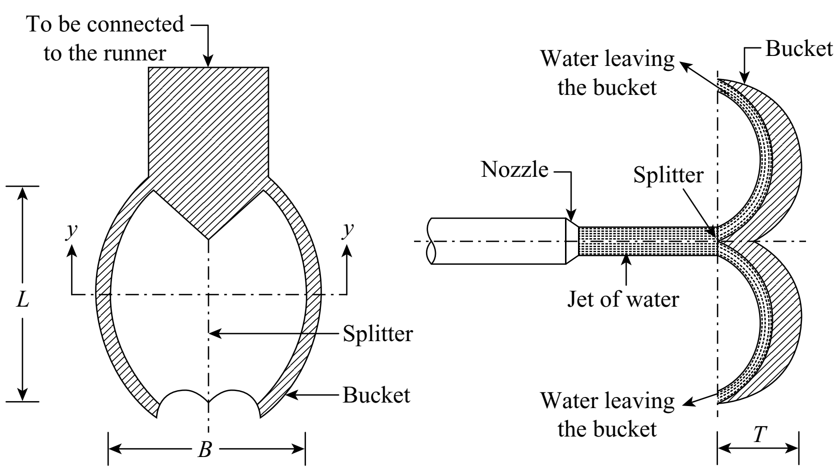


Figure 21.5 *Bucket of Pelton turbine*

For obtaining maximum change in momentum of the fluid, the deflection of water should be 180°. However, in practice the deflection is limited to about 165° so that water leaving a bucket does not hit the back of the following bucket. Depending upon the head at the inlet of turbine, the buckets are made of different materials. Generally, for low heads, the buckets are made of cast iron, whereas for higher heads these are made of cast steel, bronze or stainless steel. The buckets are properly polished to avoid erosion in its surface. In large turbines, the buckets are bolted to the runner by which the damaged bucket can be replaced easily. Many manufacturers believe that all the buckets wear out at the same time. Thus, it is more economical to cast the buckets and the disc as a single unit. Based on the specific speed, the runners may be classified as shown below with varying speeds.

(i) Slow runner: Specific speed varies from 8.5 to 20.
(ii) Normal runner: Specific speed varies from 20 to 28.
(iii) Fast runner: Specific speed varies from 28 to 35.

4. **Casing:** In Pelton turbine, the casing does not perform any hydraulic function. It is provided only to prevent splashing of water and to guide it to the tail race. It also acts as a safeguard against accidents. It is made of cast iron or fabricated steel plates. Generally, it is made in two parts so that it can be easily erected and assembled.
5. **Braking jet:** In order to shut down the turbine, the nozzle is completely closed by moving the spear in the forward direction. Thus, the amount of water striking the runner becomes zero. However, the runner still goes on revolving for a long time due to inertia. To stop the runner in a short time, a small brake nozzle is provided which directs the jet of water on the back of the buckets. This jet of water is called braking jet which is shown in Figure 21.4.

21.8 ❒ GOVERNING OF HYDRAULIC TURBINES

All modern hydraulic turbines are directly coupled to the electric generators. The generator must run at constant speed, so that the electricity is produced at constant frequency. The speed of the generator is given by the following relation.

$$\boxed{N = \frac{60f}{p}} \tag{21.11}$$

Here, f is the frequency for power generated in cycles per second and p is the number of pairs of poles for the generator. Usually, $f = 50$ Hz and therefore, from Equation (21.11), we derive the following expression.

$$\boxed{N = \frac{3000}{p}} \tag{21.11a}$$

The speed of the turbine runner has to be maintained constant so that the generator always runs at constant speed under all working conditions. This speed of the runner is called synchronous speed for which it is designed. It can be achieved by regulating the quantity of water flowing through the runner according to the changing load conditions on the turbine. Such an operation by which the speed of the turbine is kept constant under all working conditions is known as governing of a turbine. The quantity of water flowing through the runner is controlled by varying the area of flow at the turbine inlet. In Pelton turbines, the flow area is changed by moving the spear inside the nozzle and in reaction turbines, it is varied by rotating the guide vanes with the help of a governor.

21.9 ❒ GOVERNING OF PELTON TURBINES

The governing of a Pelton turbine is usually done automatically by means of oil pressure governor as shown in Figure 21.6. The main parts used in Pelton turbines are (i) relay cylinder (or servomotor), (ii) relay valve (or control valve), (iii) centrifugal governor which is driven by the turbine main shaft, (iv) oil sump, (v) oil pump (or gear pump) which is driven by belt connected to the turbine main shaft, (vi) oil supply pipes connecting the oil sump with relay valve and the relay valve with the servomotor, (vii) spear and spear rod and (viii) deflector.

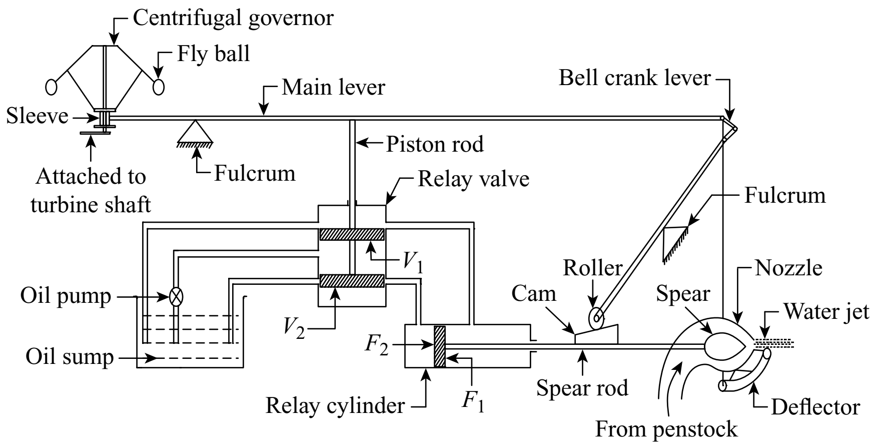


Figure 21.6 *Governing mechanism of Pelton turbine*

21.9.1 Working of the Governor

When load on the generator increases, the speed of the turbine runner decreases. As a result, the speed of the centrifugal governor connected to the turbine main shaft also decreases. Thereby, the fly balls of the governor move inwards due to the decreased centrifugal force on them. This results in the downward movement of the sleeve due to which the left hand end of the main lever gets lowered by turning about the fulcrum. This pulls the piston rod of relay valve in the upward direction. As a result, the piston of the relay valve closes the valve V_2 and opens the valve V_1. The oil pump supplies oil under pressure to the relay cylinder through the valve V_1. The oil exerts a force on the face F_1 of the piston of relay cylinder which results in the movement of the spear towards left. This increases the area of flow of water at the nozzle outlet and thereby, allows a larger quantity of water to strike the runner. Consequently, the runner speed increases till the normal speed of the turbine is restored.

When load on the generator decreases, the speed of the turbine runner increases. Due to this sudden increase, the fly balls move outward resulting in the upward movement of the sleeve. The right hand end of the main lever gets lowered which pushes the piston of the relay valve in the downward direction. This closes the valve V_1 and opens the valve V_2. The oil pump supplies oil under pressure to the relay cylinder through valve V_2. The oil exerts force on the face F_2 of the piston of relay cylinder which results in the movement of the piston towards right. This decreases the area of flow of water at the nozzle outlet and thereby, allows a smaller quantity of water to strike the runner and the normal speed of the turbine runner is thus restored.

Modern Pelton turbines are provided with double regulation, i.e., the combined spear and deflector control. Deflectors are used along with the spear to prevent water hammer issues due to rapid closing of the nozzle during sudden fall of load. Deflectors are plates simply connected to the spear rod by means of levers. A deflector deflects the jet of water so that the entire flow does not reach the bucket when the load on the turbine suddenly decreases. The motion of the fly balls is transmitted to the bell crank lever and it rotates anticlockwise. The roller on the cam is raised and the deflector is brought between the nozzle and the buckets. Thus the control of the deflector is directly linked to the governor. The deflected water goes waste into the tail race. The spear is then moved to its new position, where it reduces the rate of flow by closing the opening of nozzle gradually and therefore, it avoids the undue rise of pressure. The deflector remains engaged until the spear is adjusted to a new position of equilibrium.

21.10 ❐ VELOCITY TRIANGLES, WORK DONE AND EFFICIENCY OF THE PELTON TURBINE

The jet of water from the nozzle strikes the bucket at the splitter which splits up the jet into two parts. The parts of the jet glide over the inner surfaces and come out at the outer edge. The splitter is the inlet tip and the outer edge of the bucket is the outer tip of the bucket. Figure 21.7 shows the velocity triangles at the tips of the bucket of a Pelton turbine.

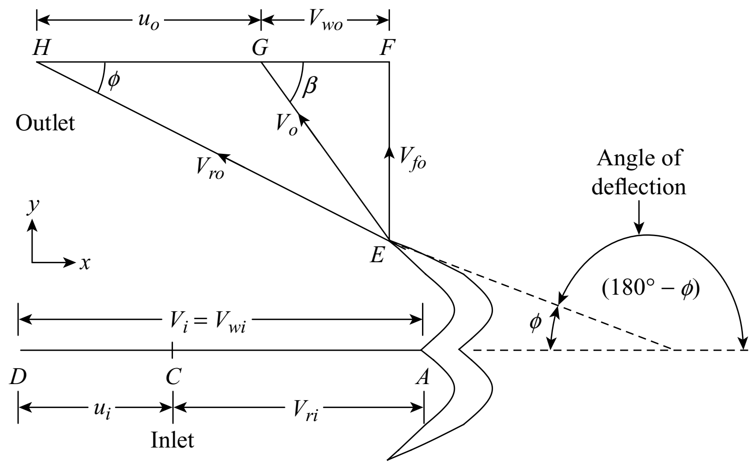


Figure 21.7 *Velocity triangles of Pelton turbine*

Let d be the diameter of the jet and $A = (\pi/4)d^2$ be the area of the jet,
D be the diameter of the wheel and N be its speed in rpm,
V_i and V_o be the absolute velocities of the jet at the inlet and outlet, respectively,
u_i and u_o be the velocities of bucket at the inlet and outlet, respectively,
V_{ri} and V_{ro} be the relative velocities of the jet and bucket at the inlet and outlet, respectively,
V_{wi} and V_{wo} be the velocities of whirl at the inlet and outlet, respectively,
V_{fi} and V_{fo} be the velocities of flow at the inlet and outlet, respectively,
α and β be the angles between the direction of jet and direction of motion of the bucket at inlet and outlet, respectively. Angle α is also known as guide blade angle.

θ and ϕ be the vane angles at inlet and outlet, respectively or the angles made by relative velocities with the direction of motion at inlet and outlet, respectively. Angle ϕ is also known as clearance angle.

Since the inlet and outlet tips of the bucket are at the same radial distance and therefore, the tangential velocity of the bucket at both the tips is same as shown below.

$$\boxed{u_i = u_o = u = \frac{\pi DN}{60}}$$

Since the velocities V_i and u_i are collinear, the velocity triangle at the inlet tip of the bucket is a straight line. Thus, $V_{wi} = V_i$, $\alpha = 0$, $\theta = 0$, and

$$V_{ri} = (V_i - u_i) = (V_{wi} - u) \tag{i}$$

Since the bucket surfaces are perfectly smooth (i.e., friction is neglected) and energy losses due to impact of jet at the splitter are neglected, $V_{ri} = V_{ro}$. However, when fiction is considered $V_{ro} < V_{ri}$ and thus, $V_{ro} = k\,V_{ri}$, here k is the bucket friction factor (or blade friction factor) and it will be slightly less than unity.

From the velocity triangle at outlet $(\beta < 90°)$, we get:

$$V_{ro} \cos\phi = (V_{wo} + u_o) = (V_{wo} + u) \tag{ii}$$

The force exerted by the jet of water in the direction of motion of bucket is given by,

$$F_x = \frac{\text{Mass}}{\text{Time}}(\text{Initial velocity in jet direction} - \text{Final velocity in jet direction})$$

$$F_x = \rho_w A V_i \times [V_{ri} - (-V_{ro}\cos\phi)] = \rho_w A V_i \times [(V_{wi} - u) - \{-(V_{wo} + u)\}]$$

$$\therefore \boxed{F_x = \rho_w A V_i \times (V_{wi} + V_{wo})} \tag{21.12}$$

Depending upon the magnitude of the peripheral speed (u), the runner may be classified as listed below.

1. Slow runner ($\beta < 90°$ and V_{wo} is negative): $F_x = \rho_w A V_i (V_{wi} + V_{wo})$
2. Medium runner ($\beta = 90°$ and $V_{wo} = 0$): $F_x = \rho_w A V_i V_{wi}$
3. Fast runner ($\beta > 90°$ and V_{wo} is positive): $F_x = \rho_w A V_i (V_{wi} - V_{wo})$

The work done by the jet on the runner per second is given by,

$$\boxed{w = F_x \times u = \rho_w A V_i (V_{wi} + V_{wo}) \times u \text{ Nm/s}} \tag{21.13}$$

Power given to the runner by the jet becomes,

$$P = \frac{w}{1000} = \frac{\rho_w A V_i (V_{wi} + V_{wo}) \times u}{1000} = \frac{\rho_w Q (V_{wi} + V_{wo}) u}{1000} \text{ kW} \tag{21.14}$$

Since weight of water striking per second $= \rho_w A V_i g$

Thus, work done per second per unit weight of water striking per second is given by the following expression.

$$w = \frac{\rho_w A V_i (V_{wi} + V_{wo}) u}{\rho_w A V_i g} = \frac{(V_{wi} + V_{wo}) u}{g} \tag{21.15}$$

Hydraulic efficiency is the ratio of work done per second by the jet on the runner to the initial kinetic energy of the jet.

$$\therefore \eta_h = \frac{\rho_w A V_i (V_{wi} + V_{wo}) u}{(1/2) \times (\rho_w A V_i) \times V_i^2} = \frac{2(V_{wi} + V_{wo}) u}{V_i^2} \tag{21.16}$$

Now $V_{wi} = V_i$, $V_{ri} = V_{ro} = (V_i - u)$, and $V_{wo} = V_{ro} \cos\phi - u = (V_i - u)\cos\phi - u$

Substituting the values of V_{wi} and V_{wo} in Equation (21.16), we get:

$$\eta_h = \frac{2[V_i + \{(V_i - u)\cos\phi - u\}]u}{V_i^2} = \frac{2[(V_i - u) + (V_i - u)\cos\phi]u}{V_i^2}$$

$$\therefore \boxed{\eta_h = \frac{2(V_i - u)(1 + \cos\phi)u}{V_i^2}} \tag{21.17}$$

When the effect of friction is considered then Equation (21.17) is derived as follows.

$$\eta_h = \frac{2(V_i - u)(1 + k\cos\phi)u}{V_i^2} \tag{21.17a}$$

For maximum efficiency, $(d\eta_h/du) = 0$.

By differentiating Equation (21.17), we get:

$$\frac{d}{du}\left[\frac{2(V_i - u)(1 + \cos\phi)u}{V_i^2}\right] = 0$$

$$\left(\frac{1 + \cos\phi}{V_i^2}\right)\frac{d}{du}(2uV_i - 2u^2) = 0$$

$$\left(\frac{1 + \cos\phi}{V_i^2}\right)(2V_i - 4u) = 0$$

Thus

$$2V_i - 4u = 0 \quad [\because (1+\cos\phi)/V_i^2 \neq 0]$$

$$\therefore \boxed{u = \frac{V_i}{2}}$$

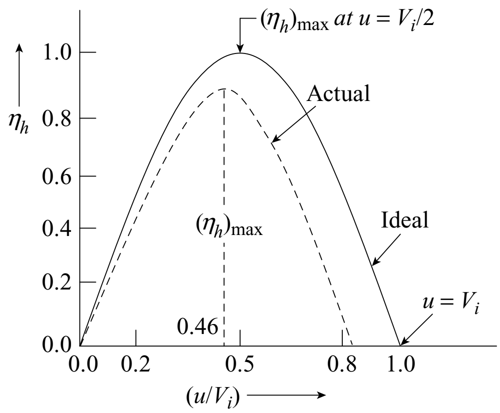


Figure 21.8 *Hydraulic efficiency versus speed ratio*

Therefore, the hydraulic efficiency is maximum when the bucket speed is half of the velocity of jet. The variation of hydraulic efficiency with speed ratio is parabolic as illustrated in Figure 21.8.

Theoretically, the maximum efficiency occurs at $(u/V_i) = 0.5$, whereas in actual practice the maximum value of efficiency occurs when $(u/V_i) = 0.46$.

By substituting $u = (V_i/2)$ in Equation (21.17), the maximum hydraulic efficiency is derived as follows.

$$(\eta_h)_{max} = \frac{2\,[V_i - (V_i/2)](1+\cos\phi)(V_i/2)}{V_i^2} = \frac{(1+\cos\phi)}{2} \qquad (21.18)$$

When the bucket friction factor (k) is considered, then we derive maximum efficiency as follows.

$$(\eta_h)_{max} = \frac{(1 + k\cos\phi)}{2} \qquad (21.18a)$$

It can be seen from Figure 21.8 that the efficiency becomes zero when $u = 0$ and $u = V_i$. In the first case, when $u = 0$, the wheel is at rest. In the second case, when $u = V_i$, the wheel runs at the highest speed. This speed of the wheel is known as runaway speed or racing speed. It is the speed at no load or when the wheel is running away from the jet with the same velocity as that of the jet. For safe design, all the rotating components must be designed for the runaway speed. The runaway speed in terms of normal working speed (N) for various types of turbines is listed below.

1. Pelton turbine: Runaway speed ranges from 1.8 to 1.9 times its normal speed.
2. Francis turbine: Runaway speed ranges from 2 to 2.2 times its normal speed.
3. Kaplan turbine: Runaway speed ranges from 2.5 to 3 times its normal speed.

The exact value of runaway speed for any turbine can be predicted from the model tests performed in the laboratory.

21.11 ❒ DESIGN ASPECTS OF THE PELTON TURBINE

21.11.1 Working Proportions of the Pelton Turbine

The following points should be considered while designing a Pelton turbine.

1. **Number of jets:** It is obtained by dividing the total water flow rate through the turbine (Q) by the water flow rate through a single jet (q). It is denoted by n and relatively expressed as shown below.

$$\boxed{n = \frac{Q}{q}}$$

Ordinarily, a Pelton turbine has one jet. However, a number of jets may be employed for providing more power with the same turbine. Such a turbine having more than one jet spaced around its runner is called multi-jet Pelton wheel. Theoretically, six jets can be used with one Pelton turbine.

2. **Jet ratio:** It is defined as the ratio of pitch diameter (D) of the Pelton wheel to the jet diameter (d). It is denoted by m and relatively expressed as shown below.

$$\boxed{m = \frac{D}{d}} \tag{21.19}$$

For maximum efficiency the jet ratio varies from 11 to 14 but in practice for most of the cases it is taken as 12.

3. **Number of buckets:** The number of buckets for a Pelton turbine should be such that no water escapes without striking the buckets. The number of buckets is usually more than 15. For determining the number of buckets, Taygun empirical formula is widely used and it is given by the following expression.

$$\boxed{Z = 15 + \frac{D}{2d} = 15 + 0.5\,m} \tag{21.20}$$

Here, Z = number of buckets, D = pitch diameter of the Pelton wheel, d = jet diameter and m = jet ratio = (D/d). The Taygun formula holds good for m varying from 6 to 35.

4. **Size of buckets:** Depth, width, and length of the bucket are expressed in terms of jet diameter as,

Depth of bucket (T) varies from 0.8 d to 1.2 d but in general it is taken as 1.2 d.

Width of bucket (B) varies from 4 d to 5 d but in general it is taken as 5 d.

Length of bucket (L) varies from 2.4 d to 3.2 d.

5. **Speed ratio:** It is defined as the ratio of peripheral (linear) velocity of buckets to the theoretical (spouting) velocity of the jet. The value of speed ratio varies from 0.43 to 0.47. It is denoted by K_u (or ϕ). Speed ratio is mathematically given by,

$$\boxed{K_u = \frac{u}{\sqrt{2gH}}} \tag{21.21}$$

Here, H = net head and $u = \pi DN/60$.

6. **Velocity of jet:** Due to friction loss, the velocity of jet at the inlet (V_i) will be slightly less than the theoretical velocity in the nozzle. The velocity of jet at the inlet is given by the following expression.

$$\boxed{V_i = C_v\sqrt{2gH}} \tag{21.22}$$

Here, C_v = coefficient of velocity (vary from 0.97 to 0.99) and H = net head.

Example 21.1 A Pelton wheel is having a mean bucket diameter of 0.9 m and is running at 900 rpm. The net head on the Pelton wheel is 600 m. If the side clearance angle is 15° and discharge is 0.09 m^3/s, then find (i) power available at the nozzle and (ii) hydraulic efficiency of the turbine. Take coefficient of velocity as 0.98.

Solution

Let $D = 0.9$ m, $N = 900$ rpm, $H = 600$ m, $\phi = 15°$, $Q = 0.09$ m^3/s and $C_v = 0.98$.

$$u = \frac{\pi DN}{60} = \frac{\pi \times 0.9 \times 900}{60} = 42.41 \text{ m/s}$$

$$V_i = C_v\sqrt{2gH} = 0.98 \times \sqrt{2 \times 9.81 \times 600} = 106.33 \text{ m/s}$$

(i) $$W.P = \frac{\rho_w gQH}{1000} = \frac{1000 \times 9.81 \times 0.09 \times 600}{1000} = \mathbf{529.74\ kW}$$

(ii) $$\eta_h = \frac{2(V_i - u)(1 + \cos\phi)u}{V_i^2}$$

$$\therefore \eta_h = \frac{2 \times (106.33 - 42.41) \times (1 + \cos 15°) \times 42.41}{106.33^2} \times 100 = \mathbf{94.27\%}$$

Example 21.2 The mean bucket speed of a Pelton turbine is 12 m/s. The rate of flow of water supplied by the jet under a head of 46 m is 850 litres per second. If the jet is deflected by the buckets at an angle of 165°, then find the power and efficiency of the turbine. Assume the coefficient of velocity as 0.985.

Solution

Refer Figure 21.9. Let $u_i = u_o = u = 12$ m/s, $H = 46$ m, $Q = 850\ l/\text{s} = 0.85\ \text{m}^3/\text{s}$, $(180° - \phi) = 165°$ and $C_v = 0.985$.

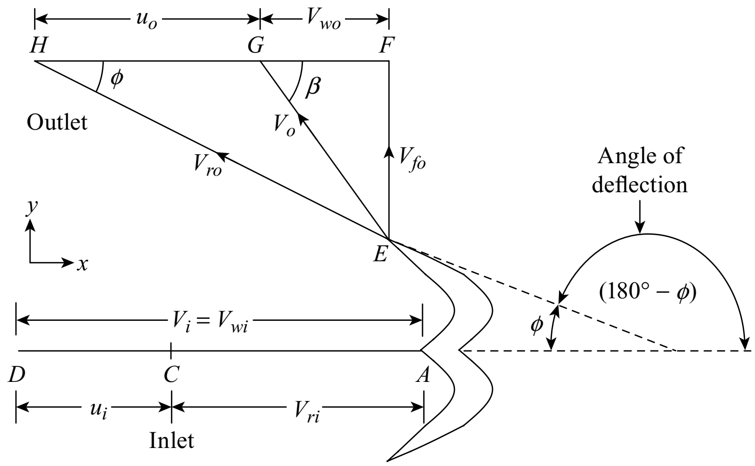


Figure 21.9

$$180° - \phi = 165° \Rightarrow \phi = 180° - 165° = 15°$$

$$V_i = C_v\sqrt{2gH} = 0.985 \times \sqrt{2 \times 9.81 \times 46} = 29.59 \text{ m/s}$$

$$V_{ri} = V_i - u = 29.59 - 12 = 17.59 \text{ m/s}$$

$$V_{wi} = V_i = 29.59 \text{ m/s}$$

$$V_{ro} = V_{ri} = 17.59 \text{ m/s}$$

$$V_{wo} = V_{ro}\cos\phi - u = 17.59\cos 15° - 12 = 4.99 \text{ m/s}$$

$$P = \frac{\rho_w Q(V_{wi} + V_{wo})u}{1000} = \frac{1000 \times 0.85 \times (29.59 + 4.99) \times 12}{1000} = \mathbf{352.716\ kW}$$

$$\eta_h = \frac{2(V_{wi} + V_{wo})u}{V_i^2} = \frac{2 \times (29.59 + 4.99) \times 12}{29.59^2} \times 100 = \mathbf{94.79\%}$$

Example 21.3 The penstock supplies water from a reservoir to the Pelton wheel with a gross head of 510 m. One third of the gross head is lost in friction in the penstock. The rate of flow through the nozzle fitted at the end of the penstock is 1.9 m^3/s. The angle of deflection of the jet is 165°. Determine the power given by the water to the runner and the hydraulic efficiency of the Pelton wheel. Take speed ratio as 0.46 and the coefficient of velocity as 0.99.

Solution

Refer Figure 21.9. Let $H_g = 510$ m, $h_f = (H_g/3)$, $Q = 1.9\ \text{m}^3/\text{s}$, $(180° - \phi) = 165°$, $K_u = 0.46$ and $C_v = 0.99$.

$$h_f = \frac{H_g}{3} = \frac{510}{3} = 170 \text{ m}$$

$$H = H_g - h_f = 510 - 170 = 340 \text{ m}$$

$$180° - \phi = 165° \Rightarrow \phi = 180° - 165° = 15°$$

$$V_i = C_v \sqrt{2gH} = 0.99 \times \sqrt{2 \times 9.81 \times 340} = 80.86 \text{ m/s}$$

$$u = u_i = u_o = K_u \sqrt{2gH} = 0.46 \times \sqrt{2 \times 9.81 \times 340} = 37.57 \text{ m/s}$$

$$V_{ri} = V_i - u = 80.86 - 37.57 = 43.29 \text{ m/s}$$

$$V_{wi} = V_i = 80.86 \text{ m/s}$$

$$V_{ro} = V_{ri} = 43.29 \text{ m/s}$$

$$V_{wo} = V_{ro} \cos\phi - u = 43.29 \cos 15° - 37.57 = 4.245 \text{ m/s}$$

$$P = \frac{\rho_w Q (V_{wi} + V_{wo}) u}{1000} = \frac{1000 \times 1.9 \times (80.86 + 4.245) \times 37.57}{1000} = \mathbf{6075.05 \text{ kW}}$$

$$\eta_h = \frac{2(V_{wi} + V_{wo})u}{V_i^2} = \frac{2 \times (80.86 + 4.245) \times 37.57}{80.86^2} \times 100 = \mathbf{97.8\%}$$

Example 21.4 A Pelton wheel is to be designed for the following specifications, such as shaft power = 11700 kW, head = 375 m, speed = 700 rpm and overall efficiency = 85%. Jet diameter is not to exceed one sixth of the wheel diameter. Determine (i) wheel diameter, (ii) diameter of the jet and (iii) number of jets required. Take $C_v = 0.99$ and $K_u = 0.46$.

Solution

Let $P = 11700$ kW, $H = 375$ m, $N = 700$ rpm, $\eta_o = 0.85$, $d = D/6$, $C_v = 0.99$ and $K_u = 0.46$.

(i) $V_i = C_v \sqrt{2gH} = 0.99 \times \sqrt{2 \times 9.81 \times 375} = 84.92 \text{ m/s}$

$$u = K_u \sqrt{2gH} = 0.46 \times \sqrt{2 \times 9.81 \times 375} = 39.46 \text{ m/s}$$

Since

$$u = \frac{\pi D N}{60}$$

$$39.46 = \frac{\pi \times D \times 700}{60}$$

$$\therefore D = \frac{39.46 \times 60}{\pi \times 700} = \mathbf{1.077 \text{ m}}$$

(ii) $d = \frac{D}{6} = \frac{1.077}{6} = \mathbf{0.1795 \text{ m}}$

(iii) $q = \frac{\pi}{4} d^2 \times V_i = \frac{\pi}{4} \times 0.1795^2 \times 84.92 = 2.149 \text{ m}^3/\text{s}$

$$Q = \frac{1000P}{\rho_w g H \eta_o} = \frac{1000 \times 11700}{1000 \times 9.81 \times 375 \times 0.85} = 3.742 \text{ m}^3/\text{s}$$

$$n = \frac{Q}{q} = \frac{3.742}{2.149} = 1.74 \approx \mathbf{2}$$

Example 21.5 A Pelton wheel develops 2.5 MW of power while operating at 260 rpm and working under a head of 250 m. The diameter of the nozzle is 15 cm and the coefficient of velocity is 0.98. The blade outlet angle is 15° and the speed ratio is 0.46. Determine (i) the turbine efficiency, (ii) wheel diameter at the pitch circle of the blades and (iii) hydraulic efficiency.

Solution

Let $P = 2.5\text{ MW} = 2.5\times10^3\text{ kW},\ N = 260\text{ rpm},\ H = 250\text{ m},\ d = 15\text{ cm} = 0.15\text{ m},\ C_v = 0.98,\ \phi = 15°$ and $K_u = 0.46$.

(i) $V_i = C_v\sqrt{2gH} = 0.98\times\sqrt{2\times9.81\times250} = 68.635\text{ m/s}$

$$Q = AV_i = \frac{\pi}{4}d^2\times V_i = \frac{\pi}{4}\times0.15^2\times68.635 = 1.213\text{ m}^3/\text{s}$$

$$\eta_o = \frac{1000P}{\rho_w gQH} = \frac{1000\times2.5\times10^3}{1000\times9.81\times1.213\times250}\times100 = \mathbf{84.04\%}$$

(ii) $K_u = \dfrac{u}{V_i} = 0.46 \Rightarrow u = 0.46V_i = 0.46\times68.635 = 31.5721\text{ m/s}$

Since

$$u = \frac{\pi DN}{60}$$

$$31.5721 = \frac{\pi\times D\times260}{60}$$

$$\therefore D = \frac{31.5721\times60}{\pi\times260} = \mathbf{2.32\text{ m}}$$

(iii) $$\eta_h = \frac{2(V_i - u)(1+\cos\phi)u}{V_i^2}$$

$$\therefore \eta_h = \frac{2\times(68.635-31.5721)\times(1+\cos15°)\times31.5721}{68.635^2}\times100 = \mathbf{97.67\%}$$

Example 21.6 Two jets strike the bucket of a Pelton wheel which is having shaft power as 16500 kW. The diameter of each jet is given as 190 mm. If the net head on the turbine is 450 m, then find the overall efficiency of the turbine. Assume coefficient of velocity as 0.985.

Solution

Let $n = 2,\ \text{S.P.} = 16500\text{ kW},\ d = 190\text{ mm} = 0.19\text{ m},\ H = 450\text{ m}$ and $C_v = 0.985$.

$$V_i = C_v\sqrt{2gH} = 0.985\times\sqrt{2\times9.81\times450} = 92.553\text{ m/s}$$

$$q = \frac{\pi}{4}d^2\times V_i = \frac{\pi}{4}\times0.19^2\times92.553 = 2.624\text{ m}^3/\text{s}$$

$$Q = n\times q = 2\times2.624 = 5.248\text{ m}^3/\text{s}$$

$$\text{W.P.} = \frac{\rho_w gQH}{1000} = \frac{1000\times9.81\times5.248\times450}{1000} = 23167.296\text{ kW}$$

$$\eta_o = \frac{\text{S.P.}}{\text{W.P.}} = \frac{16500}{23167.296}\times100 = \mathbf{71.22\%}$$

Example 21.7 A double jet Pelton wheel operates under 50 m head and develops 750 kW brake power when running at 475 rpm. Evaluate the flow rate and the diameter of the nozzle jet if the overall efficiency and coefficient of velocity are 86% and 0.98, respectively.

Solution

Let $n = 2,\ H = 50\ \text{m},\ P = 750\ \text{kW},\ N = 475\ \text{rpm},\ \eta_o = 0.86$ and $C_v = 0.98$.

$$Q = \frac{1000P}{\rho_w g H \eta_o} = \frac{1000\times 750}{1000\times 9.81\times 50\times 0.86} = \mathbf{1.778\ m^3/s}$$

$$V_i = C_v\sqrt{2gH} = 0.98\times\sqrt{2\times 9.81\times 50} = 30.694\ \text{m/s}$$

Since $$Q = n\times A\times V_i = n\times\frac{\pi}{4}d^2\times V_i$$

$$1.778 = 2\times\frac{\pi}{4}\times d^2\times 30.694$$

$$\therefore d = \sqrt{\frac{1.778\times 4}{2\times\pi\times 30.694}} = \mathbf{0.192\ m\ or\ 19.2\ cm}$$

Example 21.8 A set of data is obtained from a test on a Pelton wheel, such as head at the base of the nozzle = 40 m, discharge of the nozzle = 0.2 m^3/s, area of the jet = 7550 mm^2, power available at the shaft = 60 kW and mechanical efficiency = 94%. Calculate the power lost (i) in the nozzle, (ii) in the runner and (iii) in the mechanical friction.

Solution

Let $H = 40\ \text{m},\ Q = 0.2\ \text{m}^3/\text{s},\ A = 7550\ \text{mm}^2 = 0.00755\ \text{m}^2,\ \text{S.P.} = 60\ \text{kW}$ and $\eta_m = 0.94$.

Power at the base of nozzle is given by,

$$\text{W.P.} = \frac{\rho_w g Q H}{1000} = \frac{1000\times 9.81\times 0.2\times 40}{1000} = 78.48\ \text{kW}$$

Velocity of the nozzle is given by,

$$V_i = \frac{Q}{A} = \frac{0.2}{0.00755} = 26.49\ \text{m/s}$$

Power at the exit of nozzle is given by,

$$P_{\text{Exit}} = \text{K.E. of jet} = \frac{(1/2)\times\rho_w A V_i\times V_i^2}{1000} = \frac{\rho_w A V_i^3}{2\times 1000}\ \text{kW}$$

$$\therefore P_{\text{Exit}} = \frac{1000\times 0.00755\times 26.49^3}{2\times 1000} = 70.172\ \text{kW}$$

Power lost in the nozzle = $\text{W.P.} - P_{\text{Exit}} = 78.48 - 70.172 =$ **8.308 kW**

Power supplied to the runner = K.E. of the jet = $P_{\text{Exit}} = 70.172$ kW

$$\text{R.P.} = \frac{\text{S.P.}}{\eta_m} = \frac{60}{0.94} = 63.83\ \text{kW}$$

Power lost in the runner = $P_{\text{Exit}} - \text{R.P.} = 70.172 - 63.83 =$ **6.342 kW**

Power lost in mechanical friction = $\text{R.P.} - \text{S.P.} = 63.83 - 60 =$ **3.83 kW**

Example 21.9 A Pelton wheel of mean bucket diameter 1.2 m works under a head of 650 m. The jet deflection is 165° and its relative velocity is reduced over the buckets by 15% due to friction. If the water is to leave the bucket without any whirl, then determine (i) the rotational speed of the wheel, (ii) ratio of bucket speed to jet velocity, (iii) impulsive force and the power developed by the wheel, (iv) available power and the power input to buckets and (v) efficiency of the wheel with power input to buckets as reference input. Take coefficient of velocity as 0.97 and diameter of jet as 100 mm.

Solution

Refer Figure 21.10. Let $D = 1.2$ m, $H = 650$ m, $(180° - \phi) = 165°$, $k = 0.85$, $V_{wo} = 0$, $C_v = 0.97$ and $d = 100$ mm $= 0.1$ m.

Let N be the rotational speed of the wheel.

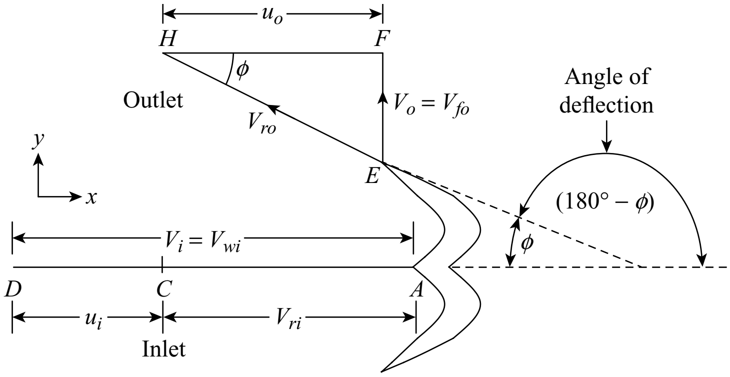


Figure 21.10

(i) $V_i = C_v\sqrt{2gH} = 0.97 \times \sqrt{2 \times 9.81 \times 650} = 109.54$ m/s

$180° - \phi = 165° \Rightarrow \phi = 180° - 165° = 15°$

From inlet velocity triangle, we get:

$$V_{ri} = V_i - u = 109.54 - u \qquad [\because u_i = u_o = u]$$

$$V_{wi} = V_i = 109.54 \text{ m/s}$$

From outlet velocity triangle, we get:

$$V_{ro} = kV_{ri} = 0.85(109.54 - u) \qquad \text{(i)}$$

$$V_{ro}\cos\phi = u \qquad \text{(ii)}$$

Thus

$$0.85(109.54 - u)\cos 15° = u \qquad \text{[Substitute (i) in (ii)]}$$

$$89.9364 - 0.821u = u$$

$$\therefore u = \frac{89.9364}{1.821} = 49.39 \text{ m/s}$$

Since

$$u = \frac{\pi DN}{60}$$

$$49.39 = \frac{\pi \times 1.2 \times N}{60}$$

$$\therefore N = \frac{49.39 \times 60}{\pi \times 1.2} = \mathbf{786.07\ rpm}$$

(ii) $\frac{u}{V_i} = \frac{49.39}{109.54} = \mathbf{0.451}$

(iii) $A = \frac{\pi}{4}d^2 = \frac{\pi}{4} \times 0.1^2 = 0.007854 \text{ m}^2$

$$F_x = \rho_w A V_i V_{wi} = 1000 \times 0.007854 \times 109.54 \times 109.54 = \mathbf{94240.24 \text{ N}}$$

$$P = \frac{F_x u}{1000} = \frac{94240.24 \times 49.39}{1000} = \mathbf{4654.525 \text{ kW}}$$

(iv) $\text{W.P.} = \frac{\rho_w gQH}{1000} = \frac{\rho_w g(AV_i)H}{1000} \text{ kW}$

$$\therefore \text{W.P.} = \frac{1000 \times 9.81 \times 0.007854 \times 109.54 \times 650}{1000} = \mathbf{5485.876 \text{ kW}}$$

Power inputs to buckets is given by,

$$P_i = \frac{(1/2) \times \rho_w A V_i \times V_i^2}{1000} = \frac{\rho_w A V_i^3}{2 \times 1000} \text{ kW}$$

$$\therefore P_i = \frac{1000 \times 0.007854 \times 109.54^3}{2 \times 1000} = \mathbf{5161.538 \text{ kW}}$$

(v) $\eta = \frac{P}{P_i} = \frac{4654.525}{5161.538} \times 100 = \mathbf{90.18\%}$

Example 21.10 A Pelton wheel having semicircular buckets functions under a head of 150 m and consumes 50 litres per second of water. If 60 cm diameter wheel turns 600 rpm, then calculate (i) the power available at the nozzle and (ii) hydraulic efficiency of the wheel. Take coefficient of velocity as 0.99.

Solution

Let $\phi = 0$ (semicircular buckets), $H = 150$ m, $Q = 50$ l/s $= 0.05 \text{ m}^3/\text{s}$, $D = 60$ cm $= 0.6$ m, $N = 600$ rpm and $C_v = 0.99$.

$$u = \frac{\pi DN}{60} = \frac{\pi \times 0.6 \times 600}{60} = 18.85 \text{ m/s}$$

(i) $\text{W.P.} = \frac{\rho_w gQH}{1000} = \frac{1000 \times 9.81 \times 0.05 \times 150}{1000} = \mathbf{73.575 \text{ kW}}$

(ii) $V_i = C_v\sqrt{2gH} = 0.99 \times \sqrt{2 \times 9.81 \times 150} = 53.71 \text{ m/s}$

Since $$\eta_h = \frac{2(V_i - u)(1 + \cos\phi)u}{V_1^2}$$

$$\therefore \eta_h = \frac{2 \times (53.71 - 18.85) \times (1 + \cos 0°) \times 18.85}{53.71^2} \times 100 = \mathbf{91.11\%}$$

Example 21.11 The jet water of diameter 150 mm strikes the buckets of the wheel which is working under a gross head of 402 m. The water is supplied from a lake to the turbine through a penstock of diameter 0.95 m and length 4000 m. The jet deflection is 165° and its relative velocity is reduced over the buckets by 15% due to friction inside surface of the bucket and water. Determine (i) the runner power, (ii) shaft power, (iii) hydraulic efficiency and (iv) overall efficiency. Assume coefficient of friction for the penstock as 0.009, velocity of buckets as 0.45 times the jet velocity at inlet and mechanical efficiency as 86%.

Solution

Refer Figure 21.11. Let $d = 150\text{ mm} = 0.15\text{ m}$, $H_g = 402\text{ m}$, $D_1 = 0.95\text{ m}$, $L = 4000\text{ m}$, $(180° - \phi) = 165°$, $k = 0.85$, $f = 0.009$, $u = 0.45V_i$ and $\eta_m = 0.86$.

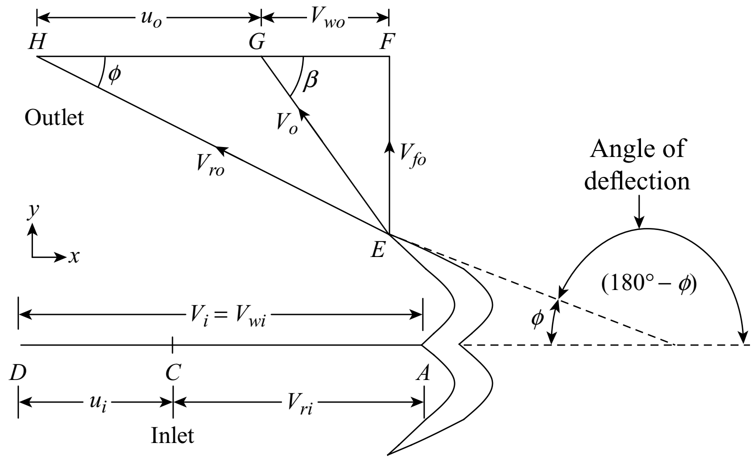


Figure 21.11

Let V_1 be the velocity of water in the penstock of diameter D_1 and V_i be the velocity of water jet of diameter d.

$$A_1V_1 = AV_i \Rightarrow \frac{\pi}{4}D_1^2V_1 = \frac{\pi}{4}d^2V_i \quad \text{[Continuity equation]}$$

$$\therefore V_1 = \left(\frac{d}{D_1}\right)^2 V_i = \left(\frac{0.15}{0.95}\right)^2 V_i = 0.024931\, V_i$$

Applying Bernoulli's equation to free surface of water in the lake and outlet of the nozzle and neglecting the head lost in nozzle, we derive the following relation.

$$\text{Head at lake} = \text{Kinetic head of water jet} + \text{Head lost due to friction in penstock}$$

$$H_g = \frac{V_i^2}{2g} + \frac{4fLV_1^2}{2gD_1}$$

Thus

$$402 = \frac{V_i^2}{2 \times 9.81} + \frac{4 \times 0.009 \times 4000 \times (0.024931\, V_i)^2}{2 \times 9.81 \times 0.95}$$

$$402 = 0.05577V_i^2 \Rightarrow V_i = \sqrt{\frac{402}{0.05577}} = 84.9\text{ m/s}$$

$$u = 0.45V_i = 0.45 \times 84.9 = 38.205\text{ m/s}$$

From inlet velocity triangle, we get:

$$V_{ri} = V_i - u = 84.9 - 38.205 = 46.695 \text{ m/s} \qquad [\because u_i = u_o = u]$$

$$V_{wi} = V_i = 84.9 \text{ m/s}$$

$$180° - \phi = 165° \Rightarrow \phi = 15°$$

From outlet velocity triangle, we get:

$$V_{ro} = kV_{ri} = 0.85 \times 46.695 = 39.691 \text{ m/s}$$

$$V_{wo} = V_{ro} \cos\phi - u = 39.691 \cos 15° - 38.205 = 0.1336 \text{ m/s}$$

$$Q = \frac{\pi}{4} d^2 \times V_i = \frac{\pi}{4} \times 0.15^2 \times 84.9 = 1.5 \text{ m}^3/\text{s}$$

(i) $\text{R.P.} = \dfrac{\rho_w Q(V_{wi} + V_{wo})u}{1000}$ kW

$$\therefore \text{R.P.} = \frac{1000 \times 1.5 \times (84.9 + 0.1336) \times 38.205}{1000} = \mathbf{4873.063 \text{ kW}}$$

(ii) $\text{S.P.} = \eta_m \times \text{R.P.} = 0.86 \times 4873.063 = \mathbf{4190.8342 \text{ kW}}$

(iii) $\eta_h = \dfrac{2(V_{wi} + V_{wo})u}{V_i^2} = \dfrac{2 \times (84.9 + 0.1336) \times 38.205}{84.9^2} \times 100 = \mathbf{90.14\%}$

(iv) $\eta_o = \eta_h \times \eta_m = 0.9014 \times 0.86 \times 100 = \mathbf{77.5204\%}$

Example 21.12 A double overhang Pelton wheel unit is coupled to a generator producing 30000 kW under an effective head of 300 m at the base of the nozzle. Find the size of the jet, mean diameter of runner, synchronous speed and specific speed of each wheel. Assume generator efficiency as 93%, overall efficiency of turbine as 85%, coefficient of nozzle velocity as 0.97, speed ratio as 0.46, frequency of generator as 50 cycles per second, pair of poles as 16 and the jet ratio as 12.

Solution

Let $n = 2,\ P_1 = 30000 \text{ kW},\ H = 300 \text{ m},\ \eta_g = 0.93,\ \eta_o = 0.85,\ C_v = 0.97,\ K_u = 0.46,\ f = 50\ Hz,\ p = 16$ and $m = 12$.

Power supplied to generator by the runners is given by,

$$P_t = \frac{P_1}{\eta_g} = \frac{30000}{0.93} = 32258.064 \text{ kW}$$

Power output by single runner is given by,

$$P = \frac{P_t}{n} = \frac{32258.064}{2} = 16129.032 \text{ kW}$$

$$Q = \frac{1000P}{\eta_o \rho_w gH} = \frac{1000 \times 16129.032}{0.85 \times 1000 \times 9.81 \times 300} = 6.448 \text{ m}^3/\text{s}$$

$$V_i = C_v \sqrt{2gH} = 0.97 \times \sqrt{2 \times 9.81 \times 300} = 74.42 \text{ m/s}$$

Since $$Q = AV_i = \frac{\pi}{4} d^2 \times V_i$$

$$6.448 = \frac{\pi}{4} d^2 \times 74.42$$

$$\therefore d = \sqrt{\frac{6.448 \times 4}{\pi \times 74.42}} = \mathbf{0.3321\ m}$$

$$D = m \times d = 12 \times 0.3321 = 3.9852 \text{ m} \quad [\because m = D/d]$$

$$u = K_u \sqrt{2gH} = 0.46 \times \sqrt{2 \times 9.81 \times 300} = 35.29 \text{ m/s}$$

Since
$$u = \frac{\pi DN}{60}$$

$$35.29 = \frac{\pi \times 3.9852 \times N}{60}$$

$$\therefore N = \frac{35.29 \times 60}{\pi \times 3.9852} = \mathbf{169.12\ rpm}$$

Speed of the generator is given by,

$$N = \frac{60f}{p} = \frac{60 \times 50}{16} = 187.5 \text{ rpm}$$

Since
$$u = \frac{\pi DN}{60} = 35.29 \text{ m/s}$$

Therefore, it remains constant and so, the revised diameter of the wheel is derived as follows.

$$D = \frac{35.29 \times 60}{\pi N} = \frac{35.29 \times 60}{\pi \times 187.5} = 3.595 \text{ m}$$

Specific speed of the wheel is given by,

$$N_s = \frac{N\sqrt{P}}{H^{5/4}} = \frac{187.5 \times \sqrt{16129.032}}{300^{5/4}} = \mathbf{19.072}$$

Example 21.13 The water available for a Pelton wheel is 4.4 m^3/s and the total head from the reservoir to the nozzle is 250 m. The turbine has two runners with two jets per runner. All the four jets have the same diameters. The pipeline is 3 km long. The efficiency of power transmission through the pipeline and the nozzle is 91% and the efficiency of each runner is 90%. The velocity coefficient of each nozzle is 0.975 and friction factor for the pipe is 0.0045. Determine (i) the power developed by the turbine, (ii) diameter of the jet and (iii) diameter of the pipeline.

Solution

Let $Q = 4.4 \text{ m}^3/\text{s},\ H_g = 250 \text{ m},\ n = 2 \times 2 = 4,\ L = 3 \text{ km} = 3000 \text{ m},\ \eta_t = 0.91,\ \eta = 0.9,\ C_v = 0.975$ and $f_f = 0.0045$.

Let P be the power developed, d be the diameter of the jet and D_1 be the diameter of the pipeline.

(i) $\eta_t = \dfrac{H_g - h_f}{H_g}$

$$0.91 = \frac{250 - h_f}{250} \Rightarrow h_f = 250 - 0.91 \times 250 = 22.5 \text{ m}$$

$$H = H_g - h_f = 250 - 22.5 = 227.5 \text{ m}$$

$$V_i = C_v \sqrt{2gH} = 0.975 \times \sqrt{2 \times 9.81 \times 227.5} = 65.14 \text{ m/s}$$

$$\text{W.P.} = \frac{(1/2) \times \rho_w Q \times V_i^2}{1000} = \frac{1000 \times 4.4 \times 65.14^2}{2 \times 1000} = 9335.083 \text{ kW}$$

Power developed by the wheel is given by,

$$P = \eta \times \text{W.P.} = 0.9 \times 9335.083 = \mathbf{8401.5747\ kW}$$

(ii) Discharge per jet is given by,

$$q = \frac{Q}{n} = \frac{4.4}{4} = 1.1\ \text{m}^3/\text{s}$$

$$q = \frac{\pi}{4} d^2 \times V_i$$

$$1.1 = \frac{\pi}{4} d^2 \times 65.14$$

$$\therefore d = \sqrt{\frac{1.1 \times 4}{\pi \times 65.14}} = \mathbf{0.1466\ m}$$

(iii) $$h_f = \frac{f_f L V_1^2}{2gD_1} = \frac{f_f L}{2gD_1}\left[\frac{Q}{A_1}\right]^2 = \frac{f_f L}{2gD_1}\left[\frac{Q}{(\pi/4)D_1^2}\right]^2 = \frac{8 f_f L Q^2}{g\pi^2 D_1^5}$$

$$\therefore D_1 = \left[\frac{8 f_f L Q^2}{g\pi^2 h_f}\right]^{1/5} = \left[\frac{8 \times 0.0045 \times 3000 \times 4.4^2}{9.81 \times \pi^2 \times 22.5}\right]^{1/5} = \mathbf{0.992\ m}$$

Example 21.14 A single jet Pelton wheel runs at 300 rpm under a head of 510 m. The jet diameter is 200 mm. The jet deflects inside the bucket by 165° and its relative velocity is reduced over the buckets by 15% due to friction. The mechanical losses are 3% of the power supplied. The values for the coefficient of velocity and speed ratio are given as 0.98 and 0.46, respectively. Determine (i) the water power, (ii) resultant force on the bucket, (iii) brake power and (iii) overall efficiency.

Solution

Refer Figure 21.12. Let $N = 300$ rpm, $H = 510$ m, $d = 200$ mm $= 0.2$ m, $(180° - \phi) = 165°$, $V_{ro} = 0.85 V_{ri}$, $\eta_m = 0.97$, $C_v = 0.98$ and $K_u = 0.46$.

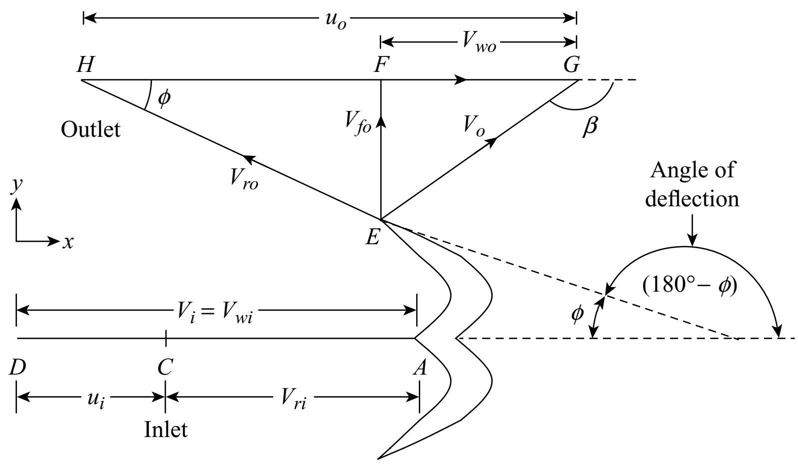


Figure 21.12

(i) $V_i = C_v\sqrt{2gH} = 0.98\times\sqrt{2\times 9.81\times 510} = 98.03$ m/s

$$Q = \frac{\pi}{4}d^2V_i = \frac{\pi}{4}\times 0.2^2\times 98.03 = 3.08 \text{ m}^3/\text{s}$$

$$\text{W.P.} = \frac{\rho_w gQH}{1000} = \frac{1000\times 9.81\times 3.08\times 510}{1000} = \mathbf{15409.548\ kW}$$

(ii) $u = K_u\sqrt{2gH} = 0.46\times\sqrt{2\times 9.81\times 510} = 46.01$ m/s

From inlet velocity triangle, we get:

$$V_{ri} = V_i - u = 98.03 - 46.01 = 52.02 \text{ m/s} \quad [\because u_i = u_o = u]$$

$$V_{wi} = V_i = 98.03 \text{ m/s}$$

$$180^\circ - \phi = 165^\circ \Rightarrow \phi = 15^\circ$$

$$V_{ro} = 0.85V_{ri} = 0.85\times 52.02 = 44.217 \text{ m/s}$$

$$V_{ro}\cos\phi = 44.217\cos 15^\circ = 42.71 \text{ m/s}$$

Since $(V_{ro}\cos\phi) < u$

$\therefore \beta > 90^\circ$ (i.e., fast runner) as shown in outlet velocity triangle in Figure 21.12.

Thus $V_{wo} = u - V_{ro}\cos\phi = 46.01 - 42.71 = 3.3$ m/s

$$F_x = \rho_w Q(V_{wi} - V_{wo}) = 1000\times 3.08\times(98.03 - 3.3) = \mathbf{291768.4\ N}$$

(iii) $\text{S.P.} = \dfrac{\eta_m F_x u}{1000} = \dfrac{0.97\times 291768.4\times 46.01}{1000} = \mathbf{13021.5362\ kW}$

(iv) $\eta_o = \dfrac{\text{S.P.}}{\text{W.P.}} = \dfrac{13021.5362}{15409.548}\times 100 = \mathbf{84.5\%}$

Example 21.15 Design a Pelton wheel working under a head of 70 m. It develops 100 kW shaft power when it runs at 220 rpm. Assume the speed ratio as 0.45, coefficient of velocity as 0.98 and overall efficiency as 85%.

Solution

Let $H = 70$ m, $S.P. = 100$ kW, $N = 220$ rpm, $K_u = 0.45$, $C_v = 0.98$ and $\eta_o = 0.85$.

(i) $V_i = C_v\sqrt{2gH} = 0.98\times\sqrt{2\times 9.81\times 70} = 36.318$ m/s

$$u = K_u\sqrt{2gH} = 0.45\times\sqrt{2\times 9.81\times 70} = 16.677 \text{ m/s}$$

Since $$u = \frac{\pi DN}{60}$$

$$16.677 = \frac{\pi\times D\times 220}{60}$$

$$\therefore D = \frac{16.677\times 60}{\pi\times 220} = \mathbf{1.448\ m}$$

(ii) $Q = \dfrac{1000P}{\rho_w g H \eta_o} = \dfrac{1000 \times 100}{1000 \times 9.81 \times 70 \times 0.85} = 0.17132 \text{ m}^3/\text{s}$

$$Q = AV_i = \frac{\pi}{4} d^2 \times V_i$$

$$0.17132 = \frac{\pi}{4} d^2 \times 36.318$$

$$\therefore d = \sqrt{\frac{0.17132 \times 4}{\pi \times 36.318}} = \mathbf{0.0775 \text{ m or } 77.5 \text{ mm}}$$

$$m = \frac{D}{d} = \frac{1.448}{0.0775} = 18.684$$

(iii) $B = 5d = 5 \times 77.5 = \mathbf{387.5 \text{ mm}}$

$T = 1.2d = 1.2 \times 77.5 = \mathbf{93 \text{ mm}}$

$L = 3.2d = 3.2 \times 77.5 = \mathbf{248 \text{ mm}}$

$Z = 15 + 0.5\text{m} = 15 + 0.5 \times 18.684 = 24.342 \approx \mathbf{25}$

Example 21.16 A Pelton wheel produces 10 MW under a head of 360 m when running at a speed of 450 rpm. If the diameter of the jet is not to exceed one tenth of the wheel diameter, then determine (i) the diameter of wheel, (ii) diameter of jet, (iii) quantity of flow, (iv) number of jets and (v) size of buckets. Assume speed ratio as 0.46, coefficient of velocity as 0.98 and overall efficiency as 88%.

Solution

Let $\text{S.P.} = 10 \text{ MW} = 10 \times 10^3 \text{ kW}$, $H = 360 \text{ m}$, $N = 450 \text{ rpm}$, $d = D/10$, $K_u = 0.46$, $C_v = 0.98$ and $\eta_o = 0.88$.

(i) $V_i = C_v \sqrt{2gH} = 0.98 \times \sqrt{2 \times 9.81 \times 360} = 82.362 \text{ m/s}$

$u = K_u \sqrt{2gH} = 0.46 \times \sqrt{2 \times 9.81 \times 360} = 38.66 \text{ m/s}$

Since $$u = \frac{\pi D N}{60}$$

$$38.66 = \frac{\pi \times D \times 450}{60}$$

$$\therefore D = \frac{38.66 \times 60}{\pi \times 450} = \mathbf{1.641 \text{ m}}$$

(ii) $d = \dfrac{D}{10} = \dfrac{1.641}{10} = \mathbf{0.1641 \text{ m}}$

(iii) $Q = \dfrac{1000P}{\rho_w g H \eta_o} = \dfrac{1000 \times 10 \times 10^3}{1000 \times 9.81 \times 360 \times 0.88} = \mathbf{3.218 \text{ m}^3/\text{s}}$

(iv) $Q = n \times A \times V_i = n \times \dfrac{\pi}{4} d^2 \times V_i$

Thus $$3.218 = n \times \frac{\pi}{4} \times 0.1641^2 \times 82.362$$

$$\therefore n = \frac{3.218 \times 4}{\pi \times 0.1641^2 \times 82.362} = 1.8474 \approx \mathbf{2}$$

Therefore, the revised jet diameter is obtained as follows.

$$3.218 = 2 \times \frac{\pi}{4} d^2 \times 82.362$$

$$\therefore d = \sqrt{\frac{3.218 \times 4}{2 \times \pi \times 82.362}} = 0.1577 \text{ m or } 157.7 \text{ mm}$$

(v) Width (B), depth (T) and radial length (L) of the buckets are derived as follows.

$$B = 5d = 5 \times 157.7 = \mathbf{788.5 \text{ mm}}$$

$$T = 1.2d = 1.2 \times 157.7 = \mathbf{189.24 \text{ mm}}$$

$$L = 3.2d = 3.2 \times 157.7 = \mathbf{504.64 \text{ mm}}$$

Example 21.17 A pipeline 1250 m long supplies water to 3 single jet Pelton wheels. The head above the nozzle is 373 m. The head lost due to friction in the pipeline is 13 m. The value of friction factor f_f for the pipeline is 0.02 and the velocity coefficient for the nozzle is 0.98. The specific speed of each turbine is 16.5 and the operating speed of each turbine is 565 rpm. If the turbine efficiency based on the head at the nozzle is 0.86, then determine (i) the total power developed, (ii) volume of water used per second, (iii) diameter of each nozzle and (iv) diameter of the pipeline.

Solution

Let $L = 1250$ m, $n = 3$, $H_g = 373$ m, $h_f = 13$ m, $f_f = 0.02$, $C_v = 0.98$, $N_s = 16.5$, $N = 565$ rpm and $\eta_o = 0.86$.

Let D_1 be the diameter of the pipe and d be the diameter of the jet.

(i) $H = H_g - h_f = 373 - 13 = 360$ m

Since $$N_s = \frac{N\sqrt{P}}{H^{5/4}}$$

$$P = \left[\frac{N_s H^{5/4}}{N}\right]^2 = \left[\frac{16.5 \times 360^{5/4}}{565}\right]^2 = 2097.14 \text{ kW}$$

Therefore, the total power developed by three turbines is derived as shown below.

$$P_t = n \times P = 3 \times 2097.14 = \mathbf{6291.42 \text{ kW}}$$

(ii) $$q = \frac{1000P}{\rho_w g H \eta_o} = \frac{1000 \times 2097.14}{1000 \times 9.81 \times 360 \times 0.86} = 0.6905 \text{ m}^3/\text{s}$$

Therefore, the total volume of water used by three turbines is derived below.

$$Q = n \times q = 3 \times 0.6905 = \mathbf{2.0715 \text{ m}^3/\text{s}}$$

(iii) $V_i = C_v\sqrt{2gH} = 0.98\times\sqrt{2\times 9.81\times 360} = 82.362$ m/s

Since
$$q = A\times V_i = \frac{\pi}{4}d^2\times V_i$$
$$0.6905 = \frac{\pi}{4}d^2\times 82.362$$
$$\therefore d = \sqrt{\frac{0.6905\times 4}{\pi\times 82.362}} = \mathbf{0.1033\ m\ or\ 103.3\ mm}$$

(iv) $$h_f = \frac{f_f L V_1^2}{2gD_1} = \frac{f_f L}{2gD}\left[\frac{Q}{A_1}\right]^2 = \frac{f_f L}{2gD_1}\left[\frac{Q}{(\pi/4)D_1^2}\right]^2 = \frac{8 f_f L Q^2}{g\pi^2 D_1^5}$$

$$\therefore D_1 = \left[\frac{8 f_f L Q^2}{g\pi^2 h_f}\right]^{1/5} = \left[\frac{8\times 0.02\times 1250\times 2.0715^2}{9.81\times\pi^2\times 13}\right]^{1/5} = \mathbf{0.9263\ m}$$

Summary

1. Hydraulic turbines convert hydraulic energy into mechanical energy.
2. Impulse turbines have only kinetic energy at its inlet, whereas reaction turbines have both kinetic and pressure energy at its inlet.
3. Gross head (H_g) is the difference between the head race and tail race levels.
4. Net head (H) is the head available at the inlet of the turbine. If h_f is the head loss due to friction in penstock, then net head is given by $H = (H_g - h_f)$.
5. The head loss due to friction is $h_f = (4fLV^2)/(2gD)$, here f = coefficient of friction, L = length of penstock, V = velocity of flow in penstock, D = diameter of penstock and g = gravitational acceleration.
6. Hydraulic efficiency (η_h) is the ratio of runner power to water power.
7. Mechanical efficiency (η_m) is the ratio of shaft power to runner power.
8. Overall efficiency (η_o) is the ratio of shaft power to water power. Also $\eta_o = \eta_m\eta_h$, when the volumetric efficiency (η_v) is also considered, $\eta_o = \eta_m\eta_h\eta_v$.
9. The Pelton turbine is a tangential flow impulse turbine. It is used for high heads generally heads in excess of 250 m.
10. In the governing of a turbine, its speed is kept constant under all working conditions with the help of an oil pressure governor.
11. The velocity of jet at inlet is $V_i = C_v\sqrt{2gH}$.
12. Discharge through the Pelton turbine is $Q = (\pi/4)d^2V_i$, here d is the jet diameter.
13. Work done per second per unit weight per second: $w = [(V_{wi} + V_{wo})u]/g$
14. Force exerted by water jet in direction of motion of bucket: $F_x = \rho_w A V_i (V_{wi} \pm V_{wo})$
15. Hydraulic efficiency of the Pelton wheel: $\eta_h = [2(V_i - u)(1+\cos\phi)u]/V_i^2$
16. Condition for maximum efficiency of the Pelton wheel: $u = V_i/2$.
17. **Maximum efficiency:** $(\eta_h)_{\max} = (1+\cos\phi)/2$, here ϕ is the clearance angle.
18. **Jet ratio:** $m = D/d$, where D is the pitch diameter and d is the jet diameter.
19. Number of buckets (Z) for a Pelton turbine: $Z = 15 + D/(2d) = 15 + 0.5\,m$
20. **Size of the buckets:** Depth of bucket = 1.2 d, width of bucket = 5 d and length of bucket = 3.2 d.

21. Runaway speed is the maximum speed attained by the runner under maximum head at full gate opening, when the external load suddenly becomes zero.

22. Speed ratio: $K_u = u/\sqrt{2gH}$

23. Specific speed is the speed of a geometrically similar turbine working under unit head and developing unit power. It is given by $N_s = (N\sqrt{P})/H^{5/4}$, here N is the normal working speed in rpm, P is the power output of the turbine in kW and H is the net head in metres.

Multiple-choice Questions

1. Pure reaction turbine is
(a) Kaplan turbine.
(b) Lawn sprinkler.
(c) Francis turbine.
(d) None of the above.

2. Pelton turbine ideally suits
(a) Low head and low discharge.
(b) High head and high discharge.
(c) High head and low discharge.
(d) Medium head and medium discharge.

3. For maximum efficiency of a Pelton turbine, the blade velocity is
(a) Half of jet velocity.
(b) Double the jet velocity.
(c) Equal to jet velocity.
(d) None of the above.

4. The number of buckets (Z) in a Pelton wheel in terms of jet ratio (m) is equal to
(a) 15 + 0.5 m.
(b) 15 – 0.5 m.
(c) 0.5 – 15 m.
(d) None of the above.

5. The coefficient of velocity in the nozzle for a Pelton wheel lies in the range of
(a) 0.5 to 0.7.
(b) 0.75 to 0.85.
(c) 0.86 to 0.95.
(d) 0.97 to 0.99.

6. In general, the sequence of dimensions for the depth, width and length of a bucket respectively in terms of jet diameter d are
(a) 1.2 d, 5 d and 3.2 d.
(b) 3.2 d, 5 d and 1.2 d.
(c) 5 d, 1.2 d and 3.2 d.
(d) None of the above.

7. The Taygun formula for number of buckets in Pelton turbine holds good for the values of jet ratio varying from
(a) 3 to 10.
(b) 0 to 3.
(c) 6 to 35.
(d) >35.

8. The hydraulic efficiency is always
(a) Greater than mechanical efficiency.
(b) Lesser than mechanical efficiency.
(c) Lesser than overall efficiency.
(d) None of the above.

9. The forces on the buckets of Pelton turbines are determined by
(a) Force balance.
(b) Energy equation.
(c) Continuity equation.
(d) Momentum equation.

10. In Pelton turbines, as flow takes place, there is change in
(a) Velocity only.
(b) Pressure only.
(c) Both velocity and pressure.
(d) None of the above.

11. In Pelton turbines, the specific speed of slow, normal and fast runners in sequence are
(a) 28–35, 8.5–20, 20–28
(b) 20–28, 28–35, 8.5–20
(c) 8.5–20, 20–28, 28–35
(d) None of the above

12. The efficiency of an impulse turbine
(a) May approach 100% for hemispherical bucket vanes.
(b) May exceed 50% with inclined flat plate vanes.
(c) May never be beyond 50% even theoretically.
(d) May approach 100% for frictionless vanes.

Review Questions

1. What do you mean by hydraulic turbines? How will you classify these turbines?

2. Give a general layout of a hydroelectric power plant. Also define the terms gross head and net head.

3. Give comparisons between impulse turbine and reaction turbine.

4. Define the hydraulic efficiency, mechanical efficiency, volumetric efficiency and overall efficiency of a turbine.

5. Explain the construction and working of a Pelton turbine with a neat diagram.
6. Define governing of turbines. Explain the governing mechanisms of impulse hydraulic turbines. Clearly state the function of a deflector in a Pelton turbine.
7. Explain the characteristic features of the cup of a Pelton turbine. What are the limitations in keeping the deflection angle of the cup as 180°?
8. Prove that the work done per second per unit weight of water in a Pelton wheel is given by $w = (1/g)\,[V_{wi} + V_{wo}]\,u$.
9. Draw the inlet and outlet velocity triangles for a Pelton turbine. Also derive an expression for maximum efficiency of the Pelton wheel giving the relationship between the jet speed and bucket speed.

Problems

1. A Pelton wheel is to be designed for the following specifications, such as shaft power = 6000 kW, head = 300 m, speed = 550 rpm, overall efficiency = 85%. Jet diameter is not to exceed one tenth of the wheel. Determine the number of jets, diameter of the jet, wheel diameter and quantity of water required. Take $C_v = 0.98$ and $K_u = 0.46$.
[**Ans.** 3, 0.1225 m, 1.225 m, 2.398 m^3/s]
2. The following data were obtained from a test on a Pelton wheel, such as head at the base of the nozzle = 70 m, discharge of the nozzle = 0.25 m^3/s, diameter of the jet = 95 mm, power available at the shaft = 140 kW, power absorbed in mechanical resistance = 2.5 kW. Calculate (i) the power lost in nozzle and (ii) power lost due to hydraulic resistance in the runner.
[**Ans.** 16.67 kW, 14.97 kW]
3. The nozzle diameter of a Pelton wheel is 200 mm and it works under a head of 220 m. The wheel operates at 250 rpm and develops 3.8 MW. The blade outlet angle is 15° and the speed ratio is 0.45. If the coefficient of velocity is 0.99, then calculate the overall efficiency of the turbine.
[**Ans.** 86.18%]
4. A Pelton wheel is having a mean bucket diameter of 1.2 m and is running at 1200 rpm. The net head on the Pelton wheel is 850 m. If the side clearance angle is 15° and discharge is 0.15 m^3/s, then find the power available at the nozzle and hydraulic efficiency of the turbine. Take coefficient of velocity as unity.
[**Ans.** 1250.77 kW, 95.56%]
5. A Pelton wheel is to be designed for the following specifications, such as shaft power =10,000 kW, head = 350 m, speed = 700 rpm, overall efficiency = 88%. Jet diameter is not to exceed one sixth of the wheel. Determine the wheel diameter, number of jets required and diameter of the jet. Take C_v = 0.985 and $K_u = 0.45$.
[**Ans.** 1.02 m, 2, 0.17 m]
6. Two jets strike the bucket of a Pelton wheel which is having shaft power as 6500 kW. The diameter of each jet is given as 100 mm. If the net head on the turbine is 500 m, then find the overall efficiency of the turbine. Assume coefficient of velocity as unity.
[**Ans.** 85.16%]
7. A double jet Pelton wheel operates under 75 m head and develops 1000 kW brake power when running at 525 rpm. Evaluate the flow rate and the diameter of the nozzle jet if the overall efficiency and coefficient of velocity are 85% and 0.98, respectively.
[**Ans.** 1.599 m^3/s, 0.1645 m]
8. A jet of water coming out of 0.2 m diameter nozzle strikes the buckets of Pelton wheel and jet is deflected through an angle 165°. Determine the force exerted by the jet of water in the direction of motion of bucket and the power developed when head = 500 m, coefficient of velocity = 0.975, speed ratio = 0.46 and reduction in relative velocity at exit of the bucket = 15%.
[**Ans.** 281813.03 N, 12839.402 kW]
9. A Pelton wheel produces 6 MW under a head of 300 m when running at a speed of 550 rpm. If the jet ratio is 10, then determine (i) the diameter of wheel, (ii) diameter of jet, (iii) quantity of water required and (iv) number of jets. Assume speed ratio as 0.46, coefficient of velocity as 0.98 and overall efficiency as 85%.
[**Ans.** 1.225 m, 0.1225 m, 2.398 m^3/s, 3, 0.1162 m]
10. A double overhang Pelton wheel unit is coupled to a generator producing 42 MW under an effective head of 400 m. Find the size of the jet, mean diameter of runner, synchronous speed and specific speed of each wheel. Assume generator efficiency and mechanical efficiency both as 96%, hydraulic efficiency of turbine as 90%, coefficient of nozzle velocity as 0.99, speed ratio as 0.46, frequency of generator as 50 cycles per second and jet ratio as 12.
[**Ans.** 0.306 m, 3.672 m, 200 rpm, 3.891 m, 16.53]
11. A Pelton wheel develops 4.2 MW at 405 rpm when operates under a head of 355 m. There are two equal jets and the bucket deflection angle is 165°. Determine the cross-sectional area of each jet, the bucket pitch circle diameter and the hydraulic efficiency of the turbine. Assume that overall efficiency is 86%, coefficient of velocity is 0.975, the speed ratio is 0.46 and the relative velocity of water at exit from the bucket is 0.85 times the relative velocity at inlet.
[**Ans.** 8.615×10^{-3} m^2, 1.81 m, 90.76%]

12. A Pelton wheel works under a head of 425 m and rotating at 720 rpm. Determine the power produced and the hydraulic efficiency when the discharge through the machine is 0.3 m^3/s and the jet is deflected by 165°. Also find the jet ratio and overall efficiency. Assume coefficient of velocity as 0.97, speed ratio as 0.46 and the blade velocity coefficient as 0.9.

[**Ans.** 1096.83 kW, 93.21%, 16.95, 87.69%]

13. Water available under a head of 275 m is delivered to the power house at a hydroelectric power plant through three pipes each 3000 m long. Friction loss through the pipes is found to be 25 m. In this project a total shaft output of 13.25 MW is to be produced by installing a number of single jet Pelton wheels whose specific speed is not to exceed 35. Determine (i) the number of Pelton wheels to be employed, (ii) diameter of wheel, (iii) jet diameter and (iv) diameter of supply pipes. The other relevant data are speed = 600 rpm, ratio of bucket speed to jet speed = 0.46, overall efficiency = 86%, for nozzle coefficient of discharge, C_d = 0.94 and coefficient of velocity, C_v = 0.97 and coefficient of friction, f = 0.006 in the formula $h_f = (4fLV^2)/(2gD)$.

[**Ans.** 4, 0.99472 m, 0.1743 m, 1.01 m]

14. A power house has five Pelton turbines and each turbine has two runners. Each runner is fitted with two nozzles. Total discharge trough the turbines is 24 m^3/s. The head on the turbine is 200 m and the length of penstock is 2500 m. By assuming the efficiency of transmission through the penstock and nozzle as 90%, the coefficient of friction as 0.009, coefficient of velocity as 0.98 and the hydraulic efficiency as 88%, determine (i) the power developed, (ii) diameter of jet and (iii) diameter of penstock.

[**Ans.** 37293.696 kW, 0.162 m, 2.925 m]

15. A Pelton wheel is to be designed for a head of 60 m while running at 200 rpm and developing 1 MW shaft power. Assume the velocity of the buckets as 0.45 times the velocity of the jet, overall efficiency as 0.85 and coefficient of the velocity as 0.97.

[**Ans.** 1.43 m, 0.2765 m, 1.3825 m, 331.8 mm, 18]

16. A Pelton turbine is required to operate under the following conditions, such as total output = 32.72 MW, gross head = 252 m, speed = 376 rpm, two jets per wheel, coefficient of velocity for nozzle = 0.97, ratio of bucket velocity to jet velocity = 0.46, overall efficiency = 82%, head loss in friction in pipeline 500 m long = 12 m, value of friction factor $4f$ = 0.025. If the specific speed of the turbine is 36, then determine (i) the number of wheel required, (ii) jet diameter, (iii) wheel diameter, (iv) pipe diameter, (v) hydraulic efficiency when the buckets deflect the jet through 165° and the blade velocity coefficient is 0.85 and (vi) power wasted with the discharge.

[**Ans.** 4, 0.201 m, 1.555 m, 1.899 m, 90.47%, 540.88 MW]

17. The three-jet Pelton turbine is required to develop 10 MW under a net head of 400 m. The blade angle at outlet is 15° and the reduction in the relative velocity while passing over the blade is 5%. If the speed ratio is 0.45, coefficient of velocity is 0.97 and the overall efficiency is 82%, then determine (i) the total flow in m^3/s, (ii) the diameter of the jet and (iii) force exerted by a jet on the buckets. Also find the speed of the turbine for a frequency of 50 cycles per second and the corresponding wheel diameter if the jet ratio is not to be less than 10.

[**Ans.** 3.11 m^3/s, 124 mm, 91.884 kN, 613.93 rpm, 1.27 m]

18. A pipeline 1225 m long supplies water to 3 single jet Pelton wheels. The head above the nozzle is 365 m. The head lost due to friction in the pipeline is 15 m. The value of friction factor $4f$ for the pipeline is 0.02 and the velocity coefficient for the nozzle is 0.985. The specific speed of each turbine is 17 and the operating speed of each turbine is 565 rpm. If the turbine efficiency based on the head at the nozzle is 0.86, then determine (i) the total power developed, (ii) diameter of each nozzle, (iii) diameter of the pipeline and (iv) volume of water used per second.

[**Ans.** 6224.31 kW, 0.1087 m, 0.7895 m, 2.2725 m^3/s]

19. Prove that the maximum efficiency of a Pelton turbine occurs when the ratio of bucket speed (u) to the jet velocity (V_i) is given by the expression $(u/V_i) = (1 - \cos\theta + k_1)/[2(1 - \cos\theta) + k_1 + k_2]$, where k_1 and k_2 are the constants and bucket angle at outlet $\theta = (180° - \phi)$. Neglect all volumetric losses. Loss due to bucket friction and shock is given by $k_1 \times [(V_i - u)^2/(2g)]$ and that due to bearing friction and windage losses as, $k_2 \times \{u^2/(2g)\}$.

20. A Pelton wheel 0.91 m in diameter works under a head of 145 m. It develops 675.2 kW when running at 500 rpm. The rate of flow of water through the nozzle is 550 litres per second, the angle of deflection of the jet is 165° and the coefficient of velocity is 0.97. Obtain an energy balance of the turbine if the friction and windage losses are 5% of the velocity head of the jet.

[**Ans.** Accountable loss = 18.5 m, unaccountable loss = 1.36 m]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (b)	**2.** (c)	**3.** (a)	**4.** (a)	**5.** (d)
6. (a)	**7.** (c)	**8.** (a)	**9.** (d)	**10.** (a)
11. (c)	**12.** (a)			

Chapter 22

Francis Turbine (Radial Flow Reaction Turbines)

22.1 ❐ INTRODUCTION

Generally, the reaction turbine differs from impulse turbine with inlet water condition. In reaction turbines, the water from the penstock passes through a row of fixed blades (guide blades) which converts a part of the total available hydraulic energy (or pressure energy) into kinetic energy before water enters the runner. Thus, the water entering the runner possesses both kinetic energy as well as pressure energy. The pressure at the inlet to the runner is higher than the pressure at the outlet. When water flows through the runner, the water is under pressure and there is a gradual conversion of pressure into kinetic energy. The rotation of the runner is partly due to impulse action and partly due to change in pressure over the runner blades. Thus, this type of turbine is called a reaction turbine.

Since the pressure inside the turbine is different than at the inlet, there is a possibility of water flowing through some passage other than the runner and it escapes without doing any work. Therefore, the runner of a reaction turbine is completely enclosed in an air-tight casing and the runner and casing completely remains full of water throughout the operation of the turbine.

After the utilization of whole pressure energy of water in the runner, it is discharged into a closed tube of gradually enlarging section called draft tube. The free end of the draft tube is submerged deep into the tail race. Thus, the entire water passage from the head race to the tail race is totally closed. Due to the gradually increasing cross section of the draft tube, the discharge velocity is partly converted into useful pressure head and the water is discharged at a relatively low velocity to the tail water.

Some of the reaction turbines are propeller turbine, Kaplan turbine, Francis turbine, Fourneyron turbine and Thomson turbine. Out of these turbines, Francis and Kaplan turbines are widely used at present. The principle of operation of reaction turbines is illustrated schematically in Figure 22.1.

The water enters the hollow wheel (or disc) through a hollow shaft. The wheel has four radial openings through tubes. The ends of the tubes are shaped as nozzles. When water escapes through these nozzles, then its pressure energy decreases and kinetic energy increases. The resulting reaction force causes the rotation of the wheel. The wheel and the shaft rotate in a direction opposite to the direction of water jet.

Hydroelectric power is a significant contributor to the sources of energy. In India, a number of hydroelectric power plants have been installed to harness the water power by using Francis turbine. Some important Francis turbine installations in India are Bhakra Dam Project (Punjab), Cauvery Hydroelectric Scheme (Karnataka), Hirakud Dam Project (Orissa), Rihand Dam Scheme (Uttar Pradesh), Chambal Hydroelectric Scheme (Rajasthan) and Ganderbal Hydro-station (Jammu and Kashmir). In this chapter, the characteristics of radial flow reaction turbines are discussed.

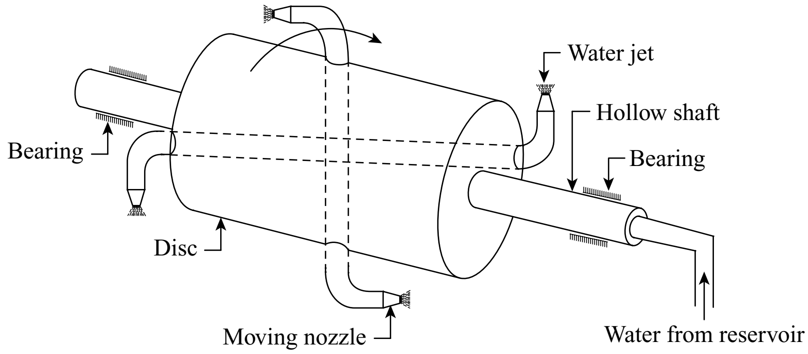


Figure 22.1 *Operation principle of reaction turbine*

22.2 ❐ RADIAL FLOW REACTION TURBINES

In radial flow turbines, the water flows in radial direction through the runner. The radial flow turbine is either inward or outward radial flow type. A schematic view of the radial flow reaction turbine is shown in Figure 22.2.

The main parts of the radial flow reaction turbine are given below.

1. **Scroll casing:** The water from the penstock is supplied to the scroll casing (or casing) of the turbine. It surrounds the runner of the turbine. It is of spiral shape in which the area decreases along the flow direction.
2. **Guide mechanism:** It consists of a number of stationary guide vanes fixed in the guide wheel around the runner. The guide vanes allow the water to enter into the moving vanes of the runner without shock at the inlet. The guide vanes are adjustable, which means the width of water passage between two consecutive vanes can be altered. Thus, the water flow rate supplied to the runner can be regulated as per the load requirement.
3. **Runner:** It is a circular wheel in which a series of smooth radial vanes are fitted. It is mounted on the turbine shaft. The vanes are so shaped that water enters and leaves the runner without shock.
4. **Draft tube:** Generally, the pressure of water coming out from the runner of a reaction turbine is less than atmospheric pressure. Thus, it cannot be directly discharged to the tail race. Therefore, a diverging tube or pipe called draft tube is fitted at the exit of the turbine. The diverging passage of the pipe increases the pressure of the exit water.

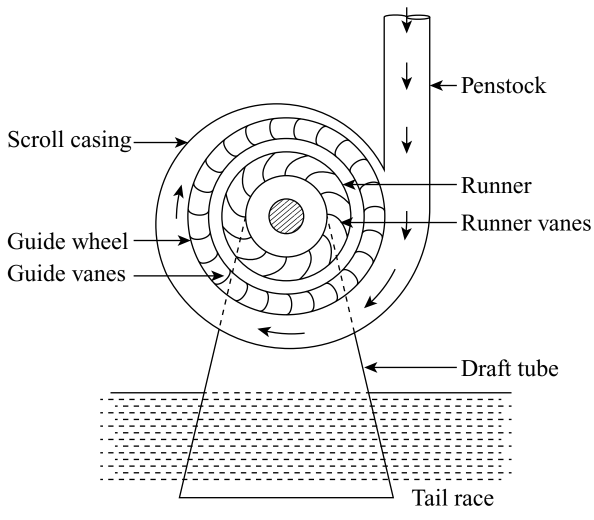


Figure 22.2 *Radial flow reaction turbine*

22.2.1 Inward Radial Flow Reaction Turbine

Water enters at the outer circumference and flows inwards radially towards the centre of the runner, for example, old Francis turbine and Thomson turbine.

The configuration of an inward radial flow reaction turbine is shown in Figure 22.3(a). The water from the casing enters the stationary guide vanes fixed on the guiding wheel. The guide vanes direct the water to enter the moving vanes of the runner. The water flows over the moving vanes in the inward radial direction and is discharged at the inner diameter of the runner. Therefore, in the inward radial flow reaction turbine, the outer diameter of the runner (D_i) is the inlet and inner diameter (D_o) is the exit for water. The hydraulic efficiency of an inward radial flow reaction turbine varies from 80% to 90%.

22.2.2 Outward Radial Flow Reaction Turbine

Water enters at the centre and flows radially outwards to the outer periphery of the runner, for example, Fourneyron turbine.

Figure 22.3(b) shows a schematic view of an outward radial flow reaction turbine. The water from the casing enters the stationary guide vanes fixed in the guide wheel. The guide vanes direct the water to enter into the runner wheel surrounding the stationary guide wheel. The water flows through the runner vanes in the outward radial direction and is discharged at the outer diameter of the runner. So, in this case the inner diameter of the runner (D_i) is the inlet and outer diameter (D_o) is the outlet for water.

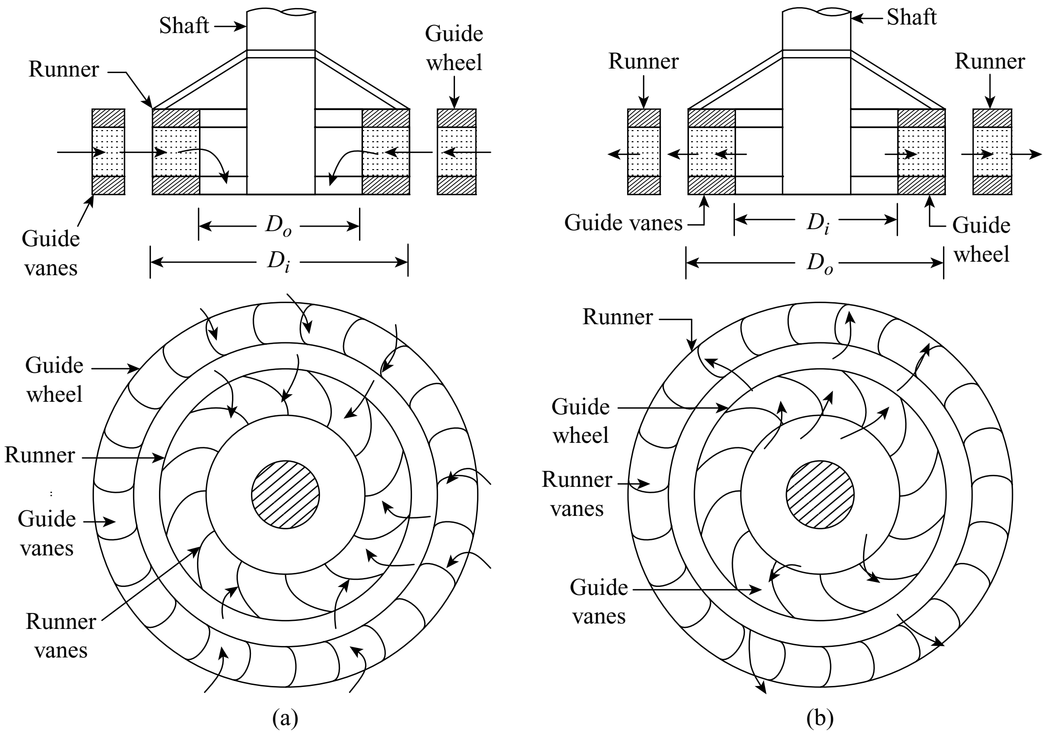


Figure 22.3 *Inward and outward radial flow reaction turbines*

22.3 ❐ COMPARISONS BETWEEN IMPULSE AND REACTION TURBINES

Impulse turbine	Reaction turbine
All the available hydraulic energy is converted into kinetic energy by a nozzle.	Only a part of the available hydraulic energy is converted into kinetic energy before the water enters the runner of the turbine.
Impulse turbines operate under atmospheric conditions and thus, the pressure remains constant throughout the action of water on the runner.	Reaction turbines operate under varying pressures different from the atmospheric pressure and the water pressure drops partly in the rotor blades and partly in the stator blades.
It is not essential to have a casing as it does not perform any hydraulic function.	Air and water tight casing is quint-essential which maintains the pressure in the turbine passage.
Water may be allowed to enter a part or whole of the wheel circumference.	Water is admitted over the whole circumference of the wheel.
Air has free access to the runner as it does not run full.	Air has no access to the runner as all the passages are completely filled by water.
It is always installed above the tail race and it does not require any draft tube.	It is connected to the tail race through a draft tube and it may be installed above or below the tail race.
Flow regulation is attained by means of a needle valve (spear) fitted into the nozzle which is possible without any loss.	Flow regulation is attained by means of guide vane assembly which is always accompanied by loss.
The relative velocity of water either remains constant or reduces slightly due to the presence of blade friction.	The relative velocity of water increases due to continuous drop in pressure during flow through the blades.
These turbines are suitable for high head, low discharge and low specific speed conditions.	These turbines are suitable for low to medium head and specific speed, and high discharge.
Some of the impulse turbines are Pelton wheel, Banki turbine, Jonval turbine, Girard turbine and Turgo-impulse wheel.	Some of the reaction turbines are propeller turbine, Kaplan turbine, Francis turbine, Fourneyron turbine and Thomson turbine.

22.4 ❐ DIFFERENCES BETWEEN INWARD AND OUTWARD RADIAL FLOW REACTION TURBINES

Inward radial flow reaction turbine	Outward radial flow reaction turbine
Water enters at the outer circumference and flows inwards radially towards the centre of the runner.	Water enters at the centre and flows radially outwards to the outer periphery of the runner.
The discharge does not increase.	The discharge increases.
Speed control is easy and effective.	It is very difficult to control the speed.
It is good for medium and high heads. It is suitable for large output units.	It is good for low or medium heads.
Centrifugal head imparted to water is negative.	Centrifugal head imparted to water is positive.
If the turbine speed increases due to any reason, then wheel tendency to race is nil. The turbine adjusts the speed by itself.	If turbine speed increases, then wheel tends to race. The turbine cannot adjust the speed by itself.
It is generally used for power projects.	It has become practically obsolete.
$D_i > D_o$ and $u_i > u_o$	$D_o > D_i$ and $u_o > u_i$

22.5 ❐ FRANCIS TURBINE

The Francis turbine is named in the honour of J. B. Francis, an American engineer, who was the first to develop an inward radial flow type of reaction turbine as shown in Figure 22.4. Later on, it was modified in which water enters the runner radially at its outer periphery and leaves axially at its centre. Thus, the modern Francis turbine is a mixed flow type turbine. It is a reaction turbine and hence, only a part of the available head is converted into the velocity head before water enters the runner. The pressure head goes on decreasing as water flows over the runner blades. The static pressure at the runner exit may be less than the atmospheric pressure and as such, water completely fills all the passages of the runner. A typical large Francis turbine can achieve a hydraulic efficiency from 90% to 95%.

Francis turbines may either have a horizontal shaft or a vertical shaft arrangement. The vertical shaft arrangement is widely used, especially in large sizes as it helps in the economy of building space. A Francis turbine is most suitable for medium heads (60 m to 250 m) and requires a relatively large quantity of water. Francis turbine is a medium specific speed turbine which varies from 50 to 255. The main parts of a Francis turbine are (1) penstock, (2) scroll casing, (3) guide mechanism, (4) runner, (5) draft tube and (6) governing mechanism.

1. **Penstock:** It is a large sized conduit which conveys water from reservoir to the runner. Trash racks are commonly provided at the inlet of penstock to obstruct the entry of debris and other foreign matter into the turbine. Penstock is also provided with the control valves to control the quantity of water. In Francis turbines, large size penstocks are required due to large volume of water flow.
2. **Scroll casing:** The water from the penstock enters the scroll casing which is of spiral shape, so it is also known as spiral casing. It completely surrounds the runner of the turbine and maintains the even distribution of water around the circumference of the runner. The area of scroll casing goes on decreasing gradually which keeps the velocity of water constant throughout its path around the runner. In Francis turbines, the casing and runner are always full of water. The casing is made of cast steel, concrete or plate steel. A plate steel scroll casing is commonly provided for turbines working under a head of about 30 m to 100 m. For heads up to about 30 m, it is made of concrete and for more than 100 m heads, it is fabricated from cast steel.
3. **Guide mechanism:** Water from the casing flow through a speed ring (or stay ring) having fixed stay vanes which direct the water to the guide vanes. Generally, the number of stay vanes is taken as half of the number of guide vanes. The stay ring is provided in big units only. From stay ring, water passes through a series of guide vanes.

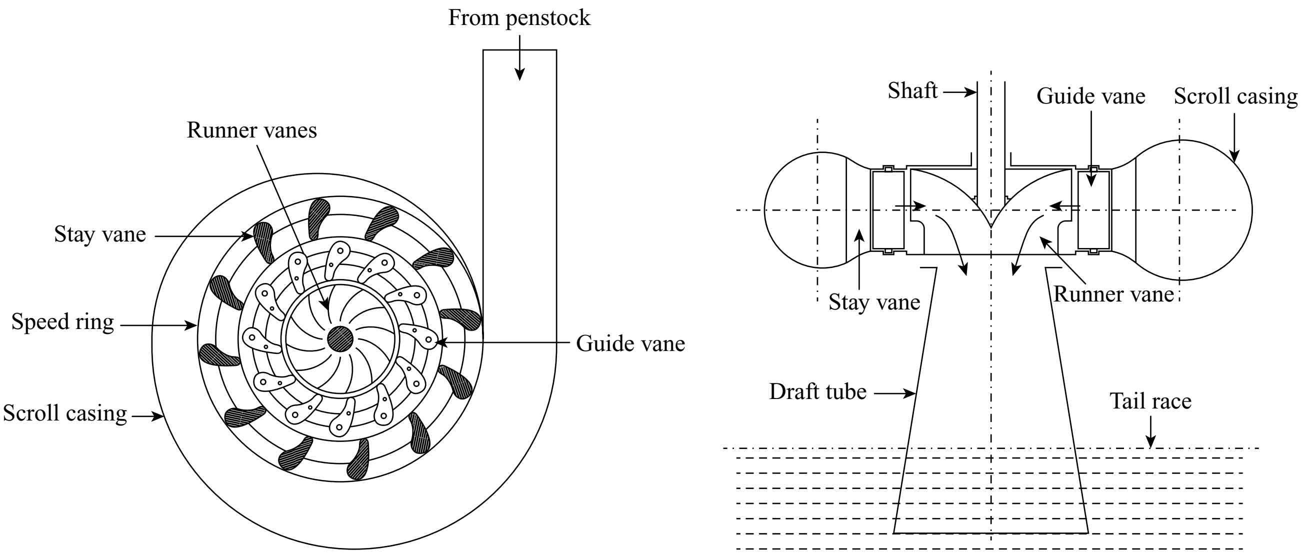


Figure 22.4 *Francis turbine*

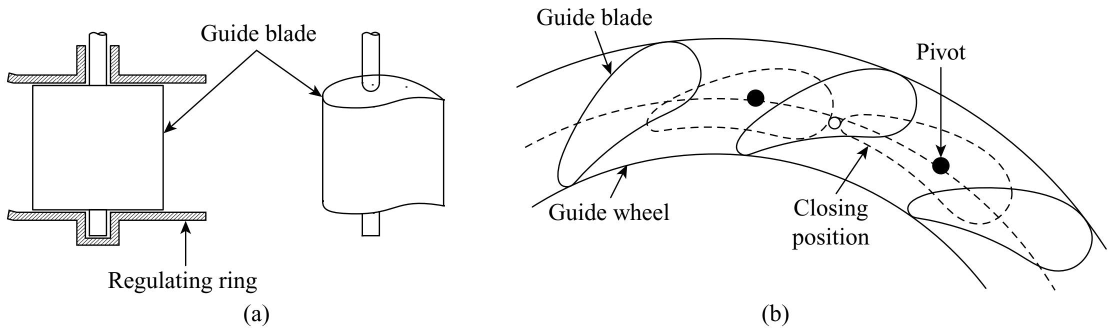


Figure 22.5 *Guide vanes and guide wheel*

The guide vanes are adjustable and are also known as wicket gates. The guide vanes are airfoil shaped (Figure 22.5(a)) so that the flow remains smooth without any separation. The guide vanes behave like nozzles. These are fixed on a stationary circular wheel (guide wheel) all around the periphery of the turbine runner. The guide vanes direct the water to strike the vanes fixed on the runner without shock at the inlet. Each guide vane can rotate about its pivot centre as shown in Figure 22.5(b). Thus, the flow cross-sectional area can be varied between two adjacent vanes either by means of a wheel or automatically by a governor. Thus, the guide vanes can also regulate the quantity of water supplied to the runner by changing the width of flow between them. The guide vanes may be made of cast steel, plate steel or stainless steel.

4. **Runner:** It is also known as rotor. Runner is a circular wheel in which a series of radial curved vanes are evenly fixed. Generally, the number of vanes varies in between 16 to 24. The radial curved vanes are so shaped that water enters the runner radially at the outer periphery and leaves axially at the inner periphery without shock. In order to minimize the frictional resistance, the surfaces of these vanes are made very smooth. The direction of water flow in the runner changes from radial to axial. Due to this change in direction of flow, a circumferential force is produced on the runner which makes the runner to rotate. The runner is keyed to the turbine shaft which is further coupled to the generator shaft. In the recent past, Francis runners as large as 7.5 m in diameter have been manufactured. The runners are made of stainless steel, cast iron or cast steel. The turbine shaft is made of forged steel.

   The width of the runner depends on the specific speed. The high specific speed runner has to work with a large quantity of water. Thus, these runners will be wider than the low specific speed runner and the runners may be classified as follows.

   (i) *Slow runner:* Specific speed varies from 60 to 120.
   (ii) *Normal runner:* Specific speed varies from 120 to 180.
   (iii) *Fast runner:* Specific speed varies from 180 to 300.

5. **Draft tube:** A draft tube is a pipe of gradually increasing cross-sectional area. It is considered as an integral part of the turbine. One end of the draft tube is connected to the runner exit, while the other end is submerged deep into the tail race. The draft tube should be drowned about one metre below the tail race level. The water after passing through the runner is discharged to the tail race through the draft tube. So, a draft tube is an outlet conduit from a turbine which acts as a diffuser. It is made of welded steel plate pipe or a concrete tunnel.
6. **Governing mechanism:** The quantity of water in a Francis turbine is controlled by varying the area of flow between adjacent guide vanes by rotating them. These guide vanes are pivoted and connected by levers and links to the regulating ring. The regulating ring is connected to the regulating shaft through regulating lever and two regulating rods as shown in Figure 22.6. The regulating shaft is connected to the servomotor. The servomotor, control valve, oil sump and piping system, etc., are similar to that in Pelton wheel. However, these components are relatively stronger as greater energy is required to move the guide vanes.

With variation in load, the speed of the turbine changes which causes the movement of servomotor piston either to the left or to the right. This rotates the regulating lever clockwise or anticlockwise. This rotation is further transmitted to the

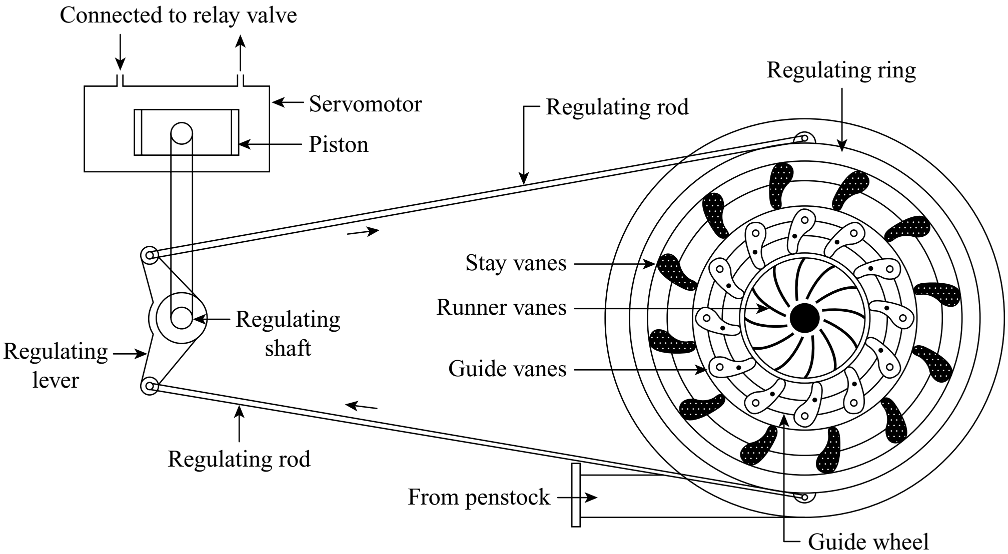


Figure 22.6 *Governing mechanism of Francis turbine*

regulating ring through regulating rods. The regulating ring rotates in the same direction as the regulating lever and thus, it closes or opens the passage between the two adjacent guide vanes as per requirement.

The rapid closure of guide vanes is not desirable because a sudden reduction of flow rate in the penstock may result in serious water hammer problems. The penstock is provided with a pressure regulator or relief valve which has to perform the same function as that of deflector in Pelton turbine. When there is a sudden decrease in load on the turbine, the relief valve opens and diverts the water to the tail race and thus, it prevents the sudden closure of guide vanes. Therefore, the double regulation, which is the simultaneous operation of two elements, is accomplished by moving the guide vanes and relief valve with the help of the governor.

22.6 ❐ VELOCITY TRIANGLES, WORK DONE AND EFFICIENCY OF RADIAL FLOW REACTION TURBINES AND FRANCIS TURBINE

The velocity triangles at the inlet and outlet for an inward radial flow reaction turbine are shown in Figure 22.7. In Chapter 20 (Section 20.9.4), the force exerted by the water on the radial curved vanes fixed on a wheel have already been discussed. However, for maintaining the continuity, this section is elaborated briefly here.

Let R_i and R_o be the radii of wheel at the inlet and outlet of the vane, respectively, ω be the angular speed of the wheel, then $u_i = \omega R_i$ and $u_o = \omega R_o$. All other notations are usual as given for Pelton Turbine.

The mass of the water striking per second for series of vanes $= \rho_w A V_i$

Momentum of water striking in the tangential direction per second at the inlet

$$= m \times \text{Component of } V_i \text{ in tangential direction}$$

$$= \rho_w A V_i (V_i \cos\alpha) = \rho_w A V_i V_{wi} \qquad [\because V_{wi} = V_i \cos\alpha]$$

Momentum of water at the outlet per second

$$= m \times \text{Component of } V_o \text{ in tangential direction}$$

$$= \rho_w A V_i (-V_o \cos\beta) = -\rho_w A V_i V_{wo} \qquad [\because V_{wo} = V_o \cos\beta]$$

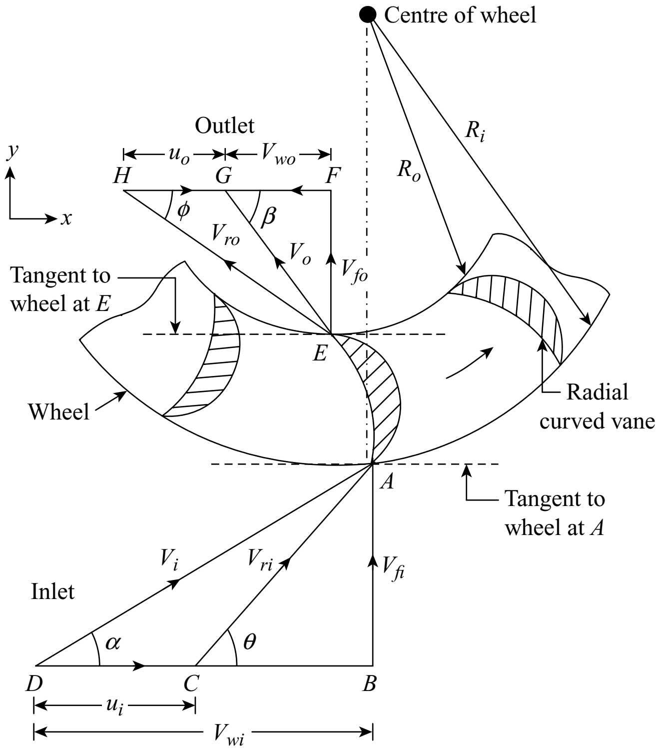


Figure 22.7 *Velocity triangles*

Negative sign is taken because V_o at the outlet is in opposite direction.

Angular momentum per second at the inlet $= \rho_w A V_i V_{wi} R_i$

Angular momentum per second at the outlet $= -\rho_w A V_i V_{wo} R_o$

Torque exerted by the water on the wheel is given by,

$$T = \text{Rate of change of angular momentum}$$

$$T = \rho_w A V_i V_{wi} R_i - (-\rho_w A V_i V_{wo} R_o) = \rho_w A V_i (V_{wi} R_i + V_{wo} R_o)$$

Work done per second on the wheel is given by,

$$w = T\omega = \rho_w A V_i (V_{wi} R_i + V_{wo} R_o)\omega = \rho_w A V_i (V_{wi} R_i \omega + V_{wo} R_o \omega)$$

$$\therefore w = \rho_w A V_i (V_{wi} u_i + V_{wo} u_o)$$

If the angle $\beta > 90°$, the work done will be given by,

$$w = \rho_w A V_i (V_{wi} u_i - V_{wo} u_o)$$

Therefore, the general expression for work done per second is given by,

$$w = \rho_w A V_i (V_{wi} u_i \pm V_{wo} u_o) = \rho_w Q (V_{wi} u_i \pm V_{wo} u_o) \qquad [\because A V_i = Q]$$

Work done per second per unit weight of water striking per second is given by,

$$\boxed{w = \frac{\rho_w Q (V_{wi} u_i \pm V_{wo} u_o)}{\rho_w Q g} = \frac{V_{wi} u_i \pm V_{wo} u_o}{g}} \tag{22.1}$$

The Equation (22.1) is known as Euler's momentum equation which is the fundamental equation of hydrodynamic machines. This equation is extensively used in fluid power engineering. It helps in the determination of torque or power exchanged between the water and the runner when water flows through the vane passages. In case of hydraulic turbines (i.e., power producing machines), the Euler's equation represents the head utilized in performing work. In case of pumps (i.e., power consuming machine), it represents the head imparted to the fluid by rotating vanes.

If the discharge is radial at outlet, then $\beta = 90°$ and the output will be maximum. Therefore, work done per second per unit weight/second is given below.

$$\boxed{w = \frac{V_{wi} u_i}{g}} \tag{22.2}$$

Hydraulic efficiency is given by,

$$\boxed{\eta_h = \frac{\rho_w Q(V_{wi} u_i \pm V_{wo} u_o)}{\rho_w Q g H} = \frac{V_{wi} u_i \pm V_{wo} u_o}{gH}} \tag{22.3}$$

The inward radial flow reaction turbine having radial discharge at the outlet is known as Francis turbine. For radial flow at outlet, $V_{wo} = 0$ and therefore, the hydraulic efficiency of Francis turbine is given below.

$$\boxed{\eta_h = \frac{V_{wi} u_i}{gH}} \tag{22.4}$$

The hydraulic efficiency of the Francis turbine varies from 85% to 90%.

22.6.1 Change of Kinetic Energy and Pressure Energy in the Runner of a Radial Flow Reaction Turbine

Work done per second per unit weight of the water striking the runner per second is given by Equation (22.1). This equation represents the total energy change per unit weight in the runner which is given in the below expression.

$$w = E_t = \frac{V_{wi}\, u_i \pm V_{wo}\, u_o}{g} \tag{22.1a}$$

From inlet velocity triangle (Figure 22.7), we get:

$$V_{ri}^2 = (V_{wi} - u_i)^2 + V_{fi}^2$$

$$(V_{wi} - u_i)^2 = V_{ri}^2 - V_{fi}^2 = V_{ri}^2 - (V_i^2 - V_{wi}^2) \qquad [\because V_{fi}^2 = V_i^2 - V_{wi}^2]$$

$$V_{wi}^2 + u_i^2 - 2V_{wi} u_i = V_{ri}^2 - V_i^2 + V_{wi}^2$$

$$2V_{wi} u_i = u_i^2 - V_{ri}^2 + V_i^2$$

$$V_{wi} u_i = \frac{1}{2}(u_i^2 - V_{ri}^2 + V_i^2) \tag{i}$$

From outlet velocity triangle (Figure 22.7), we get:

$$V_{wo}^2 = V_o^2 - V_{fo}^2$$

But $\quad V_{fo}^2 = V_{ro}^2 - (u_o + V_{wo})^2$

Thus $\quad V_{wo}^2 = V_o^2 - [V_{ro}^2 - (u_o + V_{wo})^2] = V_o^2 - V_{ro}^2 + u_o^2 + V_{wo}^2 + 2V_{wo} u_o$

$$2V_{wo} u_o = V_{ro}^2 - V_o^2 - u_o^2$$

$$V_{wo} u_o = \frac{1}{2}(V_{ro}^2 - V_o^2 - u_o^2) \tag{ii}$$

Substituting the values of the expressions (i) and (ii) in Equation 22.1(a) in which considering +ve sign, (i.e., $\beta < 90°$), we get the below expression.

$$w = E_t = \frac{1}{g}\left[\frac{1}{2}(u_i^2 - V_{ri}^2 + V_i^2) + \frac{1}{2}(V_{ro}^2 - V_o^2 - u_o^2)\right]$$

$$= \frac{1}{2g}\left[(V_i^2 - V_o^2) + (u_i^2 - u_o^2) + (V_{ro}^2 - V_{ri}^2)\right]$$

$$\therefore w = E_t = \frac{V_i^2 - V_o^2}{2g} + \frac{u_i^2 - u_o^2}{2g} + \frac{V_{ro}^2 - V_{ri}^2}{2g} \tag{22.5}$$

The Equation (22.5) gives the total energy change per unit weight in the runner. In this equation, the first term $[(V_i^2 - V_o^2)/(2g)]$ represents the change in kinetic energy of the water per unit weight. The second term $[(u_i^2 - u_o^2)/(2g)]$ represents the change of energy per unit weight due to centrifugal action which is a form of pressure energy. The third term $[(V_{ro}^2 - V_{ri}^2)/(2g)]$ represents the change in static pressure energy per unit weight. Therefore, the sum of second and third terms represents the change in pressure energy inside the runner per unit weight.

22.6.2 Degree of Reaction

Degree of reaction (R) is defined as the ratio of pressure energy change in the runner to the total energy change in the stator and the runner. In a reaction turbine, a part of pressure change occurs in the stator and remaining in the runner. Mathematically, the relation for degree of reaction is expressed as follows.

$$R = \frac{\text{Change of pressure energy in the runner}}{\text{Change of total energy in the stator and runner}}$$

The change in pressure energy of the water in the runner can be obtained by subtracting the change in its kinetic energy from the total energy released.

$$R = \frac{[1/(2g)][(u_i^2 - u_o^2) + (V_{ro}^2 - V_{ri}^2)]}{[1/(2g)][(V_i^2 - V_o^2) + (u_i^2 - u_o^2) + (V_{ro}^2 - V_{ri}^2)]}$$

$$= \frac{(u_i^2 - u_o^2) + (V_{ro}^2 - V_{ri}^2)}{(V_i^2 - V_o^2) + (u_i^2 - u_o^2) + (V_{ro}^2 - V_{ri}^2)}$$

$$= \frac{[(V_i^2 - V_o^2) + (u_i^2 - u_o^2) + (V_{ro}^2 - V_{ri}^2)] - (V_i^2 - V_o^2)}{(V_i^2 - V_o^2) + (u_i^2 - u_o^2) + (V_{ro}^2 - V_{ri}^2)}$$

$$\therefore R = 1 - \frac{(V_i^2 - V_o^2)}{(V_i^2 - V_o^2) + (u_i^2 - u_o^2) + (V_{ro}^2 - V_{ri}^2)} \tag{22.6}$$

From Equation (22.5), we get:

$$(V_i^2 - V_o^2) + (u_i^2 - u_o^2) + (V_{ro}^2 - V_{ri}^2) = 2gE_t$$

$$\therefore R = 1 - \frac{(V_i^2 - V_o^2)}{2gE_t} \tag{22.6a}$$

Case I: Degree of reaction for a Pelton turbine.
In the case of a Pelton turbine, $V_{ri} = V_{ro}$ and $u_i = u_o$. From Equation (22.6), we get:

$$R = 1 - \frac{(V_i^2 - V_o^2)}{(V_i^2 - V_o^2) + 0 + 0} = 1 - 1 = 0$$

Case II: Degree of reaction for a reaction turbine.

For radial discharge at the outlet, $\beta = 90°$ and $V_{wo} = 0$. Thus, the output will be maximum. Assuming that there is not much change in the velocity of flow, we get $V_{fi} = V_{fo}$. The velocity triangles at the inlet and outlet are shown in Figure 22.8.

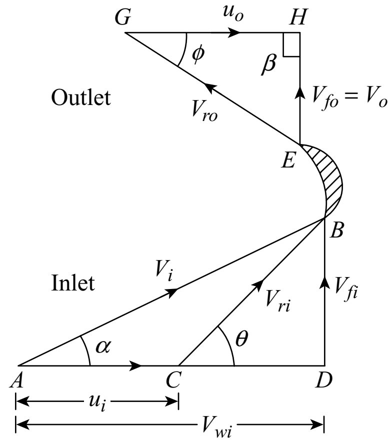


Figure 22.8 *Velocity triangles*

From Equation 22.1(a), we get:

$$w = E_t = \frac{V_{wi}\, u_i}{g} \qquad [\because V_{wo} = 0] \qquad (22.1b)$$

But $$\eta_h = \frac{V_{wi}\, u_i}{gH}$$

Thus $$w = E_t = \eta_h H$$

From Equation 22.6(a), the degree of reaction becomes,

$$R = 1 - \frac{(V_i^2 - V_o^2)}{2g\eta_h H} \qquad (22.6b)$$

From inlet velocity triangle (Figure 22.8), we get:

$$V_{wi} = V_{fi} \cot\alpha \qquad (iii)$$

$$u_i = V_{wi} - V_{fi}\cot\theta = V_{fi}\cot\alpha - V_{fi}\cot\theta \qquad (iv)$$

Substituting the expressions (iii) and (iv) in Equation 22.1(b), we get:

$$E_t = \frac{(V_{fi}\cot\alpha)(V_{fi}\cot\alpha - V_{fi}\cot\theta)}{g} = \frac{V_{fi}^2 \cot\alpha(\cot\alpha - \cot\theta)}{g} \qquad (v)$$

Now $$V_i^2 - V_o^2 = (V_{fi}\operatorname{cosec}\alpha)^2 - V_{fo}^2 \qquad [\because V_o = V_{fo}]$$

$$V_i^2 - V_o^2 = V_{fi}^2(\operatorname{cosec}^2\alpha - 1) \qquad [\because V_{fi} = V_{fo}]$$

$$V_i^2 - V_o^2 = V_{fi}^2 \cot^2\alpha \qquad (vi)$$

Substituting the expressions (v) and (vi) in Equation 22.6(a), we get:

$$\boxed{R = 1 - \frac{V_{fi}^2\cot^2\alpha \times g}{2gV_{fi}^2\cot\alpha(\cot\alpha - \cot\theta)} = 1 - \frac{\cot\alpha}{2(\cot\alpha - \cot\theta)}} \qquad (22.6c)$$

The degree of reaction lies between 0 and 1. A turbine with zero degree of reaction is called an impulse turbine, for example, Pelton turbine and a conventional wind mill. The degree of reaction of unity is not possible in the turbines. Most of the reaction turbines have degree of reaction from 0.4 to 0.6, for example, Francis and Kaplan turbines.

22.7 ❐ DEFINITIONS AND WORKING PROPORTIONS OF A FRANCIS TURBINE AND RADIAL FLOW REACTION TURBINES

1. **Speed ratio:** It is defined as the ratio of tangential velocity (peripheral speed) at the inlet of the wheel to the spouting velocity. The value of speed ratio for Francis turbine varies from 0.6 to 0.9. If u_i is the tangential velocity of the wheel at the inlet and H is the net head on the turbine, then the speed ratio (K_u) is given below.

$$\boxed{K_u = \frac{u_i}{\sqrt{2gH}}} \qquad (22.7)$$

2. **Flow ratio:** The flow ratio is defined as the ratio of the flow velocity at the inlet of the vane to the spouting velocity. The value of flow ratio for Francis turbine varies from 0.15 to 0.30. If V_{fi} is the velocity of flow at the inlet and H is the net head on the turbine, then the flow ratio (K_f) is given below.

$$\boxed{K_f = \frac{V_{fi}}{\sqrt{2gH}}} \tag{22.8}$$

3. **Ratio of width to diameter (Breadth ratio):** The ratio of width of the wheel (B_i) to its diameter (D_i) is given as $n = (B_i/D_i)$. For Francis turbine, n varies from 0.1 to 0.45.
4. **Discharge of the turbine:** The discharge through a Francis turbine is given by,

$$\boxed{Q = \pi D_i B_i V_{fi}} \tag{22.9}$$

Here, D_i = diameter of runner at the inlet, B_i = width of runner at the inlet and V_{fi} = velocity of flow at the inlet.

If the thickness of vanes (t) and its number (z) is taken into account, then discharge becomes,

$$Q = (\pi D_i - zt)B_i V_{fi} = k\pi n D_i^2 V_{fi} \quad [\because B_i = nD_i] \tag{22.10}$$

Here, k is a factor which allows for the thickness of the vanes and it is known as vane thickness coefficient.

The discharge through a radial flow reaction turbine is given by,

$$\boxed{Q = \pi D_i B_i V_{fi} = \pi D_o B_o V_{fo}} \tag{22.11}$$

Here, D_o = diameter of runner at the outlet, B_o = width of runner at the outlet and V_{fo} = velocity of flow at the outlet.

5. **Head on the turbine:** The head (H) acting on the turbine is given by,

$$H = \frac{p_i}{\rho_w g} + \frac{V_i^2}{2g} \tag{22.12}$$

Here, p_i is the pressure at the inlet.

When water flows through the vane without any energy loss and V_o is the discharge velocity, then the head is given by,

$$\boxed{H - \frac{V_o^2}{2g} = \frac{V_{wi}u_i \pm V_{wo}u_o}{g}} \tag{22.13}$$

6. **Radial discharge:** Radial discharge at the outlet means $\beta = 90°$ and $V_{wo} = 0$, while radial discharge at the inlet means $\alpha = 90°$ and $V_{wi} = 0$.

22.8 ❐ DESIGN OF FRANCIS TURBINE RUNNER

A Francis turbine runner is designed to develop a known power P, when running at a known speed N rpm under a known head H. The design of the runner involves the determination of the size and the vane angles for which the following steps are to be followed.

1. Assume the probable values of overall efficiency (η_o), hydraulic efficiency (η_h), ratio of width of the wheel to its diameter (n) and flow ratio (K_f).
2. Find out the required discharge (Q) as $Q = \dfrac{1000P}{\rho_w g H \eta_o}$
3. Determine the diameter (D) and width (B) of the runner as explained below.

Let D_i be the diameter, B_i be the width of the runner and t_i be the thickness of the vanes at the inlet as shown in Figure 22.9.

If there are z number of vanes, then the area of flow at the runner inlet is given by,

$$A_i = (\pi D_i - z\,t_i)B_i = k_{ti}\pi D_i B_i$$

Here, k_{ti} is the vane thickness factor. Its value will always be less than unity and it is usually taken as 0.95.

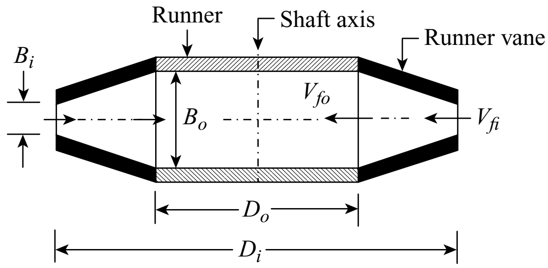


Figure 22.9 *Flow entry to runner vane*

$$V_{fi} = \frac{Q}{A_i} = \frac{Q}{k_{ti}\pi D_i B_i} = \frac{Q}{k_{ti}\pi n D_i^2} \qquad [\because B_i = nD_i]$$

Also $$V_{fi} = K_f\sqrt{2gH}$$

Thus $$\frac{Q}{k_{ti}\pi n D_i^2} = K_f\sqrt{2gH}$$

$$\therefore D_i = \left(\frac{Q}{k_{ti}\pi n K_f\sqrt{2gH}}\right)^{1/2}$$

$$B_i = nD_i$$

4. Determine the tangential velocity or rim velocity (u_i) as $u_i = \dfrac{\pi D_i N}{60}$
5. Determine the velocity of whirl at the inlet (V_{wi}) from Equation (22.4) as $V_{wi} = \dfrac{\eta_h gH}{u_i}$
6. Determine the guide vane angle (α) and the runner vane angle (θ) from the inlet velocity triangle as

$$\alpha = \tan^{-1}\left(\frac{V_{fi}}{V_{wi}}\right) \text{ and } \theta = \tan^{-1}\left(\frac{V_{fi}}{V_{wi} - u_i}\right)$$

7. The outer runner diameter D_o varies from $(D_i/3)$ to $(2D_i/3)$ and it is usually taken as $(D_i/2)$. Therefore, $D_o = (D_i/2)$ and $u_o = (u_i/2)$.
8. If B_o be the width of the runner and t_o be the thickness of the vanes at the outlet, then

$$Q = (\pi D_o - Zt_o)B_o V_{fo} = k_{to}\pi D_o B_o V_{fo}$$

Now $$Q = k_{ti}\pi D_i B_i V_{fi} = k_{to}\pi D_o B_o V_{fo} \qquad \text{[By continuity equation]}$$

$$\frac{V_{fi}}{V_{fo}} = \frac{k_{to}\pi D_o B_o}{k_{ti}\pi D_i B_i}$$

Generally, $V_{fi} = V_{fo}$ and $k_{ti} = k_{to}$ and $D_o = D_i/2$, then we have $B_o = 2B_i$.

9. Assuming radial discharge at the runner exit in which $V_{wo} = 0$ and $\beta = 90°$. Determine the runner vane angle at exit (ϕ) from the outlet velocity triangle by the following relation.

$$\phi = \tan^{-1}\left(V_{fo}/u_o\right)$$

10. Generally, the number of runner vanes varies from 16 to 24. In order to avoid periodic impulse, the number of runner vanes should be either one more or one less than the number of guide vanes.

22.8.1 Shape of Francis Turbine Runner

The power produced by a turbine is proportional to the product of discharge (Q) and the head (H). As the head decreases, the discharge must increase to produce the same power. Thus, a Francis turbine runner of a given diameter D_i working under low head should be designed so that comparatively large quantity of water is supplied to develop the required power. This can be achieved either by increasing the flow ratio (K_f) or the ratio $n = (B_i/D_i)$. Increase in the value of flow ratio means a high value of flow velocity (V_{fi}). A large value of V_{fi} at the outlet causes wastage of large quantity of kinetic energy rejected by the runner. Therefore, K_f cannot be increased to any amount and its value should be kept as low as possible. On the other hand, increase in the value of n results in a higher value of B_i for a given diameter D_i. This will result in larger inlet area which must be accompanied by a larger outlet area for discharging all the water freely. A larger outlet area can be obtained by making the discharge axial at the runner outlet. Thus, the modern Francis turbine runner is of mixed flow type.

Example 22.1 A reaction turbine works at 450 rpm under a head of 120 m. Its diameter at the inlet is 1.2 m and the flow area is 0.4 m^2. The angles made by the absolute and relative velocities at the inlet are 20° and 60°, respectively with the tangential velocity. If whirl at the outlet is zero, then determine (i) the volume flow rate, (ii) power developed and (iii) hydraulic efficiency.

Solution

Refer Figure 22.10. Let $N = 450$ rpm, $H = 120$ m, $D_i = 1.2$ m, $\pi D_i B_i = 0.4$ m^2, $\alpha = 20°$, $\theta = 60°$ and $V_{wo} = 0$.

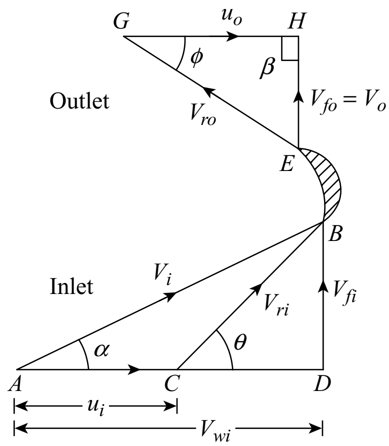


Figure 22.10

$$u_i = \frac{\pi D_i N}{60} = \frac{\pi \times 1.2 \times 450}{60} = 28.274 \text{ m/s}$$

$$V_{fi} = V_{wi} \tan\alpha = V_{wi} \tan 20° = 0.364 V_{wi}$$

Since $$\tan\theta = \frac{V_{fi}}{V_{wi} - u_i}$$

Thus $$\tan 60° = \frac{0.364 V_{wi}}{V_{wi} - 28.274}$$

$$1.732(V_{wi} - 28.274) = 0.364 V_{wi}$$

$$1.732 V_{wi} - 0.364 V_{wi} = 1.732 \times 28.274$$

$$\therefore V_{wi} = \frac{1.732 \times 28.274}{1.732 - 0.364} = 35.8 \text{ m/s}$$

$$V_{fi} = 0.364 V_{wi} = 0.364 \times 35.8 = 13.03 \text{ m/s}$$

(i) $Q = \pi D_i B_i \times V_{fi} = 0.4 \times 13.03 = \mathbf{5.212\ m^3/s}$

(ii) $P = \dfrac{\rho_w Q V_{wi} u_i}{1000} = \dfrac{1000 \times 5.212 \times 35.8 \times 28.274}{1000} = \mathbf{5275.634\ kW}$

(iii) $\eta_h = \dfrac{V_{wi} u_i}{gH} = \dfrac{35.8 \times 28.274}{9.81 \times 120} \times 100 = \mathbf{85.98\%}$

Example 22.2 An inward flow reaction turbine develops 320 kW shaft power with an overall efficiency of 85% when working under a net head of 72 m. The hydraulic efficiency of the turbine is 95% and the runner speed is 650 rpm. The ratio of wheel width to wheel diameter at the inlet is 0.1 and the ratio of inner diameter to outer diameter is 0.5. If the flow ratio is 0.17 and flow velocity is constant, then determine the dimensions and blade angles of the turbine. Assume radial discharge at the outlet and neglect area blockage by blades.

Solution

Refer Figure 22.10. Let $P = 320$ kW, $\eta_o = 0.85$, $H = 72$ m, $\eta_h = 0.95$, $N = 650$ rpm, $(B_i/D_i) = 0.1$, $(D_o/D_i) = 0.5$, $K_f = 0.17$, $V_{fi} = V_{fo}$ and $V_{wo} = 0$.

$$V_{fi} = V_{fo} = K_f\sqrt{2gH} = 0.17\times\sqrt{2\times 9.81\times 72} = 6.39 \text{ m/s}$$

$$Q = \frac{1000P}{\rho_w gH\eta_o} = \frac{1000\times 320}{1000\times 9.81\times 72\times 0.85} = 0.533 \text{ m}^3/\text{s}$$

Since $Q = \pi D_i B_i V_{fi}$

Thus $0.533 = \pi D_i \times (0.1D_i)\times 6.39 = 0.639\pi D_i^2 \qquad [\because B_i/D_i = 0.1]$

$$\therefore D_i = \left(\frac{0.533}{0.639\pi}\right)^{1/2} = \mathbf{0.5153\ m}$$

$$D_o = 0.5D_i = 0.5\times 0.5153 = \mathbf{0.25765\ m}$$

$$B_i = 0.1D_i = 0.1\times 0.5153 = \mathbf{0.05153\ m}$$

Since $\pi D_i B_i V_{fi} = \pi D_o B_o V_{fo}$

$$\therefore B_o = \frac{D_i B_i}{D_o} = \frac{D_i B_i}{0.5D_i} = 2B_i = 2\times 0.05153 = \mathbf{0.10306\ m}$$

$$u_i = \frac{\pi D_i N}{60} = \frac{\pi\times 0.5153\times 650}{60} = 17.54 \text{ m/s}$$

Since $\eta_h = \dfrac{V_{wi}u_i}{gH}$

$$\therefore V_{wi} = \frac{\eta_h gH}{u_i} = \frac{0.95\times 9.81\times 72}{17.54} = 38.256 \text{ m/s}$$

$$\alpha = \tan^{-1}\left(\frac{V_{fi}}{V_{wi}}\right) = \tan^{-1}\left(\frac{6.39}{38.256}\right) = \mathbf{9.48°}$$

$$\theta = \tan^{-1}\left(\frac{V_{fi}}{V_{wi} - u_i}\right) = \tan^{-1}\left(\frac{6.39}{38.256 - 17.54}\right) = \mathbf{17.14°}$$

$$u_o = \frac{\pi D_o N}{60} = \frac{\pi\times 0.25765\times 650}{60} = 8.77 \text{ m/s}$$

$$\phi = \tan^{-1}\left(\frac{V_{fo}}{u_o}\right) = \tan^{-1}\left(\frac{6.39}{8.77}\right) = \mathbf{36.08°}$$

Since discharge is radial, the value of $\beta = \mathbf{90°}$.

Example 22.3 A Francis turbine with an overall efficiency of 75% is required to produce 150 kW power. It is working under a head of 10 m. The peripheral velocity $= 0.25\sqrt{2gH}$ and the radial velocity of flow at inlet $= 0.95\sqrt{2gH}$. The wheel runs at 200 rpm and the hydraulic losses in the turbine are 20% of the available energy. Assuming radial discharge, determine (i) the guide blade angle, (ii) wheel vane angle at the inlet, (iii) diameter of the wheel at the inlet and (iv) width of the wheel at the inlet.

Solution

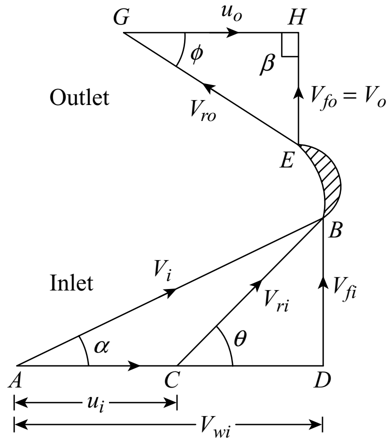


Figure 22.11

Refer Figure 22.11. Let $\eta_o = 0.75$, $P = 150$ kW, $H = 10$ m, $u_i = 0.25\sqrt{2gH}$, $V_{fi} = 0.95\sqrt{2gH}$, $N = 200$ rpm, hydraulic loss = 20% of $H = 0.2$H and $V_{wo} = 0$.

$$u_i = 0.25\sqrt{2gH} = 0.25 \times \sqrt{2 \times 9.81 \times 10} = 3.5 \text{ m/s}$$

$$V_{fi} = 0.95\sqrt{2gH} = 0.95 \times \sqrt{2 \times 9.81 \times 10} = 13.31 \text{ m/s}$$

$$\eta_h = \frac{\text{Total head at the inlet} - \text{Hydraulic loss}}{\text{Total head at the inlet}} = \frac{H - 0.2H}{H} = 0.8$$

Also $\quad \eta_h = \dfrac{V_{wi}u_i}{gH}$

$$\therefore V_{wi} = \frac{\eta_h gH}{u_i} = \frac{0.8 \times 9.81 \times 10}{3.5} = 22.423 \text{ m/s}$$

(i) $\alpha = \tan^{-1}\left(\dfrac{V_{fi}}{V_{wi}}\right) = \tan^{-1}\left(\dfrac{13.31}{22.423}\right) = \mathbf{30.69°}$

(ii) $\theta = \tan^{-1}\left(\dfrac{V_{fi}}{V_{wi} - u_i}\right) = \tan^{-1}\left(\dfrac{13.31}{22.423 - 3.5}\right) = \mathbf{35.13°}$

(iii) $\because u_i = \dfrac{\pi D_i N}{60}$

$$\therefore D_i = \frac{60u_i}{\pi N} = \frac{60 \times 3.5}{\pi \times 200} = \mathbf{0.3342 \text{ m}}$$

(iv) $Q = \dfrac{1000P}{\rho_w gH\eta_o} = \dfrac{1000 \times 150}{1000 \times 9.81 \times 10 \times 0.75} = 2.039 \text{ m}^3/\text{s}$

Since $Q = \pi D_i B_i V_{fi}$

$$\therefore B_i = \frac{Q}{\pi D_i V_{fi}} = \frac{2.039}{\pi \times 0.3342 \times 13.31} = \mathbf{0.1459 \text{ m}}$$

Example 22.4 The following data is given for a Francis turbine, such as net head = 65 m, speed = 500 rpm, shaft power = 300 kW, overall efficiency = 80%, hydraulic efficiency = 90%, flow ratio = 0.2, breadth ratio = 0.1, outer diameter of runner = 2 times inner diameter of runner, thickness of vanes occupy 6% of circumferential area of runner, velocity of flow is constant at the inlet and outlet and discharge is radial at the outlet. Determine (i) the diameter of runner at the inlet and outlet, (ii) width of wheel at the inlet, (iii) guide blade angle and (iv) runner vane angle at the inlet and outlet.

Solution

Refer Figure 22.11. Let $H = 65$ m, $N = 500$ rpm, $P = 300$ kW, $\eta_o = 0.8$, $\eta_h = 0.9$, $K_f = 0.2$, $(B_i/D_i) = 0.1$, $D_i = 2D_o$, $A_t = 0.06\pi D_i B_i$, $V_{fi} = V_{fo}$ and $V_{wo} = 0$.

(i) $V_{fi} = V_{fo} = K_f\sqrt{2gH} = 0.2 \times \sqrt{2 \times 9.81 \times 65} = 7.1423 \text{ m/s}$

$$Q = \frac{1000P}{\rho_w gH\eta_o} = \frac{1000 \times 300}{1000 \times 9.81 \times 65 \times 0.8} = 0.5881 \text{ m}^3/\text{s}$$

Since $A_t = 0.06\pi D_i B_i$

The actual area of flow is given by,

$$A_a = 0.94\pi D_i B_i = 0.94\pi D_i \times 0.1 D_i = 0.094\pi D_i^2 \qquad [\because B_i = 0.1D_i]$$

Since $$Q = A_a \times V_{fi} = 0.094\pi D_i^2 \times V_{fi}$$

$$\therefore D_i = \sqrt{\frac{Q}{0.094\pi V_{fi}}} = \sqrt{\frac{0.5881}{0.094\pi \times 7.1423}} = \mathbf{0.528\ m}$$

$$D_o = \frac{D_i}{2} = \frac{0.528}{2} = \mathbf{0.264\ m}$$

(ii) $B_i = 0.1D_i = 0.1 \times 0.528 = \mathbf{0.0528\ m\ or\ 52.8\ mm}$

(iii) $u_i = \dfrac{\pi D_i N}{60} = \dfrac{\pi \times 0.528 \times 500}{60} = 13.823$ m/s

Since $$\eta_h = \frac{V_{wi} u_i}{gH}$$

$$\therefore V_{wi} = \frac{\eta_h g H}{u_i} = \frac{0.9 \times 9.81 \times 65}{13.823} = 41.517 \text{ m/s}$$

$$\alpha = \tan^{-1}\left(\frac{V_{fi}}{V_{wi}}\right) = \tan^{-1}\left(\frac{7.1423}{41.517}\right) = \mathbf{9.76^\circ}$$

(iv) $\theta = \tan^{-1}\left(\dfrac{V_{fi}}{V_{wi} - u_i}\right) = \tan^{-1}\left(\dfrac{7.1423}{41.517 - 13.823}\right) = \mathbf{14.46^\circ}$

$$u_o = \frac{\pi D_o N}{60} = \frac{\pi \times 0.264 \times 500}{60} = 6.91 \text{ m/s}$$

$$\phi = \tan^{-1}\left(\frac{V_{fo}}{u_o}\right) = \tan^{-1}\left(\frac{7.1423}{6.91}\right) = \mathbf{45.95^\circ}$$

Example 22.5 An inward flow reaction turbine has an external and internal diameter as 1 m and 0.5 m, respectively. The turbine is running at 180 rpm and the width of turbine at the inlet is 250 mm. The velocity of flow through the runner is constant and it is equal to 2 m/s. The guide blade makes an angle of 10° to the tangent of the wheel and the discharge at the outlet is radial. If the net head on the turbine is 15 m, then determine (i) the absolute velocity of water at the inlet of runner, (ii) velocity of whirl at the inlet, (iii) relative velocity at the inlet, (iv) runner blade angles, (v) width of the runner at the outlet, (vi) mass of water flowing through the runner per second, (vii) power developed and (viii) hydraulic efficiency.

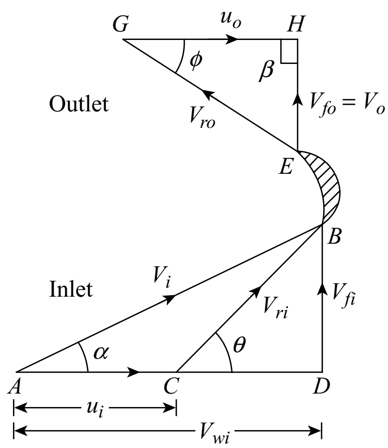


Figure 22.12

Solution

Refer Figure 22.12. Let $D_i = 1$ m, $D_o = 0.5$ m, $N = 180$ rpm, B_i = 250 mm = 0.25 m, $V_{fi} = V_{fo} = 2$ m/s, $\alpha = 10°$, $V_{wo} = 0$ and $H = 15$ m.

(i) $u_i = \dfrac{\pi D_i N}{60} = \dfrac{\pi \times 1 \times 180}{60} = 9.425 \text{ m/s}$

$u_o = \dfrac{\pi D_o N}{60} = \dfrac{\pi \times 0.5 \times 180}{60} = 4.712 \text{ m/s}$

$V_i = \dfrac{V_{fi}}{\sin\alpha} = \dfrac{2}{\sin 10°} = \mathbf{11.52\ m/s}$

(ii) $V_{wi} = V_i \cos\alpha = 11.52 \cos 10° = \mathbf{11.345\ m/s}$

(iii) $V_{ri} = \sqrt{V_{fi}{}^2 + (V_{wi} - u_i)^2} = \sqrt{2^2 + (11.345 - 9.425)^2} = \mathbf{2.772\ m/s}$

(iv) $\theta = \tan^{-1}\left(\dfrac{V_{fi}}{V_{wi} - u_i}\right) = \tan^{-1}\left(\dfrac{2}{11.345 - 9.425}\right) = \mathbf{46.17°}$

$\phi = \tan^{-1}\left(\dfrac{V_{fo}}{u_o}\right) = \tan^{-1}\left(\dfrac{2}{4.712}\right) = \mathbf{23°}$

(v) $\because \pi D_i B_i V_{fi} = \pi D_o B_o V_{fo}$

$\therefore B_o = \dfrac{D_i B_i}{D_o} = \dfrac{1 \times 0.25}{0.5} = \mathbf{0.5\ m} \qquad [\because V_{fi} = V_{fo}]$

(vi) $Q = \pi D_i B_i V_{fi} = \pi \times 1 \times 0.25 \times 2 = 1.571 \text{ m}^3\text{/s}$

Therefore, mass per second is given by,

$m = \rho_w Q = 1000 \times 1.571 = \mathbf{1571\ kg/s}$

(vii) $P = \dfrac{\rho_w Q V_{wi} u_i}{1000} = \dfrac{1000 \times 1.571 \times 11.345 \times 9.425}{1000} = \mathbf{167.982\ kW}$

(viii) $\eta_h = \dfrac{V_{wi} u_i}{gH} = \dfrac{11.345 \times 9.425}{9.81 \times 15} \times 100 = \mathbf{72.66\%}$

Example 22.6 An inward flow reaction turbine has external and internal diameters as 1 m and 0.6 m, respectively. The head on the turbine is 35 m and the velocity of flow at the outlet is 2.3 m/s. The hydraulic efficiency of the turbine is 88%. If the vane angle at the outlet is 14° and the width of the wheel is 110 mm at the inlet and outlet, then determine (i) the speed of the turbine, (ii) guide blade and vane angles at the inlet, (iii) volume flow rate of turbine and (iv) power developed. Assume radial discharge at the outlet.

Solution

Refer Figure 22.12. Let $D_i = 1$ m, $D_o = 0.6$ m, $H = 35$ m, $V_{fo} = 2.3$ m/s, $\eta_h = 0.88$, $\phi = 14°$, $B_i = B_o = 110$ mm $= 0.11$ m and $V_{wo} = 0$.

(i) $u_o = \dfrac{V_{fo}}{\tan\phi} = \dfrac{2.3}{\tan 14°} = 9.225 \text{ m/s}$

Since $u_o = \dfrac{\pi D_o N}{60}$

$\therefore N = \dfrac{60 u_o}{\pi D_o} = \dfrac{60 \times 9.225}{\pi \times 0.6} = \mathbf{293.64\ rpm}$

(ii) $u_i = \frac{\pi D_i N}{60} = \frac{\pi \times 1 \times 293.64}{60} = 15.375 \text{ m/s}$

Since $$\eta_h = \frac{V_{wi} u_i}{gH}$$

$$\therefore V_{wi} = \frac{\eta_h g H}{u_i} = \frac{0.88 \times 9.81 \times 35}{15.375} = 19.652 \text{ m/s}$$

Since $$\pi D_i B_i V_{fi} = \pi D_o B_o V_{fo}$$

$$\therefore V_{fi} = \frac{D_o V_{fo}}{D_i} = \frac{0.6 \times 2.3}{1} = 1.38 \text{ m/s} \quad [\because B_i = B_o]$$

$$\alpha = \tan^{-1}\left(\frac{V_{fi}}{V_{wi}}\right) = \tan^{-1}\left(\frac{1.38}{19.652}\right) = \mathbf{4.02°}$$

$$\theta = \tan^{-1}\left(\frac{V_{fi}}{V_{wi} - u_i}\right) = \tan^{-1}\left(\frac{1.38}{19.652 - 15.375}\right) = \mathbf{17.88°}$$

(iii) $Q = \pi D_i B_i V_{fi} = \pi \times 1 \times 0.11 \times 1.38 = \mathbf{0.4769 \text{ m}^3/\text{s}}$

(iv) $P = \frac{\rho_w Q V_{wi} u_i}{1000} = \frac{1000 \times 0.4769 \times 19.652 \times 15.375}{1000} = \mathbf{144.095 \text{ kW}}$

Example 22.7 An outward flow reaction turbine is running at 275 rpm. The internal and external diameters of the turbine are 2 m and 2.75 m, respectively. The width of the runner is constant at the inlet and outlet and it is equal to 275 mm. The head on the turbine is 180 m and the rate of flow through the turbine is 6 m^3/s. If the discharge at the outlet is radial, then determine (i) the velocity of flow at the inlet and outlet of the runner and (ii) vane angles.

Solution

Refer Figure 22.13. Let $N = 275$ rpm, $D_i = 2$ m, $D_o = 2.75$ m, $B_i = B_o = 275$ mm $= 0.275$ m, $H = 180$ m, $Q = 6 \text{ m}^3/\text{s}$, $V_{wo} = 0$ and thus, $V_o = V_{fo}$.

(i) $u_i = \frac{\pi D_i N}{60} = \frac{\pi \times 2 \times 275}{60} = 28.8 \text{ m/s}$

$$u_o = \frac{\pi D_o N}{60} = \frac{\pi \times 2.75 \times 275}{60} = 39.6 \text{ m/s}$$

Since $$Q = \pi D_i B_i V_{fi} = \pi D_o B_o V_{fo}$$

$$\therefore V_{fi} = \frac{Q}{\pi D_i B_i} = \frac{6}{\pi \times 2 \times 0.275} = \mathbf{3.472 \text{ m/s}}$$

$$\therefore V_{fo} = \frac{Q}{\pi D_o B_o} = \frac{6}{\pi \times 2.75 \times 0.275} = \mathbf{2.525 \text{ m/s}}$$

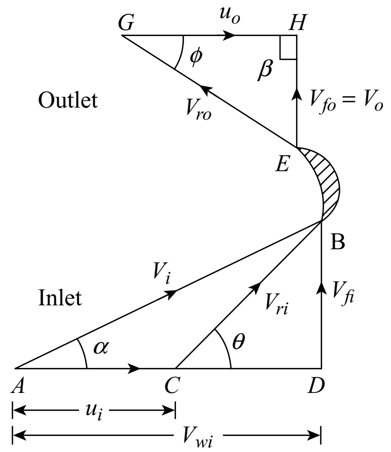


Figure 22.13

(ii) $H - \frac{V_{fo}^2}{2g} = \frac{V_{wi} u_i}{g}$ [From Equation (22.13)]

Thus $180 - \frac{2.52^2}{2 \times 9.81} = \frac{V_{wi} \times 28.8}{9.81}$

$$\therefore V_{wi} = \frac{9.81}{28.8} \times \left(180 - \frac{2.525^2}{2 \times 9.81}\right) = 61.2 \text{ m/s}$$

$$\theta = \tan^{-1}\left(\frac{V_{fi}}{V_{wi} - u_i}\right) = \tan^{-1}\left(\frac{3.472}{61.2 - 28.8}\right) = \mathbf{6.12°}$$

$$\phi = \tan^{-1}\left(\frac{V_{fo}}{u_o}\right) = \tan^{-1}\left(\frac{2.525}{39.6}\right) = \mathbf{3.65°}$$

Example 22.8 The following data pertains to an inward flow reaction turbine, where whirl velocity at the inlet to the runner = $3.05\sqrt{H}$ m/s, whirl velocity at the outlet to the runner in the same direction as at the inlet = $0.2\sqrt{H}$ m/s, flow velocity at the inlet = $1.02\sqrt{H}$ m/s, flow velocity at the outlet = $0.8\sqrt{H}$ m/s, where H is the head in metres. The inner diameter of the runner = 0.5 times the outer diameter. If hydraulic efficiency of the turbine is 85%, then determine the angles of the runner vanes at the inlet and exit and the guide blade angle.

Solution

Refer Figure 22.14. Let $V_{wi} = 3.05\sqrt{H}$ m/s, $V_{wo} = 0.2\sqrt{H}$ m/s, $V_{fi} = 1.02\sqrt{H}$ m/s, $V_{fo} = 0.8\sqrt{H}$ m/s, $D_o = 0.5D_i$ and $\eta_h = 0.85$.

Since $\dfrac{u_i}{D_i} = \dfrac{u_o}{D_o}$ [Speed ∝ Diameter]

$$\therefore u_o = \frac{D_o}{D_i} \times u_i = \frac{0.5D_i}{D_i} \times u_i = 0.5u_i$$

Since $\eta_h = \dfrac{V_{wi}u_i - V_{wo}u_o}{gH}$

Thus $$0.85 = \frac{3.05\sqrt{H} \times u_i - 0.2\sqrt{H} \times 0.5u_i}{9.81 \times H} = \frac{2.95\sqrt{H} \times u_i}{9.81H}$$

$$\therefore u_i = \frac{0.85 \times 9.81H}{2.95\sqrt{H}} = 2.827\sqrt{H}$$

$$u_o = 0.5u_i = 0.5 \times 2.827\sqrt{H} = 1.4135\sqrt{H}$$

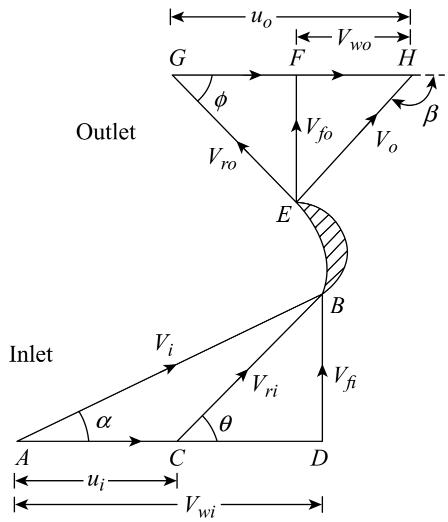


Figure 22.14

Since $u_o > V_{wo}$, $\beta > 90°$ as shown in Figure 22.14.

$$\alpha = \tan^{-1}\left(\frac{V_{fi}}{V_{wi}}\right) = \tan^{-1}\left(\frac{1.02\sqrt{H}}{3.05\sqrt{H}}\right) = \mathbf{18.49°}$$

$$\theta = \tan^{-1}\left(\frac{V_{fi}}{V_{wi} - u_i}\right) = \tan^{-1}\left(\frac{1.02\sqrt{H}}{3.05\sqrt{H} - 2.827\sqrt{H}}\right) = \mathbf{77.67°}$$

$$\phi = \tan^{-1}\left(\frac{V_{fo}}{u_o - V_{wo}}\right) = \tan^{-1}\left(\frac{0.8\sqrt{H}}{1.4135\sqrt{H} - 0.2\sqrt{H}}\right) = \mathbf{33.39°}$$

Example 22.9 A Francis turbine of specific speed 100 develops 15.2×10^3 kW under a head of 200 m. The overall efficiency is 0.86 and the velocity of flow is constant and is equal to 10 m/s. The hydraulic efficiency is 0.89, the ratio of width to diameter of wheel at the inlet is equal to 0.1 and the area occupied by the thickness of the blades is equal to 5% of the area of water way. Workout the area, guide blade angle, vane angle, peripheral velocity and velocity of whirl at the inlet. Assume axial discharge.

Solution

Refer Figure 22.15. Let $N_s = 100$, $P = 15.2\times10^3$ kW, $H = 200$ m, $\eta_o = 0.86$, $V_{fi} = V_{fo} = 10$ m/s, $\eta_h = 0.89$, $(B_i/D_i) = 0.1$, $A_t = 0.05\pi D_i B_i$ and $V_{wo} = 0$.

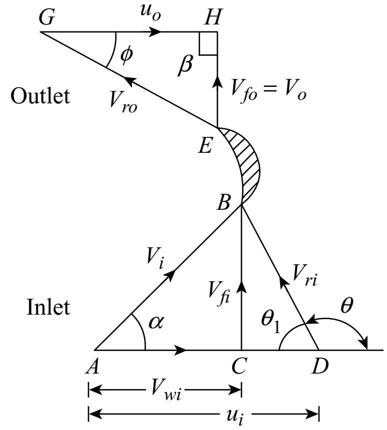


Figure 22.15

Since $$N_s = \frac{N\sqrt{P}}{H^{5/4}}$$

$$\therefore N = \frac{N_s H^{5/4}}{\sqrt{P}} = \frac{100\times200^{5/4}}{\sqrt{15.2\times10^3}} = 610.05 \text{ rpm}$$

$$Q = \frac{1000P}{\rho_w g H \eta_o} = \frac{1000\times15.2\times10^3}{1000\times9.81\times200\times0.86} = 9.01 \text{ m}^3/\text{s}$$

Since $A_t = 0.05\pi D_i B_i$

The actual area of flow is given by,

$$A_a = 0.95\pi D_i B_i = 0.95\pi D_i \times 0.1 D_i = 0.095\pi D_i^2 \qquad [\because B_i = 0.1D_i]$$

$$Q = A_a \times V_{fi} = 0.095\pi D_i^2 \times 10 = 0.95\pi D_i^2$$

$$\therefore D_i = \sqrt{\frac{Q}{0.95\pi}} = \sqrt{\frac{9.01}{0.95\pi}} = \mathbf{1.7375\ m}$$

$$B_i = 0.1D_i = 0.1\times1.7375 = 0.17375 \text{ m}$$

$$A_a = 0.095\pi D_i^2 = 0.095\pi\times1.7375^2 = \mathbf{0.901\ m^2}$$

$$u_i = \frac{\pi D_i N}{60} = \frac{\pi\times1.7375\times610.05}{60} = \mathbf{55.5\ m/s}$$

Since $$\eta_h = \frac{V_{wi}u_i}{gH}$$

$$\therefore V_{wi} = \frac{\eta_h gH}{u_i} = \frac{0.89\times9.81\times200}{55.5} = \mathbf{31.463\ m/s}$$

As $u_i > V_{wi}$, velocity triangle at the inlet will be as shown in Figure 22.15.

$$\alpha = \tan^{-1}\left(\frac{V_{fi}}{V_{wi}}\right) = \tan^{-1}\left(\frac{10}{31.463}\right) = \mathbf{17.63^\circ}$$

$$\tan\theta_1 = \tan(180^\circ - \theta) = \frac{V_{fi}}{u_i - V_{wi}} = \frac{10}{55.5 - 31.463} = 0.416$$

Thus $$(180^\circ - \theta) = \tan^{-1}(0.416) = 22.59^\circ$$

$$\therefore \theta = 180^\circ - 22.59^\circ = \mathbf{157.41^\circ}$$

Example 22.10 The following data pertains to an inward flow reaction turbine, such as shaft power = 185.5 kW, head = 10 m, speed = 250 rpm, peripheral velocity of the runner = $0.9\sqrt{2gH}$ m/s, radial velocity of flow = $0.3\sqrt{2gH}$ m/s, overall efficiency of the turbine = 78.6%, hydraulic losses = 10% of total head, where H is the effective head on the turbine in meters. If discharge at the outlet is radial, then determine (i) the diameter of the runner, (ii) width of the runner, (iii) guide blade angle and (iv) vane inlet angle of the runner.

Solution

Refer Figure 22.15. Let $P = 185.5$ kW, $H = 10$ m, $N = 250$ rpm, $u_i = 0.9\sqrt{2gH}$, $V_{fi} = 0.3\sqrt{2gH}$, $\eta_o = 0.786$, hydraulic losses = 0.1H and $V_{wo} = 0$.

(i) $u_i = 0.9\sqrt{2gH} = 0.9\times\sqrt{2\times 9.81\times 10} = 12.61$ m/s

Since $$u_i = \frac{\pi D_i N}{60}$$

$$\therefore D_i = \frac{60u_i}{\pi N} = \frac{60\times 12.61}{\pi\times 250} = \mathbf{0.9633\ m}$$

(ii) $V_{fi} = 0.3\sqrt{2gH} = 0.3\times\sqrt{2\times 9.81\times 10} = 4.2$ m/s

$$Q = \frac{1000P}{\rho_w gH\eta_o} = \frac{1000\times 185.5}{1000\times 9.81\times 10\times 0.786} = 2.406\ \text{m}^3/\text{s}$$

Since $Q = \pi D_i B_i V_{fi}$

$$\therefore B_i = \frac{Q}{\pi D_i V_{fi}} = \frac{2.406}{\pi\times 0.9633\times 4.2} = \mathbf{0.1893\ m}$$

(iii) Head supplied = Work done + Hydraulic losses [Energy balance]

$$H = \frac{V_{wi}u_i}{g} + 0.1\text{H}$$

Thus $$\frac{V_{wi}u_i}{g} = 0.9\text{H}$$

$$\therefore V_{wi} = \frac{0.9H\times g}{u_i} = \frac{0.9\times 10\times 9.81}{12.61} = 7\ \text{m/s}$$

Since $V_{wi} < u_i$, the velocity triangle will be as shown in Figure 22.15.

$$\alpha = \tan^{-1}\left(\frac{V_{fi}}{V_{wi}}\right) = \tan^{-1}\left(\frac{4.2}{7}\right) = \mathbf{30.96°}$$

(iv) $$\tan\theta_1 = \tan(180° - \theta) = \frac{V_{fi}}{u_i - V_{wi}} = \frac{4.2}{12.61 - 7} = 0.7487$$

Thus $(180° - \theta) = \tan^{-1}(0.7487) = 36.82°$

$$\therefore \theta = 180° - 36.82° = \mathbf{143.18°}$$

Example 22.11 For a Francis turbine, derive an expression for hydraulic efficiency as a function of guide blade angle α and runner vane inlet angle θ when the velocity of flow is constant and the turbine has radial discharge at the outlet. Also show that hydraulic efficiency of the turbine is given by $\eta_h = [2/(2+\tan^2\alpha)]$ if the vanes are radial at the inlet.

Solution

Refer Figure 22.16.

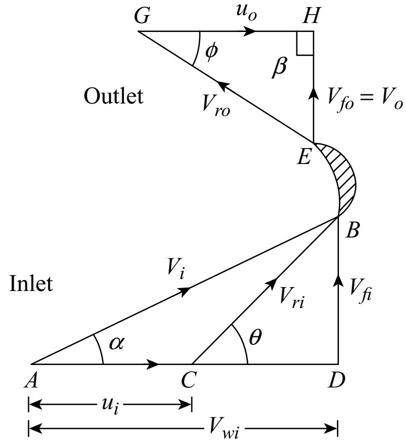


Figure 22.16

$$V_{fi} = V_{wi}\tan\alpha \qquad \text{(i)}$$

$$V_{wi} - u_i = \frac{V_{fi}}{\tan\theta} = \frac{V_{wi}\tan\alpha}{\tan\theta}$$

Thus $$u_i = V_{wi} - \frac{V_{wi}\tan\alpha}{\tan\theta} = V_{wi}\left(1-\frac{\tan\alpha}{\tan\theta}\right) \qquad \text{(ii)}$$

Since discharge is radial, $V_{wo} = 0$ and $V_o = V_{fo}$.

The velocity of flow is constant and therefore, $V_o = V_{fo} = V_{fi}$.

$$H = \frac{V_{wi}u_i}{g} + \frac{V_{fi}^2}{2g} \quad \text{[From Equation (22.13)]} \qquad \text{(iii)}$$

Substituting V_{fi} and u_i from expressions (i) and (ii) in expression (iii), we get:

$$H = \frac{V_{wi}}{g}V_{wi}\left(1-\frac{\tan\alpha}{\tan\theta}\right) + \frac{(V_{wi}\tan\alpha)^2}{2g} = \frac{V_{wi}^2}{g}\left(1-\frac{\tan\alpha}{\tan\theta}+\frac{\tan^2\alpha}{2}\right) \qquad \text{(iv)}$$

Now $$\eta_h = \frac{V_{wi}u_i}{gH} \qquad \text{(v)}$$

Substituting the values of u_i and H in expression (v), we get:

$$\eta_h = \frac{V_{wi}\times V_{wi}\left[1-(\tan\alpha/\tan\theta)\right]}{g\times(V_{wi}^2/g)[1-(\tan\alpha/\tan\theta)+(1/2)\tan^2\alpha]}$$

$$\therefore \eta_h = \frac{\left[1-(\tan\alpha/\tan\theta)\right]}{[1-(\tan\alpha/\tan\theta)+(1/2)\tan^2\alpha]} = \boldsymbol{\frac{1}{1+\dfrac{(1/2)\tan^2\alpha}{[1-(\tan\alpha/\tan\theta)]}}}$$

If vanes are radial at the inlet, then $\theta = 90°$.

Substituting this value in the above expression, we get:

$$\eta_h = \frac{1}{1+\dfrac{(1/2)\tan^2\alpha}{[1-(\tan\alpha/\tan 90°)]}} = \frac{1}{1+\dfrac{(1/2)\tan^2\alpha}{[1-0]}} = \frac{1}{1+\dfrac{\tan^2\alpha}{2}}$$

$$\therefore \eta_h = \frac{2}{2+\tan^2\alpha}$$

Hence proved.

Example 22.12 An outward flow reaction turbine works under a head of 7.5 m and running at a speed of 250 rpm. It has internal and external diameters of the runner as 0.5 m and 1 m, respectively. The guide blade angle is 20°. The velocity of flow through the runner is constant and equal to 3.5 m/s and the flow rate is 300 litres per second. If discharge at the outlet is radial, then determine (i) the runner vane angle at the inlet, (ii) runner vane angle at the outlet, (iii) work done by the runner per second per unit weight of water striking per second, (iv) power produced by the runner, (v) hydraulic efficiency and (vi) degree of reaction.

Solution

Refer Figure 22.16. Let $H = 7.5$ m, $N = 250$ rpm, $D_i = 0.5$ m, $D_o = 1$ m, $\alpha = 20°$, $V_{fi} = V_{fo} = 3.5$ m/s, $Q = 300$ l/s = 0.3 m^3/s and $V_{wo} = 0$.

$$u_i = \frac{\pi D_i N}{60} = \frac{\pi \times 0.5 \times 250}{60} = 6.545 \text{ m/s}$$

$$u_o = \frac{\pi D_o N}{60} = \frac{\pi \times 1 \times 250}{60} = 13.09 \text{ m/s}$$

$$V_{wi} = \frac{V_{fi}}{\tan\alpha} = \frac{3.5}{\tan 20^o} = 9.62 \text{ m/s}$$

(i) $\theta = \tan^{-1}\left(\dfrac{V_{fi}}{V_{wi} - u_i}\right) = \tan^{-1}\left(\dfrac{3.5}{9.62 - 6.545}\right) = \mathbf{48.7°}$

(ii) $\phi = \tan^{-1}\left(\dfrac{V_{fo}}{u_o}\right) = \tan^{-1}\left(\dfrac{3.5}{13.09}\right) = \mathbf{14.97°}$

(iii) $w = \dfrac{V_{wi} u_i}{g} = \dfrac{9.62 \times 6.545}{9.81} = \mathbf{6.418\ Nm/N}$

(iv) $P = \dfrac{\rho_w Q V_{wi} u_i}{1000} = \dfrac{1000 \times 0.3 \times 9.62 \times 6.545}{1000} = \mathbf{18.889\ kW}$

(v) $\eta_h = \dfrac{V_{wi} u_i}{gH} = \dfrac{9.62 \times 6.545}{9.81 \times 7.5} \times 100 = \mathbf{85.58\%}$

(vi) $R = 1 - \dfrac{\cot\alpha}{2(\cot\alpha - \cot\theta)} = 1 - \dfrac{\cot 20^o}{2(\cot 20^o - \cot 48.7^o)} = \mathbf{0.265}$

Example 22.13 The runner diameter of a reaction turbine is 3 m at the inlet and 2 m at the outlet. The width of the wheel is constant. The discharge through the turbine is 75 m^3/s and is radial at the outlet having a velocity of 12 m/s. The angle of the vanes at the inlet is 120°. If the working head is 160 m and the hydraulic efficiency of the turbine is 0.9, then determine its speed and power produced.

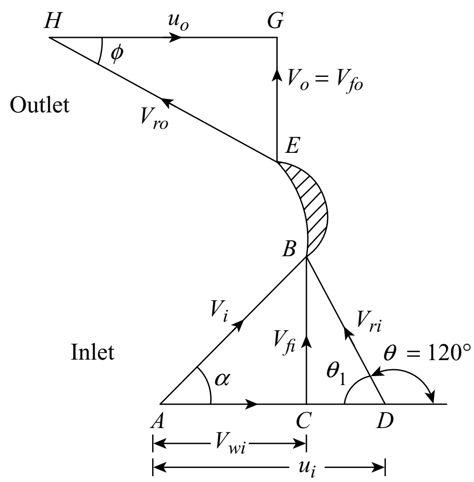


Figure 22.17

Solution

Refer Figure 22.17. Let $D_i = 3$ m, $D_o = 2$ m, $B_i = B_o$, $Q = 75$ m^3/s, $V_{wo} = 0$, $V_o = V_{fo} = 12$ m/s, $\theta = 120°$, $H = 160$ m and $\eta_h = 0.9$.

Since $\quad Q = \pi D_i B_i V_{fi} = \pi D_o B_o V_{fo}$

Thus $\quad V_{fi} = \dfrac{D_o}{D_i} \times V_{fo} = \dfrac{2}{3} \times 12 = 8$ m/s $\quad [\because B_i = B_o]$

$$\theta_1 = 180° - \theta = 180° - 120° = 60°$$

$$u_i - V_{wi} = \frac{V_{fi}}{\tan\theta_1} = \frac{8}{\tan 60°} = 4.62 \text{ m/s}$$

or $$V_{wi} = (u_i - 4.62) \text{ m/s}$$

Since $$\eta_h = \frac{V_{wi}u_i}{gH}$$

Thus $$u_i = \frac{\eta_h gH}{V_{wi}} = \frac{0.9 \times 9.81 \times 160}{u_i - 4.62}$$

$$u_i^2 - 4.62u_i - 1412.64 = 0$$

$$\therefore u_i = \frac{4.62 \pm \sqrt{4.62^2 + 4 \times 1412.64}}{2} = 39.97 \text{ m/s} \qquad \text{[Take positive value]}$$

Since $$u_i = \frac{\pi D_i N}{60}$$

$$\therefore N = \frac{60u_i}{\pi D_i} = \frac{60 \times 39.97}{\pi \times 3} = \mathbf{254.46\ rpm}$$

$$V_{wi} = u_i - 4.62 = 39.97 - 4.62 = 35.35 \text{ m/s}$$

$$P = \frac{\rho_w Q V_{wi} u_i}{1000} = \frac{1000 \times 75 \times 35.35 \times 39.97}{1000} = \mathbf{105970.4625\ kW}$$

Summary

1. **Reaction turbines:** Water entering the runner possesses both kinetic as well as pressure energy.
2. **Radial flow turbine:** Water flows in the radial direction through the runner.
3. In inward radial flow reaction turbine, water enters at the outer circumference and flows inwards radially towards the centre of the runner, whereas in outward radial flow turbine water enters at the centre and flows radially outwards.
4. **Euler's momentum equation:** $w = (V_{wi}\, u_i \pm V_{wo}\, u_o)/g$
5. **Radial discharge:** At outlet is $\beta = 90°$ and $V_{wo} = 0$. At inlet is $\alpha = 90°$ and $V_{wi} = 0$.
6. **Francis turbine:** Inward radial flow reaction turbine with radial discharge at the outlet.
7. Work done per second per unit weight of water in Francis turbine: $w = V_{wi}u_i/g$
8. Hydraulic efficiency of the Francis turbine: $\eta_h = (V_{wi}u_i)/gH$
9. **Degree of reaction (*R*):** Ratio of pressure energy change in the runner to the total energy change in the stator and the runner.
10. **Degree of reaction:** $R = 1 - [\cot\alpha/2(\cot\alpha - \cot\theta)]$
11. **Speed ratio:** $K_u = u_i/\sqrt{2gH}$ and for Francis turbine it varies from 0.6 to 0.9.
12. **Flow ratio:** $K_f = V_{fi}/\sqrt{2gH}$ and for Francis turbine it varies from 0.15 to 0.3.
13. Ratio of width to the diameter of wheel is $n = B_i / D_i$, its value varies from 0.1 to 0.45.
14. Discharge through a Francis turbine: $Q = \pi D_i B_i V_{fi} = \pi D_o B_o V_{fo}$
15. **Head on the turbine:** $H - V_o^2/(2g) = (V_{wi}u_i \pm V_{wo}u_o)/g$

Multiple-choice Questions

1. The value of speed ratio and flow ratio in case of Francis turbine, respectively ranges from
(a) 0.2 to 0.3 and 0.35 to 0.5.
(b) 0.4 to 0.5 and 0.6 to 0.9.
(c) 0.6 to 0.9 and 0.15 to 0.30.
(d) None of the above.

2. In all reaction turbines, for maximum efficiency
(a) The whirl velocity must be zero at entry.
(b) The whirl velocity must be zero at exit.
(c) The flow velocity must be zero at entry.
(d) The flow velocity must be zero at exit.

3. Specific speed of a fluid machine is
(a) Specific to the particular machine.
(b) A type number representative of its performance.
(c) The speed of a machine of unit dimensions.
(d) None of the above.

4. The reaction turbines as compared to impulse turbines have
(a) Low speed.
(b) High speed.
(c) Equal speed.
(d) None of the above.

5. Most of the reaction turbines have degree of reaction between
(a) 0.1 to 0.3.
(b) 0.4 to 0.6.
(c) 0.7 to 0.9.
(d) None of the above.

6. The degree of reaction for a Pelton turbine is
(a) Half.
(b) Equal to a Francis turbine.
(c) Equal to jet velocity.
(d) Zero.

7. Dimensionless specific speeds of Pelton, Francis and Kaplan turbines in sequence are
(a) 0.02, 0.6, 0.9. (b) 0.2, 6, 9.
(c) 2, 60, 90. (d) 20, 600, 900.

8. A low specific speed Francis turbine is respectively
(a) Tangential flow turbine and mixed flow turbine.
(b) Mixed flow turbine and radial flow turbine.
(c) Radial flow turbine and axial flow turbine.
(d) Axial flow turbine and tangential flow turbine.

Review Questions

1. Give comparisons between (i) impulse and reaction turbines and (ii) inward and outward radial flow reaction turbines.

2. Explain the construction, working and governing mechanisms of a Francis turbine.

3. Define speed and flow ratios. Also give their role in design of a Francis turbine.

4. Derive Euler's equation for hydraulic machines.

5. Derive an expression for change in kinetic and pressure energy of water in the runner of a Francis turbine.

6. Define the term degree of reaction in hydraulic turbines. In a Francis turbine, the velocity of flow through the runner is constant and discharge is radial. Show that the degree of reaction R is given by $R = 1 - [\cot\alpha/\{2(\cot\alpha - \cot\theta)\}]$, where α and θ are guide blade angle and vane angle at the inlet of the runner. Neglect the losses in runner and neglect the differences in elevation at the inlet and outlet.

7. Obtain the following expressions for hydraulic efficiency for a Francis turbine in terms of guide blade angle (α) and vane angle (θ) at the inlet as follows.

(i) $\eta_h = \left[1 + \dfrac{(1/2)\tan^2\alpha}{1 - (\tan\alpha/\tan\theta)}\right]^{-1}$ and

(ii) $\eta_h = \left[1 - \dfrac{1}{1 + 2\cot\alpha(\cot\alpha - \cot\theta)}\right]$

Problems

1. A Francis turbine works at 500 rpm under a head of 150 m. Its diameter at the inlet is 1 m and the flow area is 0.3 m^2. The guide blade and vane angles at the inlet are 15° and 60°, respectively. Determine (i) the volume flow rate, (ii) power developed and (iii) hydraulic efficiency. Assume whirl velocity zero at the outlet.
[**Ans.** 2.49 m^3/s, 2.02 MW, 55.09 %]

2. A turbine rotates at 200 rpm working under a discharge of 10 m^3/s and a head of 30 m. If the overall efficiency of the turbine is 90%, then determine (i) the specific speed, (ii) power developed under the head of 25 m and (iii) type of turbine.
[**Ans.** 146.6, 2207.25 kW, Francis turbine]

3. A Francis turbine working under a head of 7.5 m is required to produce 140 kW power with an overall efficiency of 76%. The wheel runs at 155 rpm and the hydraulic losses in the turbine are 20% of the available energy. The peripheral velocity and the radial velocity of flow at the inlet are $0.25\sqrt{2gH}$ and $0.95\sqrt{2gH}$, respectively. Determine (i) the guide blade angle, (ii) wheel vane angle at the inlet, (iii) diameter of the wheel at the inlet and (iv) width of the wheel at the inlet. Assume radial discharge at the outlet.
[Ans. 30.68°, 35.1°, 0.3733 m, 0.1853 m]

4. An outward radial flow reaction turbine is running at 200 rpm. The inner and outer diameters of the runner are 0.5 m and 1 m, respectively. The inlet guide vane angle is 15° and the discharge is radial. The head available on the turbine is 12 m. The velocity of flow is constant and it is equal to 5 m/s. If the water flow rate through the turbine is 0.25 m^3/s, then determine (i) the runner vane angle at the inlet and outlet, (ii) power developed by the turbine, (iii) and hydraulic efficiency.
[Ans. 20.42°, 25.55°, 24.4 kW, 82.9%]

5. An inward flow reaction turbine has an external diameter of 1 m and its breadth at the inlet is 0.25 m. The velocity of flow at the inlet is 1.9 m/s and 6% of the area of flow is blocked by blade thickness. If speed of the runner is 190 rpm and guide blades make an angle of 10° to the wheel tangent, then find (i) velocity of wheel at the inlet, (ii) mass of water passing per second through the runner, (iii) runner vane angle at the inlet, (iv) absolute velocity of water leaving guide vanes, and (v) relative velocity of water entering runner blade.
[Ans. 9.95 m/s, 1403 kg/s, 66.53°, 10.942 m/s, 2.07 m/s]

6. Design a Francis turbine runner with the following data, such as net head = 68.5 m, speed = 750.2 rpm, output power = 330 kW, hydraulic efficiency = 0.945, overall efficiency = 0.85, flow ratio = 0.15, breadth ratio = 0.1, and inner diameter of runner is half of outer diameter. Also assume 6% of circumferential area of the runner is occupied by the thickness of the vanes. Velocity of flow remains constant throughout and the flow is radial at exit.
[Ans. 0.597 m, 0.2985 m, 11.46°, 56.53°, 25.09°]

7. The internal and external diameters of outward radial flow reaction turbine are 1 m and 1.5 m, respectively. The water flow rate through the turbine is 3 m^3/s. The widths of the runner at the inlet and outlet are same and equal to 0.2 m. The head on the turbine is 100 m and it is running at 300 rpm. If the discharge is radial, then find out (i) the runner blade angles at the inlet and outlet, (ii) velocity of flow at the inlet and outlet, (iii) power developed by the turbine. Neglect the blade thickness.
[Ans. 5.87°, 7.69°, 4.77 m/s, 3.18 m/s, 2.93 MW]

8. An inward flow reaction turbine working under a head of 8.2 m is required to produce 150.6 kW power with an overall efficiency of 80.5%. The wheel runs at 150 rpm and the hydraulic losses in the turbine are 20% of the available energy. The peripheral velocity and the radial velocity of flow at the inlet are $0.35\sqrt{2gH}$ and $0.97\sqrt{2gH}$ respectively. If discharge is radial at the outlet, then determine (i) the guide blade angle, (ii) wheel vane angle at the inlet, (iii) diameter of the wheel at the inlet and (iv) width of the wheel at the inlet.
[Ans. 40.33°, 147.87°, 1.57 m, 0.1062 m]

9. The external and internal diameters of an inward flow reaction turbine are 1 m and 0.6 m, respectively. The hydraulic efficiency of the turbine is 90% when the head on the turbine is 36 m and discharge is radial at the outlet. The velocity of flow at the outlet is 2.5 m/s. If the vane angle at the outlet is 15° and width of the wheel is 100 mm at the inlet and outlet, then determine (i) the guide blade angle, (ii) speed of the turbine, (iii) vane angle of the runner at the inlet, (iv) volume flow rate of turbine and (v) power developed.
[Ans. 4.19°, 296.98 rpm, 17.05°, 0.4712 m^3/s, 149.76 kW]

10. The following data pertains to an inward flow reaction turbine, such as net head = 60 m, speed = 650 rpm, brake power = 275 kW, ratio of wheel width to wheel diameter at the inlet = 0.1, ratio of inner diameter to outer diameter = 0.5, flow ratio = 0.17, hydraulic efficiency of the turbine = 95% and overall efficiency = 85%. The flow velocity remains constant and the discharge is radial. Neglect area blockage by blades, workout the dimensions and blade angles of the turbine.
[Ans. 0.548 m, 0.274 m, 27.4 mm, 54.8 mm, 11°, 27.23°, 32.03°]

11. In an outward radial flow reaction turbine, the rim speed at the inlet is 22.5 m/s and the diameter ratio is 0.8. The vane angle at the inlet and exit are 90° and 20°, respectively. If radial velocity at the inlet is 7.5 m/s and discharge is radial, then determine (i) the guide blade angle, (ii) velocity of water from guide blades, (iii) pressure head at the inlet to the runner and (iv) hydraulic efficiency. Neglect the frictional losses in the runner.
[Ans. 18.44°, 23.72 m/s, 10.23 m/s, 56.94 m, 90.63%]

12. The following data pertains to a Francis turbine, such as net head = 110 m, speed = 275 rpm, brake power = 14.99 MW, ratio of wheel width to wheel diameter at the inlet = 0.15, ratio of inner diameter to outer diameter = 1.8, flow ratio = 0.2, hydraulic efficiency of the turbine = 90% and overall efficiency = 85%. The flow velocity remains constant and the discharge is radial. Neglect area blockage by blades, determine (i) the inlet and outlet diameter, (ii) guide blade angle and (iii) vane angles.
[Ans. 1.932 m, 1.073 m, 14.9°, 52.65°, 31.02°]

13. A modern Francis turbine works under a head of 182 m while rotating at 452 rpm. The runner diameter at the inlet is 1.5 m and width at the inlet is 0.15 m. If the guide blade angle is 14° and the vane angle at the inlet is 45°, then determine (i) the power developed and (ii) hydraulic efficiency.
[Ans. 13.99 MW, 94.03%]

14. An inward radial flow reaction turbine is working under a head of 18 m. The external and internal diameters of this turbine are 1.2 m and 0.6 m, respectively. The velocity of flow through the runner remains constant and it is equal to 2 m/s. The guide blade angle is given as 10° and the runner vanes are radial at the inlet. Determine (i) the speed of the turbine, (ii) vane angle at the outlet of the runner and (iii) hydraulic efficiency. Assume that discharge at the outlet is radial.
[**Ans.** 180.48 rpm, 19.43°, 72.82%]

15. The inlet and outlet diameters of the runner of an inward flow reaction turbine are 1.45 m and 0.9 m, respectively. The width of the runner is constant and it is equal to 150 mm. The hydraulic efficiency of the turbine is 90% and it works under a head of 110 m. If the discharge velocity is 5.9 m/s, then determine (i) the speed of the turbine, (ii) guide blade angle, (iii) blade angles and (iii) power produced. Take radial discharge at the outlet and blade angle at the exit as 15°.
[**Ans.** 467.28 rpm, 7.61°, 155.71°, 2.429 MW]

16. An inward radial flow reaction turbine is working under a head of 15 m and the water is supplied at a rate of 0.3 m^3/s. The inlet diameter is twice the outer diameter and the vanes are radial at the inlet. The runner completes 350 rpm and the velocity of flow is constant and it is equal to 1.8 m/s. There are no losses in the runner and the discharge is radial. If speed ratio is equal to 0.8, then determine (i) the guide vane angle, (ii) inlet and outlet diameters of the runner and (iii) width of the runner at the inlet and exit. Neglect the thickness of the vanes.
[**Ans.** 7.46°, 0.749 m, 0.374 m, 0.0708 m, 0.1416 m]

17. An inward radial flow reaction turbine works under a head of 25 m. The velocity of wheel periphery at the inlet is 10 m/s. The outlet pipe of the turbine is 0.25 m in diameter. If the turbine is supplied with 0.2 m^3/s, then determine (i) the vane angle at the inlet, (ii) guide vane angle and (iii) power of the turbine. Assume the radial velocity of flow through the wheel equal to the velocity in outlet pipe and neglect the friction.
[**Ans.** 22.146°, 16.56°, 47.382 kW]

18. The following data pertains to a vertical shaft Francis turbine, such as diameter of the runner at the inlet = 2 m, width of the runner at the inlet = 0.27 m, speed = 420 rpm, discharge = 15 m^3/s, velocity at the inlet = 10 m/s, pressure head at the inlet = 230 m, elevation above the tail race = 5 m, hydraulic efficiency = 98% and overall efficiency = 92%. Find out (i) the total head across the turbine, (ii) power output, (iii) guide blade angle and (iv) vane angle at the inlet.
[**Ans.** 240.09 m, 32.5 MW, 62.13°, 167.33°]

19. The following particulars are given for an inward radial flow reaction turbine, such as head = 24 m, discharge = 10 m^3/s, speed = 225 rpm, inlet angle of the runner vane = 110° measured from the direction of runner rotation, entry of water to the runner is without shock and with a velocity of flow = 6 m/s, and to the draft tube is without whirl and with a velocity = 5.5 m/s, discharge velocity from the exit of draft tube = 2.2 m/s, mean height of the runner entry surface = 1.5 m, and the entrance to the draft tube = 1.2 m above the tail race level. If both hydraulic and overall efficiencies are 92%, then determine (i) the diameter of the runner at entry surface, (ii) pressure head at entry to the runner and at entrance to the draft tube when the friction loss in the runner is 0.7 m and that in the draft tube is 0.5 m of water, (iii) shaft power of the turbine and (iii) specific speed of the turbine.
[**Ans.** 1.345 m, 11.14 m of water, −1.522 m of water, 2.166 MW, 197.13]

20. The following data pertains to a Francis turbine, such as head = 100 m, overall efficiency = 86%, hydraulic efficiency = 92%, speed = 500 rpm, shaft power = 2500 kW, flow ratio = 0.16, outer diameter = 2 × Inner diameter, ratio of wheel width to its diameter at the inlet = 0.1 and the flow velocity is constant. Find out the main dimensions of the given turbine.
[**Ans.** 1.153 m, 0.1153 m, 0.5765 m, 0.231 m, 13.34°, 92.3°, 25.17°]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (c)	**2.** (b)	**3.** (b)	**4.** (b)	**5.** (b)
6. (d)	**7.** (a)	**8.** (a)		

Chapter 23

Propeller and Kaplan Turbines (Axial Flow Reaction Turbines)

23.1 ❐ INTRODUCTION

When water flows parallel to the axis of the rotation of the shaft, the turbine is called axial flow turbine. The runner of axial flow turbine works under the pressure, since the head at the inlet of the turbine is the sum of pressure energy and kinetic energy. When water flows through the runner, a part of pressure energy is converted into kinetic energy. Thus, these turbines are called axial flow reaction turbines. The most widely used axial flow reaction turbines are propeller and Kaplan turbines.

In a Francis turbine, the number of blades is more due to which the contact surface with water is more and thus, a high value of frictional resistance is offered. The blades also receive water in radial direction and discharge it in axial direction, i.e., there is 90° bend to water. These are the main sources of hydraulic losses in a Francis turbine. These losses have been overcome in axial flow reaction turbines.

When the head on a turbine decreases, the discharge must increase to produce the required power. Since propeller and Kaplan turbines are low head turbines, a large quantity of water is required to flow through its runner to produce the required amount of power. The axial flow reaction turbines are designed to have minimum number of blades which reduce the frictional losses. Further, water enters the blades in axial direction from one side and leaves axially through the other side so that large quantity of water flows through the runner.

The shaft of axial flow reaction turbines is vertical. The lower end of the shaft is made larger and it is known as hub. The vanes are mounted on the hub and it acts as a runner for axial flow reaction turbines. The runner of an axial flow turbine usually has only three to eight blades. When the vanes are fixed and non-adjustable, the turbine is known as propeller turbine. The Kaplan turbine is just a propeller turbine in which the runner blades are made adjustable. Some of the important Kaplan turbine installations in India are Bhakra Nangal project (Punjab), Hirakud dam project (Orissa), Radhanagari hydroelectric scheme (Maharashtra), Nizam Sagar project (Andhra Pradesh) and Tungabhadra hydroelectric scheme (Karnataka).

In this chapter, the characteristic properties of propeller and Kaplan turbines are discussed, which are particularly suited for low head and high discharge installations. Draft tube which is an integral part of the Francis and Kaplan turbines and cavitation in reaction turbines is discussed. A brief description of the new types of turbines, namely Deriaz (or diagonal), tubular and bulb turbines is also given.

23.2 ❐ PROPELLER AND KAPLAN TURBINES

The propeller and Kaplan turbines are the important types of axial flow reaction turbines. Due to the highest specific speed (up to 860), these turbines are suitable for low heads (up to 30 m) and large flow of water. In these turbines, water flows parallel to the axis of the rotation of the shaft. The shaft of axial flow turbine is vertical. The lower end of the shaft is made

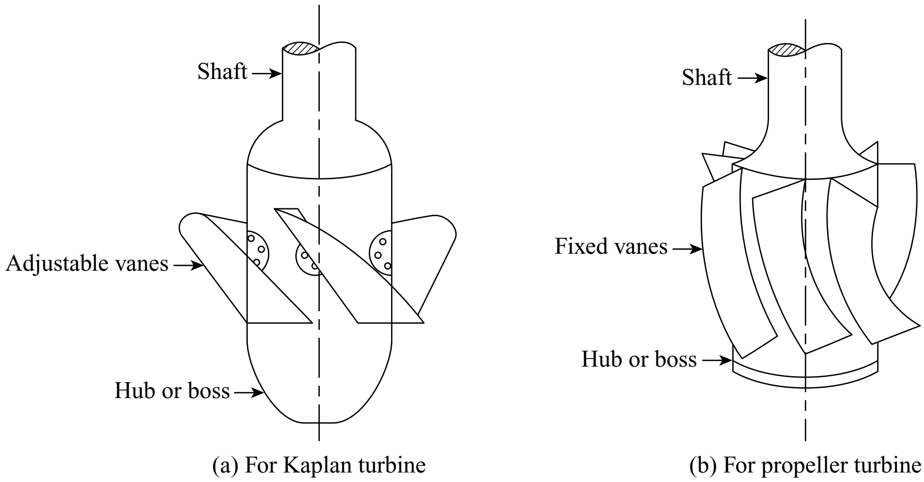


Figure 23.1 *Runners of Kaplan and propeller turbines*

larger which is known as 'hub' or 'boss'. The hub of these turbines either has fixed blades or variable pitch blades and it acts as a runner for axial flow turbine.

If the blades are fixed to the hub and non-adjustable, then it is known as propeller turbine. However, if the blades are of variable pitch and they are adjustable on the hub, then it is known as Kaplan turbine, which is named after Dr. V. Kaplan, an Austrian engineer. Due to the adjustable pitch of the blades, the Kaplan turbine can deal with a wide range of operating heads. Practically, all Kaplan turbines have single runner. The runners of a Kaplan turbine and propeller turbine are shown in Figure 23.1. The efficiency of propeller and Kaplan turbines may be achieved as high as 94%.

The propeller turbines are easy to construct but their efficiency falls sharply at reduced loads. Therefore, these turbines should be kept fully loaded for maximum efficiency. A propeller turbine is most suitable when load on the turbine remains constant. In propeller turbines, as in the Francis turbine, the runner blades are fixed. So, water enters with a shock and eddies are formed which reduces the efficiency under part load condition.

The main parts of a Kaplan turbine are (1) scroll casing, (2) stay ring (only in big units), (3) guide vanes mechanism, (4) hub with vanes or runner of the turbine and (5) draft tube. The schematic view of a Kaplan turbine is shown in Figure 23.2.

The casing, stay ring, guide mechanism and draft tube are similar to that of a Francis turbine. The Kaplan turbine also runs full, which means it is a reaction type of turbine and it operates in a closed conduit from the inlet to the tail race. A space has been provided between the ends of guide vanes and the leading edges of the runner. This space is called whirl chamber in which the direction of flow changes from radial to axial. Thus, the runner of a Kaplan turbine varies from the runner of a Francis turbine based on the following points.

1. In a Kaplan turbine, water enters the runner axially, while in a Francis turbine it enters radially.
2. The number of blades in the Kaplan turbine varies from 3 to 8, whereas in the Francis turbine it varies from 16 to 24.
3. The blades of the Kaplan turbine are made hollow and it is made of stainless steel. The blades are attached to the hub in such a way that they are able to move on their axis.

The water from the penstock enters the scroll casing and then moves to the guide vanes. From the guide vanes, water turns through 90° and flows axially through the runner as shown in Figure 23.2. The discharge through the runner is given by the following expression.

$$\boxed{Q = \frac{\pi}{4}(D_o^2 - D_b^2)\,V_{fi}} \tag{23.1}$$

Here, D_o is the outer diameter of the runner, D_b is the diameter of the hub or boss and V_{fi} is the velocity of flow at inlet.

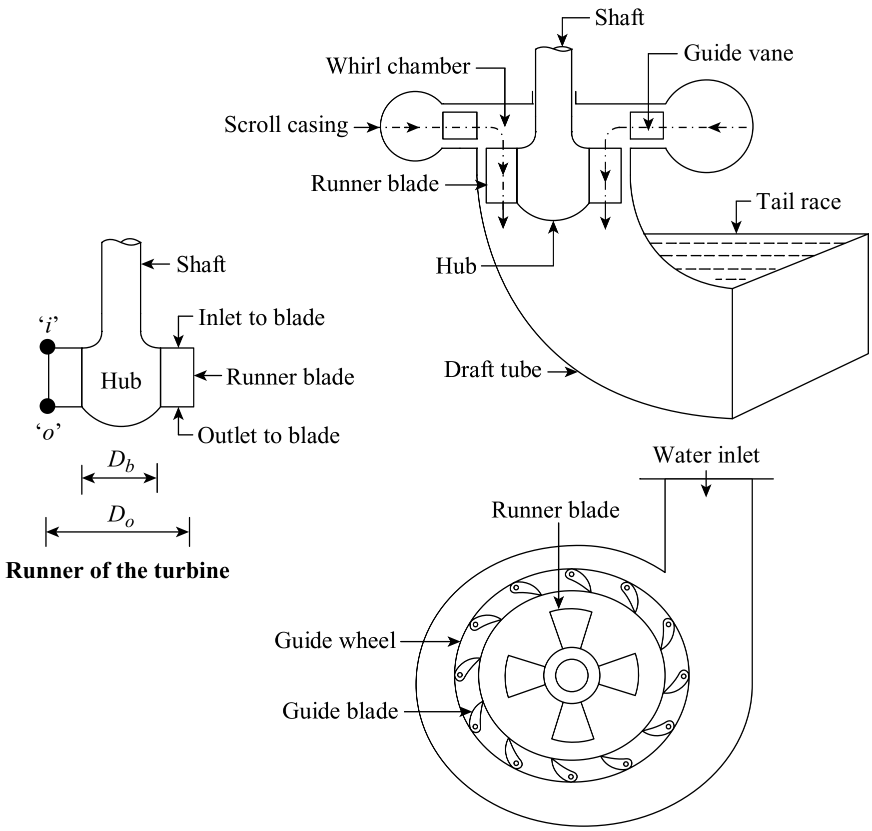


Figure 23.2 *Kaplan turbine and its runner*

The inlet and outlet velocity triangles are drawn at the extreme edge (i.e., outer periphery) of the runner blade corresponding to the points '*i*' and '*o*' as shown in Figure 23.2. It should be observed that the peripheral speed of the runner blades varies with the radial distance from the axis of rotation. Therefore, the inlet and outlet velocity triangles are different at different sections. Similarly, the vane angles at inlet and outlet are made different at different sections. Therefore, the vanes are twisted to permit shockless entry of water to the runner.

The Kaplan turbine is often known as variable pitch propeller turbine. The runner blades of a Kaplan turbine can be turned about their own axis, so that their angle of inclination may be adjusted while the turbine is in motion. The pitch of the runner blades is automatically adjusted by the governor through the action of a servomotor operating inside the hollow coupling of turbine and generator shaft. Under part load condition, when a lower discharge is flowing through a Kaplan turbine, high efficiency can be attained by the proper adjustment of blades during its operation. Due to the adjustment of blade angles, water under all working conditions flows through the runner blades without shock. Thus, the eddy losses which occur in Francis and propeller turbines are eliminated in a Kaplan turbine. Therefore, high efficiency can be attained in case of a Kaplan turbine working over a wide range of operating conditions.

23.2.1 Governing of Kaplan Turbine

In case of a Kaplan turbine, in addition to the guide vanes, the runner vanes are also adjustable. Therefore, the governor is required to operate both sets of vanes simultaneously. The runner vanes are also operated by a separate servomotor mechanism. The servomotor mechanisms for both runner and guide vanes are interconnected to ensure that for a given guide vane opening, there shall be a definite runner vane inclination. Kaplan turbines are called double regulated because the flow rate is controlled in two ways, i.e., by turning the wicket gates and by adjusting the pitch on the runner blades.

Propeller turbines are single regulated, i.e., the flow rate is regulated only by the wicket gates. The important differences between governing operation of Francis and Kaplan turbine are given below.

1. In a Kaplan turbine, there is double regulation, which means the guide vanes and the runner vanes regulation is done. Therefore, two servomotors are always required, where one servomotor controls the guide vanes and the other controls the runner vanes.
   In a Francis turbine, there is single regulation, which means the guide vanes regulation is done. Therefore, usually, one servomotor is required to control the guide vanes.
2. In a Kaplan turbine, correct disposition of the guide and moving blades is attained at any load, whereas in a Francis turbine it is attained at full load only.
3. In a Kaplan turbine, high efficiency is attained even at reduced load since both guide vanes and runner vanes are controlled simultaneously. However, in a Francis turbine, high efficiency is attained only at full load since only guide vanes are controlled.
4. In a Kaplan turbine, heavy duty governor is required, whereas in a Francis turbine ordinary governor is required.

23.3 ❐ WORKING PROPORTIONS OF KAPLAN AND PROPELLER TURBINES

The expressions for the work done, efficiency and power developed by propeller and Kaplan turbines are same as that of a Francis turbine. The main dimensions of the runner are obtained by a procedure similar to that of a Francis runner. However, the following main deviations occur.

1. Velocity of flow remains constant, i.e., $V_{fi} = V_{fo} = V_f = K_f\sqrt{2gH}$, here K_f is the flow ratio which has a value of around 0.7.
2. The ratio of the hub diameter (D_b) to runner diameter (D_o) usually varies from 0.35 to 0.6 and it is given by the relation $n = (D_b/D_o)$.
3. The discharge (Q) flowing through the runner remains constant and it is given by,

$$\boxed{Q = \frac{\pi}{4}(D_o{}^2 - D_b{}^2)V_f} \tag{23.2}$$

$$V_f = K_f\sqrt{2gH}$$

$$\boxed{Q = \frac{\pi}{4}(D_o{}^2 - D_b{}^2)K_f\sqrt{2gH}} \tag{23.2a}$$

$$n = \frac{D_b}{D_o}$$

$$Q = \frac{\pi}{4}D_o{}^2(1-n^2)K_f\sqrt{2gH} \tag{23.2b}$$

4. The peripheral velocity at inlet and outlet are equal and it is given by,

$$\boxed{u_i = u_o = \frac{\pi D_o N}{60}} \tag{23.3}$$

   Also $u_i = K_u\sqrt{2gH}$, here K_u is the speed ratio.

5. Area of flow is constant, i.e., area of flow at inlet is equal to area of flow at outlet.

$$\boxed{A_i = A_o = \frac{\pi}{4}(D_o^2 - D_b^2)} \tag{23.4}$$

23.4 ❐ DIFFERENCE BETWEEN FRANCIS AND KAPLAN TURBINES

Francis turbine	Kaplan turbine
It is a mixed flow turbine.	It is a purely axial flow turbine.
It has large number of blades in runner and it varies from 16 to 24.	It has small number of blades in runner and it varies from 3 to 8.
The runner blades are not adjustable.	The runner blades are adjustable.
The disposition of shaft may be horizontal or vertical.	The disposition of shaft is always vertical.
It is a medium head turbine and the head varies between 60 m to 250 m.	It is a low head turbine and it is capable to work under the head less than 60 m.
It works under medium discharge.	It works under high discharge.
Its specific speed varies from 50 to 255.	Its specific speed varies from 255 to 860.
Resistance to overcome is large due to large number of vanes and greater area of contact with water.	Resistance to overcome is less due to small number of vanes and less contact area.
Ordinary governor is used for its governing.	Heavy duty governor is used for its governing.
For the same power, it is less compact.	For the same power, it is more compact.

Example 23.1 A Kaplan turbine develops 50×10^3 kW under a net head of 30 m with an overall efficiency of 85%. Taking the value of speed ratio = 2, flow ratio = 0.6 and diameter of the hub = 0.35 times of the diameter of the runner, then calculate (i) the diameter of the runner, (ii) speed of the turbine and (iii) specific speed of the turbine.

Solution

Let $P = 50\times10^3$ kW, $H = 30$ m, $\eta_o = 0.85$, $K_u = 2$, $K_f = 0.6$ and $D_b = 0.35D_o$.

(i) $u_i = K_u\sqrt{2gH} = 2\times\sqrt{2\times9.81\times30} = 48.522$ m/s

$$V_{fi} = K_f\sqrt{2gH} = 0.6\times\sqrt{2\times9.81\times30} = 14.56 \text{ m/s}$$

$$Q = \frac{1000P}{\rho_w gH\eta_o} = \frac{1000\times50\times10^3}{1000\times9.81\times30\times0.85} = 199.876 \text{ m}^3/\text{s}$$

$$Q = \frac{\pi}{4}(D_o{}^2 - D_b{}^2)V_{fi}$$

$$199.876 = \frac{\pi}{4}\times[D_o{}^2 - (0.35D_o)^2]\times14.56$$

$$10.034\,D_o{}^2 = 199.876$$

$$\therefore D_o = \sqrt{\frac{199.876}{10.034}} = \mathbf{4.4632\ m}$$

(ii) $u_i = \dfrac{\pi D_o N}{60}$

$$\therefore N = \frac{60u_i}{\pi D_o} = \frac{60\times48.522}{\pi\times4.4632} = \mathbf{207.63\ rpm}$$

(iii) $N_s = \dfrac{N\sqrt{P}}{H^{5/4}} = \dfrac{207.63\times\sqrt{50\times10^3}}{30^{5/4}} = \mathbf{661.26}$

Example 23.2 A Kaplan turbine working under a head of 25 m develops 11.8 MW shaft power. The outlet diameter of the runner is 3 m and the hub diameter is 1.5 m. The guide blade angle at the extreme edge of the runner is 30°. The hydraulic and overall efficiency of the turbines are 95% and 90%, respectively. If the velocity of whirl is zero at outlet, then determine the runner vane angles at the extreme edge of the runner and speed of the turbine.

Solution

Refer Figure 23.3. Let $H = 25$ m, $P = 11.8\text{ MW} = 11.8\times10^3$ kW, $D_o = 3$ m, $D_b = 1.5$ m, $\alpha = 30°$, $\eta_h = 0.95$, $\eta_o = 0.9$ and $V_{wo} = 0$.

$$Q = \frac{1000P}{\rho_w gH\eta_o} = \frac{1000\times11.8\times10^3}{1000\times9.81\times25\times0.9} = 53.46\text{ m}^3/\text{s}$$

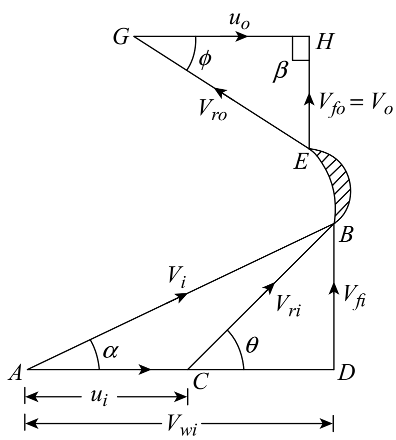


Figure 23.3

Also
$$Q = \frac{\pi}{4}(D_o^2 - D_b^2)V_{fi}$$

$$53.46 = \frac{\pi}{4}\times(3^2 - 1.5^2)\times V_{fi}$$

$$\therefore V_{fi} = V_{fo} = \frac{53.46\times4}{\pi\times(3^2-1.5^2)} = 10.08\text{ m/s}$$

$$V_{wi} = \frac{V_{fi}}{\tan\alpha} = \frac{10.08}{\tan 30°} = 17.46\text{ m/s}$$

Since
$$\eta_h = \frac{V_{wi}u_i}{gH}$$

$$\therefore u_i = u_o = \frac{\eta_h gH}{V_{wi}} = \frac{0.95\times9.81\times25}{17.46} = 13.344\text{ m/s}$$

$$\theta = \tan^{-1}\left(\frac{V_{fi}}{V_{wi}-u_i}\right) = \tan^{-1}\left(\frac{10.08}{17.46-13.344}\right) = \mathbf{67.79°}$$

$$\phi = \tan^{-1}\left(\frac{V_{fo}}{u_o}\right) = \tan^{-1}\left(\frac{10.08}{13.344}\right) = \mathbf{37.07°}$$

Since
$$u_i = \frac{\pi D_o N}{60}$$

$$\therefore N = \frac{60u_i}{\pi D_o} = \frac{60\times13.344}{\pi\times3} = \mathbf{84.95\ rpm}$$

Example 23.3 The hub diameter of a Kaplan turbine working under a head of 15 m is 0.3 times the diameter of the runner. The turbine is running at 90 rpm and the velocity of whirl at outlet is zero. If the vane angle of the extreme edge of the runner at outlet is 15° and the flow ratio is 0.6, then determine (i) the diameter of runner, (ii) the diameter of boss and (iii) discharge through the runner.

Solution

Refer Figure 23.3. Let $H = 15$ m, $D_b = 0.3D_o$, $N = 90$ rpm, $V_{wo} = 0$, $\phi = 15°$ and $K_f = 0.6$.

(i) $V_{fi} = V_{fo} = K_f \sqrt{2gH} = 0.6 \times \sqrt{2 \times 9.81 \times 15} = 10.293$ m/s

$$u_i = u_o = \frac{V_{fo}}{\tan\phi} = \frac{10.293}{\tan 15^\circ} = 38.414 \text{ m/s}$$

Since
$$u_i = \frac{\pi D_o N}{60}$$

$$\therefore D_o = \frac{60u_i}{\pi N} = \frac{60 \times 38.414}{\pi \times 90} = \mathbf{8.152\ m}$$

(ii) $D_b = 0.3D_o = 0.3 \times 8.152 = \mathbf{2.4456\ m}$

(iii) $Q = \frac{\pi}{4}(D_o{}^2 - D_b{}^2)V_{fi} = \frac{\pi}{4} \times (8.152^2 - 2.4456^2) \times 10.293 = \mathbf{488.88\ m^3/s}$

Example 23.4 A propeller turbine runner has outer diameter of 4.5 m and the diameter of the hub is 2 m. It is required to develop 25 MW when running at 160 rpm under a head of 25 m. If the hydraulic efficiency is 94% and overall efficiency is 89%, then evaluate the runner angles at inlet and exit at the mean diameter of the vanes. Also evaluate the specific speed of the turbine.

Solution

Let $D_o = 4.5$ m, $D_b = 2$ m, $P = 25$ MW $= 25 \times 10^3$ kW, $N = 160$ rpm, $H = 25$ m, $\eta_h = 0.94$ and $\eta_o = 0.89$. Let D_m be the mean diameter.

$$Q = \frac{1000P}{\rho_w g H \eta_o} = \frac{25 \times 10^3 \times 1000}{1000 \times 9.81 \times 25 \times 0.89} = 114.536 \text{ m}^3/\text{s}$$

Also
$$Q = \frac{\pi}{4}(D_o{}^2 - D_b{}^2)V_{fi}$$

$$114.536 = \frac{\pi}{4} \times (4.5^2 - 2^2) \times V_{fi}$$

$$\therefore V_{fi} = V_{fo} = \frac{114.536 \times 4}{\pi \times (4.5^2 - 2^2)} = 8.974 \text{ m/s}$$

$$D_m = \frac{D_o + D_b}{2} = \frac{4.5 + 2}{2} = 3.25 \text{ m}$$

$$u_i = u_o = \frac{\pi D_m N}{60} = \frac{\pi \times 3.25 \times 160}{60} = 27.23 \text{ m/s}$$

$$\eta_h = \frac{V_{wi} u_i}{gH}$$

$$\therefore V_{wi} = \frac{\eta_h g H}{u_i} = \frac{0.94 \times 9.81 \times 25}{27.23} = 8.47 \text{ m/s}$$

Since $u_i > V_{wi}$, the velocity triangle at inlet will be as shown in Figure 23.4.

$$\tan\theta_1 = \tan(180° - \theta) = \frac{V_{fi}}{u_i - V_{wi}} = \frac{8.974}{27.23 - 8.47} = 0.47836$$

Thus $$(180° - \theta) = \tan^{-1}(0.47836) = 25.56°$$

$$\therefore \theta = 180° - 25.56° = \mathbf{154.44°}$$

$$\phi = \tan^{-1}\left(\frac{V_{fo}}{u_o}\right) = \tan^{-1}\left(\frac{8.974}{27.23}\right) = \mathbf{18.24°}$$

$$N_s = \frac{N\sqrt{P}}{H^{5/4}} = \frac{160\times\sqrt{25\times10^3}}{25^{5/4}} = \mathbf{452.55}$$

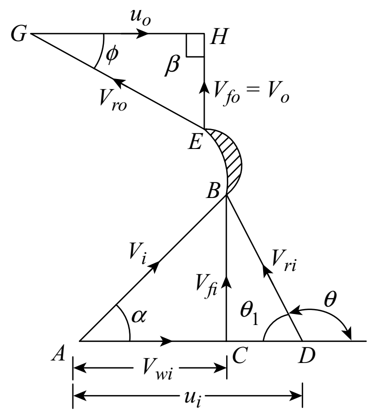


Figure 23.4

Example 23.5 A Kaplan turbine has been designed to develop 22.5 MW under a head of 20.5 m whilst running at 150 rpm. The other relevant data are overall efficiency = 87%, hydraulic efficiency = 94%, outer diameter of runner = 4.6 m and diameter of the hub = 2.1 m. If the turbine discharges without whirl at exit, then determine (i) the runner vane angles at the hub, (ii) the runner vane angles at the outer periphery and (iii) specific speed of the turbine.

Solution

Refer Figure 23.4. Let $P = 22.5$ MW $= 22.5\times10^3$ kW, $H = 20.5$ m, $N = 150$ rpm, $\eta_o = 0.87$, $\eta_h = 0.94$, $D_o = 4.6$ m, $D_b = 2.1$ m and $V_{wo} = 0$.

$$Q = \frac{1000P}{\rho_w g H \eta_o} = \frac{1000\times22.5\times10^3}{1000\times9.81\times20.5\times0.87} = 128.6 \text{ m}^3/\text{s}$$

Also $$Q = \frac{\pi}{4}(D_o^2 - D_b^2)V_{fi}$$

$$128.6 = \frac{\pi}{4}\times(4.6^2 - 2.1^2)\times V_{fi}$$

$$\therefore V_{fi} = V_{fo} = \frac{128.6\times4}{\pi\times(4.6^2 - 2.1^2)} = 9.775 \text{ m/s}$$

(i) $$u_i = u_o = \frac{\pi D_b N}{60} = \frac{\pi\times2.1\times150}{60} = 16.493 \text{ m/s}$$

$$\eta_h = \frac{V_{wi}u_i}{gH}$$

$$\therefore V_{wi} = \frac{\eta_h g H}{u_i} = \frac{0.94\times9.81\times20.5}{16.493} = 11.462 \text{ m/s}$$

Since $u_i > V_{wi}$, the velocity triangle at inlet will be as shown in Figure 23.4.

$$\tan\theta_1 = \tan(180° - \theta) = \frac{V_{fi}}{u_i - V_{wi}} = \frac{9.775}{16.493 - 11.462} = 1.943$$

$$(180° - \theta) = \tan^{-1}(1.943) = 62.77°$$

$$\therefore \theta = 180° - 62.77° = \mathbf{117.23°}$$

$$\phi = \tan^{-1}\left(\frac{V_{fo}}{u_o}\right) = \tan^{-1}\left(\frac{9.775}{16.493}\right) = \mathbf{30.65°}$$

(ii) $u_i = u_o = \dfrac{\pi D_o N}{60} = \dfrac{\pi \times 4.6 \times 150}{60} = 36.13 \text{ m/s}$

$$V_{wi} = \frac{\eta_h g H}{u_i} = \frac{0.94 \times 9.81 \times 20.5}{36.13} = 5.232 \text{ m/s}$$

$$\tan\theta_1 = \tan(180° - \theta) = \frac{V_{fi}}{u_i - V_{wi}} = \frac{9.775}{36.13 - 5.232} = 0.3164$$

$$(180° - \theta) = \tan^{-1}(0.3164) = 17.56°$$

$$\therefore \theta = 180° - 17.56° = \mathbf{162.44°}$$

$$\phi = \tan^{-1}\left(\frac{V_{fo}}{u_o}\right) = \tan^{-1}\left(\frac{9.775}{36.13}\right) = \mathbf{15.14°}$$

(iii) $N_s = \dfrac{N\sqrt{P}}{H^{5/4}} = \dfrac{150 \times \sqrt{22.5 \times 10^3}}{20.5^{5/4}} = \mathbf{515.81}$

Example 23.6 For a Kaplan turbine with a runner diameter 6.5 m, the discharge is 250 m^3/s and the hydraulic efficiency is stated to be 91%. The diameter of boss is 0.35 times the runner diameter. It develops 22.5 MW under a head of 12.5 m while running at 75 rpm. If the turbine discharges without whirl at exit, then determine (i) the overall efficiency of the turbine, (ii) flow ratio, (iii) speed ratio, (iv) degree of reaction and (v) specific speed of the turbine.

Solution

Refer Figure 23.4. Let $D_o = 6.5$ m, $Q = 250 \text{ m}^3/\text{s}$, $\eta_h = 0.91$, $D_b = 0.35 D_o$, $P = 22.5 \text{ MW} = 22.5 \times 10^3$ kW, $H = 12.5$ m, $N = 75$ rpm and $V_{wo} = 0$.

(i) $\eta_o = \dfrac{1000P}{\rho_w g Q H} = \dfrac{1000 \times 22.5 \times 10^3}{1000 \times 9.81 \times 250 \times 12.5 \times 100} \times 100 = \mathbf{73.39\%}$

(ii) $Q = \dfrac{\pi}{4}(D_o{}^2 - D_b{}^2)V_{fi} = \dfrac{\pi}{4}[D_o{}^2 - (0.35D_o)^2]V_{fi}$

Thus
$$250 = \frac{\pi}{4} \times [6.5^2 - (0.35 \times 6.5)^2] \times V_{fi}$$

$$\therefore V_{fi} = V_{fo} = \frac{250 \times 4}{\pi \times [6.5^2 - (0.35 \times 6.5)^2]} = 8.586 \text{ m/s}$$

$$K_f = \frac{V_{fi}}{\sqrt{2gH}} = \frac{8.586}{\sqrt{2 \times 9.81 \times 12.5}} = \mathbf{0.55}$$

(iii) $u_i = \frac{\pi D_o N}{60} = \frac{\pi \times 6.5 \times 75}{60} = 25.525$ m/s

$$K_u = \frac{u_i}{\sqrt{2gH}} = \frac{25.525}{\sqrt{2 \times 9.81 \times 12.5}} = \mathbf{1.63}$$

(iv) $\because \eta_h = \frac{V_{wi} u_i}{gH}$

$$\therefore V_{wi} = \frac{\eta_h gH}{u_i} = \frac{0.91 \times 9.81 \times 12.5}{25.525} = 4.372 \text{ m/s}$$

Since $u_i > V_{wi}$, the velocity triangle at inlet will be as shown in Figure 23.4.

$$V_i = \sqrt{V_{fi}^2 + V_{wi}^2} = \sqrt{8.586^2 + 4.372^2} = 9.635 \text{ m/s}$$

$$V_o = V_{fo} = V_{fi} = 8.586 \text{ m/s}$$

$$R = 1 - \frac{V_i^2 - V_o^2}{2g\eta_h H} = 1 - \frac{9.635^2 - 8.586^2}{2 \times 9.81 \times 0.91 \times 12.5} = \mathbf{0.9143}$$

(v) $N_s = \frac{N\sqrt{P}}{H^{5/4}} = \frac{75 \times \sqrt{22.5 \times 10^3}}{12.5^{5/4}} = \mathbf{478.65}$

Example 23.7 A Kaplan turbine has certain specifications, such as discharge = 67.5 m^3/s, hydraulic and mechanical efficiencies = 95%, runner diameter = 4.5 m, diameter of boss = 0.35 times the runner diameter, speed ratio = 2, there is no swirl at outlet and the discharge is free. Determine (i) the net head available on the turbine, (ii) power developed, (iii) runner speed and (iii) specific speed of the turbine.

Solution

Let $Q = 67.5 \text{ m}^3/\text{s}$, $\eta_h = \eta_m = 0.95$, $D_o = 4.5$ m, $D_b = 0.35 D_o$, $K_u = 2$ and $V_{wo} = 0$.

Since
$$Q = \frac{\pi}{4}(D_o^2 - D_b^2)V_{fi} = \frac{\pi}{4}[D_o^2 - (0.35D_o)^2]V_{fi}$$

Thus
$$67.5 = \frac{\pi}{4} \times [4.5^2 - (0.35 \times 4.5)^2] \times V_{fi}$$

$$\therefore V_{fi} = \frac{67.5 \times 4}{\pi \times [4.5^2 - (0.35 \times 4.5)^2]} = 4.84 \text{ m/s}$$

Since there is no swirl at outlet and we get $V_{fi} = V_{fo} = V_o = 4.84$ m/s.

(i) $H - \frac{V_o^2}{2g} = \frac{V_{wi} u_i}{g} = \eta_h H$

Thus
$$H - \frac{4.84^2}{2 \times 9.81} = 0.95H$$

$$H - 0.95H = 1.194$$

$$\therefore H = \frac{1.194}{0.05} = \mathbf{23.88 \text{ m}}$$

(ii) $P = \dfrac{\rho_w gQH\eta_o}{1000} = \dfrac{\rho_w gQH(\eta_h \eta_m)}{1000}$

$$\therefore P = \frac{1000\times 9.81\times 67.5\times 23.88\times 0.95\times 0.95}{1000} = \mathbf{14270.997\ kW}$$

(iii) $u_i = K_u\sqrt{2gH} = 2\times\sqrt{2\times 9.81\times 23.88} = 43.29$ m/s

Since $$u_i = \frac{\pi D_o N}{60}$$

$$\therefore N = \frac{60u_i}{\pi D_o} = \frac{60\times 43.29}{\pi\times 4.5} = \mathbf{183.73\ rpm}$$

(iv) $N_s = \dfrac{N\sqrt{P}}{H^{5/4}} = \dfrac{183.73\times\sqrt{14270.997}}{23.88^{5/4}} = \mathbf{415.78}$

Example 23.8 The following data pertain to a propeller turbine with specifications, such as power developed = 85 MW, head = 25 m, speed = 200 rpm, overall efficiency = 86%, diameter of boss = 30% of external diameter of the runner, speed ratio = 2.06 and flow ratio = 0.67. Determine (i) the diameter of the runner, (ii) discharge through the turbine, (iii) number of turbines and (iii) specific speed of the turbine.

Solution

Let $P_t = 85$ MW $= 85\times 10^3$ kW, $H = 25$ m, $N = 200$ rpm, $\eta_o = 0.86$, $D_b = 0.3D_o$, $K_u = 2.06$ and $K_f = 0.67$.

Let n_t be the number of turbines.

(i) $u_i = K_u\sqrt{2gH} = 2.06\times\sqrt{2\times 9.81\times 25} = 45.623$ m/s

Since $$u_i = \frac{\pi D_o N}{60}$$

$$\therefore D_o = \frac{60u_i}{\pi N} = \frac{60\times 45.623}{\pi\times 200} = \mathbf{4.357\ m}$$

(ii) $V_{fi} = K_f\sqrt{2gH} = 0.67\times\sqrt{2\times 9.81\times 25} = 14.84$ m/s

Since $$Q = \frac{\pi}{4}(D_o{}^2 - D_b{}^2)V_{fi} = \frac{\pi}{4}[D_o{}^2 - (0.3D_o)^2]V_{fi}$$

$$\therefore Q = \frac{\pi}{4}\times[4.357^2 - (0.3\times 4.357)^2]\times 14.84 = \mathbf{201.345\ m^3/s}$$

(iii) $P = \dfrac{\rho_w gQH\eta_o}{1000} = \dfrac{1000\times 9.81\times 201.345\times 25\times 0.86}{1000} = 42466.681$ kW

$$n_t = \frac{P_t}{P} = \frac{85\times 10^3}{42466.681} = \mathbf{2}$$

(iv) $N_s = \dfrac{N\sqrt{P}}{H^{5/4}} = \dfrac{200\times\sqrt{42466.681}}{25^{5/4}} = \mathbf{737.27}$

23.5 ❐ DRAFT TUBE

A draft tube is an airtight pipe of gradually increasing cross-sectional area. It is an important part of a reaction turbine. Here, one end of the draft tube is connected to the runner exit while the other end is sub-merged deep into the tail race under all operating conditions. The water after passing through the runner is discharged to the tail race through the draft tube. The draft tube should be submerged to about one metre below the tail race level and it is made of cast steel, plate steel or concrete. In addition to provide a passage for water discharge, a draft tube has to perform the following functions.

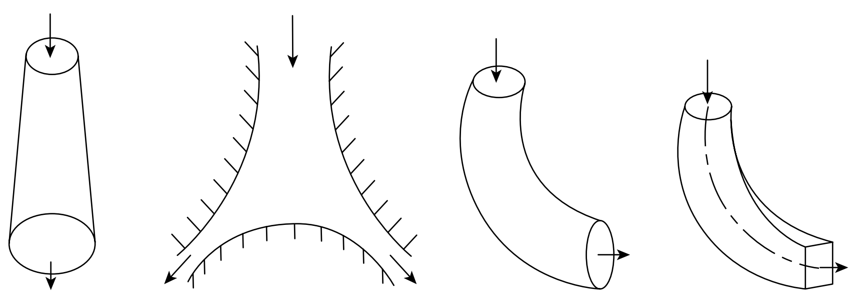
1. When water is discharged freely from the runner, the turbine works under a head equal to the height of the head race level above the runner outlet. However, when a draft tube connects the runner to the tail race, the workable head increases by an amount equal to the height of the runner outlet above the tailrace.

   Thus, the incorporation of a draft tube permits the turbine runner to be installed above the tail race without any loss of available head by maintaining a negative or suction head at the outlet of the runner. Eventually, it causes increase in the net head and thereby, the output of the turbine. It also helps to carry out inspection and repair work of the turbine easily.
2. The water leaving the runner possesses high velocity which would be lost if it is discharged freely. A draft tube reduces the velocity of the discharged water by which the loss of kinetic energy at the runner outlet minimizes and as a result, the pressure head increases. In other words, a large proportion of kinetic energy rejected at the runner exit is converted into useful pressure energy by which the efficiency of the turbine increases.

23.5.1 Types of Draft Tubes

The important type of draft tubes which are commonly employed in reaction turbines are given below.

1. **Conical draft tube or straight divergent tube:** The shape of a conical draft tube (Figure 23.5(a)) is that of the frustum of a cone. The central cone angle is kept less than 8^{o} so as to prevent the flow separation. Generally, it is employed for vertical shaft Francis turbines having low specific speed. The conical draft tube is the most efficient and its maximum efficiency varies from 85% to 90%.
2. **Moody's spreading tube or the hydracone:** Moody's spreading draft tube (Figure 23.5(b)) is provided with a solid central core of conical shape and thus, it allows a large exit area without excessive length. The central cone arrangement reduces the whirling action of discharged water and the efficiency of such a draft tube is about 85%.
3. **Simple elbow tube:** The vertical length of the draft tube is reduced to save the cost of excavation. However, simple elbow draft tube (Figure 23.5(c)) requires relatively lesser excavation for its installation. In a simple elbow tube, there is a loss of head due to bend. Thus, its efficiency is low which is of the order of 60%.
4. **Elbow tube having circular inlet and rectangular outlet:** The elbow tube having circular inlet and rectangular outlet is shown in Figure 23.5(d). Evidently, the elbow tube is widely employed in many turbine installations. It is designed to turn the water from the vertical to the horizontal direction with a minimum depth of excavation and at the same time, it gives high efficiency of the order of 85%. It also requires relatively lesser excavation for its installation.



(a) (b) (c) (d)

Figure 23.5 *Draft tubes*

23.5.2 Draft Tube Theory

Consider a turbine fitted with a conical draft tube as shown in Figure 23.6 in which section 2–2 corresponds to the runner exit (or draft tube inlet) and section 3–3 corresponds to the draft tube exit.

Let H_s be the height of runner exit above tail race level which is called the static suction head of draft tube, y be the distance of the bottom of draft tube from tail race level, p_a be the atmospheric pressure at the surface of tail race, h_f be the hydraulic energy loss in draft tube between sections 2–2 and 3–3, section 3–3 be the datum line, p_2 and V_2 be the pressure and velocity at point 2 and p_3 and V_3 be the pressure and velocity at point 3.

Applying Bernoulli's equation between sections 2–2 and 3–3, we get:

$$\frac{p_2}{\rho_w g}+\frac{V_2^2}{2g}+(H_s+y)=\frac{p_3}{\rho_w g}+\frac{V_3^2}{2g}+0+h_f \tag{i}$$

But

$$\frac{p_3}{\rho_w g}=\frac{p_a}{\rho_w g}+y$$

$$\frac{p_2}{\rho_w g}+\frac{V_2^2}{2g}+(H_s+y)=\left(\frac{p_a}{\rho_w g}+y\right)+\frac{V_3^2}{2g}+h_f$$

$$\therefore \frac{p_2}{\rho_w g}=\frac{p_a}{\rho_w g}-H_s-\left(\frac{V_2^2-V_3^2}{2g}-h_f\right) \tag{23.5}$$

If energy loss in draft tube is neglected, then Equation (23.5) is given by,

$$\frac{p_2}{\rho_w g}=\frac{p_a}{\rho_w g}-H_s-\left(\frac{V_2^2-V_3^2}{2g}\right) \tag{23.6}$$

Here, H_s is the static suction head and $[(V_2^2-V_3^2)/2g]$ is the dynamic suction head of the draft tube.

In Equation (23.6), $[p_2/(\rho_w g)]$ is less than atmospheric pressure head. It shows that by providing a draft tube, the turbine can be installed above the tail race level without any loss in the static head and a part of the kinetic energy which is discharged as a waste is recovered in the form of pressure energy.

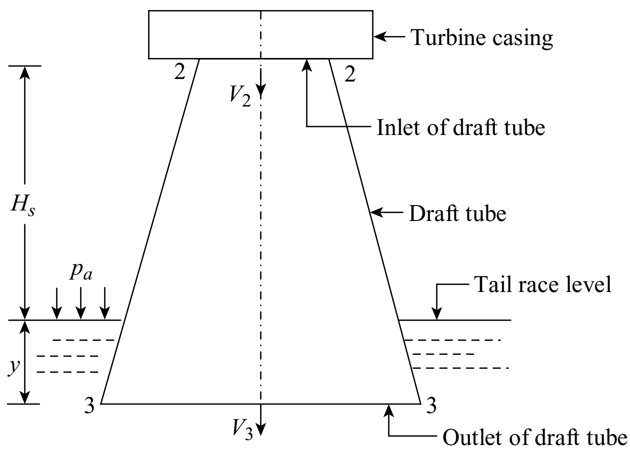


Figure 23.6 *Draft tube theory*

23.5.3 Efficiency of Draft Tube

The efficiency of a draft tube (η_d) is defined as the ratio of actual regain of pressure head to the kinetic head at entrance to the draft tube.

$$\text{Actual regain of pressure head} = \frac{V_2^2 - V_3^2}{2g} - h_f$$

$$\text{Kinetic head at entrance to draft tube} = \frac{V_2^2}{2g}$$

$$\therefore \boxed{\eta_d = \frac{[(V_2^2 - V_3^2)/(2g)] - h_f}{V_2^2/(2g)}} \tag{23.7}$$

If energy loss in draft tube is neglected, then Equation (23.7) is given by,

$$\boxed{\eta_d = \frac{[(V_2^2 - V_3^2)/(2g)]}{V_2^2/(2g)} = \frac{V_2^2 - V_3^2}{V_2^2} = 1 - \left(\frac{V_3}{V_2}\right)^2} \tag{23.8}$$

Example 23.9 A conical draft tube of length 5.5 m has a diameter of 2 m at its top. The water discharges through it with a flow rate of 20 m^3/s and 1.2 m/s velocity at the outlet. If the pressure head at the top is 6 m of water (vacuum) and atmospheric pressure head is 10.3 m of water, then determine the length of the tube immersed in water. Neglect the friction losses between the inlet and outlet of the draft tube.

Solution

Refer Figure 23.6. Let $(H_s + y) = 5.5$ m, $d_2 = 2$ m, $Q = 20\ \text{m}^3/\text{s}$, $V_3 = 1.2$ m/s, $[p_2/(\rho_w g)] = 6$ m (vac) and $[p_a/(\rho_w g)] = 10.3$ m.

Let y be the length of the tube immersed in water.

$$V_2 = \frac{Q}{A_2} = \frac{Q}{(\pi/4)d_2^2} = \frac{20}{(\pi/4)\times 2^2} = 6.37 \text{ m/s}$$

$$\frac{p_2}{\rho_w g} = 10.3 - 6 = 4.3 \text{ m (abs)}$$

$$\frac{p_2}{\rho_w g} = \frac{p_a}{\rho_w g} - H_s - \left(\frac{V_2^2 - V_3^2}{2g}\right)$$

Thus
$$4.3 = 10.3 - H_s - \left(\frac{6.37^2 - 1.2^2}{2\times 9.81}\right)$$

$$4.3 = 10.3 - H_s - 1.995$$

$$\therefore H_s = 10.3 - 1.995 - 4.3 = 4.005 \text{ m}$$

Since
$$(H_s + y) = 5.5 \text{ m}$$

$$\therefore y = 5.5 - H_s = 5.5 - 4.005 = \mathbf{1.495\ m}$$

Example 23.10 A Kaplan turbine working under a head of 5.5 m develops 2950 kW. It is fitted with a draft tube having inlet diameter 3 m and it is placed 1.6 m above the tail race level. The vacuum gauge connected to the inlet of draft tube reads 5 m of water. If the efficiency of draft tube is 75%, then determine the efficiency of the turbine. Take atmospheric pressure head as 10.3 m of water.

Solution

Refer Figure 23.6. Let $H = 5.5$ m, $P = 2950$ kW, $d_2 = 3$ m, $H_s = 1.6$ m, $[p_2/(\rho_w g)] = 5$ m (vac), $\eta_d = 0.75$ and $[p_a/(\rho_w g)] = 10.3$ m.

$$\frac{p_2}{\rho_w g} = 10.3 - 5 = 5.3 \text{ m (abs)}$$

But
$$\frac{p_2}{\rho_w g} = \frac{p_a}{\rho_w g} - H_s - \left(\frac{V_2^2 - V_3^2}{2g}\right)$$

$$5.3 = 10.3 - 1.6 - \frac{V_2^2 - V_3^2}{2g}$$

$$\frac{V_2^2 - V_3^2}{2g} = 10.3 - 1.6 - 5.3 = 3.4$$

$$\eta_d = \frac{[(V_2^2 - V_3^2)/(2g)]}{V_2^2/(2g)} = \frac{(V_2^2 - V_3^2)}{2g} \times \frac{2g}{V_2^2}$$

$$0.75 = 3.4 \times \frac{2g}{V_2^2}$$

$$\therefore V_2 = \sqrt{\frac{3.4 \times 2g}{0.75}} = \sqrt{\frac{3.4 \times 2 \times 9.81}{0.75}} = 9.43 \text{ m/s}$$

$$Q = A_2 V_2 = \frac{\pi}{4} d_2^2 \times V_2 = \frac{\pi}{4} \times 3^2 \times 9.43 = 66.657 \text{ m}^3\text{/s}$$

$$\eta_o = \frac{1000P}{\rho_w g Q H} = \frac{1000 \times 2950}{1000 \times 9.81 \times 66.657 \times 5.5} \times 100 = \mathbf{82.02\%}$$

Example 23.11 A Kaplan turbine develops 1800 kW with an overall efficiency of 85% while working under a head of 7 m. The turbine is set 2 m above the tail race level. The vacuum gauge fitted at the turbine exit reads a suction head of 3.1 m. If the inlet diameter of draft tube is 2.8 m and the loss of head due to friction in the tube is equal to 0.2 × kinetic head at the exit, then determine (i) the loss of head due to friction and (ii) the efficiency of the draft tube. Take atmospheric pressure head as 10.3 m of water.

Solution

Refer Figure 23.7. Let $P = 1800$ kW, $\eta_o = 0.85$, $H = 7$ m, $H_s = 2$ m, $p_2/(\rho_w g) = 3.1$ m (vac), $d_2 = 2.8$ m, $h_f = 0.2[V_3^2/(2g)]$ and $p_a/(\rho_w g) = 10.3$ m.

(i) $Q = \dfrac{1000P}{\rho_w g H \eta_o} = \dfrac{1000 \times 1800}{1000 \times 9.81 \times 7 \times 0.85} = 30.84 \text{ m}^3/\text{s}$

$$V_2 = \frac{Q}{A_2} = \frac{Q}{(\pi/4)d_2^2} = \frac{30.84}{(\pi/4)\times 2.8^2} = 5.01 \text{ m/s}$$

$$\frac{p_2}{\rho_w g} = 10.3 - 3.1 = 7.2 \text{ m (abs)}$$

$$\frac{p_2}{\rho_w g} = \frac{p_a}{\rho_w g} - H_s - \left(\frac{V_2^2}{2g} - \frac{V_3^2}{2g} - h_f\right)$$

Thus $\quad 7.2 = 10.3 - 2 - \left(\dfrac{5.01^2}{2\times 9.81} - \dfrac{V_3^2}{2g} - \dfrac{0.2V_3^2}{2g}\right)$

$$7.2 = 10.3 - 2 - 1.28 + \frac{1.2V_3^2}{2g}$$

$$\frac{1.2V_3^2}{2g} = 7.2 - 10.3 + 2 + 1.28 = 0.18$$

$$\therefore V_3 = \sqrt{\frac{0.18\times 2g}{1.2}} = \sqrt{\frac{0.18\times 2\times 9.81}{1.2}} = 1.715 \text{ m/s}$$

$$h_f = \frac{0.2V_3^2}{2g} = \frac{0.2\times 1.715^2}{2\times 9.81} = \mathbf{0.03\ m}$$

(ii) $\eta_d = \dfrac{[(V_2^2 - V_3^2)/(2g)] - h_f}{V_2^2/(2g)} = \left(\dfrac{V_2^2 - V_3^2}{2g} - h_f\right)\times \dfrac{2g}{V_2^2}$

$$\therefore \eta_d = \left(\frac{5.01^2 - 1.715^2}{2\times 9.81} - 0.03\right)\times \frac{2\times 9.81}{5.01^2}\times 100 = \mathbf{85.94\%}$$

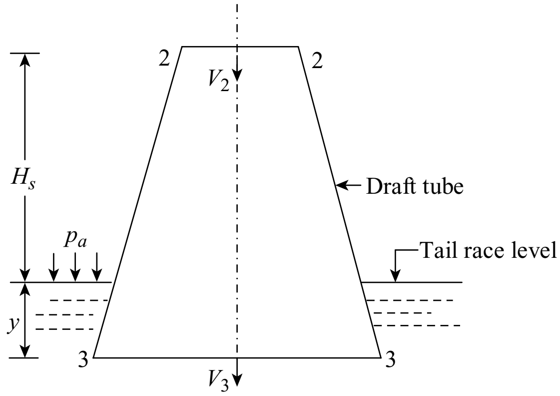


Figure 23.7

Example 23.12 A conical draft tube having inlet and outlet diameters 1.2 m and 1.8 m discharges water at outlet with a velocity of 2.8 m/s. The total length of the draft tube is 6.5 m and 1.2 m as the length of the draft tube is immersed in water. If loss of head due to friction in the draft tube is equal to 0.2 × velocity head at the outlet of the tube, then determine (i) the pressure head at inlet and (ii) efficiency of the draft tube. Take atmospheric pressure head as 10.3 m of water.

Solution

Refer Figure 23.7. Let $d_2 = 1.2$ m, $d_3 = 1.8$ m, $V_3 = 2.8$ m/s, $(H_s + y) = 6.5$ m, $y = 1.2$ m, $h_f = 0.2[V_3^2/(2g)]$ and $p_a/(\rho_w g) = 10.3$ m.

$$H_s = 6.5 - y = 6.5 - 1.2 = 5.3 \text{ m}$$

$$Q = A_3 V_3 = \frac{\pi}{4} d_3^2 \times V_3 = \frac{\pi}{4}\times 1.8^2 \times 2.8 = 7.125 \text{ m}^3/\text{s}$$

$$V_2 = \frac{Q}{A_2} = \frac{Q}{(\pi/4)d_2^2} = \frac{7.125}{(\pi/4)\times 1.2^2} = 6.3 \text{ m/s}$$

(i) $\dfrac{p_2}{\rho_w g} = \dfrac{p_a}{\rho_w g} - H_s - \left(\dfrac{V_2^2 - V_3^2}{2g} - h_f\right) = \dfrac{p_a}{\rho_w g} - H_s - \left(\dfrac{V_2^2 - V_3^2}{2g} - \dfrac{0.2V_3^2}{2g}\right)$

$$\therefore \frac{p_2}{\rho_w g} = 10.3 - 5.3 - \left(\frac{6.3^2 - 2.8^2}{2\times 9.81} - \frac{0.2\times 2.8^2}{2\times 9.81}\right) = \mathbf{3.46\ m}$$

(ii) $\eta_d = \dfrac{[(V_2^2 - V_3^2)/(2g)] - h_f}{V_2^2/(2g)} = \dfrac{[(V_2^2 - V_3^2)/(2g)] - 0.2[V_3^2/(2g)]}{V_2^2/(2g)}$

$$\eta_d = \frac{V_2^2 - 1.2V_3^2}{V_2^2} = \left[1 - 1.2\times\left(\frac{V_3}{V_2}\right)^2\right]\times 100$$

$$\therefore \eta_d = \left[1 - 1.2\times\left(\frac{2.8}{6.3}\right)^2\right]\times 100 = \mathbf{76.3\%}$$

Example 23.13 A water turbine has a velocity of 6 m/s at the entrance to the draft tube and a velocity of 1.2 m/s at the outlet. For the frictional losses of 0.1 m and tail water 5 m below the entrance to the draft tube, find the pressure head at the entrance.

Solution

Refer Figure 23.7. Let $V_2 = 6$ m/s, $V_3 = 1.2$ m/s, $h_f = 0.1$ m and $H_s = 5$ m.

$$\frac{p_2}{\rho_w g} = \frac{p_a}{\rho_w g} - H_s - \left(\frac{V_2^2 - V_3^2}{2g} - h_f\right)$$

If $p_a/(\rho_w g) = 0$, then $p_2/(\rho_w g)$ will be the vacuum pressure head at the inlet of draft tube as given below.

$$\frac{p_2}{\rho_w g} = -5 - \left[\frac{6^2 - 1.2^2}{2\times 9.81} - 0.1\right] = \mathbf{-6.66\ m}$$

If $p_a/(\rho_w g) = 10.3$ m of water, then $p_2/(\rho_w g)$ will be the absolute pressure head at the inlet of draft tube as given below.

$$\frac{p_2}{\rho_w g} = 10.3 - 5 - \left[\frac{6^2 - 1.2^2}{2\times 9.81} - 0.1\right] = \mathbf{3.64\ m}$$

Example 23.14 A Kaplan turbine operating under a head of 6 m develops 2000 kW with an overall efficiency of 86%. The draft tube has an efficiency of 75%. The inlet diameter of the tube is 2.4 m and the pressure at its entry should not fall more than 5.5 m below atmospheric pressure. Determine the height at which the runner may be set above the tail race level.

Solution

Refer Figure 23.8. Let $H = 6$ m, $P = 2000$ kW, $\eta_o = 0.86$, $\eta_d = 0.75$, $d_2 = 2.4$ m and $[(p_a - p_2)/(\rho_w g)] = 5.5$ m.

$$Q = \frac{1000P}{\rho_w g H \eta_o} = \frac{1000\times 2000}{1000\times 9.81\times 6\times 0.86} = 39.51\ \text{m}^3/\text{s}$$

$$V_2 = \frac{Q}{A_2} = \frac{Q}{(\pi/4)d_2^2} = \frac{39.51}{(\pi/4)\times 2.4^2} = 8.734\ \text{m/s}$$

Since $$\frac{V_2^2 - V_3^2}{2g} - h_f = \frac{p_a - p_2}{\rho_w g} - H_s$$

[From Equation (23.5)]

$$\frac{V_2^2 - V_3^2}{2g} - h_f = 5.5 - H_s$$

$$\eta_d = \frac{[(V_2^2 - V_3^2)/(2g)] - h_f}{V_2^2/(2g)} = \left(\frac{V_2^2 - V_3^2}{2g} - h_f\right)\frac{2g}{V_2^2}$$

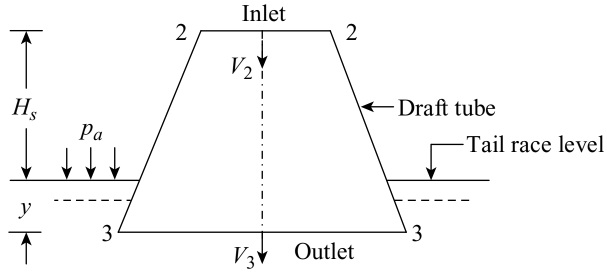


Figure 23.8

Thus $$0.75 = (5.5 - H_s) \times \frac{2 \times 9.81}{8.734^2}$$

$$\therefore H_s = 5.5 - \frac{0.75 \times 8.734^2}{2 \times 9.81} = \mathbf{2.584\ m}$$

23.6 ❐ CAVITATION IN TURBINES

The pressure at any point inside the turbine should not be less than the vapour pressure of water. Otherwise, vaporization of water starts and a number of bubbles will form in the low pressure region. These bubbles formed on account of low pressures are carried away by the stream to higher pressure zones where the vapours condense and the bubbles suddenly collapse. This results in the formation of cavity and the surrounding liquid rushes to fill it. The liquid moving from all directions collides at the centre of cavity and it creates very high local pressure which may be as high as 7000 atmospheres. This cycle of cavity formation and high pressure is repeated with a high frequency (about 2500 cycles per second). The metallic surface in the vicinity of this region is also subjected to this intense pressure. This may cause severe damage to the surface which ultimately fails by fatigue and the surface becomes badly pitted and scored. Therefore, this phenomenon is called cavitation.

Generally, cavitation occurs in reaction turbines at the runner exit or at the inlet of the draft tube. The turbine parts should be properly designed to avoid cavitation. Due to cavitation, the metallic surfaces are damaged and cavities are formed. The pitting alters the streamline pattern of the blade contours which reduces the torque produced and also the power. Due to cavitation, the metal of the runner vanes and the draft tube is gradually eaten away, which eventually lowers the efficiency of the turbine. Cavitation also causes a considerable vibration and noise.

The effects of cavitation can be reduced by following specific methods, such as (i) setting the turbine near the tail race level, (ii) by using cavitation resistant materials, like aluminium-bronze, stainless steel and nickel steel for manufacturing of blades and (iii) by applying coatings of cavitation resistant materials over the place where cavitation is likely to occur.

In order to determine the zone where turbine can work without being affected from cavitation, Prof. D. Thoma of Germany suggested a dimensionless parameter called Thoma's cavitation factor, which is denoted by σ and its value is given below.

$$\boxed{\sigma = \frac{H_b - H_s}{H} = \frac{(H_a - H_v) - H_s}{H} = \frac{H_{ts}}{H}} \quad (23.9)$$

Here, H_b is the barometric pressure head in m of water, H_a is the atmospheric pressure head in m of water, H_v is the vapour pressure head in m of water, H_s is the suction pressure head in m of water, (or height of the runner outlet above tail race), H is the working head of the turbine in m of water and H_{ts} is the total suction head.

The value of cavitation factor σ depends on the specific speed of the turbine denoted by N_s. For a turbine of known N_s, the factor σ can be reduced up to a certain value up to which its efficiency remains constant. Further decrease in its value causes sharp fall in efficiency. This limiting value of σ is called critical cavitation factor and it is denoted by σ_c, which is different for different turbines. The value of σ_c for different turbines may be calculated from the following empirical relationships.

For Francis turbine:

$$\boxed{\sigma_c = 0.044 \times (0.01 N_s)^2} \tag{23.10}$$

For propeller turbine:

$$\boxed{\sigma_c = [0.3 + 0.0032 \times (0.01 N_s)^{2.73}]} \tag{23.11}$$

For Kaplan turbine:

$$\boxed{\sigma_c = 1.1 \times [0.3 + 0.0032 \times (0.01 N_s)^{2.73}]} \tag{23.12}$$

Example 23.15 A Francis turbine works under a head of 25 m and produces 11750 kW while operating at 120 rpm. The turbine has been installed at a suction where atmospheric pressure is 10 m of water and vapour pressure is 0.25 m of water. Determine the maximum height of the straight draft tube for the turbine.

Solution

Let $H = 25$ m, $P = 11750$ kW, $N = 120$ rpm, $H_a = 10$ m and $H_v = 0.25$ m. Let H_s be the maximum height of the straight draft tube for the turbine.

$$N_s = \frac{N\sqrt{P}}{H^{5/4}} = \frac{120 \times \sqrt{11750}}{25^{5/4}} = 232.69$$

$$\sigma_c = 0.044(0.01 N_s)^2 = 0.044 \times (0.01 \times 232.69)^2 = 0.238$$

To avoid cavitation, the cavitation factor (σ) must be equal to at least σ_c as expressed below.

$$\sigma_c = \frac{(H_a - H_v) - H_s}{H}$$

Thus

$$0.238 = \frac{10 - 0.25 - H_s}{25}$$

$$\therefore H_s = 10 - 0.25 - 0.238 \times 25 = \mathbf{3.8\ m}$$

23.7 ❐ NEW TYPES OF TURBINES

Some of the new types of turbines are (i) Deriaz or diagonal turbine, (ii) tubular turbine, and (iii) bulb turbine. The characteristic properties of these turbine types are discussed below.

23.7.1 Deriaz or Diagonal Turbine

Deriaz turbine is named in the honour of its inventor P. Deriaz. He utilized the idea of Kaplan turbine in which maximum efficiency is attained under variable load conditions by the use of movable blade runner. The Deriaz turbine is an intermediate between the mixed flow and the axial flow turbines. In a Deriaz turbine, the blades are inclined to the hub at an angle of 45° and thus, it is also known as diagonal turbine. The blades are adjustable like a Kaplan turbine and its numbers vary from 10 to 12. The blades are movable, so its runner has no outer rim to connect them. A schematic view of a Deriaz turbine is shown in Figure 23.9.

Its blades are simpler than the Kaplan turbine but the basic difference lies in its design. In closed position, the vanes of Deriaz turbine come in contact with each other at the periphery as well as near the hub. Thereby, the whole cross-sectional area can be recovered. However, in Kaplan turbines, the vanes come in contact with each other at the periphery only and a

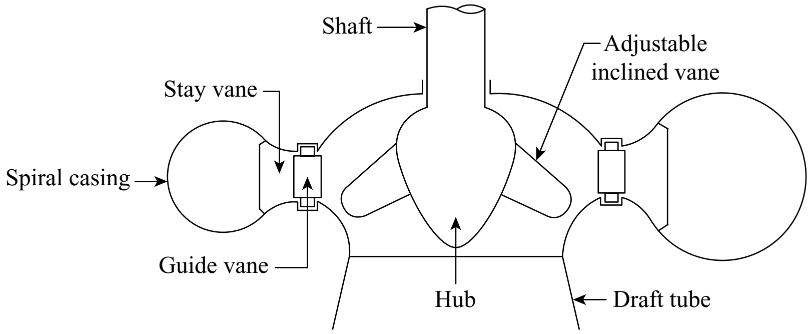


Figure 23.9 *Deriaz (or diagonal) turbine*

large gap remains between them near the hub. The feature of complete closure of the vanes causes simplification in operation, reduction in the overall size of the machine, easy starting and reduction in cost of power house structure. The other parts of this turbine, such as scroll casing, stay vanes, guide vanes and draft tubes are similar to Kaplan turbine (reaction turbine).

A Deriaz turbine is suitable for the head range between Kaplan and Francis turbine. It is particularly suitable for medium head ranging from 30 m to 150 m. Its runner is so shaped that it can be used both as a turbine as well as a pump. Thus, it is also known as a reversible type turbine. It is economical to use this turbine as a turbine or a pump working in pump storage plants. Some of the advantages of this turbine are low starting torque, over load capacity, improved part load efficiency and stability of operation during starting.

23.7.2 Tubular Turbine

The efficiency of a Kaplan turbine under low heads is less due to excessive loss at the bends. Moreover, deep excavation increases overall cost of the plant. These two facts lead to the development of low head turbines, such as tubular and bulb turbines. Tubular turbine was developed by Arno Fischer (Germany) in 1937. This turbine is a modified axial flow turbine and it has no scroll casing. The runner is fitted in a tube extending from the head water to the tail water, so it is given the name tubular turbine. Its blades may be adjustable or non-adjustable. Thus, it is similar to Kaplan and propeller turbines. These turbines may have either vertical or horizontal or inclined disposition of shaft and are capable to work under the heads ranging from 3 m to 15 m. The schematic views of vertical shaft and inclined shaft tubular turbines are shown in Figure 23.10.

23.7.3 Bulb Turbine

A turbo-generator set having a tubular turbine and the generator housed in a bulb shaped watertight casing is called a bulb set which remains submerged in the stream of water. The tubular turbine with horizontal disposition of shaft used for the set is called a bulb turbine. A schematic view of a bulb turbine is shown in Figure 23.11.

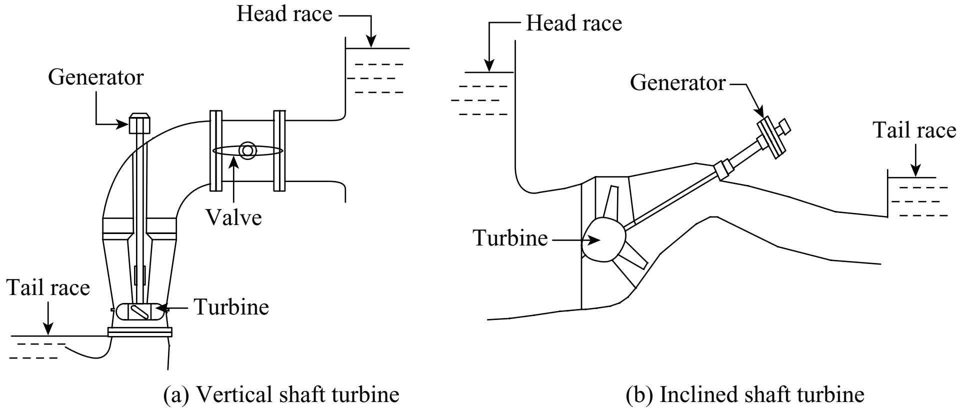


Figure 23.10 *Tubular turbine*

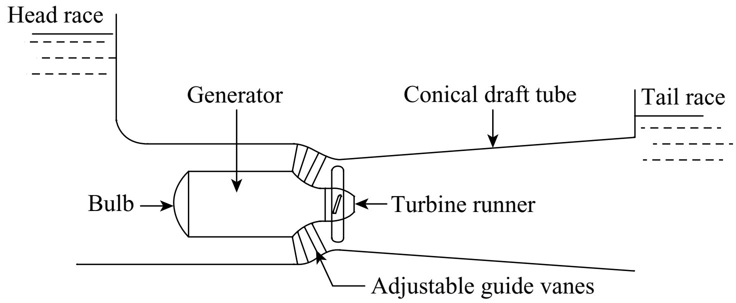


Figure 23.11 *Bulb turbine*

The horizontal positioning of the shaft results in a reduced size when compared to the vertical shaft arrangement. The bulb could be either upstream or downstream of the runner. Mostly, the bulb remains upstream of the runner and the axis of rotation coincides with the axis of passage of water, which generally remains straight. The outer surface of the bulb is streamlined to minimize the loss of energy during the flow of water from the head pond to the turbine. Generally, there are two types of bulb sets, namely large bulb and small bulb sets. Large bulb sets are utilized to harness tidal power, whereas small bulb sets are employed for irrigation channels.

In a bulb turbine, the straight line flow appreciably increases the hydraulic efficiency in comparison with a Kaplan turbine. On the intake side of a bulb turbine, the spiral casing is replaced by straight convergent intake duct which is favourable for the runner operation. In its downstream side, the straight conical draft tube replaces the elbow type draft tube. Therefore, much more of the kinetic energy can be recovered at the discharge side of the turbine and thus, the turbine efficiency is improved.

The bulb turbine is a small axial flow reaction turbine which is used for extremely low heads varying from 3 m to 15 m. A bulb turbine has less advantage in comparison to the Kaplan turbine beyond the mentioned head. It has higher full-load efficiency and higher flow capacity as compared to the Kaplan turbine. The bulb turbine is quite suitable for tidal power plants. It is considered as the best selection for exploitation of hydraulic power with extremely low water head and extremely large discharge. It has high specific speed, high efficiency and large discharge. Moreover, it needs less excavation in civil works.

In a bulb turbine, the bulb set is completely submerged under pressure. This leads to water leakage into the generator chamber and condensation, which are the main source of trouble. In this turbine, the erection technique also involves considerable amount of time. In India, bulb turbines are being used in power stations, such as in Gandak Western Canal House and the Kosi East Canal Power House.

Summary

1. **Axial flow turbines:** Water flows parallel to the axis of rotation of the vertical shaft.
2. **Propeller turbine:** The runner blades are fixed and non-adjustable.
3. **Kaplan turbine:** The runner blades are made adjustable. It has twisted runner vanes due to which its part load efficiency is higher than Francis turbine.
4. **Discharge (Q) through Kaplan and propeller turbines:** $Q = (\pi/4)(D_o^2 - D_b^2)V_{fi}$
5. **Velocity of flow:** $V_{fi} = V_{fo} = V_f = K_f\sqrt{2gH}$
6. A draft tube is an airtight pipe of gradually increasing cross-sectional area. It is used in reaction turbines for discharging water. One end of the draft tube is connected to the runner exit and the other is submerged into the tail race.
7. **Efficiency of a draft tube:** $\eta_d = \dfrac{[(V_2^2 - V_3^2)/(2g)] - h_f}{V_2^2/(2g)}$
8. The phenomenon where the formation of vapour bubbles in the low pressure zone and sudden collapsing of these bubbles in the high pressure zone may cause severe damage to the surfaces is called cavitation.

9. **Thoma's cavitation factor:** $\sigma = (H_a - H_v - H_s)/H$. The limiting value of σ up to which it can be decreased but efficiency remains constant is called critical cavitation factor (σ_c).

$\sigma_c = 0.044 \times (0.01N_s)^2$ (For Francis turbine)

$\sigma_c = [0.3 + 0.0032 \times (0.01N_s)^{2.73}]$ (For propeller turbine)

$\sigma_c = 1.1 \times [0.3 + 0.0032 \times (0.01N_s)^{2.73}]$ (For Kaplan turbine)

10. The Deriaz turbine is an intermediate between the mixed flow and the axial flow turbines and it is also known as diagonal turbine.
11. Tubular turbine is a modified axial flow turbine whose runner is fitted in a tube extending from the head water to the tail water.
12. The tubular turbine with horizontal disposition of shaft used for the bulb set is called a bulb turbine. It is used for extremely low heads ranging from 3 to 15 m.

Multiple-choice Questions

1. The dimensionless specific speed of a Kaplan turbine is about
   (a) 9.
   (b) 0.
   (c) 0.9.
   (d) None of the above
2. The value of flow ratio in case of Kaplan turbine is
   (a) 0.4 and 0.5.
   (b) 0.5 and 0.
   (c) 0.6 and 1.
   (d) 0.7 and 1.5.
3. The number of blades in a Kaplan turbine are
   (a) 2 to 4.
   (b) 3 to 8.
   (c) 8 to 12.
   (d) 12 to 15.
4. In Kaplan turbines, the ratio of the hub diameter to runner diameter usually varies from
   (a) 0.1 to 0.3.
   (b) 0.35 to 0.6.
   (c) 0.65 to 0.8.
   (d) None of the above.
5. All reaction turbines become susceptible to cavitation when
   (a) Pressure becomes high.
   (b) Velocity becomes high.
   (c) Pressure falls below the vapour pressure.
   (d) None of the above.
6. The degree of reaction of a Kaplan turbine is
   (a) 1.
   (b) Less than 1.
   (c) 0.
   (d) Less than 1 but greater than 0.5.
7. Which of the following statement is incorrect for a Kaplan turbine?
   (a) It has blades of small chamber to avoid separation.
   (b) It has adjustable blades.
   (c) It has large guide blade angles than of a Francis turbine.
   (d) It has mixed flow velocity.
8. The pressure of water acting on the runner vanes of a reaction turbine is
   (a) Below atmospheric.
   (b) Equal atmospheric.
   (c) Above atmospheric.
   (d) None of the above.
9. The following draft tube is the most efficient
   (a) Moody's spreading.
   (b) Conical type.
   (c) Elbow type.
   (d) Elbow tube having circular inlet and rectangular outlet.

Review Questions

1. How is a Kaplan turbine different from a propeller turbine? Explain the characteristic features of the Kaplan turbine.
2. Draw a schematic view of a Kaplan turbine and explain briefly its construction and working principle. Also compare Francis and Kaplan turbines.
3. Explain the governing mechanism of a Kaplan turbine. Also clearly state how it differs from the governing mechanism of a Francis turbine.
4. What is a draft tube? Also derive an expression for its efficiency.
5. Give functions of a draft tube and discuss some typical draft tubes with diagrams.
6. Describe the Deriaz turbine with a suitable sketch and also give its advantages.
7. Write short note on the following: (i) tubular turbine and (ii) bulb turbine.
8. Define cavitation. Why does it occur and what are its effects?

Problems

1. A Kaplan turbine develops 20 MW at an average head of 30 m with an overall efficiency of 90%. Taking the value of speed ratio = 2, flow ratio = 0.6 and diameter of the hub = 0.35 times of the diameter of the runner, calculate (i) the diameter of the runner, (ii) speed of the turbine and (iii) specific speed of the turbine.
[**Ans.** 2.743 m, 337.83 rpm, 680.47]

2. A propeller turbine has been designed to develop 22 MW under a head of 20 m whilst running at 150 rpm. The other relevant data are overall efficiency = 88.3%, hydraulic efficiency = 95.2%, outer diameter of runner = 5 m, diameter of the hub = 2.5 m. If the turbine discharges without whirl at exit, then determine the runner vane angles at the hub and at the outer periphery. Also determine the specific speed of the turbine.
[**Ans.** 139.58°, 23.71°, 165.98°, 12.38°, 526.03]

3. A Kaplan turbine runner is to be designed to develop 9 MW under a net head of 5.5 m with an overall efficiency of 88%. If the diameter of the boss is one third of the diameter of the runner, the speed ratio is 2.09 and flow ratio is 0.68, then calculate (i) the diameter of the runner, (ii) speed of the runner and (iii) specific speed of the turbine.
[**Ans.** 6.2 m, 66.9 rpm, 753.52]

4. The hub diameter of a Kaplan turbine working under a head of 10 m is 0.3 times the diameter of the runner. The turbine is running at 100 rpm. The velocity of whirl at outlet is zero. If the vane angle of the extreme edge of the runner at outlet is 15° and the flow ratio is 0.62, then determine the diameter of runner and the boss and discharge through the runner.
[**Ans.** 6.19 m, 1.86 m, 237.62 m^3/s]

5. A Kaplan turbine runner is to be designed to develop 7350 kW shaft power. The net available head is 5.6 m. Assume that the speed ratio is 2.05 and flow ratio is 0.67, and the overall efficiency is 60.5%. The diameter of the boss is one third of the diameter of the runner. Find the diameter of the runner and the boss, runner speed and specific speed.
[**Ans.** 6.72 m, 2.24 m, 61.07 rpm, 607.77]

6. A Kaplan turbine runner has outer diameter of 4.6 m and the diameter of the hub is 2.1 m. It is required to develop 20.62 MW when running at 150.2 rpm, under a head of 21.2 m. If the hydraulic efficiency is 94.5% and overall efficiency is 88.5%, then evaluate (i) the runner angles at inlet and exit at the mean diameter of the vanes and (ii) the runner angles at inlet and exit at two sections near the hub and the outer periphery. Also evaluate the specific speed of the turbine.
[**Ans.** 155.72°, 17.92°, 121.44°, 24.54°, 166.22°, 11.77°, 474.13]

7. The runner and boss diameters of a Kaplan turbine are 6 m and 2 m, respectively. The discharge through the turbine is 200 m^3/s. The hydraulic and mechanical efficiencies of the turbine are 90% and 95%, respectively. Assuming that discharge is free and there is no swirl at outlet, determine (i) the head, (ii) brake power developed by the turbine, (iii) runner speed if the speed ratio is 2.09 and (iv) specific speed.
[**Ans.** 32.29 m, 54166.798 kW, 167.43 rpm, 506.25]

8. A propeller turbine of runner diameter 4.5 m is running at 45 rpm. The guide blade angle at inlet is 145° and the runner blade angle at outlet is 25° to the direction of vane. The axial flow area of water through runner is 30 m^2. If the runner blade angle at inlet is radial, then determine (i) the hydraulic efficiency of the turbine, (ii) discharge through the turbine and (iii) power developed by the turbine.
[**Ans.** 57.32%, 222.66 m^3/s, 1252.136 kW]

9. A Kaplan turbine has the following specifications, such as discharge = 60.5 m^3/s, hydraulic efficiency = 90.4%, mechanical efficiency = 94.5%, runner diameter = 4 m, diameter of boss = 0.3 times the runner diameter, speed ratio = 2, there is no swirl at outlet and discharge is free. Determine (i) the net head available on the turbine, (ii) power developed, (iii) runner speed and (iii) specific speed of the turbine.
[**Ans.** 14.86 m, 7534.31 kW, 163.05 rpm, 485.09]

10. An axial flow bulb turbine operates at 6 MW generator. The other relevant data are speed = 140 rpm, head = 6 m, runner tip diameter = 5.5 m, hub diameter = 3 m, generator efficiency = 95%, overall efficiency = 89%, hydraulic efficiency = 93%. If there is no exit whirl, then determine the runner vane angles at the inlet and exit at the outer periphery. Also determine the specific speed of the turbine.
[**Ans.** 169.5°, 10.15°, 1184.82]

11. The following data are given for a propeller turbine, such as head = 14.5 m, speed = 150 rpm, runner mean diameter = 1.6 m, water leaves the guide vanes at an angle = 15° with the peripheral speed, moving blade outlet angle = 25°, kinetic energy coefficient for the moving blades = 0.94, and losses in the penstock and the guide vane account for = 10% of the total head available. Determine (i) the inlet angle of the moving blade, (ii) axial and whirl velocities at inlet and outlet, (iii) magnitude and direction of the absolute velocity at outlet and (iv) hydraulic efficiency.
[**Ans.** 55.17°, 4.14 m/s, 15.45 m/s, 2.07 m/s, 8.4 m/s, 14.27°, 64.6%]

12. A Kaplan turbine develops 2.8 MW while operating under a head of 5.2 m. It is fitted with a draft tube with its inlet set 1.8 m above the tail race level. A vacuum gauge connected to the draft tube indicates a reading of 5.2 m of water. If the inlet

diameter of the draft tube is 3 m and its efficiency is 75%, then determine the overall efficiency of the turbine.

[**Ans.** 82.35%]

13. A conical draft tube of 5 m height and 2 m in diameter at the top discharges water with a velocity of 1.5 m/s with a rate of 30 m^3/s. Find the height of tube immersed in water if the pressure head at the inlet is 7.5 m of water (vacuum), the atmospheric pressure is 10.3 m of water and there is no pressure loss.

[**Ans.** 2.03 m]

14. Determine the efficiency of a Kaplan turbine developing 3000 kW under a net head of 5.2 m. It is fitted with a draft tube with its inlet diameter 3 m set placed 1.5 m above the tail race level. A vacuum gauge connected to the inlet of draft tube indicates a reading of 5 m of water. Assume the draft tube efficiency as 80%.

[**Ans.** 87.39%]

15. A Kaplan turbine develops 1500 kW under a head of 6 m. The turbine is set at 2.5 m above water level. A vacuum gauge inserted at the turbine outlet records a suction head of 3.1 m. If the turbine efficiency is 85%, then what is the efficiency of the draft tube having inlet diameter of 3 m?

[**Ans.** 65.48%]

16. A Francis turbine provided with a cylindrical draft tube of diameter 2 m works under a static head of 5 m. It develops 300 kW at an overall efficiency of 86%. If a tapered draft tube with inlet and outlet diameters as 2 m and 3.5 m, respectively and having an efficiency of conversion of 92% is substituted for the cylindrical one, then determine the increase in efficiency and power of the turbine. It is given that head, speed and discharge remains constant.

[**Ans.** 4.3%, 15 kW]

17. A Kaplan turbine is fitted with an elbow type draft tube with a circular inlet of 2 m diameter. It develops 1900 kW under a net head of 8.5 m. The inlet is set at a height of 1.2 m above the tail race level. A vacuum gauge connected to draft tube inlet measures a reading of $35.02\ \text{kN/m}^2$. If the efficiency of the draft tube is 76%, then determine the efficiency of the turbine. If the ratio of area of circular inlet and rectangular exit of the draft tube is 1 : 4, then determine the power lost due to friction in the tube. If the turbine output is reduced to 900 kW and speed remains unchanged, then determine the vacuum gauge reading.

[**Ans.** 92.93%, 0.55 m, −1.303 m]

18. A Kaplan turbine develops 1550 kW under a head of 6.2 m. The turbine is set 1.5 m above the tail race level. A vacuum gauge inserted at the turbine outlet records a suction head of 3 m. If the hydraulic efficiency is 85%, then what would be the efficiency of draft tube having inlet diameter of 2.5 m? What will be the reading of suction gauge if power developed is reduced to 775 kW, where the head, efficiency and speed remains constant.

[**Ans.** 79.35%, 1.876 m]

19. The axis of a Francis turbine and its draft tube is vertical. The pressure head in the spiral casing at inlet is 42 m above atmosphere and the speed of water is 5.5 m/s. The water flow rate through the turbine is 2 m^3/s. The hydraulic and overall efficiencies of the turbine are 86% and 82%, respectively. The top of the draft tube is 1 m below the centre line of the casing and the tail race is 3.5 m below the top of the draft tube. If the diameter of the draft tube at its exit is 1.15 m, then determine (i) the total head across the turbine, (ii) power output, (iii) head lost in friction in the turbine and draft tube, and (iv) the power lost in mechanical friction.

[**Ans.** 48.04 m, 772.89 kW, 6.5356 m, 37.692 kW]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (c)	**2.** (d)	**3.** (b)	**4.** (b)	**5.** (c)
6. (d)	**7.** (d)	**8.** (c)	**9.** (a)	

Chapter 24

Performances of Hydraulic Turbines

24.1 ❒ INTRODUCTION

The turbines are designed for specific operating conditions also known as the design conditions. The turbines produce maximum efficiency while operating in these conditions. However, turbines are required to work under varying conditions of head, speed, power output and gate openings. The variations involved in the operating conditions of a turbine are as follows. In such conditions, (i) the power output may vary by the movements of the wicket gates or the spear while the head and speed remains constant, (ii) the head and the power output of the turbine may vary, the speed is adjusted to maintain the same efficiency and the gate opening remains constant, (iii) the head and speed may vary which generally happens in low head units and (iv) the speed may vary by adjusting the load on the turbine, while the head and gate opening remains constant.

In order to predict the behaviour and to establish a comparison between the performances of the turbines of the same type operating under varying conditions, the results are presented in terms of unit quantities, such as unit speed, unit discharge and unit power. These quantities are obtained by reducing the head to unity. Similarly, to establish a comparison between different types of turbines irrespective of their sizes and specific quantities, such as specific speed will be helpful.

The exact behaviour of the turbines operating under varying conditions can be determined by performing various tests either on the prototypes or on their small scale models. The test results are graphically plotted and the resulting curves are called characteristic curves. These curves are plotted in terms of unit quantities. The characteristic curves are of three types, namely constant head characteristic curves (or main characteristic curves), constant speed characteristic curves (or operating characteristic curves) and constant efficiency curves (or Muschel curves).

24.2 ❒ UNIT QUANTITIES

A turbine designed for specific operating conditions is often required to work under varying conditions of head, speed, output and gate opening. To predict the behaviour of a turbine and to establish a comparison between the performances of the turbines of the same type but of different sizes, the results are expressed in terms of unit quantities. The unit quantities are obtained when the head on the turbine is reduced to unity. For obtaining unit quantities, the efficiency of the turbine is assumed as constant, which is possible when the velocity triangles under working head and unit head are geometrically similar so that the water enters the turbine without shock. The important unit quantities are unit speed, unit power and unit discharge.

24.2.1 Unit Speed

It is defined as the speed of the turbine working under unit head and it is denoted by N_u. Let H be the head under which the turbine is working, N be the speed of the turbine and u be the tangential velocity.

Since $u \propto V \propto \sqrt{H}$ $[\because V = C_v\sqrt{2gH}]$

$\therefore u \propto \sqrt{H}$

Also $u \propto N$, $D =$ Constant for a given turbine $[\because u = \pi DN/60]$

$\therefore N \propto \sqrt{H}$

or $N = k_1\sqrt{H}$ $[k_1 = \text{Constant}]$ (i)

If head on the turbine becomes unity $(H = 1\text{ m})$, then $N = N_u$. Thus, from Equation (i), we get the following expression.

$$N_u = k_1\sqrt{1} = k_1 \quad \text{(ii)}$$

$$N = N_u\sqrt{H} \quad \text{[Substitute (ii) in (i)]}$$

$$\therefore \boxed{N_u = \frac{N}{\sqrt{H}}} \quad (24.1)$$

24.2.2 Unit Discharge

It is defined as the discharge passing through a turbine which is working under a unit head and it is denoted by Q_u. Let H be the head under which turbine is working and Q be the discharge through the turbine. The discharge through the turbine is given by the product of area and velocity as given below.

$$Q = AV$$

Since $V \propto \sqrt{H}$ and $A =$ Constant for a given turbine

Thus $Q \propto \sqrt{H}$

or $Q = k_2\sqrt{H}$ $[k_2 = \text{Constant}]$ (i)

If head on the turbine becomes unity $(H = 1\text{ m})$, then $Q = Q_u$. Thus, from Equation (i), we get the following expression.

$$Q_u = k_2\sqrt{1} = k_2 \quad \text{(ii)}$$

$$Q = Q_u\sqrt{H} \quad \text{[Substitute (ii) in (i)]}$$

$$\therefore \boxed{Q_u = \frac{Q}{\sqrt{H}}} \quad (24.2)$$

24.2.3 Unit Power

It is defined as the power developed by a turbine working under a unit head and it is denoted by P_u. Let H be the head under which turbine is working, P be the power developed by the turbine and Q be the discharge through the turbine.

$$P = \frac{\eta_o \rho_w g Q H}{1000}$$

or $$P \propto QH \quad [\rho_w, g, \eta_o = \text{Constant}]$$

or $$P \propto \sqrt{H} \times H \quad [\because Q \propto \sqrt{H}]$$

or $$P \propto H^{3/2}$$

or $$P = k_3 H^{3/2} \quad [k_3 = \text{Constant}] \tag{i}$$

If head on the turbine becomes unity $(H = 1\text{ m})$, then $P = P_u$. Thus, from Equation (i), we get the following expression.

$$P_u = k_3 (1)^{3/2} = k_3 \tag{ii}$$

$$P = P_u H^{3/2} \quad [\text{Substitute (ii) in (i)}]$$

$$\therefore \boxed{P_u = \frac{P}{H^{3/2}}} \tag{24.3}$$

24.2.4 Use of Unit Quantities

When a turbine works under different heads, its behaviour can be easily known from the value of the unit quantities. Let H_1 and H_2 be the heads under which turbine works, Q_1 and Q_2 be the corresponding discharges, N_1 and N_2 be the corresponding speeds, and P_1 and P_2 be the corresponding powers developed. By using Equations (24.1), (24.2), and (24.3), respectively, we get the following expressions.

$$\boxed{N_u = \frac{N_1}{\sqrt{H_1}} = \frac{N_2}{\sqrt{H_2}}} \tag{24.4}$$

$$\boxed{Q_u = \frac{Q_1}{\sqrt{H_1}} = \frac{Q_2}{\sqrt{H_2}}} \tag{24.5}$$

$$\boxed{P_u = \frac{P_1}{H_1^{3/2}} = \frac{P_2}{H_2^{3/2}}} \tag{24.6}$$

Example 24.1 A turbine is to operate under a head of 25 m at 200 rpm. The discharge is 9 m^3/s. If the efficiency is 90%, then determine the performance of turbine under a head of 20 m.

Solution

Let $H_1 = 25$ m, $N_1 = 200$ rpm, $Q_1 = 9\ \text{m}^3/\text{s}$, $\eta_o = 0.9$ and $H_2 = 20$ m.

$$P_1 = \frac{\eta_o \rho_w g Q_1 H_1}{1000} = \frac{0.9 \times 1000 \times 9.81 \times 9 \times 25}{1000}$$

$$= 1986.525\ \text{kW}$$

Since $$\frac{N_1}{\sqrt{H_1}} = \frac{N_2}{\sqrt{H_2}}$$

$$\therefore N_2 = \frac{N_1\sqrt{H_2}}{\sqrt{H_1}} = \frac{200\times\sqrt{20}}{\sqrt{25}} = \mathbf{178.885\ rpm}$$

Since $$\frac{Q_1}{\sqrt{H_1}} = \frac{Q_2}{\sqrt{H_2}}$$

$$\therefore Q_2 = \frac{Q_1\sqrt{H_2}}{\sqrt{H_1}} = \frac{9\times\sqrt{20}}{\sqrt{25}} = \mathbf{8.05\ m^3/s}$$

Since $$\frac{P_1}{{H_1}^{3/2}} = \frac{P_2}{{H_2}^{3/2}}$$

$$\therefore P_2 = \frac{P_1{H_2}^{3/2}}{{H_1}^{3/2}} = \frac{1986.525\times 20^{3/2}}{25^{3/2}} = \mathbf{1421.4416\ kW}$$

24.3 ❐ SPECIFIC SPEED

It is defined as the speed of a geometrically similar turbine (i.e., a turbine which is identical in shape, dimensions, blade angles, gate openings, etc.) to the actual turbine but of such a size that under corresponding conditions it will develop unit power when working under unit head. The value of specific speed (N_s) for a turbine is given below.

$$P = \frac{\eta_o \rho_w g Q H}{1000} \qquad \text{(Shaft power of the actual turbine)}$$

or $$P \propto QH \qquad [\rho_w, g, \eta_o = \text{Constant}] \qquad \text{(i)}$$

Since $$u \propto V \propto \sqrt{H}$$

Thus $$u \propto \sqrt{H}$$

Also $$u \propto DN \qquad [\because u = \pi DN/60]$$

Thus $$DN \propto \sqrt{H}$$

or $$D \propto \frac{\sqrt{H}}{N} \qquad \text{(ii)}$$

The discharge through the turbine is given by,

$$Q = AV$$

But $$A \propto BD \propto D^2 \qquad [\because B \propto D]$$

and $$V \propto \sqrt{H}$$

Thus $$Q \propto D^2\sqrt{H} \qquad \text{(iii)}$$

$$Q \propto \left(\frac{\sqrt{H}}{N}\right)^2 \sqrt{H} \quad \text{[Substitute (ii) in (iii)]}$$

or

$$Q \propto \frac{H^{3/2}}{N^2}$$

Substituting the value of Q in expression (i), we get:

$$P \propto \frac{H^{3/2}}{N^2} \times H$$

or

$$P \propto \frac{H^{5/2}}{N^2}$$

Thus

$$P = k\frac{H^{5/2}}{N^2} \qquad [k = \text{Constant}] \tag{iv}$$

By definition: When $H = 1$ and $P = 1$, then $N = N_s$. By substituting these values in Equation (iv), we get the following result.

$$1 = k\frac{1^{5/2}}{N_s^{\,2}}$$

$$N_s^{\,2} = k \tag{v}$$

Thus

$$P = N_s^{\,2}\,\frac{H^{5/2}}{N^2} \quad \text{[Substitute (v) in (iv)]}$$

$$N_s^{\,2} = \frac{N^2 P}{H^{5/2}}$$

$$\therefore \boxed{N_s = \frac{N\sqrt{P}}{H^{5/4}}} \tag{24.7}$$

It is observed from Equation (24.7) that the specific speed is independent of the dimensions or size of the turbines (model or prototype). Therefore, all geometrically similar turbines working under the same head, having the same speed ratio (K_u) and flow ratios (K_f) and thus, having the same efficiency will have the same specific speed irrespective of its sizes and powers developed under different heads. Usually, the specific speed is evaluated for working conditions corresponding to maximum efficiency. It is pertinent to mention here that specific speed signifies the shape rather than the size of a machine. Thus, it is evident that actual turbines of different shapes will have different specific speeds.

24.3.1 Significance of Specific Speed

The specific speed plays a significant role in the selection of the turbine types. The performance of a turbine can also be predicted by knowing the specific speed of the turbine. The higher specific speed of a turbine results in the reduction of runner diameter as well as the overall size of the runner. Thus, the weight and cost of the runner can also be reduced. Therefore, from economic perspective, a turbine runner with the highest possible specific speed should be selected. The types of turbine for different specific speeds are given below.

1. Pelton wheel with single jet: 8.5 to 30
2. Pelton wheel with two or more jets: 30 to 50
3. Francis turbine: 51 to 255
4. Propeller and Kaplan turbine: 255 to 860

24.4 ❐ SUCTION SPECIFIC SPEED

The suction specific speed (N_{su}) may be defined as the speed of geometrically similar turbine such that when it is developing one kilowatt power, the total suction head is equal to one metre. The expression for suction specific speed may be obtained by replacing the total head or working head of the turbine (H) in Equation (24.7) by the total suction head (H_{ts}). Therefore, suction specific speed is given in the following expression.

$$\boxed{N_{su} = \frac{N\sqrt{P}}{H_{ts}^{5/4}}} \tag{24.8}$$

The suction specific speed provides very useful criterion for establishing similarity with regard to cavitation in the turbines in addition to Thoma's cavitation factor (σ).

Now recalling Equation (23.9), the total suction head is given by,

$$H_{ts} = \sigma H \tag{24.9}$$

By combining equations (24.8) and (24.9), we get:

$$N_{su} = \frac{N\sqrt{P}}{(\sigma H)^{5/4}}$$

Thus

$$\boxed{\sigma = \left(\frac{N_s}{N_{su}}\right)^{4/5}} \tag{24.10}$$

Equation (24.10) is useful to establish a similarity with respect to cavitation in the model and prototype turbines.

Example 24.2 A turbine is to operate under a head of 30 m at 190 rpm and the discharge is 8 m^3/s. If the efficiency is 85%, then determine (i) the power generated, (ii) specific speed of the turbine, (iii) type of turbine and (iv) the performance of turbine under a head of 20 m.

Solution

Let $H_1 = 30$ m, $N_1 = 190$ rpm, $Q_1 = 8\ \text{m}^3/\text{s}$, $\eta_o = 0.85$ and $H_2 = 20$ m.

(i) $P_1 = \dfrac{\eta_o \rho_w g Q_1 H_1}{1000} = \dfrac{0.85 \times 1000 \times 9.81 \times 8 \times 30}{1000} = \mathbf{2001.24\ kW}$

(ii) $N_s = \dfrac{N_1\sqrt{P_1}}{H_1^{5/4}} = \dfrac{190 \times \sqrt{2001.24}}{30^{5/4}} = \mathbf{121.06}$

(iii) Since specific speed lies in the range of 51 to 255, it is a Francis turbine.

(iv) From Equations (24.4), (24.5) and (24.6), respectively, we get the following results.

$$N_2 = \frac{N_1\sqrt{H_2}}{\sqrt{H_1}} = \frac{190\times\sqrt{20}}{\sqrt{30}} = \mathbf{155.13\ rpm}$$

$$Q_2 = \frac{Q_1\sqrt{H_2}}{\sqrt{H_1}} = \frac{8\times\sqrt{20}}{\sqrt{30}} = \mathbf{6.532\ m^3/s}$$

$$P_2 = \frac{P_1 H_2^{3/2}}{H_1^{3/2}} = \frac{2001.24\times 20^{3/2}}{30^{3/2}} = \mathbf{1089.34\ kW}$$

Example 24.3 In a hydroelectric station, water is available at the rate of 175 m^3/s under a head of 18 m. The turbine runs at a speed of 150 rpm with an overall efficiency of 82%. Find the number of turbines required if they have maximum specific speed of 460.

Solution

Let $Q = 175\ \text{m}^3/\text{s}, H = 18\ \text{m},\ N = 150\ \text{rpm},\ \eta_o = 0.82$ and $N_s = 460$. Let n be the number of turbines.

Since $$N_s = \frac{N\sqrt{P}}{H^{5/4}}$$

$$\therefore P = \left(\frac{N_s H^{5/4}}{N}\right)^2 = \left(\frac{460\times 18^{5/4}}{150}\right)^2 = 12927.5\ \text{kW}$$

The total power available from these turbines is given by,

$$P_t = \frac{\eta_o \rho_w g Q H}{1000} = \frac{0.82\times 1000\times 9.81\times 175\times 18}{1000} = 25339.23\ \text{kW}$$

$$\therefore n = \frac{P_t}{P} = \frac{25339.23}{12927.5} = 1.96 \approx \mathbf{2}$$

24.5 ❒ SPECIFIC SPEED IN TERMS OF KNOWN COEFFICIENTS

24.5.1 Specific Speed of Pelton Turbine

The specific speed of the Pelton turbine may be expressed in terms of C_v (or K_v), K_u (or ϕ), η_o and m as given below.

Since $$u = K_u\sqrt{2gH} = \frac{\pi D N}{60}$$

Thus $$N = \frac{60K_u\sqrt{2gH}}{\pi D} = \frac{84.6K_u\sqrt{H}}{D} \quad \text{(i)}$$

Since $$P = \frac{\eta_o \rho_w g H Q}{1000} = \frac{\eta_o \rho_w g H (AV)}{1000} = \frac{\eta_o \rho_w g H[(\pi/4)d^2 \times C_v\sqrt{2gH}]}{1000}$$

$$\therefore P = 34.13(\eta_o d^2 C_v H^{3/2})\ kW \quad \text{(ii)}$$

$$N_s = \frac{N\sqrt{P}}{H^{5/4}} = \frac{84.6K_u\sqrt{H}\times(34.13)^{1/2}(\eta_o d^2 C_v H^{3/2})^{1/2}}{D\times H^{5/4}} \quad \text{[Substitute (i) and (ii)]}$$

$$\therefore N_s = \frac{494.24K_u(\eta_o C_v)^{1/2}}{(D/d)} = \frac{494.24K_u(\eta_o C_v)^{1/2}}{m} \tag{24.7.1}$$

Considering usual values of the coefficients $K_u = 0.46$, $C_v = 0.98$ and $\eta_o = 0.85$, we get:

$$N_s = \frac{494.24\times0.46\times(0.85\times0.98)^{1/2}}{m} = \frac{207.5}{m} \tag{24.7.1a}$$

Equation (24.7.1a) signifies the relationship between specific speed (N_s) and jet ratio (m) for a single jet Pelton wheel. For maximum efficiency, the jet ratio varies from 11 to 14, but in practice for most of the cases it is taken as 12. However, in some exceptional cases, an abnormally low value of $m = 7$ has also been used. The value of specific speed is observed to have a narrow range of variation which varies from about 30 to 17 for the values of jet ratio varying from 7 to 12. For multiple jets Pelton turbine with n jets, the Equation (24.7.1a) for specific speed is given below.

$$\boxed{N_s = \frac{207.5\sqrt{n}}{m}} \tag{24.7.1b}$$

24.5.2 Specific Speed of Francis Turbine

The specific speed of the Francis turbine may be expressed in terms of K_u, K_f, η_o and n as given below.

Since
$$u_i = K_u\sqrt{2gH} = \frac{\pi D_i N}{60}$$

Thus
$$N = \frac{60K_u\sqrt{2gH}}{\pi D_i} = \frac{84.6K_u\sqrt{H}}{D_i} \tag{i}$$

Since
$$P = \frac{\eta_o\rho_w gHQ}{1000} = \frac{\eta_o\rho_w gH(AV)}{1000} = \frac{\eta_o\rho_w gH(k\pi nD_i^2\times K_f\sqrt{2gH})}{1000}$$

$$P = 136.51(\eta_o knD_i^2 K_f H^{3/2})\ kW \tag{ii}$$

$$N_s = \frac{N\sqrt{P}}{H^{5/4}} = \frac{84.6K_u\sqrt{H}\times(136.51)^{1/2}[\eta_o knD_i^2 K_f H^{3/2}]^{1/2}}{D_i\times H^{5/4}} \quad \text{[Substitute (i) and (ii)]}$$

$$\therefore \boxed{N_s = 988.45K_u\sqrt{knK_f\eta_o}} \tag{24.7.2}$$

Generally, the overall efficiency (η_o) and the vane thickness factor (k) have constant values. Thus, the Equation (24.7.2) indicates that specific speed N_s for the Francis turbines depends upon the speed ratio (K_u), flow ratio (K_f) and breadth ratio (n). In general, K_u ranges from 0.6 to 0.9, K_f ranges from 0.15 to 0.30 and n ranges from 0.10 to 0.45. The variation in any or all of these parameters change N_s and hence, a much greater range of N_s is available for Francis turbines (51 to 255) when compared to that for Pelton wheel (8.5 to 50).

24.5.3 Specific Speed of Kaplan and Propeller Turbines

The specific speed of the Kaplan and propeller turbine may be expressed similarly as that of Francis turbines. It may be obtained in terms of K_u and K_f as given below.

Since $$u_i = u_o = K_u\sqrt{2gH} = \frac{\pi D_o N}{60}$$

Thus $$N = \frac{60K_u\sqrt{2gH}}{D_o} = \frac{84.6K_u\sqrt{H}}{D_o} \quad \text{(i)}$$

$$P = \frac{\eta_o\rho_w gHQ}{1000} = \frac{\eta_o\rho_w gH(AV)}{1000} = \frac{\eta_o\rho_w gH\times(\pi/4)D_o^2(1-n^2)\times K_f\sqrt{2gH}}{1000}$$

$$P = 34.13[\eta_o D_o^2(1-n^2)K_f H^{3/2}] \quad \text{(ii)}$$

$$N_s = \frac{N\sqrt{P}}{H^{5/4}} = \frac{84.6K_u\sqrt{H}\times(34.13)^{1/2}[\eta_o D_o^2(1-n^2)K_f H^{3/2}]^{1/2}}{D_o\times H^{5/4}} \quad \text{[Substitute (i) and (ii)]}$$

$$\therefore N_s = 494.24K_u\sqrt{(1-n^2)K_f\,\eta_o} \quad (24.7.3)$$

If $\eta_o = 0.9$ and $n = (D_b/D_o) = 0.35$, then Equation (24.7.3) is given by,

$$\boxed{N_s = 494.24K_u\sqrt{(1-0.35^2)K_f\times 0.9} = 439.22K_u\sqrt{K_f}} \quad (24.7.3a)$$

24.6 ❐ MODEL RELATIONSHIP AND TESTING OF TURBINES

The model is a small scale replica of the actual machine or the prototype. For complete similarity to exist between the model and the prototype turbines, the following conditions may be satisfied. The subscripts *m* and *p* used in the following discussion denote the model and the prototype turbines, respectively.

24.6.1 Head Coefficient

Since $$u \propto V \propto \sqrt{H}$$

$$u \propto \sqrt{H}$$

Also $$u \propto DN \qquad [\because u = \pi DN/60]$$

$$ND \propto \sqrt{H} \quad \text{(i)}$$

$$\frac{ND}{\sqrt{H}} = \text{Constant} \quad (24.11)$$

$$\therefore \boxed{\frac{H}{N^2D^2} = \text{Constant}} \quad (24.11a)$$

Thus $$\left(\frac{H}{N^2D^2}\right)_m = \left(\frac{H}{N^2D^2}\right)_p \tag{24.11b}$$

The parameter $[H/(N^2D^2)]$ is called head coefficient.

The scale ratio is the ratio of diameter of turbine model and the prototype of turbine. The scale ratio for a turbine may be expressed as shown below.

From Equation (i), we get:

$$\frac{D_m N_m}{D_p N_p} = \frac{\sqrt{H_m}}{\sqrt{H_p}}$$

$$\therefore \boxed{\frac{D_m}{D_p} = \frac{N_p}{N_m}\sqrt{\frac{H_m}{H_p}}} \tag{24.12}$$

24.6.2 Capacity or Flow Coefficient

The discharge through the turbine is given by the following expression.

$$Q = AV$$

Since $$A \propto D^2 \text{ and } V \propto \sqrt{H}$$

$$\therefore Q \propto D^2\sqrt{H}$$

$$Q \propto D^2 (ND) \quad [\because ND \propto \sqrt{H}]$$

$$Q \propto ND^3$$

$$\therefore \boxed{\frac{Q}{ND^3} = \text{Constant}} \tag{24.13}$$

Thus $$\left(\frac{Q}{ND^3}\right)_m = \left(\frac{Q}{ND^3}\right)_p \tag{24.13a}$$

The parameter $[Q/(ND^3)]$ is called the capacity or flow coefficient.

24.6.3 Power Coefficient

The power at the shaft of a turbine is given by the following expression.

$$P = \frac{\eta_o \rho_w g Q H}{1000}$$

$$P \propto QH \quad [\because \rho_w, g, \eta_o = \text{Constant}]$$

Since $$Q \propto ND^3 \text{ and } \sqrt{H} \propto ND$$

$$P \propto (ND^3)(N^2D^2)$$

$$P \propto N^3D^5$$

$$\therefore \boxed{\frac{P}{N^3 D^5} = \text{Constant}} \tag{24.14}$$

Thus
$$\left(\frac{P}{N^3 D^5}\right)_m = \left(\frac{P}{N^3 D^5}\right)_p \tag{24.14a}$$

The parameter $[P/(N^3 D^5)]$ is called the power coefficient.

24.6.4 Model Testing of Turbines

The numerous variables involved in the analysis of model testing of turbines are discharge (Q), head (H), speed (N), runner diameter (D), power output (P), mass density (ρ) and viscosity (μ) of the fluid used. With the help of dimensional analysis, these variables may be grouped into the following dimensionless numbers.

$$\frac{Q}{ND^3}, \frac{gH}{N^2 D^2}, \frac{P}{\rho gHND^3}, \frac{\mu}{\rho ND^2}$$

The last term is the expression of Reynolds number which does not affect the performance of turbines when the flow is turbulent. In most of the problems, the value of acceleration due to gravity and the density are same in the model and the prototype. Therefore, we have the following groups.

$$\frac{Q}{ND^3} \text{ (1}^{\text{st}} \text{ group)}, \frac{H}{N^2 D^2} \text{ (2}^{\text{nd}} \text{ group)}, \frac{P}{HND^3} \text{ (3}^{\text{rd}} \text{ group)}$$

By combining 1^{st} and 2^{nd} groups, we get:

$$\frac{1^{\text{st}} \text{ group}}{\sqrt{2^{\text{nd}} \text{ group}}} = \frac{Q}{ND^3} \times \sqrt{\frac{N^2 D^2}{H}} = \frac{Q}{D^2 \sqrt{H}}$$

Thus
$$\left(\frac{Q}{D^2 \sqrt{H}}\right)_m = \left(\frac{Q}{D^2 \sqrt{H}}\right)_p \tag{24.15}$$

By combining 2^{nd} and 3^{rd} groups, we get:

$$\frac{3^{\text{rd}} \text{ group}}{\sqrt{2^{\text{nd}} \text{ group}}} = \frac{P}{HND^3} \times \sqrt{\frac{N^2 D^2}{H}} = \frac{P}{D^2 H^{3/2}}$$

Thus
$$\left(\frac{P}{D^2 H^{3/2}}\right)_m = \left(\frac{P}{D^2 H^{3/2}}\right)_p \tag{24.16}$$

By combining 2^{nd} and 3^{rd} groups, the specific speed for turbines can be obtained as given below.

$$\frac{\sqrt{3^{\text{rd}} \text{ group}}}{(2^{\text{nd}} \text{ group})^{3/4}} = \frac{\sqrt{P}}{\sqrt{HND^3}} \times \left(\frac{N^2 D^2}{H}\right)^{3/4} = \frac{N\sqrt{P}}{H^{5/4}} = N_s$$

Thus
$$\left(\frac{N\sqrt{P}}{H^{5/4}}\right)_m = \left(\frac{N\sqrt{P}}{H^{5/4}}\right)_p \text{ or } (N_s)_m = (N_s)_p \tag{24.17}$$

With the use of above relations, it is possible to present the behaviour of a prototype from the test conducted on geometrically similar model which is assumed to have the same values of speed ratio, flow ratio and specific speed. Geometrically, similar machines have the same values of head, capacity and power coefficients or their combinations.

24.6.5 Scale Effect

For complete similarity to exist between the model and the prototype turbines, it is assumed that their efficiencies are equal. However, the efficiencies of the model and the prototype turbines are not equal. This is due to the surface roughness which causes more energy losses in the model than its prototype. Moreover, disproportionate leakage, mechanical and exit losses tend to cause different efficiencies. Thus, the efficiency of the model turbine is generally lower than that of its prototype. This aspect is referred to as scale effect that measures the error in predicting the performance of prototype turbine on the basis of the model test results. The overall efficiency of the prototype turbine is given by the following expression.

$$\eta_{op} = \left(\frac{H - h_L}{H}\right)_p$$

Here, H and h_L are the head and the loss of head, respectively.

Thus, the net effective head available for the prototype turbine is given by,

$$(H - h_L)_p = \eta_{op}(H)_p$$

The overall efficiency of the model turbine is given by,

$$\eta_{om} = \left(\frac{H - h_L}{H}\right)_m$$

Thus, the net effective head available for the model turbine is given by,

$$(H - h_L)_m = \eta_{om}(H)_m$$

By considering the above values of the head in place of H in Equations (24.11b), (24.15), (24.16) and (24.17), the following expressions can be obtained.

$$\left(\frac{\eta_o H}{N^2 D^2}\right)_m = \left(\frac{\eta_o H}{N^2 D^2}\right)_p \tag{24.11c}$$

$$\left(\frac{Q}{D^2\sqrt{\eta_o H}}\right)_m = \left(\frac{Q}{D^2\sqrt{\eta_o H}}\right)_p \tag{24.15a}$$

$$\left[\frac{P}{D^2(\eta_o H)^{3/2}}\right]_m = \left[\frac{P}{D^2(\eta_o H)^{3/2}}\right]_p \tag{24.16a}$$

$$\left[\frac{N\sqrt{P}}{(\eta_o H)^{5/4}}\right]_m = \left[\frac{N\sqrt{P}}{(\eta_o H)^{5/4}}\right]_p \tag{24.17a}$$

It is important to note that Lewis Ferry Moody suggested an expression which is generally used to determine the efficiency of a prototype turbine from the efficiency obtained for its model given by the following expression.

$$\boxed{\frac{1-\eta_{op}}{1-\eta_{om}} = \left(\frac{D_m}{D_p}\right)^{0.2}} \tag{24.18}$$

Here, D_p and D_m are the respective diameters of their runners.

Equation (24.18) is applicable to the reaction turbines working under a head less than 150 *m*. The following equation is applicable for head more than 150 m.

$$\frac{1-\eta_{op}}{1-\eta_{om}} = \left(\frac{D_m}{D_p}\right)^{0.25}\left(\frac{H_m}{H_p}\right)^{0.1} \tag{24.19}$$

Here, H_p and H_m are the heads acting on the prototype and model turbines, respectively.

No scale effect is observed for the impulse turbine which means the efficiency of an impulse turbine is equal to that of its model.

Example 24.4 A one fifth scale model of a Francis turbine develops 4.5 kW under a head of 2 m and 425 rpm. Find the speed and power of the Francis turbine working under a head of 30 m.

Solution

Let $D_m/D_p = 1/5,\ P_m = 4.5\text{ kW},\ H_m = 2\text{ m},\ N_m = 425\text{ rpm}$ and $H_p = 30\text{ m}.$

$$\frac{D_m}{D_p} = \frac{N_p}{N_m}\times\sqrt{\frac{H_m}{H_p}} \Rightarrow N_p = N_m\times\frac{D_m}{D_p}\times\sqrt{\frac{H_p}{H_m}}$$

$$\therefore N_p = 425\times\frac{1}{5}\times\sqrt{\frac{30}{2}} = \mathbf{329.2\ rpm}$$

$$\left(\frac{N\sqrt{P}}{H^{5/4}}\right)_m = \left(\frac{N\sqrt{P}}{H^{5/4}}\right)_p \Rightarrow P_p = \left[\frac{N_m}{N_p}\times\left(\frac{H_p}{H_m}\right)^{5/4}\times\sqrt{P_m}\right]^2$$

$$\therefore P_p = \left[\frac{425}{329.2}\times\left(\frac{30}{2}\right)^{5/4}\times\sqrt{4.5}\right]^2 = \mathbf{6535.8\ kW}$$

Example 24.5 A hydraulic turbine generates 0.13 MW at 230 rpm while operating under a head of 16 m. Calculate the scale ratio and the speed of a similar turbine which will develop 0.65 MW when operating under a head of 25 m.

Solution

Let $P_1 = 0.13\text{ MW},\ N_1 = 230\text{ rpm},\ H_1 = 16\text{ m},\ P_2 = 0.65\text{ MW}$ and $H_2 = 25\text{ m}.$

$$\frac{P_1}{D_1^2H_1^{3/2}} = \frac{P_2}{D_2^2H_2^{3/2}} \Rightarrow \frac{D_2}{D_1} = \left(\frac{P_2H_1^{3/2}}{P_1H_2^{3/2}}\right)^{1/2}$$

$$\therefore \frac{D_2}{D_1} = \left(\frac{0.65\times16^{3/2}}{0.13\times25^{3/2}}\right)^{1/2} = \mathbf{1.6}$$

$$\frac{H_1}{N_1^2 D_1^2} = \frac{H_2}{N_2^2 D_2^2} \Rightarrow N_2 = \left[\frac{H_2 N_1^2 D_1^2}{H_1 D_2^2}\right]^{1/2}$$

$$\therefore N_2 = \left[\frac{25 \times 230^2 D_1^2}{16(1.6D_1)^2}\right]^{1/2} = \mathbf{179.69\ rpm}$$

Example 24.6 A hydraulic turbine delivering 10 MW power is to be tested with the help of a geometrically similar 1 : 8 model, which runs as the same speed as the prototype. Determine (i) the power developed by the model assuming that the efficiencies of the model and the prototype are equal and (ii) the ratio of the heads and the ratio of mass flow rates between the prototype and the model.

Solution

Let $P_p = 10$ MW, $D_m/D_p = 1/8$ and $N_m = N_p$. Let m be the mass flow rate.

(i) $$\left(\frac{P}{N^3 D^5}\right)_m = \left(\frac{P}{N^3 D^5}\right)_p \Rightarrow P_m = P_p \times \left(\frac{N_m}{N_p}\right)^3 \times \left(\frac{D_m}{D_p}\right)^5$$

$$\therefore P_m = 10 \times 10^6 \times (1)^3 \times \left(\frac{1}{8}\right)^5 = \mathbf{305.176\ W}$$

(ii) $$\left(\frac{H}{N^2 D^2}\right)_m = \left(\frac{H}{N^2 D^2}\right)_p \Rightarrow \frac{H_p}{H_m} = \left(\frac{N_p}{N_m}\right)^2 \times \left(\frac{D_p}{D_m}\right)^2$$

$$\therefore \frac{H_p}{H_m} = (1)^2 \times \left(\frac{8}{1}\right)^2 = \mathbf{64}$$

$$\left(\frac{Q}{ND^3}\right)_m = \left(\frac{Q}{ND^3}\right)_p \Rightarrow \frac{Q_p}{Q_m} = \left(\frac{N_p}{N_m}\right) \times \left(\frac{D_p}{D_m}\right)^3$$

$$\therefore \frac{m_p}{m_m} = \frac{Q_p}{Q_m} = (1) \times \left(\frac{8}{1}\right)^3 = \mathbf{512}$$

Example 24.7 A hydraulic turbine is to develop 1000 kW when running at 150 rpm under a head of 10 m. Determine the specific speed and maximum flow rate for the turbine if the overall efficiency is 90%. In order to predict its performance, a 1 : 10 scale model is tested under a head of 6 m. Determine the speed, water consumption and power output of the model if it runs under the conditions similar to the prototype?

Solution

Let $P_p = 1000$ kW, $N_p = 150$ rpm, $H_p = 10$ m, $\eta_{op} = 0.9$, $D_m/D_p = 1/10$ and $H_m = 6$ m.

$$Q_p = \frac{1000 P_p}{\rho_w g H_p \eta_{op}} = \frac{1000 \times 1000}{1000 \times 9.81 \times 10 \times 0.9} = \mathbf{11.33\ m^3/s}$$

$$(N_s)_p = \frac{N_p \sqrt{P_p}}{H_p^{5/4}} = \frac{150 \times \sqrt{1000}}{10^{5/4}} = \mathbf{266.74}$$

The two runners will be similar if head coefficient, flow coefficient and power coefficient are equal.

$$\left(\frac{H}{N^2D^2}\right)_m = \left(\frac{H}{N^2D^2}\right)_p \Rightarrow N_m = N_p \times \left(\frac{D_p}{D_m}\right) \times \left(\frac{H_m}{H_p}\right)^{1/2}$$

$$\therefore N_m = 150 \times \left(\frac{10}{1}\right) \times \left(\frac{6}{10}\right)^{1/2} = \mathbf{1161.9\ rpm}$$

$$\left(\frac{Q}{ND^3}\right)_m = \left(\frac{Q}{ND^3}\right)_p \Rightarrow Q_m = Q_p \times \left(\frac{N_m}{N_p}\right) \times \left(\frac{D_m}{D_p}\right)^3$$

$$\therefore Q_m = 11.33 \times \left(\frac{1161.9}{150}\right) \times \left(\frac{1}{10}\right)^3 = \mathbf{0.088\ m^3/s}$$

$$\left(\frac{P}{N^3D^5}\right)_m = \left(\frac{P}{N^3D^5}\right)_p \Rightarrow P_m = P_p \times \left(\frac{N_m}{N_p}\right)^3 \times \left(\frac{D_m}{D_p}\right)^5$$

$$\therefore P_m = 1000 \times \left(\frac{1161.9}{150}\right)^3 \times \left(\frac{1}{10}\right)^5 = \mathbf{4.65\ kW}$$

Example 24.8 A hydraulic turbine with specific speed 180 is to develop 25000 kW when running at 150 rpm. An experimental model is prepared to work under a head of 5 m with a flow rate of 0.5 m^3/s. Determine the speed, power and scale ratio for the model if the efficiency of turbine and its model is given 86%. Also determine the flow rate of the turbine.

Solution

Let $(N_s)_p = 180$, $P_p = 25000$ kW, $N_p = 150$ rpm, $H_m = 5$ m, $Q_m = 0.5$ m^3/s and $\eta_o = 0.86$.

$$P_m = \frac{\eta_o \rho_w g Q_m H_m}{1000} = \frac{0.86 \times 1000 \times 9.81 \times 0.5 \times 5}{1000} = \mathbf{21.0915\ kW}$$

Since $$\left(\frac{N\sqrt{P}}{H^{5/4}}\right)_m = \left(\frac{N\sqrt{P}}{H^{5/4}}\right)_p = 180$$

$$\therefore N_m = \frac{180 H_m^{5/4}}{\sqrt{P_m}} = \frac{180 \times 5^{5/4}}{\sqrt{21.0915}} = \mathbf{293.04\ rpm}$$

$$\therefore H_p = \left(\frac{N_p\sqrt{P_p}}{180}\right)^{4/5} = \left(\frac{150 \times \sqrt{25000}}{180}\right)^{4/5} = 49.64\text{ m}$$

$$\frac{D_m}{D_p} = \frac{N_p}{N_m} \times \sqrt{\frac{H_m}{H_p}} = \frac{150}{293.04} \times \sqrt{\frac{5}{49.64}} = \mathbf{0.1624}$$

$$Q_p = \frac{1000 P_p}{\rho_w g H_p \eta_o} = \frac{1000 \times 25000}{1000 \times 9.81 \times 49.64 \times 0.86} = \mathbf{59.6954\ m^3/s}$$

Example 24.9 A model of Francis turbine that is one-fourth of full size develops 2.5 kW at 300 rpm working under a head of 1.6 m. Determine the speed and power of full size turbine operating under a head of 5.5 m, if (i) the efficiency of the model and the full size turbine are same and (ii) the efficiency of the model turbine is 75% and the scale effect is considered.

Solution

Let $D_m/D_p = 1/4,\ P_m = 2.5\text{ kW},\ N_m = 300\text{ rpm},\ H_m = 1.6\text{ m},\ H_p = 5.5\text{ m},\ \eta_{om} = \eta_{op}$ and $\eta_{om} = 0.75$.

(i) $$\left(\frac{H}{N^2D^2}\right)_m = \left(\frac{H}{N^2D^2}\right)_p \Rightarrow N_p = N_m \times \left(\frac{D_m}{D_p}\right) \times \left(\frac{H_p}{H_m}\right)^{1/2}$$

$$\therefore N_p = 300 \times \left(\frac{1}{4}\right) \times \left(\frac{5.5}{1.6}\right)^{1/2} = \mathbf{139.05\ rpm}$$

$$\left(\frac{P}{D^2H^{3/2}}\right)_m = \left(\frac{P}{D^2H^{3/2}}\right)_p \Rightarrow P_p = P_m \times \left(\frac{D_p}{D_m}\right)^2 \times \left(\frac{H_p}{H_m}\right)^{3/2}$$

$$\therefore P_p = 2.5 \times \left(\frac{4}{1}\right)^2 \times \left(\frac{5.5}{1.6}\right)^{3/2} = \mathbf{254.932\ kW}$$

(ii) $$\frac{1-\eta_{op}}{1-\eta_{om}} = \left(\frac{D_m}{D_p}\right)^{0.2} \Rightarrow \frac{1-\eta_{op}}{1-0.75} = \left(\frac{1}{4}\right)^{0.2}$$

$$1-\eta_{op} = 0.25 \times 0.25^{0.2} = 0.25^{1.2}$$

$$\therefore \eta_{op} = [1-0.25^{1.2}] \times 100 = \mathbf{81.05\%}$$

$$\left(\frac{\eta_o H}{N^2D^2}\right)_m = \left(\frac{\eta_o H}{N^2D^2}\right)_p \Rightarrow N_p = N_m \times \left(\frac{D_m}{D_p}\right) \times \left(\frac{\eta_{op}H_p}{\eta_{om}H_m}\right)^{1/2}$$

$$\therefore N_p = 300 \times \left(\frac{1}{4}\right) \times \left(\frac{0.8105 \times 5.5}{0.75 \times 1.6}\right)^{1/2} = \mathbf{144.55\ rpm}$$

$$\left[\frac{P}{D^2(\eta_o H)^{3/2}}\right]_m = \left[\frac{P}{D^2(\eta_o H)^{3/2}}\right]_p \Rightarrow P_p = P_m \times \left(\frac{D_p}{D_m}\right)^2 \times \left(\frac{\eta_{op}H_p}{\eta_{om}H_m}\right)^{3/2}$$

$$\therefore P_p = 2.5 \times \left(\frac{4}{1}\right)^2 \times \left(\frac{0.8105 \times 5.5}{0.75 \times 1.6}\right)^{3/2} = \mathbf{286.392\ kW}$$

Example 24.10 In a hydroelectric generating plant there are four similar turbines of total output 220000 kW. Each turbine is 90% efficient and it runs at 100 rpm under a head of 65 m. It is proposed to test the model of the above turbine in a flume where discharge is 0.4 m^3/s under a head of 4 m. Determine the size (scale ratio) of the model and also calculate the model speed and power results expected from the model.

Solution

Let $n = 4, (P_p)_t = 220000\text{ kW}, \eta_{op} = 0.9,\ N_p = 100\text{ rpm},\ H_p = 65\text{ m},\ Q_m = 0.4\text{ m}^3/\text{s}$ and $H_m = 4\text{ m}$.

$$P_p = \frac{(P_p)_t}{n} = \frac{220000}{4} = 55000 \text{ kW}$$

$$Q_p = \frac{1000 P_p}{\rho_w g H_p \eta_{op}} = \frac{55000 \times 1000}{1000 \times 9.81 \times 65 \times 0.9} = 95.838 \text{ m}^3/\text{s}$$

$$\left(\frac{Q}{D^2 \sqrt{H}}\right)_m = \left(\frac{Q}{D^2 \sqrt{H}}\right)_p \Rightarrow \frac{D_m}{D_p} = \left(\frac{Q_m \times \sqrt{H_p}}{Q_p \times \sqrt{H_m}}\right)^{1/2}$$

$$\therefore \frac{D_m}{D_p} = \left(\frac{0.4 \times \sqrt{65}}{95.838 \times \sqrt{4}}\right)^{1/2} = \mathbf{0.1297}$$

$$\left(\frac{H}{N^2 D^2}\right)_m = \left(\frac{H}{N^2 D^2}\right)_p \Rightarrow N_m = N_p \times \left(\frac{D_p}{D_m}\right) \times \left(\frac{H_m}{H_p}\right)^{1/2}$$

$$\therefore N_m = 100 \times \left(\frac{1}{0.1297}\right) \times \left(\frac{4}{65}\right)^{1/2} = \mathbf{191.26 \text{ rpm}}$$

$$\left(\frac{P}{N^3 D^5}\right)_m = \left(\frac{P}{N^3 D^5}\right)_p \Rightarrow P_m = P_p \times \left(\frac{N_m}{N_p}\right)^3 \times \left(\frac{D_m}{D_p}\right)^5$$

$$\therefore P_m = 55000 \times \left(\frac{191.26}{100}\right)^3 \times (0.1297)^5 = \mathbf{14.123 \text{ kW}}$$

24.7 ❐ CHARACTERISTIC CURVES

Generally, the hydraulic turbines are designed for specific conditions given by six important parameters, namely (i) head (H), (ii) discharge (Q), (iii) speed (N), (iv) power (P), (v) gate opening (G.O.) and (vi) efficiency (η_o) which are known as the design conditions. Turbines produce maximum efficiency while operating at design conditions, but these are required to operate under different conditions. Thus, it becomes necessary to determine the exact behaviour of the turbines working under varying conditions by conducting tests on the prototypes or their models. The results obtained from the experiments are graphically plotted by means of curves which are termed as characteristic curves. These curves are plotted in terms of unit quantities. Among the six given parameters, H, Q and N are termed as independent parameters. Out of the three independent parameters (H, Q and N), one of the parameters (assume H) is kept constant and the variations of the remaining parameters with respect to any one of the two independent parameters (assume Q and N) are plotted and thus, various characteristic curves can be obtained. The important characteristic curves of a turbine are listed below.

1. **Main characteristic curves (or constant head characteristic curves):** The turbines are tested at constant head. Here, Q and P are plotted against varying N, for a fixed G.O.
2. **Operating characteristic curves (or constant speed characteristic curves):** The turbines are tested at constant speed. Here, P is measured against varying Q. H remains constant but it may vary.
3. **Muschel curves (or constant efficiency curves or iso-efficiency curves)**.

24.7.1 Main Characteristic Curves (or Constant Head Characteristic Curves)

Main characteristic curves are obtained by maintaining a constant head and a constant gate opening on the turbine. The speed of the turbine is varied by changing the load on the turbine. For each value of speed, the corresponding values of discharge and output power are measured. A series of such tests are conducted by varying the gate opening but keeping the head constant at the previous value. The values of N_u, Q_u, P_u and η_o for each gate opening is calculated. By taking N_u as abscissa, the values of Q_u, P_u and η_o are plotted for each gate opening. The main characteristic curves for four different gate openings of the Pelton turbine and reaction turbines (Francis and Kaplan turbines) are shown in Figures 24.1 and 24.2, respectively.

The main characteristics curves provide the following information:

1. Q_u versus N_u curves for Pelton turbine are horizontal straight lines which indicate that Q_u depends only on the gate opening and it is independent of N_u.
2. Q_u versus N_u curves for Francis turbine are drooping curves. This indicates that as the speed increases, the discharge through the turbine decreases. This is due to the presence of centrifugal head acting against the flow which increases with speed and thus, it reduces the flow.
3. Q_u versus N_u curves for Kaplan turbine are rising curves which indicate that unit discharge increases with increase in unit speed.
4. P_u versus N_u and η_o versus N_u curves are parabolic in nature for different turbines.
5. The maximum efficiency of a Pelton turbine for each gate opening occurs at the same speed which corresponds to $(u/V_i) = 0.46$.
6. The maximum efficiency for a reaction turbine for each gate opening attains at different values of speed.

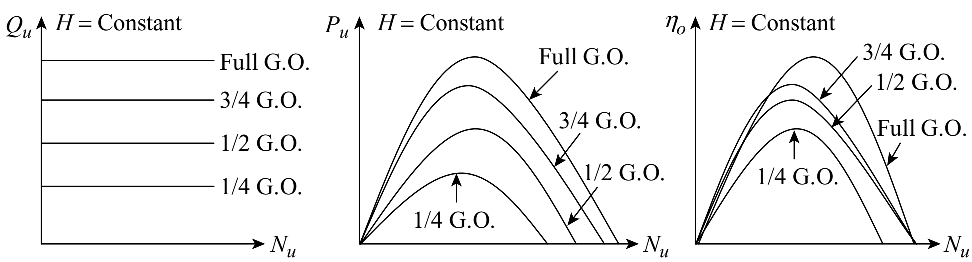


Figure 24.1 *Main characteristic curves for the Pelton turbine*

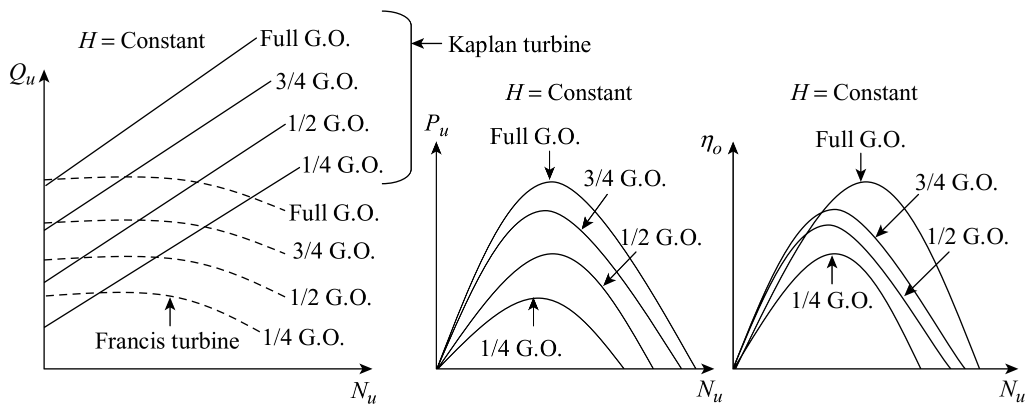


Figure 24.2 *Main characteristic curves for reaction turbines*

24.7.2 Operating Characteristic Curves (or Constant Speed Characteristic Curves)

The operating characteristic curves are obtained as given below.

1. The turbines are operated at constant speed which is maintained by regulating the gate opening. Thus, the discharge flowing through the turbines varies as the load varies. The head may remain constant. The power developed for each gate opening is measured by means of a dynamometer and the corresponding values of overall efficiency are calculated. The ratio of measured power to full load known as percentage of full load is calculated. The results are then graphically illustrated by plotting η_o versus full load.

   Figure 24.3(a) shows the curves between percentages of full load versus η_o for the four different types of turbines. The overall efficiency increases as the percentage full load increases and it is near about maximum at 100 per cent full load for all the turbines. The maximum overall efficiency is observed about 85% in all cases. It is also observed that the Kaplan and Pelton turbines produce high efficiency over a long range of the part load in comparison to the Francis and propeller turbines.
2. For obtaining operating characteristics curves, N and H are maintained at constant and the variation of P and η_o with respect to Q are also plotted as shown in Figure 24.3(b). The power and efficiency curves are slightly away from the origin as some discharge is required to initiate the motion of the runner from its state of rest. It is observed that P versus Q is a straight line which shows that the power output is directly proportional to the discharge when head is constant. The plot between η_o and Q is curvilinear. The overall efficiency increases with discharge and it remains constant beyond a particular value of discharge.

24.7.3 Muschel Curves (or Constant Efficiency Curves or Iso-efficiency Curves)

Muschel curves are obtained from η_o versus N_u and Q_u (or P_u) versus N_u curves plotted at different gate openings. These curves are also known as universal characteristic curves of the turbine as it shows the efficiencies of a turbine for all conditions of running. From η_o versus N_u curves, it can be seen that there exists two speeds for one value of efficiency for each G.O. Corresponding to these speed values, there are two values of discharge at Q_u versus N_u curves for a particular G.O. and for a given efficiency. Thus, there are two values of speeds and two values of discharge for a particular G.O. and a given efficiency except for maximum efficiency that occurs at one speed only.

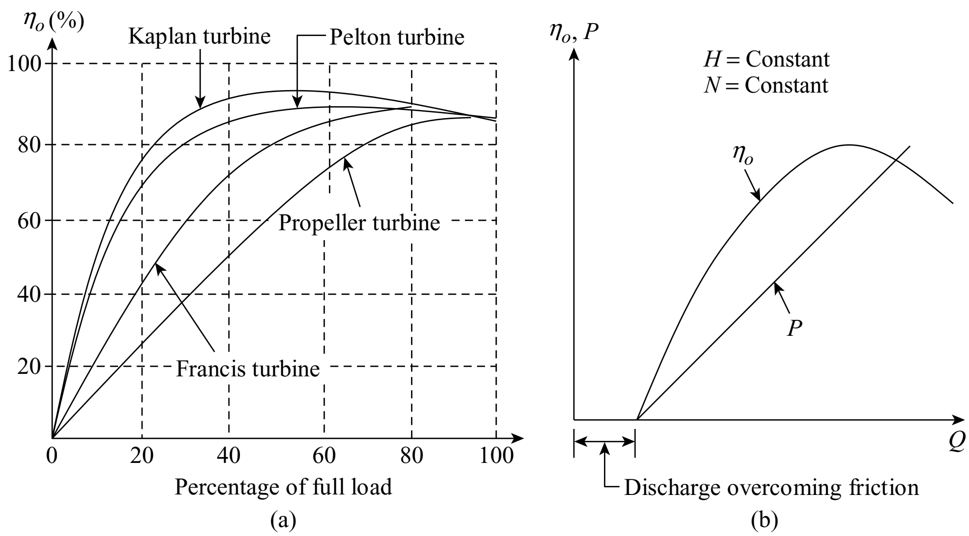


Figure 24.3 *Operating characteristic curves*

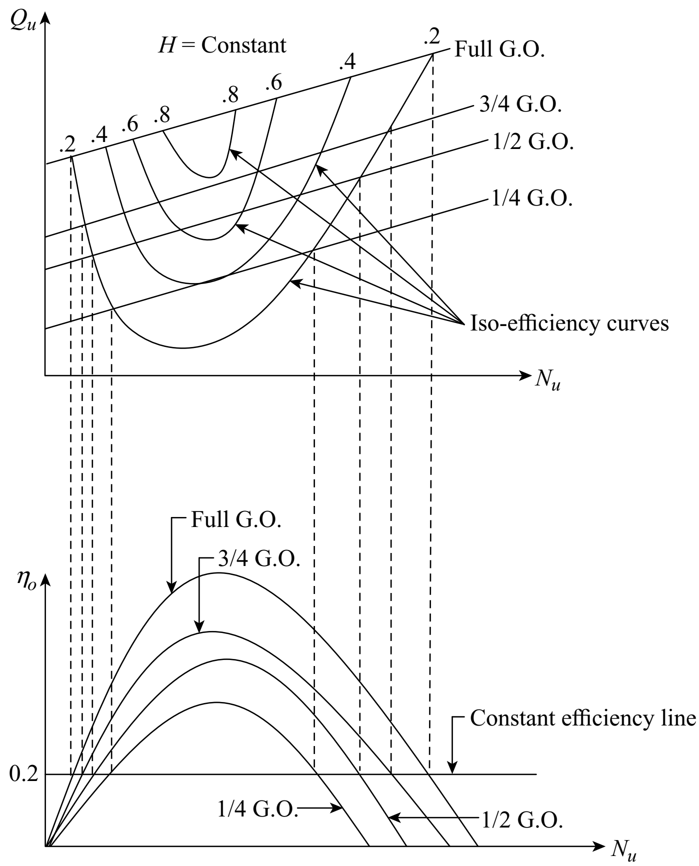


Figure 24.4 *Constant efficiency curves*

A horizontal line is drawn for a given value of efficiency (assume 0.2) which intersects η_o versus N_u curves for different gate openings. Thus, two speeds for one value of efficiency are obtained from the points of intersection. These values of speeds are then transferred to the main curve Q_u versus N_u for the corresponding gate openings. The points having the same efficiency are joined by a smooth curve to get a constant efficiency curve as shown in Figure 24.4. This procedure is repeated for different gate openings and thus, other constant efficiency curves are obtained. These smooth curves represent the iso-efficiency curves. By joining the peak points of these iso-efficiency curves, a curve for best performance is obtained. The constant efficiency curves are helpful in locating the zones where the turbine would work with highest efficiency. Sometimes, the diagram showing iso-efficiency curves is also known as Hill diagram.

24.8 ❐ SELECTION OF TURBINES

The following factors should be considered while selecting the right type of hydraulic turbine.

1. **Specific speed:** As a general rule, it may be stated that as far as possible a turbine with highest permissible specific speed should be chosen. It is not only inexpensive but its relatively small size and high rotational speed will reduce the size of the generator as well as power house. However, higher specific speed is responsible for cavitation which should be avoided. The specific speed should be high when the head is low and output is large. The types of turbine for different specific speed are (i) Pelton wheel: 8.5 to 50, (ii) Francis turbine: 51 to 255 and (iii) propeller and Kaplan turbine: 255 to 860.

Francis turbine runs at a higher speed than the Pelton turbine. Therefore, under similar operating conditions, the size of a Francis turbine will be smaller than those of the Pelton turbine and it should be preferred.

2. **Rotational speed:** The turbines are directly coupled to the generator which has to operate at its synchronous speed. High synchronous speed of generator reduces the number of poles and thus its size. Therefore, the value of the specific speed of turbines should be such that it gives synchronous speed of the generator. High rotational speed results in smaller size of the turbine and the generator. Thus, the overall cost of the plant reduces.
3. **Head:** The selection of the turbine depends upon the power and speed desired as well as the head also. The range of head for which each type of turbine is suitable is listed below.
   (i) Pelton wheel single or multiple jets (very high head turbines): 300 m or more
   (ii) Pelton or Francis turbine (high head turbines): 150 m to 300 m
   (iii) Francis turbine (medium head turbines): 60 m to 150 m
   (iv) Kaplan or propeller turbine (low heads turbines): Less than 60 m
   (v) Bulb turbines (very low heads turbines): 2 m to 15 m

   The given range of heads is flexible. It simply illustrates the general idea of the ranges of heads to which a particular turbine is considered.
4. **Part load operation:** The load at which a turbine provides maximum efficiency is called full load. Any load that is above is called overload and below than that is called part load. The turbines are required to work under variable load conditions. As the load varies from the normal working load, the efficiency would also vary. At part load, the performance of Kaplan and Pelton turbines is better in comparison to that of Francis and propeller turbines. For higher range of heads (i.e., 150 m to 300 m), Pelton turbine is preferable for part load operation in comparison to Francis turbine. For heads below 30 m, Kaplan turbine is preferable in comparison to propeller turbine.
5. **Disposition of turbine shaft:** Generally, vertical shaft arrangement is preferable for large sized reaction turbine which is almost universally adopted. In case of a large sized impulse turbine, horizontal shaft arrangement is employed.
6. **Overall cost:** The overall cost which includes the initial cost and the running cost should also be considered while selecting a turbine. The turbine should be designed to generate the power with minimum cost.
7. **Cavitation:** The alternate formation and collapse of vapour bubbles in a flowing fluid due to local fall in fluid pressure is called cavitation. It is likely to occur when the pressure at the runner outlet equals the vapour pressure. Cavitation may cause severe damage to the surface. The surface becomes badly scored and pitted which ultimately fails by fatigue. A turbine should be installed closer to the tail race with a minimum cost of excavation for the draft tube.

Pelton turbine is free from cavitation because the pressure at runner outlet is atmospheric. However, there may be a possibility of cavitation occurring at the nozzle. In reaction turbines, the cavitation commences at the top portion of the draft tubes. The installation of hydraulic reaction turbines over the tail race is affected by cavitation. Due to cavitation, the Francis turbines cannot be employed for very high heads.

According to Prof. Thoma, cavitation can be avoided if the turbine is installed in such a way that the cavitation factor (σ) is always greater than its critical value (σ_c). Here, σ_c is found to be a function of the specific speed of the turbine and its value for different turbines may be calculated by different empirical relations.

24.9 ❒ SURGE TANKS

A surge tank is a reservoir in the form of a large diameter tank which is generally open to atmosphere at the top. It is fitted at some opening before the turbine to receive the rejected flow when the pipeline is suddenly closed by a valve at its steep end. Rapid velocity fluctuation of water in a penstock is due to (i) sudden closure and opening of valves or wicket gates and (ii) the start and shut down of a turbine that set up large magnitude pressure transients. The excessive pressures may cause water hammer which may lead to bursting of the pipe. The sudden surge of water in a penstock is taken by the surge tank when the water requirement reduces suddenly. Surge tank also supplies additional water required by the turbine due to the sudden increase in demand, before the water comes from the reservoir.

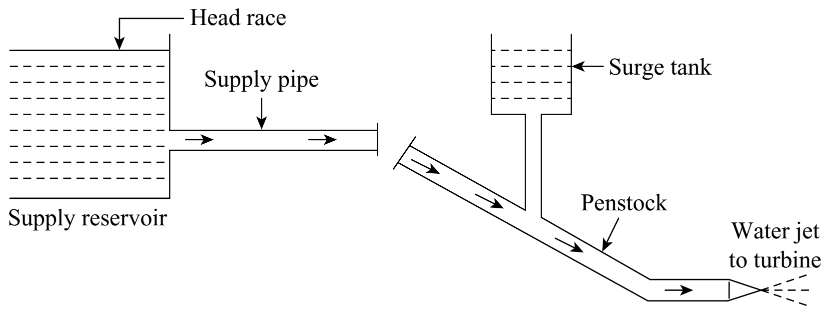


Figure 24.5 *Surge tank and its location*

Ideally, a surge tank should be fitted near to the turbine and its height should be more than the available head race level to avoid any water spilling. In practice, the surge tank is located at the junction of penstock and supply pipe as shown in Figure 24.5.

When the turbine works under steady load, the flow through the penstock remains uniform. The water level in the surge tank remains lower than that in the supply reservoir by an amount equal to the friction head loss in the pipe connecting the reservoir and the surge tank.

When the load on turbine decreases, the governor mechanism partially closes the gate openings of the turbine for reducing the flow of water to the runner to maintain constant speed. The rejected quantity of water get stored in the surge tank and its water level rises. Thus, the surge tank decelerates (or retards) the flow from supply reservoir and it reduces the velocity of flow in the pipeline corresponding to the reduced discharge required by the turbine.

When the load on turbine increases, the governor opens the gate openings of the turbine to increase the flow of water to the runner to maintain constant speed. The increased demand by the turbine is partly fulfilled by water stored in the surge tank and thus, its water level falls. Thus, the surge tank accelerates the flow from supply reservoir and it increases the velocity of flow in the pipeline to a value corresponding to the increased discharge required by the turbine.

24.9.1 Types of Surge Tanks

The various types of surge tanks schematically shown in Figure 24.6 are discussed below.

1. **Open conical surge tank:** It remains open to atmosphere at the top and is directly connected to the penstock. It is slow in action and nowadays, it is seldom used.
2. **Closed cylindrical surge tank:** Its top is closed and the space above water contains air supplied from a compressor. It has an internal bell mouth spillway which permits the overflow to be easily disposed of. The size of a closed cylindrical surge tank is smaller than that of an open one under same working conditions.

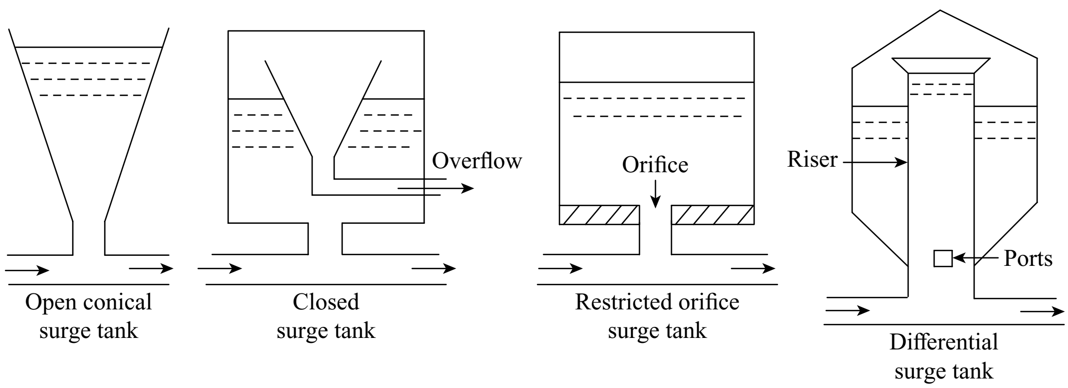


Figure 24.6 *Surge tanks*

3. **Restricted orifice surge tank:** It is connected to the penstock through an orifice fitted at the base of the tank and it is also called throttled surge tank. It is less effective in speed regulation under small and rapid changes. Its design is much complicated and hence, it is not much popular.
4. **Differential surge tank:** This surge tank is provided with a small vertical pipe called central riser having small holes (ports) at its lower end. It quickly develops the accelerating and retarding heads required. For the same stabilizing effects, its capacity is less than the cylindrical surge tanks and no water is spilled to waste from it.

Summary

1. **Unit speed:** $N_u = N/\sqrt{H}$, **Unit discharge:** $Q_u = Q/\sqrt{H}$ and **Unit power:** $P_u = P/H^{3/2}$.
2. **Specific speed:** $N_s = (N\sqrt{P})/H^{5/4}$.
3. **Head coefficient:** $H/(N^2D^2)$, **Flow coefficient:** $Q/(ND^3)$ and **Power coefficient:** $P/(N^3D^5)$.
4. Due to surface roughness, the efficiency of prototype will be different from the corresponding model efficiency which is referred to as scale effect.
5. The results obtained from the tests conducted on turbines or their models are graphically plotted by means of curves called characteristic curves.
6. The important characteristic curves are main characteristic curves (or constant head characteristic curves), operating characteristic curves (or constant speed characteristic curves) and Muschel curves (or constant efficiency curves or iso-efficiency curves).
7. A surge tank is a reservoir in the form of a large diameter tank which takes the sudden surge of water in a penstock when the water requirement reduces suddenly and it also supplies additional water required by the turbine due to sudden increase in demand.

Multiple-choice Questions

1. If H is the available head for a hydraulic turbine, the power, speed and discharge, respectively are proportional to
   (a) $H^{3/2}, H^{5/2}, H^{1/2}$
   (b) $H^{3/2}, H^{1/2}, H^{1/2}$
   (c) $H^{1/2}, H^{5/2}, H^{1/2}$
   (d) $H^{5/2}, H^{3/2}, H^{1/2}$
2. Which of the following two relations are necessary for homologous turbines when P is the power, Q is the discharge, H is the head and C is the constant?
   (a) $P/(QH) = C$ and $H/(N^2D^2) = C$.
   (b) $H/(ND^3) = C$ and $Q/(N^2D^2) = C$.
   (c) $(N\sqrt{Q})/H^{1.5} = C$ and $(N\sqrt{P})/H^{3/4} = C$.
   (d) None of the above.
3. If n is the number of jets in a Pelton wheel, then its specific speed is proportional to
   (a) n
   (b) $n^{5/2}$
   (c) $n^{3/2}$
   (d) $\sqrt{n}$
4. The unit discharge and unit speed curves for the Kaplan, Francis and Pelton turbines in sequence are
   (a) Rising curves, drooping curves and straight line.
   (b) Drooping curves, straight line and rising curves.
   (c) Straight line, rising curves and drooping curves.
   (d) None of the above.
5. Which of the following statement is correct?
   (a) Curves at constant efficiency are called main characteristic curves.
   (b) Curves at constant efficiency are called operating characteristic curves.
   (c) Curves at constant head are called main characteristics curves.
   (d) Curves at constant speed are called main characteristics curves.
6. For a given diameter, the ratio between the model and prototype, the relation between power (P) and head (H) is given by
   (a) $P \alpha H^{1/2}$
   (b) $P \alpha H^{3/2}$
   (c) $P \alpha H^{5/2}$
   (d) None of the above.

7. The specific speed of a turbomachine
   (a) Is the speed of a machine having unit dimension.
   (b) Has the dimension of rotational speed.
   (c) Remains unchanged under different conditions of operation.
   (d) Relates the shape rather than the size of the machine.

8. For a hydraulic turbine, the shape number also known as dimensionless form of specific speed given by the following expression when N is the speed, H is the head, P is the power, ρ_w is the density of water and g is acceleration due to gravity is
   (a) $(N\sqrt{P})/H^{5/4}$. (b) $(N\sqrt{P/\rho_w})/H^{5/4}$.
   (c) $(N\sqrt{P/g})/H$. (d) $(N\sqrt{P})/\rho_w g H^2$.

Review Questions

1. Define the terms unit speed, unit discharge and unit power for a hydraulic turbine. Also derive expressions for each of them.
2. Define and derive an expression for specific speed of a turbine. What is the physical significance of it?
3. Derive expressions for specific speed of the Pelton, Francis and Kaplan turbines in terms of known coefficients.
4. Explain various dimensionless numbers used to show the relationship of model and prototype.
5. Write short notes on: (i) model testing of turbines and (ii) scale effect.
6. Discuss the performance characteristics of hydraulic turbines.
7. Discuss the important points which should be considered while selecting hydraulic turbines for a hydroelectric power plant.
8. What is a surge tank? Briefly explain its necessity and working operation in the penstock of a turbine. Also discuss its types.

Problems

1. In a hydroelectric station, water is available at the rate of 30.6 m^3/s under a head of 60.2 m. The turbine runs at a speed of 550.5 rpm with an efficiency of 85.4%. If maximum specific speed is 210, then find the number of turbines required.
   [**Ans.** 4]
2. If a turbine has specific speed of 240 operating under a head of 25 m at a speed of 135 rpm, then determine the power developed by it.
   [**Ans.** 9876.54 kW]
3. A Francis turbine runner of 1.2 m diameter working under a head of 5.2 m produces 75.6 kW power at a speed of 212 rpm when the water flow rate is 1.85 m^3/s. If the head is raised to 16.5 m, then find its new speed, discharge and power.
   [**Ans.** 377.64 rpm, 3.295 m^3/s, 427.31 kW]
4. A hydraulic turbine develops 13.5 MW power operating under a head of 300 m at a speed of 425 rpm. Determine the specific speed, speed of the turbine and power when it operates under a head of 140 m.
   [**Ans.** 39.55, 290.33 rpm, 4303.72 kW]
5. A Francis turbine works under a head of 30 m and produces 2.7 kW while running at 300 rpm. Determine the power and speed of this turbine under unit head.
   [**Ans.** 0.0164 kW, 54.77 rpm]
6. A hydraulic turbine develops 100 kW power under a head of 50 m. Determine the percentage increase in its speed when the head is increased by 100 m.
   [**Ans.** 41.42%]
7. A Kaplan turbine develops 9 MW when running at 100 rpm. The head on the turbine is 30 m. If the head on the turbine is reduced to 20 m, then determine the speed and power developed by the turbine.
   [**Ans.** 81.65 rpm, 4898.98 kW]
8. A water turbine works under a head of 25.6 m and runs at 100.2 rpm. It develops 6.65 MW power. Determine the speed and power when it works under a head of 15 m. Also suggest the type of turbine.
   [**Ans.** 76.7 rpm, 2982.62 kW]
9. A turbine is running at a speed of 150 rpm and it develops 2 MW power while operating under a head of 36 m. Determine its speed and power output if operating head is changed to 25 m.
   [**Ans.** 125 rpm, 1157.41 kW]
10. A hydraulic turbine is to develop 1025 kW when running at 120 rpm under a head of 12 m. If the overall efficiency is given 92%, then find the specific speed and maximum flow rate for the turbine. In order to predict its performance, a 1 : 10 scale model is tested under a head of 7.2 m. Determine

the speed, power output and water consumption of the model if it runs under the conditions similar to the prototype.

[**Ans.** 172.01, 9.464 m^3/s, 929.5 rpm, 4.763 kW, 0.0733 m^3/s]

11. A Kaplan turbine develops 60.2 MW when running at 90.5 rpm. The head on the turbine is 16.4 m and discharge is 600.5 m^3/s. Determine the unit speed, unit discharge, unit power and specific speed of the turbine.

[**Ans.** 22.35 rpm, 148.28 m^3/s, 906.42 kW, 672.81]

12. Determine the unit discharge, unit power and unit speed when a Pelton turbine develops 5 MW under a head of 230 m at an overall efficiency of 82% and revolving at a speed of 205 rpm. If the head on the turbine falls to 140 m, then determine discharge, power and speed for this turbine. Assume peripheral coefficient as constant.

[**Ans.** 0.178 m^3/s, 1.433 kW, 13.52 rpm, 2.108 m^3/s, 2374.49 kW, 159.94 rpm]

13. A prototype turbine develops 40 MW at 81 m head while running at a speed of 150 rpm. Its model develops 36 kW at 8 m head. Determine the speed of the model runner and the scale ratio between the prototype and the model.

[**Ans.** 276.83 rpm, 5.88]

14. A Pelton turbine develops 5800 kW at a speed of 205 rpm under a head of 225 m. If overall efficiency of the turbine at best operating point is 85%, then determine the unit speed, unit discharge and unit power. It is tested at a site where the maximum supply head is 150 m. Find the discharge, power and speed for this turbine.

[**Ans.** 13.67 rpm, 0.206 m^3/s, 1.718 kW, 2.523 m^3/s, 3157.12 kW, 171.46 rpm]

15. A model of Francis turbine that is one-fifth of full size develops 3 kW at 320 rpm under a head of 1.5 m. Determine the speed and power of full size turbine operating under a head of 5 m if (i) efficiency of the model and the full size turbine are same and (ii) efficiency of the model turbine is 75% and the scale effect is considered.

[**Ans.** 116.85 rpm, 456.46 kW, 149.53 rpm, 498.33 kW]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (b) **2.** (a) **3.** (d) **4.** (a) **5.** (c)
6. (b) **7.** (d) **8.** (b)

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Chapter 25

Centrifugal Pumps

25.1 ❒ INTRODUCTION

Pumps are mechanical devices which convert mechanical energy into hydraulic energy in the form of pressure energy. Pump lifts liquids from a lower level to a higher level by absorbing power. Therefore, a pump is a power absorbing device which is used to increase the pressure energy of a liquid and it is subsequently converted into potential energy as the liquid is raised from a lower level to a higher level.

Centrifugal pumps are rotodynamic type of pumps in which a dynamic pressure is generated which lifts the liquid to a higher level. A centrifugal pump works on the principle of forced vortex flow. The pump works when a certain mass of liquid is rotated by an external force where it is thrown away from the central axis of rotation and a centrifugal head acts on the liquid which raises it to a higher level. The rise in pressure head can be given by $V^2/2g = (\omega^2 r^2)/2g$, here V is the tangential velocity of liquid. When more liquid is constantly added to the centre of rotation, a continuous supply of liquid is maintained at a higher level.

The rotating component called impeller (or rotor) is the main element of a centrifugal pump which imparts momentum to the liquid. In these pumps, the liquid is lifted due to centrifugal action, and hence, they are called centrifugal pumps. In addition to centrifugal action, when liquid passes through the impeller, its angular momentum changes which also increase the liquid pressure. The centrifugal pumps closely resemble reaction turbines. It acts as reverse of inward radial flow reaction turbines. Thus, the flow in centrifugal pumps is in the radial outward direction.

The liquid is more often water in the domestic and agriculture domains. Generally, pump is widely used in many applications, such as in the fields of agriculture and irrigation works, water supply plants, sewage and drainage system, steam power plants, oil refineries, chemical plants and steel mills, food processing industries, transport, mines, and many other utility services and industries in which fluids are pumped.

This chapter highlights the key components, classification, basic terminology, theoretical analysis for determining power requirements and other associated problems related to centrifugal pumps.

25.2 ❒ BRIEF HISTORICAL DEVELOPMENT OF CENTRIFUGAL PUMPS

The idea of lifting water by centrifugal force was given by Leonardo Da Vinci (1452–1519). The first centrifugal pump having impeller with blades and a volute was built by Denis Papin (French physicist in 1705). However, reciprocating pumps were popular at that time. The first centrifugal pump in USA was built by Massachusetts pump factory. The first three stage pump was built in 1846 by Johnson (USA). The guide vanes were developed in 1850 by James Thomson (UK). A pump with diffusion vanes was built by Osborne Reynolds in 1875 and the manufacturing of such pumps were started in

UK by Mather and Platt in 1893. The multistage pump by using the impellers in series was first introduced by W. J. Johnson (America). After systematic and scientific investigation, Sulzer brothers (Switzerland) started manufacturing pumps in 1890 and thereby, the design of mixed flow and axial flow pumps were evolved.

25.3 ❐ CLASSIFICATION OF PUMPS

Pumps can be broadly classified into two categories, namely rotodynamic pumps (or dynamic pressure pumps) and positive displacement pumps.

25.3.1 Rotodynamic Pumps (or Dynamic Pressure Pumps or Rotary Pumps)

The rotodynamic pumps have a rotating element called impeller. When liquid flows through the impeller, its angular momentum changes, as a result, pressure energy of the liquid increases. According to the general flow direction of liquid within the passage of the impeller, the rotodynamic pumps can be classified into three types, (i) centrifugal flow pumps, (ii) axial flow (or propeller) pumps and (iii) mixed flow (half axial or screw) pumps.

1. **Centrifugal flow pumps:** In a centrifugal flow pump, liquid enters axially (i.e., in the same direction as the axis of the rotating shaft) in the centre of the pump and it is discharged radially (or tangentially) along the outer radius of the pump casing (Figure 25.1a). Thus, these pumps are also known as radial flow pumps. These pumps handle lower volumes at higher pressures. The centrifugal pumps are the most common examples of rotodynamic pumps in which liquid flows in the outward radial direction. It means that the action of a centrifugal pump is the reverse of a radially inward flow reaction turbine.
2. **Axial flow pumps:** In axial flow pumps, liquid flows in axial direction only (Figure 25.1b). The action of axial flow pumps is the reverse of propeller or Kaplan turbines. Axial flow pumps can handle very large volume but at limited pressures.
3. **Mixed flow pumps:** Mixed flow pumps are intermediate between centrifugal and axial flow pumps. It means, in mixed flow pumps, liquid flows through the impeller axially as well as radially (Figure 25.1c). The action of mixed flow pumps is the reverse of Francis turbines or mixed flow type turbines. Mixed flow pumps handle comparatively larger volumes at medium range of pressures.

Also there are some non-rotary dynamic pumps, such as jet pumps and electromagnetic pumps.

25.3.2 Positive Displacement Pumps

In positive displacement pumps, liquid is sucked and actually pushed or bodily displaced due to the thrust exerted on it by a moving member which lifts the liquid to the required height. The liquid inside the positive displacement pumps may be

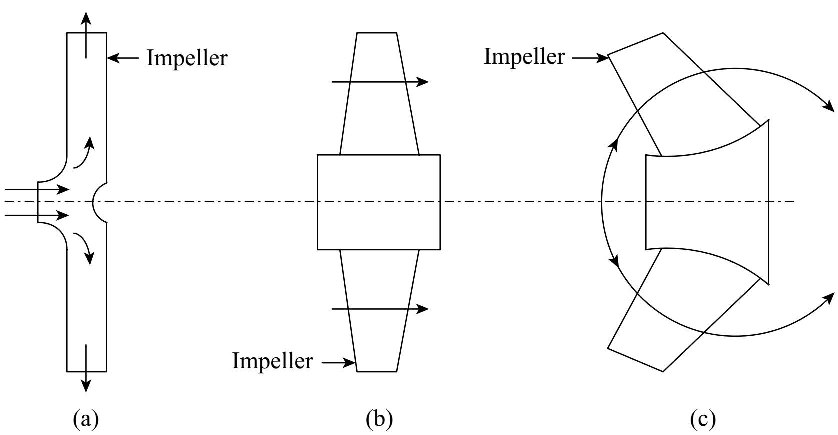


Figure 25.1

subjected either to a reciprocating motion or to a rotary (or circular) motion. So, based on the motion of the liquid, these pumps are classified into two types, namely reciprocating pumps and rotary positive displacement pumps.

1. **Reciprocating pumps:** The most common example of the positive displacement pumps is reciprocating pump. Reciprocating pumps first trap the liquid in a cylinder by suction and then push the liquid against pressure. Thus, the discharge of liquid pumped by these pumps fully depends on the speed. These pumps are limited by the low speed of operation and small volume it handles.
2. **Rotary positive displacement pumps:** Rotary positive displacement pumps also trap the liquid in a volume and push the same out against pressure. These pumps are limited by lower pressures of operation and small volume it handles. Its main types are gear pump, vane pump, screw pump and lobe pump.

25.3.3 Classification of Centrifugal Pumps

Centrifugal pumps may be classified into several ways, such as on the basis of their characteristic features, utility, design and constructional features. Centrifugal pumps may be classified into eight ways as given below. First classification is a commercial classification from the point of utility of the pumps. However, the second to seventh classifications are practical considerations which govern important constructional features of the pump. Eventually, the last classification is a theoretical aspect which provides a sound basis for absolute classification of the pumps.

1. On the basis of working head.
   (a) **Low head pump:** These pumps work against heads up to 15 m and below.
   (b) **Medium head pump:** These pumps work between ranges from 15 m to 40 m.
   (c) **High head:** These pumps are used to build up heads more than 40 m.
2. On the basis of casing type as (a) volute pump, (b) volute pump with vortex chamber and (c) diffuser or turbine pumps.
3. On the basis of relative direction of flow through impeller.
   (a) **Radial flow pump:** Liquid flows through the impeller in radial direction only.
   (b) **Axial flow pump:** Liquid flows through the impeller in axial direction only.
   (c) **Mixed flow pump:** Liquid flows through the impeller axially as well as radially.
4. On the basis of number of stages as single stage pump and multistage pump.
   If a centrifugal pump consists of two or more impellers, then the pump is called multistage centrifugal pump. The impellers may be mounted on the same shaft or on different shafts. Multistage pumps are required to produce high head or to discharge large quantity of water. To develop a high head, the impellers are connected in series or mounted on the same shaft, whereas for discharging large quantity of water, the impellers are connected in parallel.
   These pumps are used essentially for high heads. The number of stages depends on the head required and may be employed up to 10.
5. On the basis of liquid handled as (a) closed impeller pump, (b) semi-open impeller pump and (c) open impeller pump.
6. On the basis of number of entrances to the impeller (Figure 25.2).
   (a) **Single entry or single suction pump:** Liquid is admitted from a suction pipe from one side of the impeller as shown in Figure 25.2(a).
   (b) **Double entry or double suction pump:** Liquid enters from both sides of the impeller as shown in Figure 25.2(b).
7. On the basis of disposition of shaft.
   (a) **Horizontal shafts:** Generally, pumps have horizontal shafts.
   (b) **Vertical shafts:** For deep wells and mines, pumps have vertical shafts.
8. On the basis of specific speed.
   (a) **Low speed radial flow pump:** Specific speed varies from 10 to 30.
   (b) **Medium speed radial flow pump:** Specific speed varies from 30 to 50.

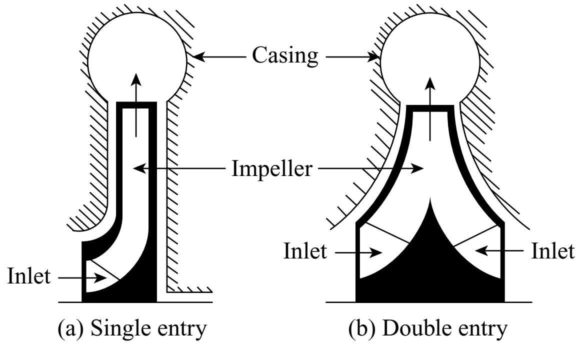


Figure 25.2 *Single and double suction pumps*

(c) **High speed radial flow pump:** Specific speed varies from 50 to 80.
(d) **Mixed flow pump:** Specific speed varies from 80 to 160.
(e) **Axial flow pump:** Specific speed varies from 100 to 450.

25.4 ❐ CONSTRUCTION AND WORKING OF CENTRIFUGAL PUMPS

The function of centrifugal pumps is mainly considered because energy is imparted to the fluid by centrifugal action of moving blades from the inner radius to the outer radius. In general, all the rotodynamic pumps resemble the reaction type of hydraulic turbines. Therefore, these types of pumps may be known as reversed reaction turbines. The action of the centrifugal pump is just the reverse of an inward flow reaction turbine. Hence, the flow in centrifugal pumps is in the radial outward direction.

25.4.1 Main Parts of a Centrifugal Pump

The Figure 25.3 illustrates the main parts of a centrifugal pump.

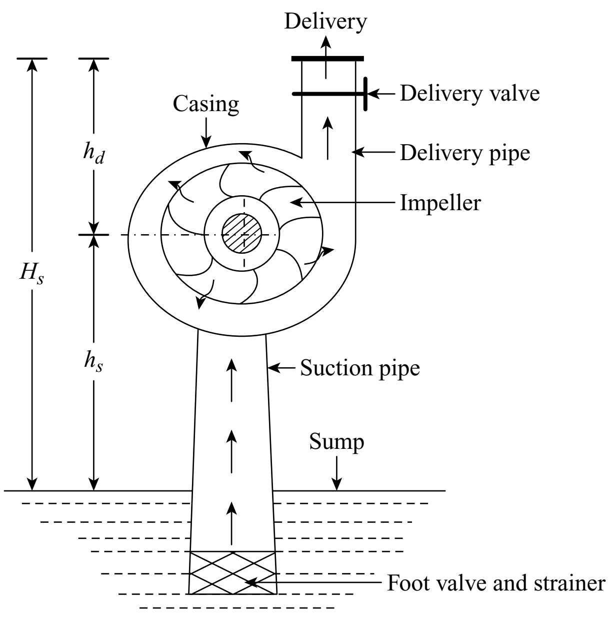


Figure 25.3 *Main parts of a centrifugal pump*

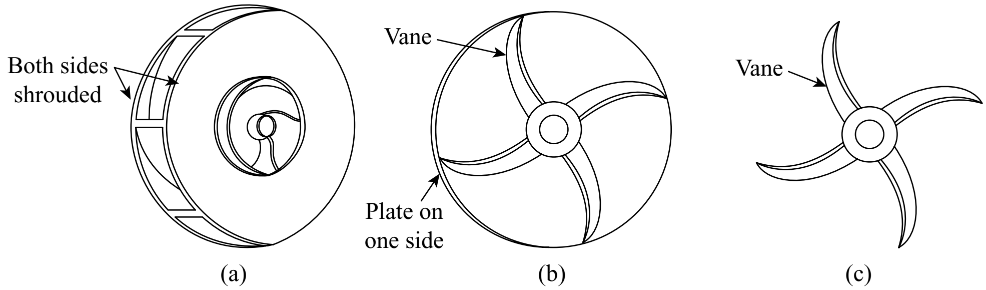


Figure 25.4 *Types of Impeller*

The main parts of a centrifugal pump are (i) impeller, (ii) casing, (iii) suction pipe with a foot valve and a strainer, and (iv) delivery pipe and delivery valve.

1. **Impeller:** It is a wheel or rotor having a series of backward curved vanes (or blades) which usually varies from 6 to 12. Thus, impeller is a rotating part of a centrifugal pump. It is mounted on a shaft which is coupled to the shaft of an electric motor. The impellers are of three types (Figure 25.4), namely (i) shrouded or closed impeller, (ii) semi-open impeller and (iii) open impeller.

   (i) **Shrouded or closed impeller:** A shrouded or closed impeller is that whose vanes are covered on both sides with metal plates as shown in Figure 25.4(a). These metal plates or shrouds are known as crown plate and base plate. The closed impeller is more efficient and provides better guidance to the liquid. This type of impeller is most suited when the liquid to be pumped is pure, free from debris and have low viscosity, such as ordinary water, hot water, hot oils and acids. The materials of the impeller are selected based on the type of liquid used.

   (ii) **Semi-open impeller:** If the vanes have only the base plate and no crown plate, then the impeller is known as semi-open type impeller as shown in Figure 25.4(b). Such an impeller can be used even if the liquid contains some debris, such as sewage water, paper pulp and sugar molasses. In order to avoid any clogging of the impeller, its number of vanes is reduced and their height is increased.

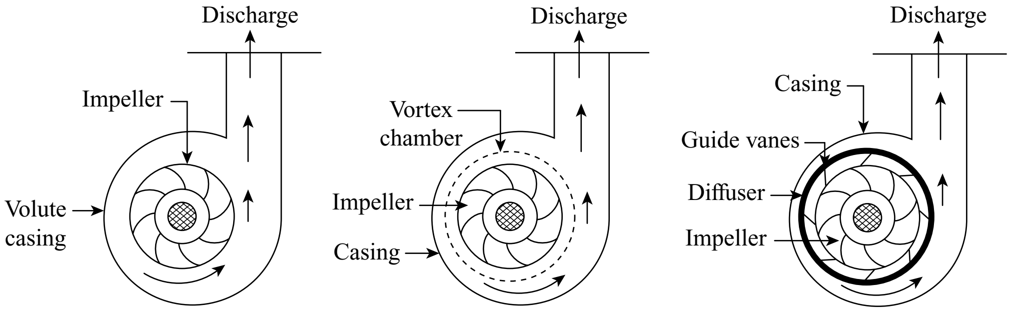
   (iii) **Open impeller:** An open impeller is that whose vanes have neither the crown plate nor the base plate, i.e., vanes are open on both sides as shown in Figure 25.4(c). Such impellers are used for pumping the liquids containing suspended solid matter, such as paper pulp, sewage and water containing sand, grit, pebbles and clay. Generally, this impeller is made of forged steel, as it has to perform very rough duty.

2. **Casing:** It is an airtight chamber which surrounds the impeller. It is similar to the casing of a reaction turbine. It is designed in such a way that the kinetic energy of the water discharged at the outlet of the impeller is converted into pressure energy before the water leaves the casing and enters the delivery pipe. The three types of casing, namely volute casing, vortex casing and casing with guide blades (Figure 25.5) are commonly used and the pump is named after the casing it uses.

   (i) **Volute casing:** The volute casing (Figure 25.5(a)) is of spiral shape in which the area of flow increases gradually from the impeller outlet to the delivery pipe. The increase in area of flow decreases the velocity of flow with corresponding increase in the pressure of water flowing through the casing. Single stage pumps are mostly having volute casing. The volute casing has higher eddy losses which results in lower overall efficiency. The pumps having volute casing are known as volute pump.

   (ii) **Vortex casing:** If a circular chamber is provided between the impeller and the casing as shown in Figure 25.5(b), then such casing is known as vortex casing. The circular chamber is also known as vortex or whirlpool chamber. The vortex chamber reduces the eddy formation to a considerable extent. Thus, the efficiency of these pumps will be more than that of volute casing pumps. The pumps using vortex casing are known as volute pump with vortex chamber.

   (iii) **Casing with guide blades:** The impeller is surrounded by a series of fixed guide blades which are mounted on a ring known as diffuser as shown in Figure 25.5(c). The guide vanes are designed in such a way that the water from the impeller enters in it without shock. The liquid leaving the impeller passes through the passage between



(a) Volute casing (b) Vortex casing (c) Casing with guide blades

Figure 25.5 *Types of casing*

the guide vanes whose area increases. This reduces the velocity of flow through guide vanes and consequently, increases the pressure of water. The water from the guide vanes then passes through the casing which is in most of the cases will be concentric with the impeller. The pumps having guide vanes are called diffuser pumps. The guide vanes resemble a reversed turbine and hence, they are also called turbine pumps. Casing with guide blades pumps may be either vertical or horizontal shaft type. The vertical shaft type occupies very less space and is suitable in deep well installations and in narrow wells and mines, etc. These pumps have maximum efficiency but are less satisfactory when a wide range of operating condition is required.

3. **Suction pipe with foot valve and a strainer:** It is a pipe whose upper end is connected to the inlet of the pump (or to the centre of the impeller which is known as eye) and the lower end dips into liquid in the sump from which the liquid is to be pumped. The lower end of the suction pipe is fitted with a foot valve and a strainer. The foot valve is a non-return or one way valve which opens in the upward direction only. It does not allow the liquid to flow in downward direction back to the sump. The liquid first enters the strainer which removes the debris. The suction pipe should always be fitted in such a way that water always flows in the upward direction and all the pipe fittings must be air tight so that no air pockets are formed and the pump works smoothly.
4. **Delivery pipe and delivery valve:** It is a pipe whose lower end is connected to the outlet of the pump and its upper end delivers liquid to the required height. Just near the outlet of the pump on delivery pipe, a regulating valve is fitted which controls the flow of liquid from the pump into the delivery pipe.

25.4.2 Working of a Centrifugal Pump

The stepwise working of a centrifugal pump is explained below.

1. The first step in the working operation of a centrifugal pump is priming. During the priming operation, the delivery valve is kept closed. Priming is the operation in which the suction pipe, casing of the pump and the portion of the delivery pipe up to the delivery valve are completely filled with the liquid which is to be pumped so that no air pocket is left. The presence of even a small air pocket in any part of the pump may cause no delivery of the liquid from the pump. The pressure generated in a centrifugal pump is proportional to the density of the fluid it handles. Very small pressure will develop if the impeller rotates in the presence of air. Thus, no liquid will be lifted up by the pump. Therefore, it is essential to properly prime a pump before it can be started.
2. The electric motor is started to rotate the impeller and the delivery valve is still kept closed to reduce the starting torque.
3. The rotation of the impeller in the casing full of liquid produces a forced vortex which provides a centrifugal head to the liquid and thus, it results in increase of pressure throughout the liquid. The rise in pressure head at any point of the rotating liquid is proportional to the square of the tangential velocity of the liquid at that point and the distance of the point from the axis of rotation. Thus, if the speed of the impeller of the pump is high enough, then the pressure of the liquid surrounding the impeller increases considerably. When the delivery valve is opened, the liquid flows in an outward radial direction and leaves the vanes of the impeller at the outer radius with high velocity and pressure.

4. The rotation of the impeller due to centrifugal action causes a partial vacuum at its eye which causes the suction of the liquid from the sump through the suction pipe. The sucked liquid replaces the liquid which is being discharged from the whole circumference of the impeller.
5. The high pressure of the liquid leaving the impeller is utilized in lifting the liquid to the required height.

25.4.3 Priming Devices

Small pumps are primed by pouring liquid into the casing through a funnel. The air vent is kept open during the filling operation to escape the air through it. The air vent is closed after ensuring that all the air has escaped from the suction pipe, impeller and casing. Large pumps are primed by evacuating the casing and the suction pipe with the help of a vacuum pump or a steam ejector. Thus, the liquid is sucked into the suction pipe and the pump is filled with it. However, some pumps are self-primed pumps in which a special arrangement containing a supply of liquid is provided in the suction pipe which does the automatic priming of the pump.

25.5 ❒ VELOCITY TRIANGLES AND WORK DONE BY CENTRIFUGAL PUMP

The expression for work done by the impeller in water is obtained by drawing velocity triangles at the inlet and outlet of the impeller in the same way as for a turbine. For drawing the velocity triangles, the same notations are used as that of turbines. The velocity triangles at the inlet and outlet tips of a vane of the impeller are shown in Figure 25.6.

The velocity triangles also known as Euler's velocity triangles have been drawn by assuming that there are infinite numbers of blades in the impeller. The other assumptions are (i) flow is steady and one dimensional, (ii) there is no energy loss in the impeller due to friction and eddy formation and (iii) there is no energy loss due to shock at entry of the impeller.

Let N be the speed in rpm,

$\omega = (2\pi N/60)$ rad/s be the angular velocity,

R_i and R_o be the radii of the impeller at the inlet and outlet, respectively,

D_i and D_o be the diameters of impeller at the inlet and outlet, respectively,

B_i and B_o be the widths of impeller at the inlet and outlet, respectively,

$A_i = \pi D_i B_i$ and $A_o = \pi D_o B_o$ be the areas of impeller at the inlet and outlet, respectively,

$u_i = 2\pi R_i N/60 = \omega R_i$ and $u_o = 2\pi R_o N/60 = \omega R_o$ be the tangential velocities of the impeller at the inlet and outlet, respectively,

V_i and V_o be the absolute velocities of the water at the inlet and outlet, respectively,

V_{ri} and V_{ro} be the relative velocities of the water at the inlet and outlet, respectively,

V_{wi} and V_{wo} be the velocities of whirl at the inlet and outlet, respectively,

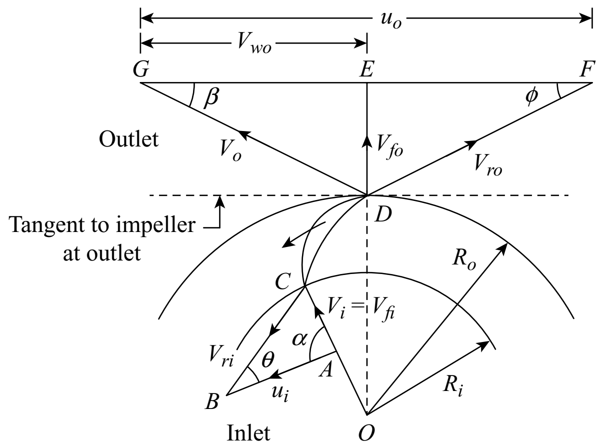


Figure 25.6 *Velocity triangles at inlet and outlet of a centrifugal pump*

V_{fi} and V_{fo} be the velocities of flow at the inlet and outlet, respectively,

α and β be the angles made by absolute velocities at the inlet and outlet, respectively,

θ and ϕ be the vane angles at the inlet and outlet, respectively.

For best efficiency of the pump, the water is assumed to enter the impeller radially at the inlet. This means the whirl component V_{wi} is equal to zero and the flow component V_{fi} equals the absolute velocity V_i. In other words, $\alpha = 90°$, $V_{wi} = 0$ and $V_{fi} = V_i$. A centrifugal pump is the reverse of a radially inward flow reaction turbine. In the case of radially inward flow reaction turbine, the work done by water on the runner per second per unit weight is given by Euler's equation as follows.

$$w_{\text{turbine}} = \frac{V_{wi}u_i - V_{wo}u_o}{g}$$

Therefore, work done by impeller in water per second per unit weight of water striking per second is given below.

$$w = -w_{\text{turbine}} = -\left(\frac{V_{wi}u_i - V_{wo}u_o}{g}\right) = \frac{V_{wo}u_o - V_{wi}u_i}{g} \tag{25.1}$$

Since water enters radially, $V_{wi} = 0$ and therefore, Equation (25.1) is written as follows.

$$\boxed{w = \frac{V_{wo}u_o}{g}} \tag{25.2}$$

Equation (25.2) represents the head imparted by the impeller to the water or energy imparted by impeller to the liquid per unit weight per second which is also known as Euler head or theoretical head (H_e). It tells that for delivering water at high heads, the peripheral velocity (u_o) must be high and whirl velocity (V_{wo}) must also be large. For obtaining high values of u_o, the impeller diameter and its speed of rotation should be increased. For large values of V_{wo} the number of vanes should be adequate and should be of suitable size and shape.

Work done by impeller in water per second is given by,

$$\boxed{w = \frac{W}{g}(V_{wo}u_o) = \rho_w Q V_{wo}u_o} \tag{25.3}$$

Here, W = weight of water per second = $\rho_w gQ$ and Q is the volume of water per second that is given as $Q = \pi D_i B_i V_{fi} = \pi D_o B_o V_{fo}$.

Torque exerted by the impeller on the water is equal to the rate of change of angular momentum as expressed below.

$$T = \rho_w Q V_{wo} R_o \tag{25.4}$$

Thus, power at the impeller (P_{im}) or work done by impeller per second in water is given below.

$$P_{im} = T\omega = \rho_w Q V_{wo} R_o \omega = \frac{\rho_w Q V_{wo} u_o}{1000}\ kW \qquad [\because u_o = \omega R_o] \tag{25.5}$$

From outlet velocity triangle, we get:

$$V_{fo}^2 = V_{ro}^2 - (u_o - V_{wo})^2 \tag{i}$$

Also

$$V_{fo}^2 = V_o^2 - V_{wo}^2 \tag{ii}$$

From expressions (i) and (ii), we get:

$$V_{ro}^2 - (u_o - V_{wo})^2 = V_o^2 - V_{wo}^2$$

$$V_{ro}^2 - (u_o^2 + V_{wo}^2 - 2u_oV_{wo}) = V_o^2 - V_{wo}^2$$

Thus

$$u_oV_{wo} = \frac{1}{2}(V_o^2 + u_o^2 - V_{ro}^2) \tag{iii}$$

By drawing inlet velocity triangle considering whirl velocity component at the inlet, similarly, we get the below expression.

$$u_i V_{wi} = \frac{1}{2}(V_i^2 + u_i^2 - V_{ri}^2) \quad \text{(iv)}$$

Substituting expressions (iii) and (iv) in Equation (25.1) and rearranging, we get:

$$w = H_e = \frac{V_o^2 - V_i^2}{2g} + \frac{u_o^2 - u_i^2}{2g} + \frac{V_{ri}^2 - V_{ro}^2}{2g} \quad (25.6)$$

Equation (25.6) gives the work done on the liquid per second per unit weight of liquid. Equations (25.2) and (25.6) represent the head imparted by the impeller to the water only if there are infinite numbers of blades in the impeller and it is generally termed as Euler head (or theoretical head). Usually, Equation (25.6) is known as Euler's equation and sometimes it is also called the fundamental equation of centrifugal pump. In this equation, the first term $[(V_o^2 - V_i^2)/(2g)]$ represents the dynamic head or an increase in kinetic energy. The second term $[(u_o^2 - u_i^2)/(2g)]$ represents the effect of centrifugal head. The third term $[(V_{ri}^2 - V_{ro}^2)/(2g)]$ represents the change in static pressure energy. Here, the losses in the impeller and the effect of difference in elevations of the inlet and outlet points of the impeller are neglected.

25.6 ❐ HEAD OF A CENTRIFUGAL PUMP

The head of a centrifugal pump are expressed in the following ways.

1. **Suction head:** It is the vertical height of the centre line of the pump shaft above the water surface in the sump from which water is being lifted (Figure 25.3). It is also known as static suction lift and it is denoted by h_s.
2. **Delivery head:** It is the vertical height of the water surface in the tank to which the water is delivered above the centre line of the pump shaft (Figure 25.3). It is also known as static delivery lift and it is denoted by h_d.
3. **Static head:** It is the vertical distance between the water surface in the sump and the tank to which the water is being delivered by the pump. Thus, static head is the sum of suction head and delivery head. It is denoted by H_s and it is given by the below expression.

$$H_s = h_s + h_d \quad (25.7)$$

4. **Manometric head:** Manometric head (H_m) is the head against which a centrifugal pump has to work. It is measured across the pump inlet and outlet flanges. If there are no energy losses in the pump (i.e., in the impeller and casing), then manometric head will be equal to the energy given to water by the impeller, i.e., ($H_m = H_e$). Thus, the expression for manometric head is given below.

$$\boxed{H_m = \frac{V_{wo} u_o}{g}} \quad (25.8)$$

If the loss of head (h_f) in the impeller and casing of the pump are considered, then we get the below expression.

$$H_m = \frac{V_{wo} u_o}{g} - h_f \quad (25.9)$$

The manometric head may also be given by the following expressions.

(i) $$H_m = h_s + h_d + h_{fs} + h_{fd} + \frac{V_d^2}{2g} \quad (25.10)$$

Here, h_s is the suction head, h_d is the delivery head, h_{fs} is the frictional head loss in the suction pipe, h_{fd} is the frictional head loss in the delivery pipe and V_d is the velocity of water in the delivery pipe.

If the velocity head in the delivery pipe $[V_d^2/(2g)]$ is relatively small, it may be neglected and Equation (25.10) can be written as follows.

$$H_m = h_s + h_d + h_{fs} + h_{fd} \tag{25.10a}$$

(ii) H_m = Total head at the outlet of the pump – Total head at the inlet of the pump

Thus
$$H_m = \left(\frac{p_o}{\rho_w g} + \frac{V_o^2}{2g} + z_o\right) - \left(\frac{p_i}{\rho_w g} + \frac{V_i^2}{2g} + z_i\right) \tag{25.11}$$

Here, $p_o/(\rho_w g) = h_d$ = pressure head at the outlet of the pump, $V_o^2/(2g) = V_d^2/(2g)$ = velocity head at the outlet of the pump = velocity head in the delivery pipe and z_o= datum head at the outlet of the pump or vertical height of the pump outlet from the datum line, and $p_i/(\rho_w g)$, $V_i^2/(2g)$ and z_i are the corresponding values of pressure head, velocity head and datum head at the inlet of the pump, respectively.

25.7 ❐ PRESSURE RISE IN THE IMPELLER

Assuming radial entry of water and there is no gravitational and frictional losses. Applying Bernoulli's theorem between the inlet and outlet edges of the impeller. Let *i* and *o* be the inlet and outlet points for the impeller.

Energy at the inlet = Energy at the outlet – Work input

$$\frac{p_i}{\rho_w g} + \frac{V_i^2}{2g} = \left(\frac{p_o}{\rho_w g} + \frac{V_o^2}{2g}\right) - \frac{V_{wo} u_o}{g}$$

Therefore, pressure rise is given by,

$$\frac{p_o - p_i}{\rho_w g} = \frac{V_i^2}{2g} - \frac{V_o^2}{2g} + \frac{V_{wo} u_o}{g} \tag{i}$$

From inlet velocity triangle (Figure 25.6), we get:

$$V_i = V_{fi} \tag{ii}$$

From outlet velocity triangle (Figure 25.6), we get:

$$V_o^2 = V_{fo}^2 + V_{wo}^2 \tag{iii}$$

and
$$V_{wo} = u_o - V_{fo} \cot\phi \tag{iv}$$

Thus
$$V_o^2 = V_{fo}^2 + (u_o - V_{fo} \cot\phi)^2$$

or
$$V_o^2 = V_{fo}^2 + V_{fo}^2 \cot^2\phi + u_o^2 - 2u_o V_{fo} \cot\phi \tag{v}$$

or
$$V_o^2 = V_{fo}^2(1 + \cot^2\phi) + u_o^2 - 2u_o V_{fo} \cot\phi$$

or
$$V_o^2 = V_{fo}^2 \operatorname{cosec}^2\phi + u_o^2 - 2u_o V_{fo} \cot\phi \tag{vi}$$

Substituting the expressions (ii), (iv) and (vi) in expression (i), we get:

$$\frac{p_o - p_i}{\rho_w g} = \frac{V_{fi}^2}{2g} - \frac{1}{2g}(V_{fo}^2 \text{cosec}^2\phi + u_o^2 - 2u_o V_{fo}\cot\phi) + \frac{(u_o - V_{fo}\cot\phi)u_o}{g}$$

$$= \frac{1}{2g}(V_{fi}^2 - V_{fo}^2\text{cosec}^2\phi - u_o^2 + 2u_o V_{fo}\cot\phi + 2u_o^2 - 2u_o V_{fo}\cot\phi)$$

$$\therefore \frac{p_o - p_i}{\rho_w g} = \frac{1}{2g}(V_{fi}^2 + u_o^2 - V_{fo}^2\text{cosec}^2\phi) \quad (25.12)$$

The manometric head (H_m) is given by the pressure rise through the impeller with a certain percentage of kinetic head at the impeller exit which is recovered in the volute chamber when any loss of head in the pump is also neglected.

Thus
$$H_m = \frac{p_o - p_i}{\rho_w g} + \frac{kV_o^2}{2g} \quad \text{(vii)}$$

Substituting Equation (25.12) and expression (v) in expression (vii), we get:

$$H_m = \frac{1}{2g}(V_{fi}^2 + u_o^2 - V_{fo}^2\text{cosec}^2\phi) + \frac{k}{2g}(V_{fo}^2 + V_{fo}^2\cot^2\phi + u_o^2 - 2u_o V_{fo}\cot\phi)$$

$$H_m = \frac{1}{2g}[u_o^2(1+k) - 2ku_o V_{fo}\cot\phi + V_{fo}^2(k\cot^2\phi - \text{cosec}^2\phi) + V_{fi}^2]$$

Assuming flow velocity as constant, i.e., $V_{fi} = V_{fo} = V_f$, we get:

$$H_m = \frac{1}{2g}[u_o^2(1+k) - 2ku_o V_f\cot\phi + V_f^2(k\cot^2\phi - \text{cosec}^2\phi) + V_f^2]$$

$$H_m = \frac{1}{2g}(xu_o^2 + yu_o V_f + zV_f^2)$$

Here, $x = (1+k)$, $y = -2k\cot\phi$ and $z = (1 + k\cot^2\phi - \text{cosec}^2\phi)$.

Since $u_o \,\alpha\, N$ and $V_f \,\alpha\, Q$ $\quad [\because u_o = \pi D_o N/60 \text{ and } V_f = Q/A]$

$$\therefore H_m = \frac{1}{2g}(AN^2 + BNQ + CQ^2) \quad (25.13)$$

Here, A, B and C are constants. The Equation (25.13) gives the head delivery law for a particular pump at a particular speed.

Example 25.1 The external and internal diameters of the impeller of a centrifugal pump are 0.4 m and 0.2 m, respectively. The centrifugal pump runs at 1200 rpm and its vanes at the exit are set back at an angle of 25°. If a constant radial flow through the impeller is maintained at 2.5 m/s, then determine (i) the inlet vane angle, (ii) angle made by absolute velocity at the outlet and (iii) work done by the impeller per unit weight of water.

Solution

Refer Figure 25.7. Let $D_o = 0.4$ m, $D_i = 0.2$ m, $N = 1200$ rpm, $\phi = 25°$ and $V_{fi} = V_{fo} = 2.5$ m/s. Let w be the work done by impeller per unit weight of water.

(i) $u_i = \dfrac{\pi D_i N}{60} = \dfrac{\pi \times 0.2 \times 1200}{60} = 12.57\ \text{m/s}$

$$\theta = \tan^{-1}\left(\frac{V_{fi}}{u_i}\right) = \tan^{-1}\left(\frac{2.5}{12.57}\right) = \mathbf{11.25^\circ}$$

(ii) $u_o = \dfrac{\pi D_o N}{60} = \dfrac{\pi \times 0.4 \times 1200}{60} = 25.13\ \text{m/s}$

$$V_{wo} = u_o - \frac{V_{fo}}{\tan\phi} = 25.13 - \frac{2.5}{\tan 25^\circ} = 19.77\ \text{m/s}$$

$$\beta = \tan^{-1}\left(\frac{V_{fo}}{V_{wo}}\right) = \tan^{-1}\left(\frac{2.5}{19.77}\right) = \mathbf{7.2^\circ}$$

(iii) $w = \dfrac{V_{wo} u_o}{g} = \dfrac{19.77 \times 25.13}{9.81} = \mathbf{50.644\ Nm/N}$

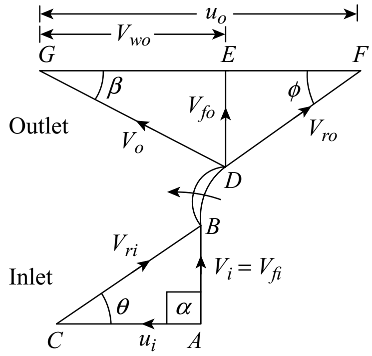


Figure 25.7

Example 25.2 The internal and external diameters of a centrifugal pump are 10 cm and 20 cm, respectively. It runs at 2800 rpm and delivers 0.105 m^3/s of water. The widths of impeller at the inlet and outlet are 2 cm and 1 cm, respectively. The water enters the impeller radially at the inlet and impeller blade angle at the exit is 45°. Determine the pressure rise in the impeller by assuming that flow velocity as constant and neglecting losses through it.

Solution

Let $D_i = 10\ \text{cm} = 0.1\ \text{m},\ D_o = 20\ \text{cm} = 0.2\ \text{m},\ N = 2800\ \text{rpm},\ Q = 0.105\ \text{m}^3/\text{s},\ B_i = 2\ \text{cm} = 0.02\ \text{m},\ B_o = 1\ \text{cm} = 0.01\ \text{m},$ $V_{wi} = 0,\ \phi = 45^\circ$ and $V_{fi} = V_{fo}$.

Since
$$Q = \pi D_o B_o V_{fo}$$

$$\therefore V_{fo} = \frac{Q}{\pi D_o B_o} = \frac{0.105}{\pi \times 0.2 \times 0.01} = 16.71\ \text{m/s}$$

Thus
$$V_{fi} = V_{fo} = 16.71\ \text{m/s}$$

$$u_o = \frac{\pi D_o N}{60} = \frac{\pi \times 0.2 \times 2800}{60} = 29.32\ \text{m/s}$$

Since
$$\frac{p_o - p_i}{\rho_w g} = \frac{(V_{fi}^2 + u_o^2 - V_{fo}^2 \operatorname{cosec}^2\phi)}{2g}$$

$$\therefore \frac{p_o - p_i}{\rho_w g} = \frac{(16.71^2 + 29.32^2 - 16.71^2 \operatorname{cosec}^2 45^\circ)}{2 \times 9.81} = \mathbf{29.584\ m}$$

25.8 ❐ LOSSES, POWER AND EFFICIENCIES OF CENTRIFUGAL PUMPS

25.8.1 Losses in Centrifugal Pumps

The various losses occurring during the operation of a centrifugal pump are hydraulic losses, mechanical losses and leakage losses. These losses are schematically illustrated in Figure 25.8.

1. **Mechanical losses:** The mechanical losses occur due to (a) disc friction between the impeller and the water which fills the clearance space between the impeller and the casing and (b) mechanical friction of the main bearing and glands.

2. **Hydraulic losses:** The hydraulic losses decreases the head developed by the impeller. Hydraulic losses in the pump include the shock and friction losses. For the given values (or design values) of the blade angles and speed of rotation, there will be only one rate of discharge which ensures tangential entry to the impeller and tangential exit from it. Generally, the pumps operate at off design conditions which results in the variation in the rate of discharge. Thus, shock losses occur at the entry and exit of the impeller. At the exit from the impeller, energy loss also occurs due to change in angles of the water as it enters the casing. Friction losses in the impeller and friction and eddy losses in the guide vanes and casing are also considered as hydraulic losses.

   The other hydraulic losses include friction and other minor losses in the suction and delivery pipes.

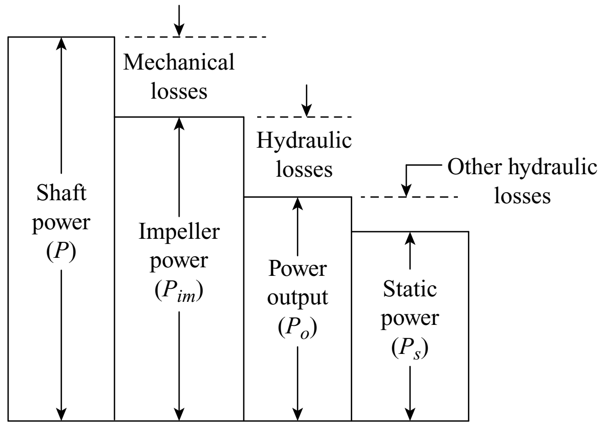


Figure 25.8 *Losses in centrifugal pumps*

3. **Leakage loss:** A certain amount of water always leaks from the high pressure region to the low pressure region and wastes its energy in the form of eddies. The loss of energy due to leakage of water is known as the leakage loss.

25.8.2 Power of Centrifugal Pumps

1. **Shaft power:** It is the power supplied by the motor (or prime mover) to the pump shaft and it is denoted by P.
2. **Impeller power:** It is the power available at the impeller and will be equal to the work done per second by the impeller on water. It is denoted by P_{im} and from Equation (25.5), it is given below.

$$\boxed{P_{im} = \frac{\rho_w Q V_{wo} u_o}{1000} \text{ kW}}$$

3. **Power output:** It is the power output from the pump that is available at casing exit. It is denoted by P_o and it is given below.

$$\boxed{P_o = \frac{\text{Weight of water lifted per second } \times H_m}{1000} = \frac{WH_m}{1000} = \frac{\rho_w gQ \times H_m}{1000} \text{ kW}} \tag{25.14}$$

4. **Static power:** It is the power available at delivery exit of the pump. It is denoted by P_s and it is expressed below.

$$\boxed{P_s = \frac{\text{Weight of water lifted per second } \times H_s}{1000} = \frac{WH_s}{1000} = \frac{\rho_w gQ \times H_s}{1000} \text{ kW}} \tag{25.15}$$

25.8.3 Efficiencies of Centrifugal Pumps

1. **Manometric efficiency:** It is defined as the ratio of manometric head developed by the pump to the head imparted by the impeller to the water. Manometric efficiency takes into account the hydraulic losses in the pump. It is denoted by η_{man} and the expression is given below.

$$\boxed{\eta_{\text{man}} = \frac{\text{Manometric head developed}}{\text{Head imparted by impeller to water}} = \frac{H_m}{(V_{wo} u_o)/g} = \frac{gH_m}{V_{wo} u_o}} \tag{25.16}$$

   The manometric efficiency is also defined as the ratio of power given to water at the outlet of the pump (P_o) to the power available at the impeller (P_{im}).

Thus $$\eta_{man} = \frac{P_o}{P_{im}} = \frac{(\rho_w gQH_m)/1000}{(\rho_w QV_{wo}u_o)/1000} = \frac{gH_m}{V_{wo}u_o} \quad \text{[Same as Equation (25.16)]}$$

Manometric efficiency becomes equal to hydraulic efficiency (η_h) when the vane efficiency (ε) approaches unity (For details refer Section 25.10).

2. **Volumetric efficiency:** A certain amount of liquid say q slips (or leaks) from the outlet of the impeller (high pressure zone) to the eye of the impeller (low pressure zone) through the clearances between the impeller and the casing. Thus, volumetric efficiency is defined as the ratio of actual discharge (Q) from the pump to the total discharge per second through the impeller. It is denoted by η_v and its expression is given below.

$$\boxed{\eta_v = \frac{\text{Actual discharge}}{\text{Total discharge}} = \frac{Q}{Q+q}} \tag{25.17}$$

Here, q is the amount of water leakage per second from the impeller.

Generally, the value of volumetric efficiency of centrifugal pumps ranges from 97% to 98%.

3. **Mechanical efficiency:** An electric motor is used to give the power input to the pump shaft which is more than the power delivered by the impeller to the water. Mechanical efficiency is defined as the ratio of the power available at the impeller (P_{im}) to the power at the shaft (P) of the centrifugal pump. It is denoted by η_m and its expression is given below.

$$\boxed{\eta_m = \frac{\text{Power at the impeller}}{\text{Power at the shaft}} = \frac{P_{im}}{P} = \frac{\rho_w QV_{wo}u_o}{1000P}} \tag{25.18}$$

But when q is considered, Equation (25.18) becomes,

$$\eta_m = \frac{\rho_w (Q+q)V_{wo}u_o}{1000P} \tag{25.18a}$$

Generally, the mechanical efficiency of centrifugal pump ranges from 95% to 98%.

4. **Overall efficiency:** It is defined as the ratio of power output of the pump (P_o) to the power input to the pump (P). It is denoted by η_o and its expression is given below.

$$\boxed{\eta_o = \frac{P_o}{P} = \frac{WH_m}{1000P} = \frac{\rho_w gQ \times H_m}{1000P}} \tag{25.19}$$

Also $$\eta_o = \eta_{\text{man}} \times \eta_m = \frac{gH_m}{V_{wo}u_o} \times \frac{\rho_w QV_{wo}u_o}{1000P} = \frac{\rho_w gQH_m}{1000P} \tag{25.19a}$$

When volumetric efficiency is also considered, then overall efficiency becomes,

$$\eta_o = \eta_{\text{man}} \times \eta_m \times \eta_v = \frac{gH_m}{V_{wo}u_o} \times \frac{\rho_w (Q+q)V_{wo}u_o}{1000P} \times \frac{Q}{Q+q} = \frac{\rho_w gQH_m}{1000P} \tag{25.19b}$$

Generally, the overall efficiency of a centrifugal pump ranges from 70% to 86%.

25.9 ❐ EFFECT OF OUTLET VANE ANGLE ON MANOMETRIC EFFICIENCY

The energy supplied to the impeller is $(V_{wo}u_o)/g$. The water leaving the impeller has a pressure energy (H_m) and kinetic energy $V_o^2/(2g)$. If the loss of head in the pump is neglected, then we get the below expression.

$$\frac{V_{wo}u_o}{g} = H_m + \frac{V_o^2}{2g}$$

Thus $$H_m = \frac{V_{wo}u_o}{g} - \frac{V_o^2}{2g} \tag{25.20}$$

From the outlet velocity triangle shown in Figure 25.6, we get:

$$V_o^2 = V_{wo}^2 + V_{fo}^2 \text{ and } V_{wo} = u_o - V_{fo}\cot\phi$$

Thus $$H_m = \frac{(u_o - V_{fo}\cot\phi)u_o}{g} - \frac{(u_o - V_{fo}\cot\phi)^2 + V_{fo}^2}{2g} = \frac{u_o^2 - V_{fo}^2\text{cosec}^2\phi}{2g} \tag{25.21}$$

Thus under ideal conditions the manometric efficiency of the centrifugal pump becomes,

$$\eta_{\text{man}} = \frac{gH_m}{V_{wo}u_o} = \frac{(u_o^2 - V_{fo}^2\text{cosec}^2\phi)}{2u_o(u_o - V_{fo}\cot\phi)} \tag{25.22}$$

The value of η_{man} evaluated from Equation (25.22) is observed to increase from 47% to 73% for different values of ϕ decreasing from 90° to 20° and for flow ratio K_f is equal to 0.25. It is also observed that a further decrease in the value of ϕ increases the efficiency. However, it is unfeasible to have the value of ϕ less than 20° because it results in the long and narrow passages which produce high frictional losses. Thus, the value of ϕ is not decreased below 20°. However, by employing guide vanes, a part of the velocity energy is converted into useful pressure energy and efficiency of the pump is increased.

25.10 ❐ EFFECT OF NUMBER OF VANES OF IMPELLER ON HEAD AND EFFICIENCY

The vanes are designed on the basis of Euler's velocity triangles which are drawn by assuming infinite number of vanes in the impeller. As the impeller has a finite number of vanes, the actual velocity triangles differ than the Euler's velocity triangles. Due to secondary flow (or circulatory flow), the actual velocity of whirl (V_{wo}) is always less than that in the Euler's velocity triangles. As a result, the actual head (H_i) imparted by the impeller with finite number of vanes to the water is always less than the Euler's head (H_e). The ratio of actual head to the Euler's head is known as vane efficiency (or vane effectiveness). It is denoted by ε and it is mathematically expressed as follows.

$$\varepsilon = \frac{H_i}{H_e} \tag{25.23}$$

Experimentally, it is found that as the number of vanes is increased, the value of vane efficiency (ε) increases and approaches unity. It means that the actual head imparted by the impeller to the water approaches Euler's head as the number of vanes are increased. The value of vane efficiency also depends on the shape of vane and the outlet vane angle (ϕ). Generally, for radial flow pumps, the vane efficiency is found to increase from 60% to 80% as the number of vanes is increased from 4 to 12. For impellers having vanes more than 24, the value of ε is taken as unity. Unless otherwise mentioned, the value of ε is also taken as unity.

A portion of the actual head imparted by the impeller with finite number of vanes to the water is lost in the pump say h_f. Thus, the manometric head available from the pump is given below.

$$H_m = H_i - h_f$$

The ratio of manometric head to the actual head is known as hydraulic efficiency (η_h) of the pump and its expression is given below.

$$\boxed{\eta_h = \frac{H_m}{H_i}} \tag{25.24}$$

The manometric efficiency in Equation (25.16) and Equation (25.2) is given by,

$$\boxed{\eta_{\text{man}} = \frac{H_m}{(V_{wo}u_o)/g} = \frac{H_m}{H_e}} \tag{25.25}$$

Thus, by combining the Equations (25.23), (25.24), and (25.25), we get:

$$\eta_{man} = \frac{H_m}{H_i} \times \frac{H_i}{H_e} = \eta_h \varepsilon \tag{25.26}$$

For $\varepsilon = 1$, $H_i = H_e$ and $\eta_{\text{man}} = \eta_h$.

25.11 ❐ SLIP FACTOR

The slip factor (σ) is defined as the ratio of actual outlet whirl velocity to the blade velocity at the outlet. The slip factor is mathematically expressed as follows.

$$\sigma = \frac{\text{Actual outlet whirl velocity}}{\text{Blade tip velocity at outlet}} = \frac{V_{wo}}{u_o} \tag{25.27}$$

The value of slip factor depends on the number of vanes. The greater the number of vanes, the smaller the slip, i.e., more nearly V_{wo} approaches u_o and thus, larger the slip factor. Stanitz (in 1952) suggested an appropriate empirical formula to determine the slip factor for a radial vane impeller in terms of number of vanes (n) as given below.

$$\sigma = 1 - \frac{0.63\pi}{n} \tag{25.28}$$

Generally, the value of slip factor is about 0.9 when the number of vanes varies from 19 to 21.

25.12 ❐ LOSS OF HEAD DUE TO REDUCED OR INCREASED FLOW

When a pump operates at its designed values of discharge and speed, it gives maximum efficiency. Due to increased or decreased flow rate, there will be loss of head due to shock at the entrance to the impeller. The head loss lowers the efficiency of the pump. Let *ABC* be the inlet velocity triangle for a centrifugal pump when it runs under normal conditions as shown in Figure 25.9(a).

The vanes at inlet tip will remain along the relative velocity represented by *AC*. When the flow rate decreases (or increases) the flow velocity decreases (or increases) from *BC* to *BD*. While the pump speed remains the same and the velocity triangle is shown by *ABD*. The new relative velocity *AD* no longer remains parallel to the vane and thus, shock

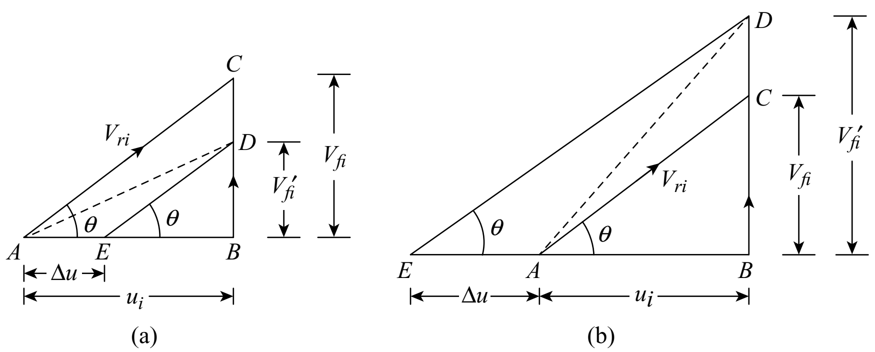


Figure 25.9 *Inlet velocity triangles with decreased and increased flow*

occurs at entry. Now flow velocity *BD* is fixed and also the water flow along the vane. Thus, the velocity triangle will become *EBD* and *ED* is parallel to *AC*. A sudden tangential change of velocity *AE* (Δu) results in a shock causing the head loss. The head loss (h_f) due to sudden change in velocity is given below.

$$h_f = \frac{(\Delta u)^2}{2g} = \frac{(u_i - V'_{fi}\cot\theta)^2}{2g} \tag{25.29}$$

Effect of the increased flow is shown in Figure 25.9(b) and the loss of head becomes,

$$h_f = \frac{(V'_{fi}\cot\theta - u_i)^2}{2g} \tag{25.30}$$

Example 25.3 A centrifugal pump is to discharge 0.118 m^3/s at a speed of 1450 rpm against a head of 25 m. The impeller diameter is 250 mm, its width at the outlet is 50 mm and manometric efficiency is 75%. Determine the vane angle at the outer periphery of the impeller.

Solution

Refer Figure 25.10. Let $Q = 0.118\ m^3/s$, $N = 1450$ rpm, $H_m = 25$ m, $D_o = 250$ mm $= 0.25$ m, $B_o = 50$ mm $= 0.05$ m and $\eta_{man} = 0.75$.

$$u_o = \frac{\pi D_o N}{60} = \frac{\pi \times 0.25 \times 1450}{60} = 18.98 \text{ m/s}$$

Since $$Q = \pi D_o B_o V_{fo}$$

$$\therefore V_{fo} = \frac{Q}{\pi D_o B_o} = \frac{0.118}{\pi \times 0.25 \times 0.05} = 3 \text{ m/s}$$

Since $$\eta_{man} = \frac{gH_m}{V_{wo}u_o}$$

$$\therefore V_{wo} = \frac{gH_m}{\eta_{man}u_o} = \frac{9.81 \times 25}{0.75 \times 18.98} = 17.23 \text{ m/s}$$

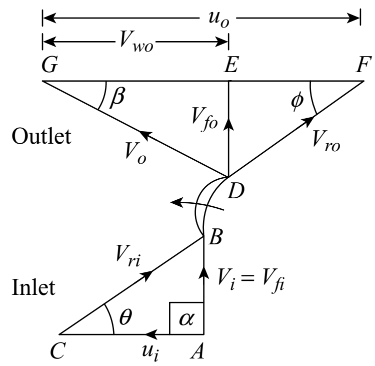


Figure 25.10

$$\phi = \tan^{-1}\left(\frac{V_{fo}}{u_o - V_{wo}}\right) = \tan^{-1}\left(\frac{3}{18.98 - 17.23}\right) = \mathbf{59.74°}$$

Example 25.4 The following data are given for a centrifugal pump, such as outer diameter = 2 × internal diameter, speed = 3000 rpm, internal diameter = 0.1 m, impeller width at outlet = 0.02 m, vane angle at outlet = 30°, constant flow velocity = 3 m/s, manometric efficiency = 0.8 and overall efficiency = 0.7. Calculate (i) vane angle at the inlet, (ii) rate of discharge, (iii) manometric head, (iv) shaft power and (v) torque.

Solution

Refer Figure 25.10. Let $D_o = 2D_i$, $N = 3000$ rpm, $D_i = 0.1$ m, $B_o = 0.02$ m, $\phi = 30°$, $V_{fi} = V_{fo} = 3$ m/s, $\eta_{man} = 0.8$ and $\eta_o = 0.7$.

(i) $$u_i = \frac{\pi D_i N}{60} = \frac{\pi \times 0.1 \times 3000}{60} = 15.71 \text{ m/s}$$

$$\theta = \tan^{-1}\left(\frac{V_{fi}}{u_i}\right) = \tan^{-1}\left(\frac{3}{15.71}\right) = \mathbf{10.81°}$$

(ii) $D_o = 2D_i = 2\times0.1 = 0.2$ m

$$Q = \pi D_o B_o V_{fo} = \pi\times0.2\times0.02\times3 = \mathbf{0.0377\ m^3/s}$$

(iii) $$u_o = \frac{\pi D_o N}{60} = \frac{\pi\times0.2\times3000}{60} = 31.416 \text{ m/s}$$

$$V_{wo} = u_o - \frac{V_{fo}}{\tan\phi} = 31.416 - \frac{3}{\tan 30°} = 26.22 \text{ m/s}$$

Since $\eta_{man} = \dfrac{gH_m}{V_{wo}u_o}$

$$\therefore H_m = \frac{\eta_{man}V_{wo}u_o}{g} = \frac{0.8\times26.22\times31.416}{9.81} = \mathbf{67.174\ m}$$

(iv) $\because \eta_o = \dfrac{\rho_w gQH_m}{1000P}$

$$\therefore P = \frac{\rho_w gQH_m}{1000\eta_o} = \frac{1000\times9.81\times0.0377\times67.174}{1000\times0.7} = \mathbf{35.491\ kW}$$

(v) $$P_o = \frac{\rho_w gQH_m}{1000} = \frac{1000\times9.81\times0.0377\times67.174}{1000} = 24.843 \text{ kW}$$

Since $P_o = \dfrac{2\pi NT}{60}$

$$\therefore T = \frac{60P_o}{2\pi N} = \frac{60\times24.843\times10^3}{2\times\pi\times3000} = \mathbf{79.08\ Nm}$$

Example 25.5 The impeller of a centrifugal pump is of 0.3 m diameter, 0.05 m width at the periphery and has blades whose tip angle inclines backwards 60° from the radius. The pump delivers 15 m^3/min and the impeller rotates at 1000 rpm. Assume that the pump is designed to admit radially and calculate (i) the speed and direction of water as it leaves the impeller, (ii) torque exerted by the impeller in water, (iii) shaft power required and (iv) lift of the pump. Take mechanical efficiency as 95% and hydraulic efficiency as 75%.

Solution

Refer Figure 25.11. Let $D_o = 0.3$ m, $B_o = 0.05$ m, $\phi = 60°$, $Q = 15\ m^3/\text{min} = 0.25\ m^3/s$, $N = 1000$ rpm, $V_{wi} = 0$, $\eta_m = 0.95$ and $\eta_h = 0.75$.

$$u_o = \frac{\pi D_o N}{60} = \frac{\pi\times0.3\times1000}{60} = 15.71 \text{ m/s}$$

Since $$Q = \pi D_o B_o V_{fo}$$

$$\therefore V_{fo} = \frac{Q}{\pi D_o B_o} = \frac{0.25}{\pi\times0.3\times0.05} = 5.3 \text{ m/s}$$

Since $$\tan\phi = \frac{V_{fo}}{u_o - V_{wo}}$$

Thus $$V_{wo} = u_o - V_{fo}\cot\phi = 15.71 - 5.3\cot 60° = 12.65 \text{ m/s}$$

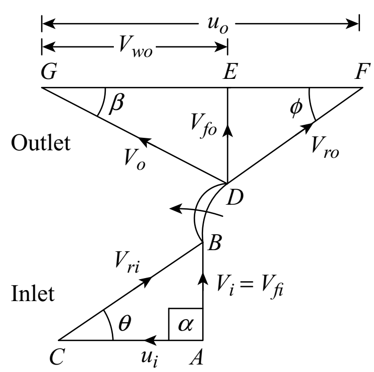


Figure 25.11

(i) $V_o = \sqrt{V_{fo}^2 + V_{wo}^2} = \sqrt{5.3^2 + 12.65^2} = \mathbf{13.715\ m/s}$

$$\therefore \beta = \tan^{-1}\left(\frac{V_{fo}}{V_{wo}}\right) = \tan^{-1}\left(\frac{5.3}{12.65}\right) = \mathbf{22.73°}$$

(ii) Torque exerted by the impeller on the water is given by,

$$T = \rho_w Q V_{wo} R_o = 1000 \times 0.25 \times 12.65 \times 0.15 = \mathbf{474.375\ Nm}$$

(iii) $P_{im} = T\omega = \dfrac{2\pi NT}{60} = \dfrac{2\pi \times 1000 \times 474.375}{60 \times 1000} = 49.676$ kW

$$\therefore P = \frac{P_{im}}{\eta_m} = \frac{49.676}{0.95} = \mathbf{52.29\ kW}$$

(iv) $\eta_h = \dfrac{P_o}{P_{im}} = \dfrac{\rho_w g Q H_m}{1000 P_{im}}$ $\quad [\because \eta_h = \eta_{man}]$

$$\therefore H_m = \frac{1000\eta_h P_{im}}{\rho_w g Q} = \frac{1000 \times 0.75 \times 49.676}{1000 \times 9.81 \times 0.25} = \mathbf{15.1914\ m}$$

Example 25.6 A centrifugal pump is required to discharge 60 litres per second water against a head of 12 m, when running at a speed of 750 rpm. The manometric efficiency is to be 80%, the loss of head in the pump being assumed as $0.025V_o^2$ of water, where V_o is the absolute velocity of water leaving the impeller. Water enters the impeller without whirl and the velocity of flow at the exit is 3 m/s. Determine (i) the impeller diameter and outlet area, (ii) vane angle at the outlet edge of the impeller and (iii) angle made by absolute velocity of water leaving the vane with the direction of motion at the outlet.

Solution

Refer Figure 25.12. Let $Q = 60\ l/\text{s} = 0.06\ \text{m}^3/\text{s}$, $H_m = 12$ m, $N = 750$ rpm, $\eta_{\text{man}} = 0.8$, head loss $= 0.025V_o^2$, $V_{wi} = 0$ and $V_{fo} = 3$ m/s.

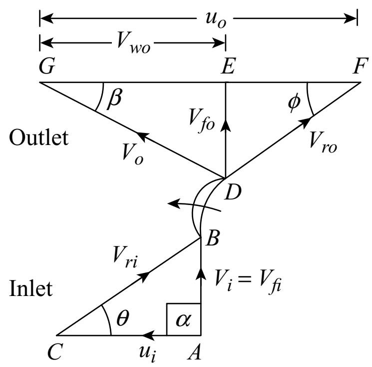


Figure 25.12

$$u_o = \frac{\pi D_o N}{60} = \frac{\pi \times D_o \times 750}{60} = 39.27 D_o \text{ m/s}$$

$$\text{Head developed} = \frac{H_m}{\eta_{\text{man}}} = \frac{12}{0.8} = 15 \text{ m}$$

$$\text{Head loss} = 15 - 12 = 3 \text{ m}$$

Thus $\quad 0.025V_o^2 = 3$

$$\therefore V_o = \sqrt{\frac{3}{0.025}} = 10.954 \text{ m/s}$$

$$V_{wo} = \sqrt{V_o^2 - V_{fo}^2} = \sqrt{10.954^2 - 3^2} = 10.535 \text{ m/s}$$

Since $\quad \eta_{man} = \dfrac{gH_m}{V_{wo}u_o}$

Thus $\quad 0.8 = \dfrac{9.81 \times 12}{10.535 \times 39.27 D_o}$

$$\therefore D_o = \frac{9.81\times 12}{10.535\times 39.27\times 0.8} = \mathbf{0.356\ m}$$

$$u_o = 39.27D_o = 39.27\times 0.356 = 13.98\ \text{m/s}$$

$$A_o = \frac{Q}{V_{fo}} = \frac{0.06}{3} = \mathbf{0.02\ m^2}$$

$$\phi = \tan^{-1}\left(\frac{V_{fo}}{u_o - V_{wo}}\right) = \tan^{-1}\left(\frac{3}{13.98-10.535}\right) = \mathbf{41.05°}$$

$$\beta = \tan^{-1}\left(\frac{V_{fo}}{V_{wo}}\right) = \tan^{-1}\left(\frac{3}{10.535}\right) = \mathbf{15.89°}$$

Example 25.7 A centrifugal pump has an impeller of 0.75 m diameter and it delivers 1000 litres per second against a head of 65 m. The impeller runs at 1000 rpm and the width at the outlet is 6 cm. If the leakage loss is 3.5% of the discharge, then the external mechanical loss is 15 kW and the manometric efficiency is 85%. Determine (i) the blade angle at the outlet, (ii) power required and (iii) efficiency of the pump.

Solution

Refer Figure 25.13. Let $D_o = 0.75$ m, $Q_a = 1000\ l/\text{s} = 1\ \text{m}^3/\text{s}$, $H_m = 65$ m, $N =$ 1000 rpm, $B_o = 6$ cm $= 0.06$ m, leakage loss $= 3.5\%$ of Q_a, mechanical loss $=15$ kW and $\eta_{man} = 0.85$.

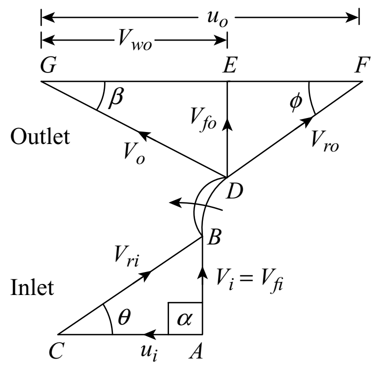


Figure 25.13

(i) Let Q_{th} and Q_a be the theoretical and actual discharges, respectively.

$$Q_{th} = Q_a + 3.5\%\times Q_a = 1.035Q_a = 1.035\times 1 = 1.035\ \text{m}^3/\text{s}$$

Since $\quad Q_a = \pi D_o B_o V_{fo}$

$$\therefore V_{fo} = \frac{Q_a}{\pi D_o B_o} = \frac{1}{\pi\times 0.75\times 0.06} = 7.07\ \text{m/s}$$

$$u_o = \frac{\pi D_o N}{60} = \frac{\pi\times 0.75\times 1000}{60} = 39.27\ \text{m/s}$$

Since $\quad \eta_{\text{man}} = \dfrac{gH_m}{V_{wo}u_o}$

$$\therefore V_{wo} = \frac{gH_m}{\eta_{man}u_o} = \frac{9.81\times 65}{0.85\times 39.27} = 19.1\ \text{m/s}$$

$$\phi = \tan^{-1}\left(\frac{V_{fo}}{u_o - V_{wo}}\right) = \tan^{-1}\left(\frac{7.07}{39.27-19.1}\right) = \mathbf{19.32°}$$

(ii) Theoretical power $= \dfrac{\rho_w g Q_{th} H_m}{1000} = \dfrac{1000\times 9.81\times 1.035\times 65}{1000} = 659.968$ kW

Actual power input = Theoretical power + Mechanical loss

$\therefore$ Actual power input $= 659.968 + 15 =$ **674.968 kW**

(iii) Actual power output $= \dfrac{\rho_w g Q_a H_m}{1000} = \dfrac{1000 \times 9.81 \times 1 \times 65}{1000} = 637.65$ kW

$$\eta = \frac{\text{Actual power output}}{\text{Actual power input}} = \frac{637.65}{674.968} \times 100 = \mathbf{94.47\%}$$

Example 25.8 A centrifugal pump is required to deliver 0.03 m^3/s of water to a height of 25 m through a 12 cm diameter pipe and 110 m long. Determine the power required to drive the pump if its overall efficiency is 72%. Take coefficient of friction $f = 0.01$ for the pipe line.

Solution

Let $Q = 0.03\ \text{m}^3/\text{s}$, $(h_s + h_d) = 25$ m, $D = 12\ \text{cm} = 0.12$ m, $L = 110$ m, $\eta_o = 0.72$ and $f = 0.01$. Let $V = V_s = V_d$ be the velocity of water in pipe and P be the power required by the pump.

$$V = V_d = \frac{Q}{A} = \frac{Q}{(\pi/4)D^2} = \frac{0.03}{(\pi/4) \times 0.12^2} = 2.652 \text{ m/s}$$

$$(h_{fs} + h_{fd}) = \frac{4fLV^2}{2gD} = \frac{4 \times 0.01 \times 110 \times 2.652^2}{2 \times 9.81 \times 0.12} = 13.144 \text{ m}$$

$$H_m = h_s + h_d + h_{fs} + h_{fd} + \frac{V_d^2}{2g} = 25 + 13.144 + \frac{2.652^2}{2 \times 9.81} = 38.5 \text{ m}$$

$$P = \frac{\rho_w g Q H_m}{1000\eta_o} = \frac{1000 \times 9.81 \times 0.03 \times 38.5}{1000 \times 0.72} = \mathbf{15.737\ kW}$$

Example 25.9 The discharge from a centrifugal pump running at 750 rpm is 0.25 m^3/s and the head developed is 12 m. The inner and outer diameters of the impeller are 0.2 m and 0.4 m, respectively. The blade outlet angle is 30° to the tangent. The flow area is constant as 0.08 m^2. If the flow at the inlet is radial, then determine (i) the manometric efficiency of the pump, (ii) vane angle at the inlet and (iii) loss of head at the inlet to impeller when the discharge is reduced by 40% without changing the speed.

Solution

Refer Figure 25.14(a). Let $N = 750$ rpm, $Q = 0.25\ \text{m}^3/\text{s}$, $H_m = 12$ m, $D_i = 0.2$ m, $D_o = 0.4$ m, $\phi = 30°$, $A = 0.08\ \text{m}^2$, $V_{wi} = 0$ and discharge reduced by 40%. Let h_f be the loss of head.

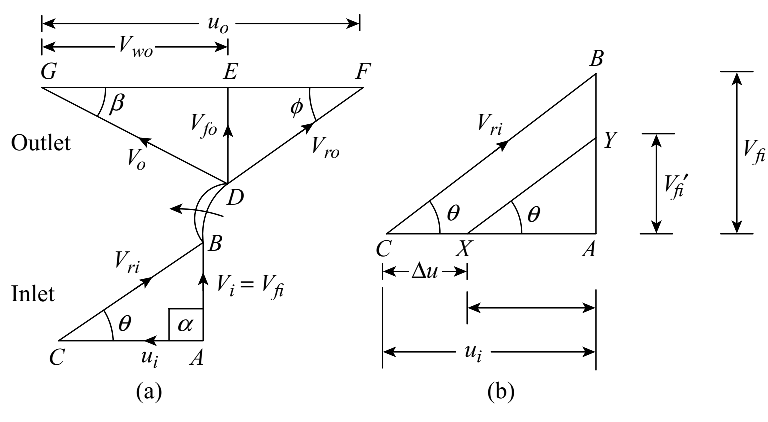


Figure 25.14

(i) $V_{fi} = V_{fo} = \dfrac{Q}{A} = \dfrac{0.25}{0.08} = 3.125 \text{ m/s}$

$$u_i = \frac{\pi D_i N}{60} = \frac{\pi \times 0.2 \times 750}{60} = 7.854 \text{ m/s}$$

$$u_o = \frac{\pi D_o N}{60} = \frac{\pi \times 0.4 \times 750}{60} = 15.71 \text{ m/s}$$

$$V_{wo} = u_o - \frac{V_{fo}}{\tan\phi} = 15.71 - \frac{3.125}{\tan 30°} = 10.297 \text{ m/s}$$

$$\eta_{\text{man}} = \frac{gH_m}{V_{wo}u_o} = \frac{9.81 \times 12}{10.297 \times 15.71} \times 100 = \mathbf{72.77\%}$$

(ii) $\theta = \tan^{-1}\left(\dfrac{V_{fi}}{u_i}\right) = \tan^{-1}\left(\dfrac{3.125}{7.854}\right) = \mathbf{21.7°}$

(iii) When the flow rate is reduced, the velocity of flow is reduced. However, the blade angle remains same. The water suddenly drops in velocity by CX as shown in Figure 25.14(b).

New discharge is given by,

$$Q_{\text{new}} = 0.6 \times 0.25 = 0.15 \text{ m}^3/\text{s}$$

$$\therefore V'_{fi} = \frac{Q_{\text{new}}}{A} = \frac{0.15}{0.08} = 1.875 \text{ m/s}$$

Drop in velocity is given by,

$$\Delta u = u_i - \frac{V'_{fi}}{\tan\theta} = 7.854 - \frac{1.875}{\tan 21.7°} = 3.142 \text{ m/s}$$

$$\therefore h_f = \frac{\Delta u^2}{2g} = \frac{3.142^2}{2 \times 9.81} = \mathbf{0.5032 \text{ m}}$$

Example 25.10 The head capacity characteristics of a centrifugal pump running at constant speed is given by $H = 30 + 40Q - 750.98Q^2$, where H is the total head generated in m and Q is the discharge in m^3/s. The pump is required to discharge water through a pipeline 1000 m long and 30 cm diameter. The static lift is 20 m. Determine the operating head and discharge rate of the pump. Also determine the power required to drive the pump if it has an overall efficiency of 76% for the particular operating point. Neglect velocity head and take pipe friction coefficient as $f = 0.002$.

Solution

Let $H = 30 + 40Q - 750.98Q^2$, $L = 1000$ m, $D = 30$ cm $= 0.3$ m, static lift = 20 m, $\eta_o = 0.76$ and $f = 0.002$. Head developed by the pump is equal to the sum of static head and the head lost in pipe friction.

Thus
$$30 + 40Q - 750.98Q^2 = 20 + \frac{4fLV^2}{2gD}$$

Since
$$V = \frac{Q}{A} = \frac{Q}{(\pi/4)D^2} = \frac{4Q}{\pi D^2}$$

$$30 + 40Q - 750.98Q^2 = 20 + \frac{4fL}{2gD} \times \left(\frac{4Q}{\pi D^2}\right)^2$$

$$30 + 40Q - 750.98Q^2 = 20 + \frac{4 \times 0.002 \times 1000 \times 16Q^2}{2 \times 9.81 \times 0.3 \times \pi^2 \times 0.3^4}$$

$$30 + 40Q - 750.98Q^2 = 20 + 272.02Q^2$$

$$1023Q^2 - 40Q - 10 = 0$$

$$\therefore Q = \frac{40 \pm \sqrt{40^2 + 4 \times 1023 \times 10}}{2 \times 1023} = \mathbf{0.1203\ m^3/s}$$

Since $$H = 30 + 40Q - 750.98Q^2$$

$$\therefore H = 30 + 40 \times 0.1203 - 750.98 \times 0.1203^2 = \mathbf{23.944\ m}$$

$$P = \frac{\rho_w g Q H}{1000\eta_o} = \frac{1000 \times 9.81 \times 0.1203 \times 23.944}{1000 \times 0.76} = \mathbf{37.181\ kW}$$

25.13 ❐ MINIMUM STARTING SPEED

Recalling the fundamental equation (Euler equation) of centrifugal pump [Equation (25.6)], we get the below expression.

$$w = H_e = \frac{V_o^2 - V_i^2}{2g} + \frac{u_o^2 - u_i^2}{2g} + \frac{V_{ri}^2 - V_{ro}^2}{2g}$$

The water velocities at the start time of a centrifugal pump are negligible. Thus, the heads due to kinetic energy and relative velocity are not present. Therefore, only centrifugal or pressure head $[(u_o^2 - u_i^2)/(2g)]$ caused by the centrifugal force on the rotating water will be available. The water will start flowing only when the centrifugal head is more than or equal to manometric head (H_m). Thus, the pump starting condition will be as follows.

$$\frac{(u_o^2 - u_i^2)}{2g} \geq H_m \qquad (25.31)$$

Thus, for minimum speed, we must have,

$$\frac{(u_o^2 - u_i^2)}{2g} = H_m \qquad (25.32)$$

Since $$H_m = \frac{\eta_{man} V_{wo} u_o}{g}$$ [From Equation (25.16)]

Thus $$\frac{u_o^2 - u_i^2}{2g} = \frac{\eta_{man} V_{wo} u_o}{g}$$

$$\frac{1}{2g}\left[\left(\frac{\pi D_o N}{60}\right)^2 - \left(\frac{\pi D_i N}{60}\right)^2\right] = \frac{\eta_{man} V_{wo}}{g}\left(\frac{\pi D_o N}{60}\right)$$

$$\frac{\pi N}{120}(D_o^2 - D_i^2) = \eta_{man} V_{wo} D_o$$

$$\boxed{\therefore N = \frac{120\eta_{man} V_{wo} D_o}{\pi(D_o^2 - D_i^2)}} \qquad (25.33)$$

The Equation (25.33) represents minimum starting speed of a centrifugal pump to maintain continuous discharge of water.

Example 25.11 A centrifugal pump has diameters at the outlet and inlet as 0.5 m and 0.25 m, respectively. The vanes outlet angle is 45°. If the velocity of flow at the outlet is 2.5 m/s and the manometric efficiency of the pump is 75%, then determine its minimum starting speed.

Solution

Refer Figure 25.15. Let $D_o = 0.5$ m, $D_i = 0.25$ m, $\phi = 45°$, $V_{fo} = 2.5$ m/s and $\eta_{\text{man}} = 0.75$. Let N be the minimum starting speed.

$$u_o = \frac{\pi D_o N}{60} = \frac{\pi \times 0.5 \times N}{60} = 0.0262\text{N}$$

$$V_{wo} = u_o - \frac{V_{fo}}{\tan\phi} = 0.0262N - \frac{2.5}{\tan 45°} = (0.0262N - 2.5) \text{ m/s}$$

$$N = \frac{120\eta_{man}V_{wo}D_o}{\pi(D_o^2 - D_i^2)} = \frac{120 \times 0.75 \times (0.0262N - 2.5) \times 0.5}{\pi(0.5^2 - 0.25^2)} = 2N - 190.986$$

$\therefore N =$ **190.986 rpm**

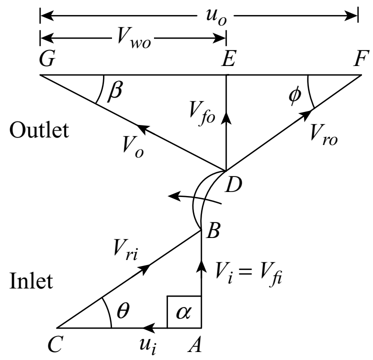


Figure 25.15

Example 25.12 A centrifugal pump has the following particulars, such as outer diameter = 1.2 m, inlet diameter = 0.6 m, speed = 200 rpm, discharge = 1.88 m^3/s, average lift and head = 6 m, vane outlet angle = 26° and velocity of flow = 2.5 m/s. Determine manometric efficiency, least speed to start pumping action and power required to drive the pump impeller if mechanical efficiency is given as 95%.

Solution

Refer Figure 25.15. Let $D_o = 1.2$ m, $D_i = 0.6$ m, $N = 200$ rpm, $Q = 1.88$ m^3/s, $H_m = 6$ m, $\phi = 26°$, $V_{fo} = 2.5$ m/s and $\eta_m = 0.95$.

$$u_o = \frac{\pi D_o N}{60} = \frac{\pi \times 1.2 \times 200}{60} = 12.566 \text{ m/s}$$

$$V_{wo} = u_o - \frac{V_{fo}}{\tan\phi} = 12.566 - \frac{2.5}{\tan 26°} = 7.44 \text{ m/s}$$

$$\eta_{man} = \frac{gH_m}{V_{wo}u_o} = \frac{9.81 \times 6}{7.44 \times 12.566} \times 100 = \mathbf{62.96\%}$$

Since $$\frac{(u_o^2 - u_i^2)}{2g} = H_m \quad \text{[For minimum speed]}$$

But $$u_i = \frac{u_o}{2} \quad [\because D_i = D_o/2]$$

Thus $$\frac{u_o^2 - (u_o/2)^2}{2 \times 9.81} = 6$$

$$\frac{3}{4}u_o^2 = 6 \times 2 \times 9.81$$

$$u_o = \sqrt{\frac{6 \times 2 \times 9.81 \times 4}{3}} = 12.528 \text{ m/s}$$

Since $$u_o = \frac{\pi D_o N}{60}$$

$$\therefore N = \frac{60u_o}{\pi D_o} = \frac{60 \times 12.528}{\pi \times 1.2} = \mathbf{199.39\ rpm}$$

$$P = \frac{\rho_w Q V_{wo} u_o}{1000\eta_m} = \frac{1000 \times 1.88 \times 7.44 \times 12.566}{1000 \times 0.95} = \mathbf{185.014\ kW}$$

25.14 ❒ DESIGN CONSIDERATIONS

1. **Speed ratio (K_u):** The ratio of tangential velocity at the outlet to the theoretical jet velocity corresponding to manometric head is known as speed ratio. Generally, its value for impellers varies from 0.95 to 1.25. The expression for speed ratio is given below.

$$K_u = \frac{u_o}{\sqrt{2gH_m}} \tag{25.34}$$

2. **Flow ratio (K_f):** The flow ratio is defined as the ratio of flow velocity at the outlet (V_{fo}) to the theoretical jet velocity corresponding to manometric head. The usual range of K_f for impeller is 0.1 to 0.25. The expression for flow ratio is given below.

$$K_f = \frac{V_{fo}}{\sqrt{2gH_m}} \tag{25.35}$$

3. **Outlet diameter of impeller (D_o):**

$$u_o = \frac{\pi D_o N}{60}, \text{ also } u_o = K_u\sqrt{2gH_m}$$

Thus $$\frac{\pi D_o N}{60} = K_u\sqrt{2gH_m}$$

$$\therefore D_o = \frac{60K_u\sqrt{2gH_m}}{\pi N} \tag{25.36}$$

If D_o and N are known, then by Equation (25.36), the head which can be developed by a pump can be determined. This will serve as a check for the given pump.

4. **Inlet diameter of impeller (D_i):** Based on the specific speed or manometric head, the inlet diameter D_i is kept in the range of $(D_o/3)$ to $(2D_o/3)$. However, it is usually taken as $D_i = D_o/2$.

5. **Least diameter of impeller:** To determine the least diameter of the pump impeller, we have to consider that the water will start flowing only when the centrifugal head is equal to manometric head (H_m).

Thus $$\frac{(u_o^2 - u_i^2)}{2g} = H_m$$

or $$\left(\frac{\pi D_o N}{60}\right)^2 - \left(\frac{\pi D_i N}{60}\right)^2 = 2gH_m$$

Taking $D_i = D_o/2$ and rearranging, we get:

$$D_o = \frac{97.68\sqrt{H_m}}{N} \tag{25.37}$$

6. **Diameter of suction pipe (d_s):** The discharge of a centrifugal pump is given by,

$$Q = \frac{\pi}{4} d_s^2 \times V_s$$

$$\therefore d_s = \sqrt{\frac{4Q}{\pi V_s}} \tag{25.38}$$

Here, V_s is the velocity of flow in the suction pipe which varies from 1.5 to 3 m/s.

7. **Diameter of delivery pipe (d_d):** The diameter of the delivery pipe is given by,

$$d_d = \sqrt{\frac{4Q}{\pi V_d}} \tag{25.39}$$

Here, V_d is the velocity of flow in the delivery pipe which varies from 1.5 to 3.5 m/s.

25.15 ❐ MULTISTAGE PUMPS

A multistage pump consists of two or more identical impellers mounted on the same shaft or on different shafts. A multistage pump performs the following functions.

1. To produce heads greater than that permissible with a single impeller, discharge remaining constant. This task is accomplished by series arrangement in which two or more impellers are mounted on the same shaft and enclosed in the same casing. A multistage pump is known as two stage, three stage, etc., according to the number of impellers fitted in the casing. A two stage pump is shown in Figure 25.16.

   The discharge with increased pressure from the first impeller passes through the connecting passages to the inlet of the second impeller and so on. Finally, the discharge from the last impeller passes on the delivery pipe. Impellers in series are employed for delivering a relatively small quantity of liquid against very high heads (>100 m).

   Let n be the number of identical impellers mounted on the same shaft and H_m be the head developed by each impeller, the total head (H_t) developed by multistage pump is given below.

$$\boxed{H_t = nH_m} \tag{25.40}$$

   The discharge of a multistage pump is same as the discharge capacity of one impeller.

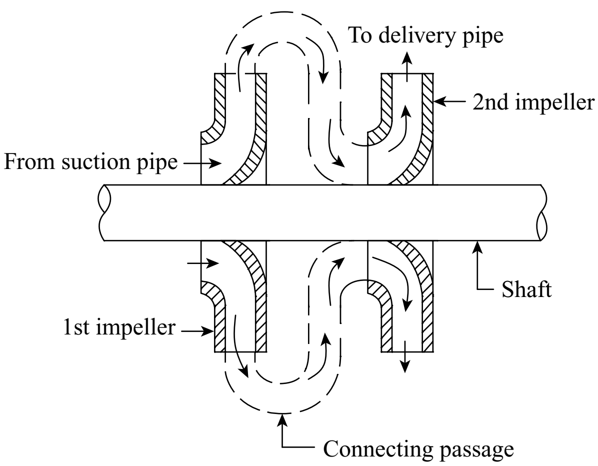


Figure 25.16 *Impellers in series*

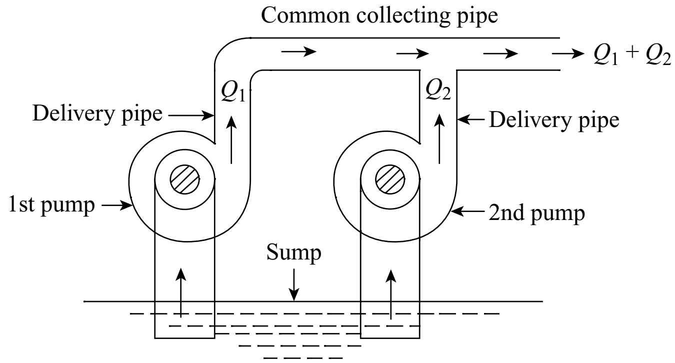


Figure 25.17 *Impellers in parallel*

2. To discharge a large quantity of water, head remaining constant. When a large quantity of liquid is required to pump against a relatively small heads, two or more pumps mounted on separate shafts are used which separately lifts the liquid from a common sump and delivers it to a common collecting pipe as shown in Figure 25.17.

   Let n be the number of identical impellers mounted on separate shafts and arranged in parallel and Q be the discharging capacity which is same for each pump, the total discharge (Q_t) delivered by this arrangement is given below.

$$\boxed{Q_t = nQ} \tag{25.41}$$

   In this case, each of the pumps delivers the liquid against the same head.

Example 25.13 A three stage centrifugal pump delivers water at the rate of 0.06 m^3/s. Each impeller is 0.42 m in diameter and 0.024 m wide at the outlet. The speed of the impellers is 950 rpm. The vanes are curved back at the outlet at an angle of 45° and reduce the circumferential area by 10%. The overall efficiency is 78% and the manometric efficiency is 88%. Determine the head generated and the power consumed.

Solution

Refer Figure 25.18. Let $n = 3$, $Q = 0.06 \text{ m}^3/\text{s}$, $D_o = 0.42$ m, $B_o = 0.024$ m, $N = 950$ rpm, $\phi = 45°$, $k = 1 - 0.1 = 0.9$, $\eta_o = 0.78$ and $\eta_{man} = 0.88$.

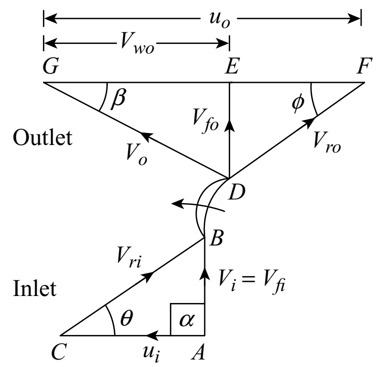


Figure 25.18

Since $$Q = k\pi D_o B_o V_{fo}$$

$$\therefore V_{fo} = \frac{Q}{k\pi D_o B_o} = \frac{0.06}{0.9\pi \times 0.42 \times 0.024} = 2.105 \text{ m/s}$$

$$u_o = \frac{\pi D_o N}{60} = \frac{\pi \times 0.42 \times 950}{60} = 20.892 \text{ m/s}$$

$$V_{wo} = u_o - \frac{V_{fo}}{\tan\phi} = 20.892 - \frac{2.105}{\tan 45°} = 18.787 \text{ m/s}$$

Since $$\eta_{man} = \frac{gH_m}{V_{wo}u_o}$$

$$\therefore H_m = \frac{\eta_{\text{man}} V_{wo} u_o}{g} = \frac{0.88 \times 18.787 \times 20.892}{9.81} = 35.21 \text{ m}$$

$$H_t = nH_m = 3\times 35.21 = \mathbf{105.63\ m}$$

$$P = \frac{\rho_w g Q H_m}{1000\eta_o} = \frac{1000\times 9.81\times 0.06\times 105.63}{1000\times 0.78} = \mathbf{79.71\ kW}$$

25.16 ❐ SPECIFIC SPEED OF CENTRIFUGAL PUMPS

The specific speed (N_s) of a centrifugal pump is defined as the speed of a geometrically similar pump which would deliver 1 m^3 of liquid per second against a head of 1 m. Specific speed is used for the classification of pumps on the basis of their performance and dimensions regardless of their actual size or speed at which they operate.

Specific speed may be derived as follows:

Since $$Q = \pi D B V_f$$

Thus $$Q \propto D B V_f$$

But $$B \propto D$$

Thus $$Q \propto D^2 V_f \qquad \text{(i)}$$

Since $$u = \frac{\pi D N}{60}$$

Thus $$u \propto DN$$

Also $$u \propto V_f \propto \sqrt{H_m}$$

Thus $$DN \propto \sqrt{H_m} \Rightarrow D \propto \frac{\sqrt{H_m}}{N}$$

Substituting the values of D and V_f in expression (i), we get:

$$Q \propto \left(\frac{\sqrt{H_m}}{N}\right)^2 \sqrt{H_m}$$

$$Q \propto \frac{H_m^{3/2}}{N^2}$$

$$Q = k\frac{H_m^{3/2}}{N^2} \qquad [k = \text{Constant}] \qquad \text{(ii)}$$

As per definition: If $H_m = 1\ \text{m}$, $Q = 1\ \text{m}^3/\text{s}$, then $N = N_s$ and expression (ii) can be written as follows.

$$1 = k\frac{1^{3/2}}{{N_s}^2}$$

Thus $$k = {N_s}^2$$

Substituting the value of k in expression (ii), we get:

$$Q = N_s^2 \frac{H_m^{3/2}}{N^2}$$

$$N_s^2 = \frac{N^2 Q}{H_m^{3/2}}$$

$$\boxed{\therefore N_s = \frac{N\sqrt{Q}}{H_m^{3/4}}} \tag{25.42}$$

It is to be noted that in Equation (25.42), the value of H_m for a multistage pump is obtained by dividing the total head developed by the number of stages. For a double suction pump, half the actual discharge delivered by the pump is taken as Q.

25.17 ❐ MODEL TESTING OF CENTRIFUGAL PUMPS

The model is a small scale replica of the actual machine or the prototype. The pumps are manufactured only after testing their small scale models. From the test results, the performance of the prototype can be predicted in advance and the required changes in the prototype can also be made. For complete similarity to exist between the model and the prototype pumps, the following conditions may be satisfied. The subscripts m and p used in the following discussion specify the model and the prototype pumps, respectively. For geometrically similar pumps, the subscripts m and p have been replaced by 1 and 2, respectively.

1. **Capacity or flow coefficient**

Since $\quad Q = \pi D B V_f$

$$Q \propto DBV_f \propto D^2 V_f \qquad [\because B \propto D]$$

Since $\quad V_f \propto u \propto DN$

Thus $\quad Q \propto D^2 \times DN$

$$Q \propto D^3 N$$

$$\boxed{\therefore \frac{Q}{ND^3} = \text{Constant}}$$

The parameter $[Q/(ND^3)]$ called the capacity or flow coefficient of the model and prototype is equal as given below.

$$\left(\frac{Q}{ND^3}\right)_m = \left(\frac{Q}{ND^3}\right)_p \tag{25.44}$$

$$\left(\frac{Q}{ND^3}\right)_1 = \left(\frac{Q}{ND^3}\right)_2 \quad \text{(For geometrically similar pumps)} \tag{25.44a}$$

2. **Head coefficient**

Since $\quad u = \dfrac{\pi DN}{60}$

Thus $\quad u \propto DN$

Also $\quad u \propto \sqrt{H_m}$

Thus $$\sqrt{H_m} \propto ND$$

$$\frac{\sqrt{H_m}}{ND} = \text{Constant}$$

Thus $$\boxed{\frac{H_m}{N^2 D^2} = \text{Constant}} \tag{25.45}$$

The parameter $[H_m/(N^2D^2)]$ called the head coefficient of the model and prototype is equal as given below.

$$\left(\frac{H_m}{N^2 D^2}\right)_m = \left(\frac{H_m}{N^2 D^2}\right)_p \tag{25.46}$$

$$\left(\frac{H_m}{N^2 D^2}\right)_1 = \left(\frac{H_m}{N^2 D^2}\right)_2 \quad \text{(For geometrically similar pumps)} \tag{25.46a}$$

3. **Power coefficient**

$$\eta_o = \frac{\rho_w g Q H_m}{1000P} \Rightarrow P = \frac{\rho_w g Q H_m}{1000\eta_o}$$

Thus $$P \propto Q \times H_m \propto D^3 N \times N^2 D^2 \quad [\because \rho_w, g \text{ and } \eta_o = \text{Constant}]$$

or $$P \propto N^3 D^5$$

Thus $$\boxed{\frac{P}{N^3 D^5} = \text{Constant}}$$

The parameter $[P/(N^3D^5)]$ called the power coefficient of the model and prototype is equal as given below.

$$\left(\frac{P}{N^3 D^5}\right)_m = \left(\frac{P}{N^3 D^5}\right)_p \tag{25.47}$$

$$\left(\frac{P}{N^3 D^5}\right)_1 = \left(\frac{P}{N^3 D^5}\right)_2 \quad \text{(For geometrically similar pumps)} \tag{25.47a}$$

For determining the performance of one particular pump, the capacity, head and power coefficients are simplified as follows.

$$\left(\frac{Q}{N}\right) = \text{Constant or } \left(\frac{Q}{N}\right)_1 = \left(\frac{Q}{N}\right)_2 \tag{25.48}$$

$$\left(\frac{H_m}{N^2}\right) = \text{Constant or } \left(\frac{H_m}{N^2}\right)_1 = \left(\frac{H_m}{N^2}\right)_2 \tag{25.49}$$

$$\left(\frac{P}{N^3}\right) = \text{Constant or } \left(\frac{P}{N^3}\right)_1 = \left(\frac{P}{N^3}\right)_2 \tag{25.50}$$

4. Specific speed of model = Specific speed of prototype

$$(N_s)_m = (N_s)_p$$

$$\left(\frac{N\sqrt{Q}}{H_m^{3/4}}\right)_m = \left(\frac{N\sqrt{Q}}{H_m^{3/4}}\right)_p \tag{25.51}$$

Example 25.14 Determine the specific speed of a centrifugal pump which delivers water at the rate of 2 m^3/s under a head of 20 m while running at 3500 rpm and operating at a maximum efficiency of 85%. Also determine the discharge, head and power input to the pump at the speed of 2500 rpm assuming that the efficiency remains constant at all the speeds.

Solution

Let $Q_1 = 2\ m^3/s,\ H_m = H_1 = 20\ m,\ N_1 = 3500\ rpm,\ \eta_o = 0.85$ and $N_2 = 2500$ rpm.

$$N_s = \frac{N_1\sqrt{Q_1}}{H_m^{3/4}} = \frac{3500\times\sqrt{2}}{20^{3/4}} = \mathbf{523.372\ rpm}$$

$$P_1 = \frac{\rho_w g Q_1 H_1}{1000\eta_o} = \frac{1000\times 9.81\times 2\times 20}{1000\times 0.85} = 461.647\ kW$$

Since $$\left(\frac{Q}{N}\right)_1 = \left(\frac{Q}{N}\right)_2$$

$$\therefore Q_2 = \frac{Q_1 N_2}{N_1} = \frac{2\times 2500}{3500} = \mathbf{1.4286\ m^3/s}$$

Since $$\left(\frac{H}{N^2}\right)_1 = \left(\frac{H}{N^2}\right)_2$$

$$\therefore H_2 = \frac{H_1 N_2^{\ 2}}{N_1^{\ 2}} = \frac{20\times 2500^2}{3500^2} = \mathbf{10.204\ m}$$

Since $$\left(\frac{P}{N^3}\right)_1 = \left(\frac{P}{N^3}\right)_2$$

$$\therefore P_2 = \frac{P_1 N_2^{\ 3}}{N_1^{\ 3}} = \frac{461.647\times 2500^3}{3500^3} = \mathbf{168.239\ kW}$$

Example 25.15 A five stage centrifugal pump delivers water at the rate of 6.5 m^3 per minute against a net pressure rise of 4500 kN/m^2. Determine the specific speed of the pump if it runs at 1500 rpm. Also comment upon the type of the impeller.

Solution

Let $n = 5$, $Q = 6.5\ m^3/min = 0.1083\ m^3/s$, $p = 4500\ kN/m^2$ and $N = 1500$ rpm. Let H_m be the head developed per stage.

$$H_t = \frac{p}{\rho_w g} = \frac{4500\times 10^3}{1000\times 9.81} = 458.715\ \text{m of water}$$

$$H_m = \frac{H_t}{n} = \frac{458.715}{5} = 91.743\ \text{m of water}$$

$$N_s = \frac{N\sqrt{Q}}{H_m^{3/4}} = \frac{1500\times\sqrt{0.1083}}{91.743^{3/4}} = \mathbf{16.652}$$

Since the specific speed lies in the range of 10 to 30, the pump is a slow speed radial flow impeller.

Example 25.16 A multistage centrifugal pump lifts water through a total head of 100 m. It delivers water at the rate of 0.25 m^3 per second while running at 900 rpm. Find the number of stages required when specific speed of each stage is 30. Also comment upon the arrangement of impellers.

Solution

Let $H_t = 100$ m, $Q = 0.25$ m^3/s, $N = 900$ rpm and $N_s = 30$. Let H_m be the head developed per stage and n be the number of stages.

Since
$$N_s = \frac{N\sqrt{Q}}{H_m^{3/4}}$$

$$\therefore H_m = \left(\frac{N\sqrt{Q}}{N_s}\right)^{4/3} = \left(\frac{900\times\sqrt{0.25}}{30}\right)^{4/3} = 36.993 \text{ m}$$

$$n = \frac{H_t}{H_m} = \frac{100}{36.993} = 2.7 \approx \mathbf{3}$$

The total head is more than the head developed by one pump, so the pumps are to be connected in series.

Example 25.17 Two geometrically similar pumps run at the same speed of 1200 rpm. One pump with impeller diameter of 0.4 m delivers water at the rate of 0.03 m^3/s against the head of 20 m. Determine the diameter and head delivered by the other pump if it has to deliver 50% discharge of the first pump.

Solution

Let $N_1 = N_2 = 1200$ rpm, $D_1 = 0.4$ m, $Q_1 = 0.03$ m^3/s, $H_1 = 20$ m and $Q_2 = 0.5Q_1$.

Since
$$\left(\frac{Q}{ND^3}\right)_1 = \left(\frac{Q}{ND^3}\right)_2$$

$$\therefore D_2 = \left(\frac{Q_2}{Q_1}\right)^{1/3} \times D_1 = 0.5^{1/3} \times 0.4 = \mathbf{0.3175\ m}$$

Since
$$\left(\frac{H}{N^2D^2}\right)_1 = \left(\frac{H}{N^2D^2}\right)_2$$

$$\therefore H_2 = \left(\frac{D_2}{D_1}\right)^2 \times H_1 = \left(\frac{0.3175}{0.4}\right)^2 \times 20 = \mathbf{12.6\ m}$$

25.18 ❒ PERFORMANCE CHARACTERISTICS OF CENTRIFUGAL PUMPS

A pump provides maximum efficiency when it operates at designed values of speed, discharge and head. In actual practice, a pump has to operate at different conditions than the designed ones under which the behaviour of the pump may be different. In order to predict the behaviour and performance of a pump under varying conditions, various tests are performed and the results of the tests are plotted in the form of curves. These curves are known as the characteristic curves of the pump. The important characteristic curves of a pump are (i) main characteristic curves, (ii) operating characteristic curves, (iii) constant efficiency or Muschel curves, and (iv) constant head and constant discharge characteristic curves.

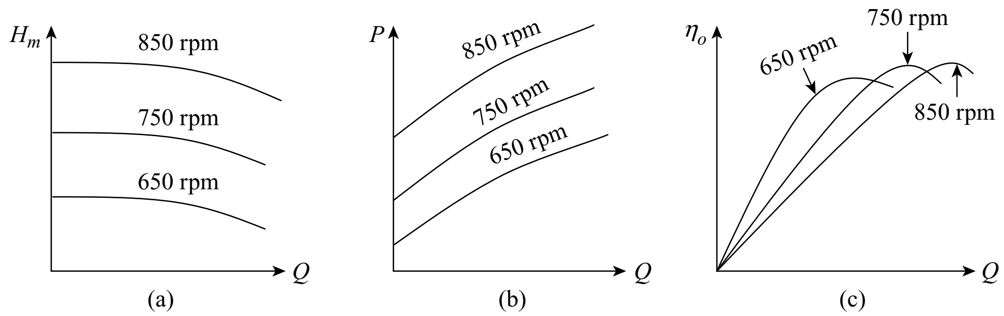


Figure 25.19 *Main characteristic curves*

25.18.1 Main Characteristic Curves

In order to obtain the test data for main characteristic curves, the pump is operated at a constant speed and the discharge is varied over the required range by means of a valve. For each value of discharge (Q), the corresponding values of head (H_m) and shaft power (P) are measured and the overall efficiency (η_o) of the pump is calculated. This procedure is repeated by keeping the speed constant at different values. The curves are then plotted for H_m, P and η_o versus Q for different speeds of the pump. These curves represent the main characteristics of a pump, which indicate the performance of a pump at different speed. The main characteristics of a pump for an arbitrary set of speeds varying from 650 rpm to 850 rpm are shown in Figure 25.19.

From P versus Q curves (Figure 25.19b), it can be seen that as the discharge increases, the power input also increases. These curves do not pass through the origin as some power is used in overcoming the mechanical losses. However, η_o versus Q curves (Figure 25.19c) pass through the origin as efficiency is zero when there is no discharge.

25.18.2 Operating Characteristic Curves

During the operation, a pump is generally required to run at constant speed, which is its designed speed (same as the speed of the driving motor). In order to obtain maximum efficiency, a pump is required to run at its designed speed. The values of the head and discharge corresponding to the maximum efficiency are known as the normal (or designed) head and discharge of the pump. A particular set of main characteristic curves corresponding to the design speed are called the operating characteristic curves (Figure 25.20). These curves give an idea about the size of the prime mover required to drive the pump. The head corresponding to zero or no discharge is known as the shut-off head of the pump. The input power curve is away from the origin, since even at zero discharge some power is required to overcome the mechanical losses.

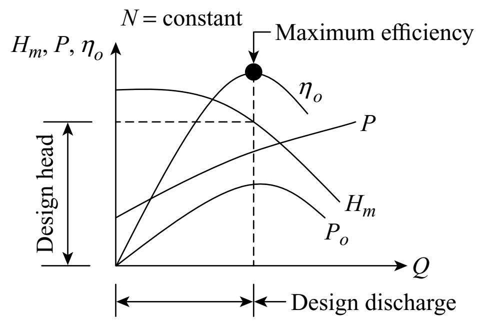


Figure 25.20 *Operating characteristic curves*

25.18.3 Constant Efficiency Curves (Muschel Curves)

The constant efficiency or iso-efficiency curves are also known as Muschel curves. These curves can be drawn from H_m versus Q and η_o versus Q curves. For a given efficiency, there will be two values of discharge at a constant speed and corresponding to these discharge values, there will be two different values of head. For a particular efficiency, a horizontal line is drawn which intersects the curves for different pump speeds on the η_o versus Q curves and two values of discharge are obtained for each speed. These values are transferred to H_m versus Q curves for the corresponding speeds. The points corresponding to the same efficiency are then joined by a smooth curve which represents the iso-efficiency curve. Similarly for other values of efficiency, the points are obtained and projected. The points corresponding to the same efficiency are smoothly joined to obtain iso-efficiency curves. From these curves, the line of maximum efficiency is obtained by joining the peak points of various iso-efficiency curves as shown in Figure 25.21. These curves help in locating the regions where the pump operates with maximum efficiency.

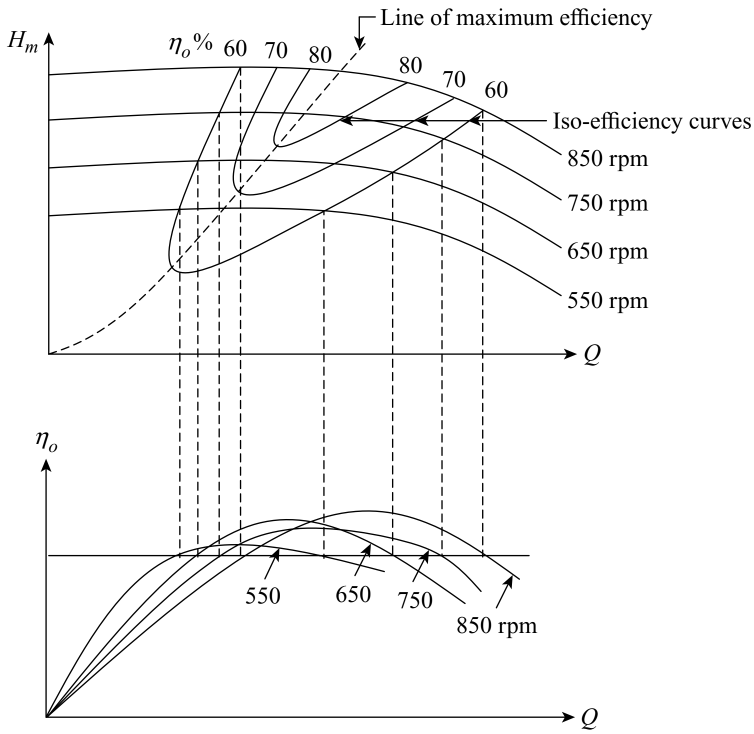


Figure 25.21 *Constant efficiency curves*

25.18.4 Constant Head and Constant Discharge Characteristics

These curves help in determining the performance of variable speed pump for which speed is constantly varying. The following curves may be plotted as shown in Figure 25.22.

1. Q **versus** N **curves:** When manometric head (H_m) is maintained constant and speed varies, the discharge also varies. A plot between Q and N can be plotted which helps in determining the speed required to give varying discharge at a constant pressure head. From Equation (25.48), it can be observed that $Q \propto N$ and thus, Q versus N curve is linear as shown in Figure 25.22.
2. H_m **versus** N **curves:** When discharge is maintained constant and speed varies, the manometric head also varies. A plot between N and H_m can be plotted which helps in determining the speed required to give certain amount of discharge at different pressure heads. From Equation (25.49), it is observed that $H_m \propto N^2$ and thus, H_m versus N plot is a parabolic curve as shown in Figure 25.22.

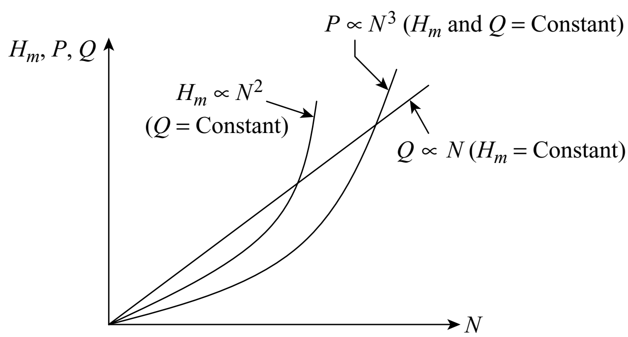


Figure 25.22 *Constant head, discharge, and power curves*

3. ***P* versus *N* curves:** When manometric head and discharge are maintained constant and speed varies, the power also varies. From Equation (25.50), it is observed that $P \propto N^3$ and thus, P versus N plot is a cubic curve as shown in Figure 25.22.

25.19 ❐ MAXIMUM SUCTION LIFT (OR SUCTION HEIGHT)

Consider a centrifugal pump which lifts liquid from a sump open to atmosphere as shown in Figure 25.23. Let h_s be the suction height (or lift) which is the vertical distance between the centre line of the pump and the free liquid surface of the sump and V_s be the velocity of liquid in the suction pipe.

Applying Bernoulli's equation to the free surface of liquid and centre of the impeller (i.e., section 1–1), and taking the free surface of liquid as datum, we get the below expression.

$$\frac{p_a}{\rho g}+\frac{V_a^2}{2g}+Z_a=\frac{p_1}{\rho g}+\frac{V_1^2}{2g}+Z_1+h_l$$

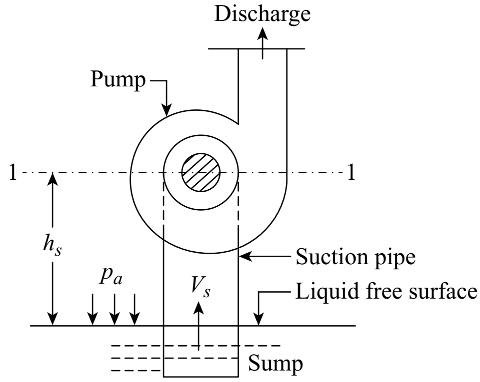


Figure 25.23

Here, p_a is the atmospheric pressure, V_a is the velocity of liquid at the free surface = 0, Z_a is the height of free surface from datum line = 0, p_1 is the absolute pressure at the inlet of the pump (section 1–1), $V_1 = V_s$ is the velocity of liquid through the suction pipe, $Z_1 = h_s$ is the height of inlet of the pump from datum line and $h_l = h_{fs}$ is the loss of head in the suction pipe. Therefore, we get the following expression.

$$\frac{p_a}{\rho g}=\frac{p_1}{\rho g}+\frac{V_s^2}{2g}+h_s+h_{fs} \tag{25.52}$$

To avoid cavitation, the pressure at the inlet of the pump (p_1) should not fall below the vapour pressure of the liquid (p_v). Thus, for a limiting case, taking $p_1 = p_v$, we get the following expression.

$$\frac{p_a}{\rho g}=\frac{p_v}{\rho g}+\frac{V_s^2}{2g}+h_s+h_{fs}$$

or

$$H_a = H_v+\frac{V_s^2}{2g}+h_s+h_{fs}$$

Here, $H_a = p_a/(\rho g)$ and $H_v = p_v/(\rho g)$ are the atmospheric and vapour pressure heads in terms of meters of liquid, respectively. Therefore, the expression for maximum suction lift of a centrifugal pump is given below.

$$h_s = H_a - H_v - \frac{V_s^2}{2g} - h_{fs} \tag{25.53}$$

Thus, the suction height of any pump should not be greater than that given by Equation (25.53). A greater value of suction lift may result in a rapid vaporization of the liquid due to the reduction of pressure, which may ultimately lead to cavitation and there will not be any flow of the liquid. Generally, the suction lift of centrifugal pumps is restricted to about 6 m to 8 m.

25.20 ❐ NET POSITIVE SUCTION HEAD (NPSH)

Net positive suction head (NPSH) is defined as the available suction head at the pump inlet above the vapour pressure head corresponding to the temperature of the liquid pumped. The term NPSH is very commonly used in pump industry. Generally, the minimum suction conditions for the pumps are specified in terms of NPSH. The NPSH is given as the absolute pressure head at the inlet to the pump minus the vapour pressure head in absolute units plus the velocity head. The mathematical expression for NPSH is given below.

NPSH = Abssolute pressure head at inlet – Vapour pressure head in absolute units + Velocity head

Thus
$$\text{NPSH} = \frac{p_1}{\rho g} - \frac{p_v}{\rho g} + \frac{V_s^2}{2g} \tag{25.54}$$

But
$$\frac{p_1}{\rho g} = \frac{p_a}{\rho g} - \left(\frac{V_s^2}{2g} + h_s + h_{fs}\right) \quad \text{[From Equation (25.52)]}$$

Thus
$$\text{NPSH} = \frac{p_a}{\rho g} - \left(\frac{V_s^2}{2g} + h_s + h_{fs}\right) - \frac{p_v}{\rho g} + \frac{V_s^2}{2g}$$

$$\text{NPSH} = \frac{p_a}{\rho g} - \frac{p_v}{\rho g} - h_s - h_{fs} = H_a - H_v - h_s - h_{fs}$$

$$\boxed{\therefore \text{NPSH} = (H_a - h_s - h_{fs}) - H_v} \tag{25.55}$$

The right hand side of Equation (25.55) also represents the total suction head H_s and its expression is given below.

$$H_s = (H_a - h_s - h_{fs}) - H_v$$

$$\therefore \text{NPSH} = H_s \tag{25.55a}$$

Therefore, NPSH may also be defined as the head required to make the liquid flow through the suction pipe to the impeller.

For any pump installation, a distinction is made between the required NPSH and the available NPSH. The available NPSH for a pump is calculated from Equation (25.55). The required NPSH varies with the pump design, speed of the pump and its capacity. The required NPSH is usually provided by the manufacturer of the pump which may also be determined experimentally. In order to have cavitation free operation of a centrifugal pump, the available NPSH should always be greater than the required NPSH.

25.21 ❒ CAVITATION IN CENTRIFUGAL PUMPS

When the pressure at the suction side of the pump impeller falls below the vapour pressure of the liquid, some of the liquid vaporizes and bubbles of the vapour is carried along with the liquid. These vapour bubbles condense and collapse rapidly on reaching to high pressure zone (near the impeller exit). This process continues and creates high pressure which may damage the impeller. This phenomenon is called cavitation which is highly undesirable. At the inlet of the impeller, the pressure remains lowest on the underside of vanes from where cavitation commences and the vanes tips at impeller exit are the most common site for cavitation attack. The cavitation can be noticed by a sudden drop in efficiency and head.

To indicate whether cavitation will occur, the Thoma's cavitation factor (σ) is used and the expression is given below.

$$\boxed{\sigma = \frac{H_a - H_v - h_s - h_{fs}}{H_m} = \frac{\text{NPSH}}{H_m} = \frac{H_s}{H_m}} \tag{25.56}$$

When the value of σ is less than the critical value (σ_c), then cavitation occurs in the pumps. The value of σ_c depends on the specific speed (N_s) of a pump. The value of σ_c can be determined by the following relation.

$$\boxed{\sigma_c = 1.03 \times 10^{-3} (N_s)^{4/3}} \tag{25.57}$$

Suction specific speed Suction specific speed (N_{su}) is a cavitation parameter which is also used to know about the occurrence of cavitation. It is obtained by replacing the manometric head in the expression of specific speed of pump [i.e., Equation (25.42)] by the total suction head (H_s). Therefore, the expression for suction specific speed is given below.

$$N_{su} = \frac{N\sqrt{Q}}{H_s^{3/4}} \tag{25.58}$$

By combining Equations (25.42), (25.56) and (25.58), we get the below expression.

$$\sigma = \left(\frac{N_s}{N_{su}}\right)^{4/3} \tag{25.59}$$

Generally, in centrifugal pumps, for cavitation free flow, the limiting value of the suction speed is 175.

The cavitation in pumps can be avoided by the following factors

1. By reducing the suction lift that increases the value of σ which ensures sufficient availability of NPSH.
2. By reducing the velocity in the suction pipe.
3. By avoiding the bends.
4. By reducing h_{fs} in suction pipe.
5. By selecting the pump of lower specific speed.

Cavitation is undesirable due to the following harmful effects

1. A large number of vapour bubbles formed suddenly collapse in a high pressure region which causes the rush of surrounding liquid and results in shock, noise and vibration. This phenomenon is called water hammer.
2. The continuous water hammering action of collapsing bubbles causes pitting and erosion of the surface.
3. The water hammer causes fatigue of the metal parts and reduces their lifetime.
4. Cavitation causes sudden drop in head and efficiency.

25.22 ❒ TROUBLES IN CENTRIFUGAL PUMPS AND THEIR CAUSES

Some of the common troubles with their causes in centrifugal pumps are given below.

1. Pump fails to start pumping when (i) pump is not properly primed, (ii) suction height is too high, (iii) total static head is much higher than the designed value, (iv) wrong direction of the pump impeller, (v) impeller, strainer or suction line may be clogged and (vi) low impeller speed.
2. Pump delivers less liquid than the desired quantity when (i) leakage occurs in pump, (ii) foot valve is not submerged fully in the liquid or it is of smaller size and (iii) impeller is damaged or bearings worn out.
3. Pump does not develop enough pressure when (i) air in liquid, (ii) speed is too low, (iii) wrong direction of the pump impeller and (iv) some of the parts are damaged due to wear and tear.
4. Pump works for a while and stops pumping when (i) the pump is not properly primed or there is leakage in the suction line, (ii) air pockets in the suction line and (iii) suction lift is too high.
5. Pump consumes much power, noisy in operation and has low efficiency when (i) cavitation occurs, (ii) head may be too low and pump delivers too much liquid, (iii) pump may be operating in wrong direction, (iv) liquid may have too high specific gravity, (v) suction lift is too high and (vi) rotating parts are loose, impeller may rubbing on casing, bearing worn out and misalignment of shaft.

Example 25.18 Determine the height from water surface a centrifugal pump should be installed to avoid cavitation when atmospheric pressure (abs) is 101.325 kPa, vapour pressure is 2.5 kPa (abs), the inlet and other losses in suction pipe are 1.5 m, effective head of the pump is 50 m and cavitation factor is 0.115.

Solution

Let $p_a = 101.325$ kPa, $p_v = 2.5$ kPa, $h_{fs} = 1.5$ m, $H_m = 50$ m and $\sigma = 0.115$.

Since
$$\sigma = \frac{H_a - H_v - h_s - h_{fs}}{H_m}$$

$$\therefore h_s = H_a - H_v - h_{fs} - \sigma H_m$$

or
$$h_s = \frac{p_a}{\rho_w g} - \frac{p_v}{\rho_w g} - h_{fs} - \sigma H_m$$

$$\therefore h_s = \frac{101.325\times10^3}{1000\times9.81} - \frac{2.5\times10^3}{1000\times9.81} - 1.5 - 0.115\times50 = \mathbf{2.824\ m}$$

Example 25.19 The following particulars are given for a centrifugal pump, such as discharge = 0.15 m^3/s of water, manometric head = 35 m, speed of the pump = 1150 rpm, atmospheric pressure (abs) = 1.01325 bar, vapour pressure at the temperature of water pumped = 3.5 kPa (abs), inlet and other losses in suction pipe = 0.25 m of water. Determine minimum NPSH and maximum allowable height of the pump from the free surface of water in the pump.

Solution

Let $Q = 0.15\ \text{m}^3/\text{s}$, $H_m = 35$ m, $N = 1150$ rpm, $p_a = 1.01325$ bar, $p_v = 3.5$ kPa and $h_{fs} = 0.25$ m.

Since
$$\sigma_c = \frac{(\text{NPSH})_{\min}}{H_m}$$

Thus
$$(\text{NPSH})_{\min} = \sigma_c H_m$$

But
$$\sigma_c = 1.03\times10^{-3}(N_s)^{4/3}$$

Thus
$$(\text{NPSH})_{\min} = 1.03\times10^{-3}(N_s)^{4/3} H_m = 1.03\times10^{-3}\left(\frac{N\sqrt{Q}}{H_m^{3/4}}\right)^{4/3} H_m$$

$$\therefore (\text{NPSH})_{\min} = 1.03\times10^{-3}\left(\frac{1150\times\sqrt{0.15}}{35^{3/4}}\right)^{4/3}\times35 = \mathbf{3.503\ m}$$

The maximum suction height (h_s) can be obtained when NPSH is minimum and, thus, we get the below expression.

$$(\text{NPSH})_{\min} = (H_a - h_s - h_{fs}) - H_v$$

Thus
$$h_s = H_a - H_v - h_{fs} - (\text{NPSH})_{\min}$$

or
$$h_s = \frac{p_a}{\rho_w g} - \frac{p_v}{\rho_w g} - h_{fs} - (\text{NPSH})_{\min}$$

$$\therefore h_s = \frac{1.01325\times10^5}{1000\times9.81} - \frac{3.5\times10^3}{1000\times9.81} - 0.25 - 3.503 = \mathbf{6.22\ m}$$

25.23 ❒ AXIAL FLOW PUMP

The axial flow pumps do not utilize centrifugal forces. The impeller blades behave like the wing of an airplane producing lift by changing the momentum of the fluid as they rotate. These pumps are often known as propeller pumps because its impeller somewhat resembles a marine propeller. The impeller comprises of a central hub in which a number of vanes (2 to 6) are mounted. The impeller rotates inside a casing. Generally, for the sake of compactness, the axial flow pumps are designed to operate with vertical shaft. The guide vane sets are provided at the inlet and the outlet of the impeller. Figure 25.24 shows a schematic view of an axial flow pump. The axial flow pumps are the converse of propeller or Kaplan turbines.

The flow ratio for axial flow pumps varies from 0.25 to 0.6 and the speed ratio varies from 2 to 2.7. The hub to tip ratio lies in the range of 0.3 to 0.6. These pumps have high specific speed (100 to 450) and are used where large discharge at low delivery head (under 12 m) is required. They are suitable for irrigation, drainage, sewage, flood control, purification, etc.

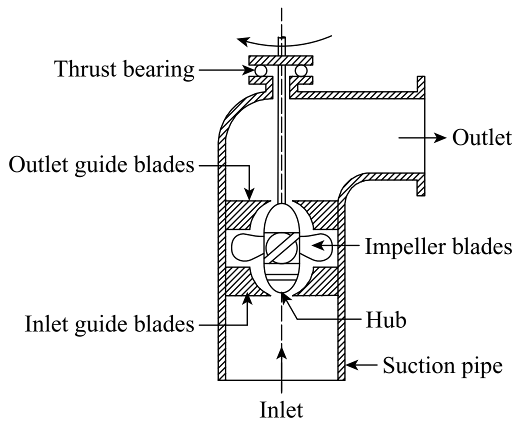


Figure 25.24 *Axial flow pump*

25.24 ❒ DEEP WELL (VERTICAL TURBINE PUMP) AND SUBMERSIBLE PUMPS

Deep well pumps are generally multistage centrifugal pumps assembled in series and are keyed to the lower end of a vertical shaft which is further coupled to a line shaft. The line shaft refers to sections of shaft between the impeller shaft and the inner shaft passing through the driven hollow shaft. The number of stages depends on the head required. All the impellers and at least three metres of suction pipe with a strainer and foot valve at the end are placed below the water level. This is the sole reason for such pumps to be known as deep well pumps or borehole pumps. The pump is connected to an electric motor usually placed above the ground level. A schematic view of deep well pump is shown in Figure 25.25(a). The water conducts to the surface through the rising main pipe which connects the impeller with the outlet. In these pumps, only closed or semi-open types of impellers are used. The pump shaft is aligned with bronze bearings placed at suitable interval along the shaft which prevents vibration and whip. The bearings may be lubricated either by oil or water.

Deep well pumps belong to the category of rotodynamic pumps. These pumps are used to lift drinking water which is available at a depth of about 100 metres or more below the ground. The action of these pumps is reverse of a reaction turbine and thus, it is also known as turbine pumps.

Vertical turbine pumps driven by submersible motors fitted at the bottom of the pumps are known as submersible pumps. The motor is completely insulated, enclosed and oil filled. The pump suction is through a perforated strainer (or inlet screen) located between the motor and first stage impeller as shown in Figure 25.25(b). There is no suction pipe and also there is no shaft above the pump. The pump unit is supported by the discharge pipe only. Submersible pumps are self-primed, if they do not run dry. The power is delivered through a heavily insulated electricity cable.

Submersible centrifugal pumps are used for residential, commercial, municipal and industrial water extraction, water wells and in oil wells. The range of depth for submersible pumps varies from 7 m to 200 m or more. It is able to produce efficiency in the range of 40% to 70%. The main advantage of this type of pump is that it prevents cavitation, a problem associated with a high elevation difference between pump and the fluid surface. The main limitations of submersible centrifugal pumps are its price, the need to maintain a reliable supply of electricity and high level of technology involved.

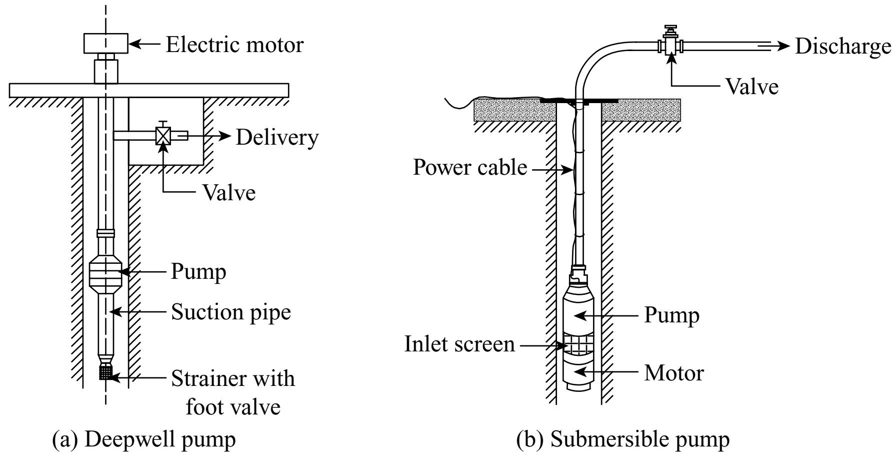


Figure 25.25

Summary

1. **Work done by impeller per second per unit weight of water:** $w = (V_{wo}u_o)/g$
2. **Euler's equation:**

$$w = H_e = \frac{V_o^2 - V_i^2}{2g} + \frac{u_o^2 - u_i^2}{2g} + \frac{V_{ri}^2 - V_{ro}^2}{2g}$$

3. **Pressure rise in a centrifugal pump:**

$$\frac{p_o - p_i}{\rho_w g} = \frac{V_{fi}^2 + u_o^2 - V_{fo}^2 \text{cosec}^2\phi}{2g}$$

4. **Manometric head (H_m):** Head against which a centrifugal pump has to work.
   (i) $H_m = V_{wo}u_o/g$,
   (ii) $H_m = h_s + h_d + h_{fs} + h_{fd} + V_d^2/(2g)$
   (iii) $H_m = \left[\frac{p_o}{\rho_w g} + \frac{V_o^2}{2g} + z_o\right] - \left[\frac{p_i}{\rho_w g} + \frac{V_i^2}{2g} + z_i\right]$
5. Shaft power (P) is the power supplied by the motor to the pump shaft.
6. **Impeller power:** $P_{im} = \frac{\rho_w Q V_{wo} u_o}{1000}$ kW
7. **Power output:**

$$P_o = \frac{\text{Weight of water lifted per second} \times H_m}{1000} = \frac{WH_m}{1000} = \frac{\rho_w g Q H_m}{1000} \text{ kW}$$

8. **Manometric efficiency:** $\eta_{man} = \frac{P_o}{P_{im}} = \frac{gH_m}{V_{wo}u_o}$
9. **Mechanical efficiency:**

$$\eta_m = \frac{\text{Power at the impeller}}{\text{Power at the shaft}} = \frac{P_{im}}{P} = \frac{\rho_w Q V_{wo} u_o}{1000P}$$

10. **Volumetric efficiency:** $\eta_v = \frac{\text{Actual discharge}}{\text{Total discharge}} = \frac{Q}{Q+q}$, q is the leakage per second.
11. **Overall efficiency:** $\eta_o = \frac{P_o}{P} = \frac{\rho_w g Q H_m}{1000P}$ also $\eta_o = \eta_{\text{man}}\eta_m\eta_v$
12. **Minimum starting speed:** $N = \frac{120\,\eta_{man} V_{wo} D_o}{\pi(D_o^2 - D_i^2)}$
13. **Specific speed of a centrifugal pump:** $N_s = (N\sqrt{Q})/H_m^{3/4}$
14. For complete similarity to exist between the model and the prototype pumps, the capacity coefficient $[Q/(ND^3)]$, head coefficient $[H_m/(N^2D^2)]$ and power coefficient $[P/(N^3D^5)]$ of the model and prototype are equal.
15. Maximum suction lift of a centrifugal pump is $h_s = H_a - H_v - V_s^2/(2g) - h_{fs}$.
16. $\text{NPSH} = (H_a - h_s - h_{fs}) - H_v$
17. When the pressure at the suction side of the pump impeller falls below the vapour pressure of the liquid, some of the liquid vaporizes and bubbles of the vapour is carried along with the liquid to high pressure zone (near the impeller exit), where these vapour bubbles condense and collapse rapidly to create high pressure. This phenomenon is called cavitation.

Multiple-choice Questions

1. Generally, the vanes of a centrifugal pump are
(a) Twisted.
(b) Curved forward.
(c) Radial.
(d) Curved backward.

2. The series operation of a pump results in
(a) Reduced power.
(b) Higher discharge.
(c) Low speed.
(d) High head.

3. Which of the following two relations are necessary for homologous pumps, when Q is the discharge, N is the speed, P is the power, D is the diameter and C is the constant?
(a) $[Q/(ND^3)] = C$ and $[H_m/(N^2D^2)] = C$.
(b) $[H_m/(ND^3)] = C$ and $[Q/(N^2D^2)] = C$.
(c) $(N\sqrt{Q})/H_m{}^{1.5} = C$ and $(N\sqrt{P})/H_m{}^{3/4} = C$.
(d) None of the above.

4. For a given centrifugal pump, which of the following one is correct when Q is the discharge, N is the speed, P is the power and H is the head?
(a) $Q \alpha N$.
(b) $Q \alpha N^2$.
(c) $H_m \alpha (1/N^2)$.
(d) $P \alpha N^5$.

5. A pump with specific speed of 100 to 450 indicates that pump is
(a) Radial flow.
(b) Mixed flow.
(c) Axial flow.
(d) None of the above.

6. For a centrifugal pump,
(a) Discharge varies inversely as the speed.
(b) Pressure gain in the diffusion section is more than that in the impeller.
(c) Slip factor is minimum when the vanes are radial.
(d) None of the above.

7. When the diameter of a centrifugal pump impeller is doubled but the discharge is to remain same, then the head needs to be reduced by
(a) 4 times.
(b) 16 times.
(c) 8 times.
(d) None of the above.

8. If discharge is kept constant, the variation of power (P) with speed (N) and manometric head (H_m) with speed (N) is respectively given as
(a) $P \propto N$ and $H_m \propto N$.
(b) $P \propto N^2$ and $H_m \propto N^3$.
(c) $P \propto N^3$ and $H_m \propto N^2$.
(d) $P \propto N^5$ and $H_m \propto (1/N)$.

9. An impeller with backward curved vanes
(a) Has a falling head characteristic.
(b) Has a rising head characteristic.
(c) Has a constant head characteristic.
(d) None of the above.

10. A fast centrifugal pump impeller has
(a) Propeller type blade.
(b) Radial blades.
(c) Forward facing blades.
(d) Backward facing blades.

11. In a centrifugal pump, the inlet angle is designed to have
(a) Absolute velocity in the radial direction.
(b) Relative velocity in the radial direction.
(c) Flow velocity is zero.
(d) Tangential velocity is zero.

12. Which of the following type of impeller is used in centrifugal pump to deal with mud?
(a) Two sides shrouded.
(b) One sided shrouded.
(c) Double suction.
(d) Open.

13. The flow in the volute casing outside the running impeller is
(a) Axial flow.
(b) Free vortex flow.
(c) Radial flow.
(d) Forced vortex flow.

Review Questions

1. Describe the principle, constructional and working details of a centrifugal pump.
2. Derive Euler's equation applied to centrifugal pumps.
3. Derive an expression for pressure rise in the impeller of a centrifugal pump.
4. Define heads, losses, power and efficiencies associated with centrifugal pumps.

5. Discuss the effect of number of vanes of impeller of a centrifugal pump on head and efficiency. Also define slip factor.
6. Derive an expression for minimum starting speed of a centrifugal pump to maintain continuous discharge of water.
7. Define specific speed of a centrifugal pump and derive its expression.
8. Explain the properties for a centrifugal pump, such as (i) main characteristics curves, (ii) operating characteristics curves and (iii) Muschel curves.
9. Derive an expression for maximum possible suction lift.
10. Define NPSH of a centrifugal pump? How is it related to cavitation in pumps?
11. What is cavitation its causes? How it can be prevented in centrifugal pumps?
12. What is priming? Why is it necessary? Also briefly discuss the priming devices.

Problems

1. The following data are given for a centrifugal pump, such as outer diameter = 500 mm, internal diameter = 250 mm, speed = 1250 rpm, manometric head = 45 m, constant velocity of flow through impeller = 2.75 m/s, vanes exit angle = 30°, impeller width at outlet = 0.05 m. Calculate (i) vane angle at the inlet, (ii) work done by impeller in water per second and (iii) manometric efficiency.
[**Ans.** 9.54°, 197.52 kNm/s, 48.25%]
2. The outer diameter of an impeller of a centrifugal pump is 0.4 m and outlet width is 0.05 m. The pump is working against a total head of 15 m and running at 800 rpm. If its manometric efficiency is 0.8 and the vanes angle at the outlet is 40°, then find (i) velocity of flow at the outlet, (ii) velocity of water leaving the vane and angle made by the absolute velocity at the outlet with the direction of motion at the outlet and (iii) discharge of the pump.
[**Ans.** 4.84 m/s, 11.99 m/s, 23.79°, 0.3041 m^3/s]
3. The internal and external diameters of the impeller of a centrifugal pump are 0.5 m and 1 m, respectively. The velocity of flow is given constant through the impeller of the pump. The vane angles at the inlet and outlet are 30° and 45°, respectively. If the impeller is running at 1200 rpm and flow through the pump is 0.25 m^3/s, then determine the minimum power required to run the pump.
[**Ans.** 701.32 kW]
4. A centrifugal pump has inlet and outlet diameters as 0.2 m and 0.4 m, respectively. The vane angles at the inlet and outlet are 0.4 radians and 0.2 radians, respectively. The width of the impeller at the inlet and outlet is same and equal to 0.05 m. If the impeller is running at 109.9 radians per second, then find (i) the discharge and (ii) head developed by the pump. Assume shockless entry to the pump.
[**Ans.** 0.1461 m^3/s, 23.55 m]
5. An impeller of a centrifugal pump has inlet and outlet diameters as 0.2 m and 0.5 m, respectively. The exit vane angle is 30°. The impeller is running at 1000 rpm. Take radial flow through the impeller and flow velocity as 3 m/s. Find (i) the inlet vane angle, (ii) outlet angle of water and (iii) power required to run the impeller, if the mechanical efficiency is 90% and water flow through the impeller is 1.65 m^3 per minute.
[**Ans.** 15.99°, 8.14°, 16.783 kW]
6. An impeller of a centrifugal pump has inlet and outlet diameters as 0.15 m and 0.3 m, respectively. The both inlet and exit vane angles are 30°. The water flow rate is 0.05 m^3/s and the impeller inlet area is 0.025 m^2. Take radial flow through the impeller and flow velocity as constant. Find (i) the speed of the impeller and (ii) torque produced.
[**Ans.** 440.54 rpm, 25.875 Nm]
7. The following particulars are related to a centrifugal pump, such as impeller diameter = 250 mm, width of impeller at its periphery = 50 mm, outlet vane angle = 60°, discharge = 0.25 m^3/s, speed of the impeller = 900 rpm, mechanical efficiency = 0.9, and hydraulic efficiency = 0.75. Determine (i) the speed and direction of water as it leaves the impeller, (ii) torque exerted by the impeller, (iii) shaft power required and (iv) lift of the pump.
[**Ans.** 10.3 m/s, 38.18°, 253.125 Nm, 26.511 kW, 7.3 m]
8. A centrifugal pump running at 1000 rpm has internal and external diameters as 0.25 m and 0.5 m, respectively. The angle of backward curved vanes at the outlet is 30°. The radial velocity of flow is 2 m/s and it remains constant throughout the impeller. Find the angle of vanes at the inlet, the velocity and direction of water at the outlet and rise in pressure head in the impeller.
[**Ans.** 8.69°, 5.03°, 60.6 m]
9. The internal and external diameters of an impeller of a centrifugal pump are 0.25 m and 0.5 m, respectively. The discharge through the pump is 0.05 m^3/s. It runs at 950 rpm and the velocity of flow is constant and equal to 2.5 m/s. The diameters of the suction and delivery pipes are 16 cm and 10 cm, respectively. The suction and delivery heads are 7 m (abs) and 32 m (abs) of water, respectively. The outlet vane angle is 45° and power required to drive the pump is 20.5 kW. Find (i) the vane angle of the impeller at the inlet, (ii) overall efficiency of the pump and (iii) manometric efficiency of the pump.
[**Ans.** 11.37°, 64%, 47.16%]

10. The following particulars are related to a centrifugal pump, such as impeller diameter = 300 mm, width of impeller at its periphery = 50 mm, outlet vane angle = 30°, discharge = 0.2 m^3/s, speed of the impeller = 1000 rpm, thickness of vanes occupies = 12% of the peripheral area and flow velocity = constant. Determine (i) the pressure rise in the impeller and (ii) percentage of total work converted to kinetic head.

[**Ans.** 9.03 m of water, 33.45%]

11. A centrifugal pump runs at 1000 rpm and discharge 250 litres per second. It raises the head of water by 30.2 m. The outlet angle of the backward curved vanes is 30° and the velocity of flow at the outlet is 2.85 m/s. If the hydraulic efficiency of the pump is 80%, find the diameter and width of the impeller at the outlet.

[**Ans.** 0.4177 m, 0.0668 m]

12. The discharge through a centrifugal pump is 0.2 m^3/s. It works against a head of 20.5 m. The pump runs at a speed of 900 rpm. The vane angle at the exit of the impeller is 45° and flow velocity at the outlet is 2.5 m/s. If the manometric efficiency of the pump is 80.5%, find the diameter and width of the impeller at its outlet.

[**Ans.** 0.3629 m, 0.0702 m]

13. A centrifugal pump is required to discharge 0.25 m^3/s against a head of 25 m when the impeller rotates at a speed of 1440 rpm. The loss of head in pump in metres due to fluid resistance is $0.025\,V_o^2$, where V_o m/s is the absolute velocity of water leaving the impeller, the area of the impeller outlet surface is $1.25\,D_o^2$, where D_o is the impeller diameter in m, and water enters the impeller without whirl. If the manometric efficiency is given 80%, determine (i) the impeller diameter and (ii) vane angle at the outlet edge of the impeller.

[**Ans.** 0.2619 m, 34.74°]

14. A three-stage centrifugal pump has impeller 0.4 m in diameter and 2 cm wide. The vane angle at outlet is 45° and the area occupied by the thickness of the vanes is 8% of the total area. The other particulars are mechanical efficiency = 88%, manometric efficiency = 77%, discharge = 0.06 m^3/s and speed = 925 rpm. Find the manometric head and power of the pump.

[**Ans.** 76.53 m, 66.48 kW]

15. A centrifugal pump discharges 0.15 m^3/s of water against a head of 13 m. The speed of rotation of the impeller is 600 rpm. The outer and inner diameters of the impeller are 0.5 m and 0.25 m, respectively. The vanes are bent back at an angle of 35° to the tangent at exit. The area of flow is constant and equal to 0.07 m^2 from inlet to outlet of the impeller. Determine (i) the manometric efficiency, (ii) vane angle at inlet, and (iii) loss of head at the inlet to the impeller when discharge is reduced by 40%.

[**Ans.** 64.26%, 15.25°, 0.502 m of water]

16. The impeller of a centrifugal pump has an outer diameter of 0.25 m and an effective area of 0.017 m^2. The vanes are bent backwards so that the direction of outlet relative velocity makes an angle of 148° with the tangent drawn in the direction of impeller rotation. The diameters of suction and delivery pipes are 0.15 m and 0.1 m, respectively. The pump delivers 31 litres per second at 1450 rpm when the gauge points on the suction and delivery pipes close to the pump show heads of 4.6 m below and 18 m above atmosphere, respectively. The head losses in the suction and delivery pipes are 2 m and 2.9 m, respectively. The motor driving the pump delivers 8.67 kW. If the water enters the pump without shock and whirl, then determine the manometric efficiency and the overall efficiency of the pump.

[**Ans.** 74.7%, 81.48%]

17. Water flows through an impeller of a centrifugal pump at the rate of 0.015 m^3/s. The outlet and inlet diameters of the impeller are 0.4 m and 0.2 m, respectively and the widths of the impeller at outlet and inlet are 0.5 cm and 1 cm, respectively. The pump is running at a speed of 1200 rpm. Neglect losses through the impeller. If the water enters the impeller radially at the inlet and impeller vane angle at the outlet is 45°, then determine the rise in pressure in the impeller.

[**Ans.** 31.89 m]

18. Calculate the discharge given by the centrifugal pump which runs at a speed of 1200 rpm and works against a head of 20 m. The diameter and width of the impeller at the outlet is 450 mm and 50 mm, respectively. The vanes of the impeller are curved back at an angle of 25° and manometric efficiency of the pump is 85%.

[**Ans.** 0.663 m^3/s]

19. A centrifugal pump has the following particulars, such as outer diameter of the impeller = 0.8 m, width of impeller vanes at outlet = 0.1 m, angle of impeller vanes at outlet = 45°, speed = 550 rpm, discharge = 1 cubic metres of water per second, effective head = 35 m. If water enters the impeller vanes radially at the inlet and a 500 kW motor is used to drive the pump, then determine the manometric, mechanical and overall efficiencies of the pump.

[**Ans.** 78.19%, 87.82%, 68.67%]

20. A centrifugal pump with 1 m diameter runs at 200 rpm. It pumps 1.88 m^3/s and the average lift is 6 m. The angle which the vanes make at exit with the tangent to the impeller is 30°. The radial velocity of flow is 2.5 m/s. Determine the manometric efficiency and the least speed to start pumping action against a head of 6 m when the inner diameter of the impeller being 0.5 m.

[**Ans.** 91.56%, 239.3 rpm]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (d)	**2.** (d)	**3.** (a)	**4.** (a)	**5.** (c)
6. (c)	**7.** (b)	**8.** (c)	**9.** (a)	**10.** (d)
11. (a)	**12.** (d)	**13.** (b)		

Chapter 26

Reciprocating Pumps

26.1 ❒ INTRODUCTION

The reciprocating pumps are positive displacement pumps in which a certain volume of liquid is taken in an enclosed volume and then it is forced out against pressure to the required application. The mechanical energy is converted into pressure energy by sucking the liquid into a cylinder in which a piston is reciprocating which exerts the thrust on the liquid and increases its pressure energy. The cylinder is alternately filled and emptied by drawing and forcing the liquid by mechanical motion.

Reciprocating pumps are now obsolete due to their high capital and maintenance cost relative to the centrifugal pumps. Reciprocating pumps are being replaced by centrifugal pumps where large quantities of liquid are to be handled. However, reciprocating pumps are preferred where low quantity and high pressure are required. Reciprocating pumps are still used for oil drilling industries, light oil pumping and where there is no electricity. These are also used for pneumatic pressure systems.

Reciprocating pumps, its classification, basic terminology, working principle, indicator diagrams, theoretical analysis of air vessels, effects of acceleration, and friction, and its characteristic curves are discussed briefly in this chapter. A brief introduction to rotary positive displacement pumps, namely vane pump, lobe pump, axial piston pump, gear pump, screw pump and radial piston pump is also given.

26.2 ❒ CLASSIFICATION OF RECIPROCATING PUMPS

The reciprocating pumps may be classified into two categories as given below.

1. According to the water being in contact with piston.
   (i) **Single acting pump:** If the water (liquid) is in contact with one side of piston, then the pump is known as single acting pump.
   (ii) **Double acting pump:** If the water is in contact with both sides of the piston, then the pump is known as double acting pump.
2. According to the number of cylinders provided.
   (i) **Single cylinder pump:** A reciprocating pump having only one cylinder is known as single cylinder pump. A single cylinder pump may be single acting or double acting.
   (ii) **Multi-cylinder pump:** Generally, the reciprocating pumps having more than one cylinder are known as multi-cylinder pumps. For example, double cylinder pump, triple cylinder pump, duplex double acting pump and quintuplex pump.

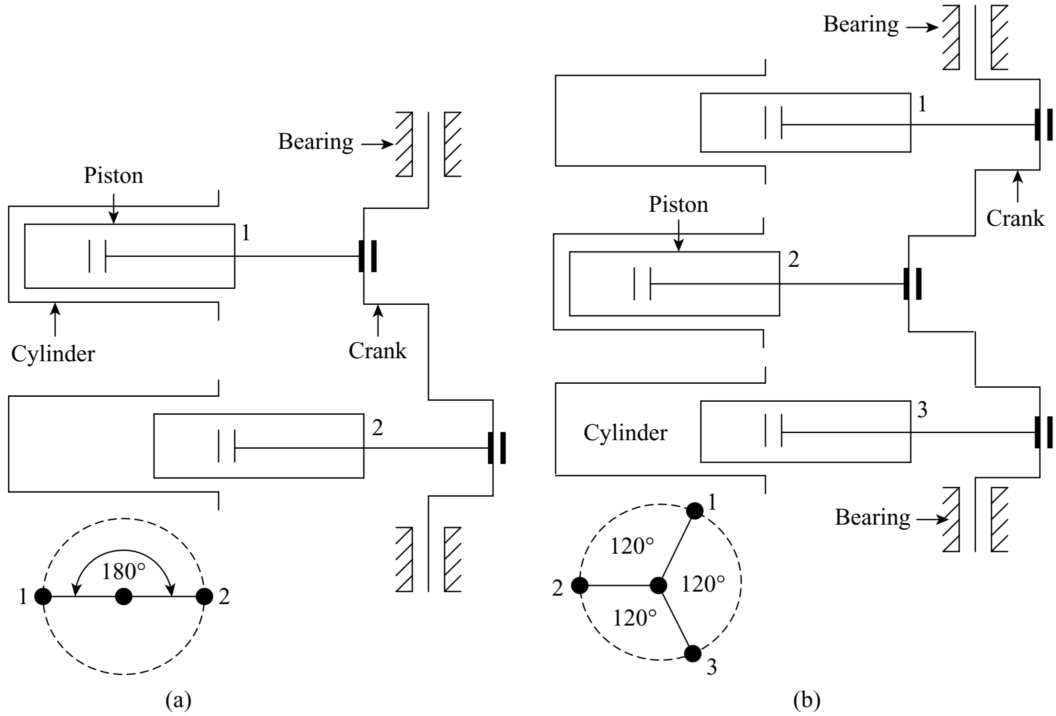


Figure 26.1 *(a) Double cylinder pump (b) Triple cylinder pumps*

(a) **Double cylinder pumps:** A double cylinder pump (or two throw pump) has two single acting cylinders and two pistons working with two connecting rods fitted to the crank at 180°. Each cylinder is equipped with a suction pipe and a delivery pipe with appropriate valves. The schematic view of a double cylinder pump is shown in Figure 26.1(a).

(b) **Triple cylinder pumps:** In the case of triple cylinder pumps (or three throw pumps), there will be three cylinders and three pistons working with three connecting rods fitted to the crank at 120°. The schematic view of three cylinder pump is shown in Figure 26.1(b). Double and triple cylinder pumps provide more uniform discharge in comparison to a single cylinder pump.

(c) **Duplex double acting pumps:** Duplex double acting pumps (or four throw pumps) are a combination of in line two double acting single cylinder pumps driven by a crank or in line two double acting double cylinder pumps driven by two cranks set at 90°.

(d) **Quintuplex pump:** A quintuplex (or five throw pump) has five single acting cylinders driven from cranks set at 72°.

26.3 ❐ MAIN PARTS AND WORKING OF A RECIPROCATING PUMP

26.3.1 Main Parts of a Reciprocating Pump

The main parts of a reciprocating pump are (i) cylinder, (ii) piston (or plunger), (iii) piston rod, (iv) crank, (v) connecting rod, (vi) suction pipe and suction valve, and (vii) delivery pipe and delivery valve which are shown in Figure 26.2(a).

26.3.2 Working of a Single Acting Reciprocating Pump

Figure 26.2(a) shows a single acting reciprocating pump in which water is acting on one side of the piston only. It consists of a piston (or plunger) which moves to and fro in a close fitting cylinder. The cylinder is connected to the suction and

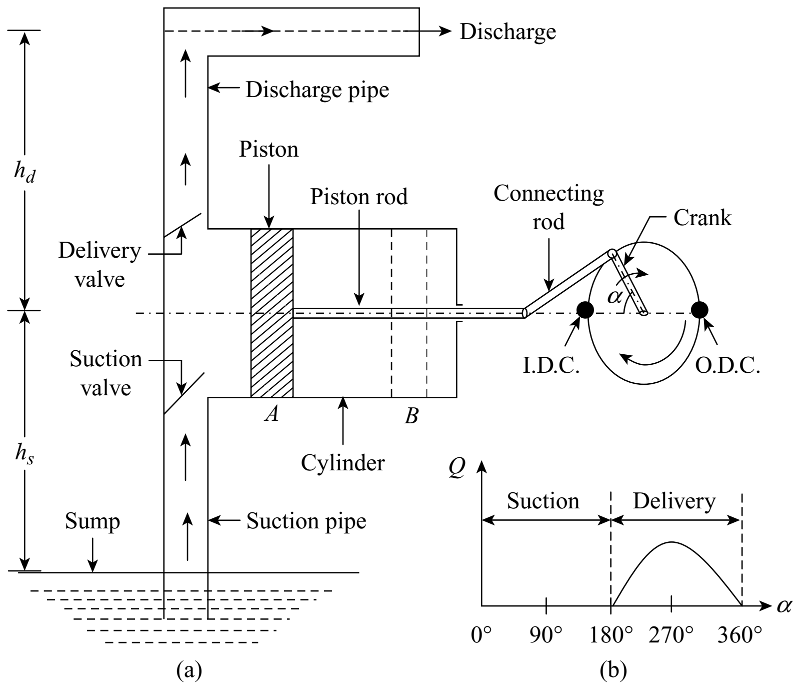


Figure 26.2 *(a) Main parts of a reciprocating pump (b) Variation of discharge*

delivery pipes, each of which is provided with a one way valve called suction valve and delivery valve, respectively. The suction valve allows water from the suction pipe to the cylinder while delivery valve allows water from the cylinder to delivery pipe. The piston is connected to a crank by means of a connecting rod. The action of reciprocating pump is similar to that of reciprocating engines. When the crank is rotated by a driving engine or motor, the piston moves to and fro in the cylinder. When the crank rotates from $\alpha = 0°$ [i.e., crank at the inner dead centre (I.D.C.)] to $\alpha = 180°$ [i.e., crank at the outer dead centre (O.D.C.)], the piston which is at the extreme left position '*A*' moves to extreme right position '*B*'. This outward movement of the piston from position '*A*' to position '*B*' is called suction stroke. Due to outward movement of the piston, partial vacuum is created inside the cylinder. Since atmospheric pressure is acting on the surface of liquid in the sump which is more than the pressure inside the cylinder. Thus, liquid is forced in the suction pipe which opens the suction valve and fills the cylinder.

When crank is further rotated from $\alpha = 180°$ (i.e., crank at the O.D.C.) to $\alpha = 360°$ (i.e., crank at the I.D.C.), the piston moves inwardly from position 'B' to position '*A*'. This inward movement of the piston raises the pressure of the liquid inside the cylinder above the atmospheric pressure due to which the suction valve closes and the delivery valve opens. Thus, the liquid is then forced into delivery pipe and raised to the required height. This inward movement of the piston which causes the delivery of the liquid to the required height is known as delivery stroke. At the end of the delivery stroke, the piston is at position '*A*' and the crank is at $\alpha = 0°$ or $360°$ (i.e., crank is at the I.D.C.). Thus, the crank has completed one full revolution and both the valves are closed. This cycle is repeated as the crank rotates and pump works continuously.

There are intermittent delivery strokes and thus, a single acting reciprocating pump gives non-uniform discharge. The variation of discharge (Q) through delivery pipe with crank angle (α) is shown in Figure 26.2(b).

26.3.3 Discharge, Work Done and Power Required for Driving a Single Acting Reciprocating Pump

Consider a single acting reciprocating pump as shown in Figure 26.2(a). Let r be the radius of crank, N be the crank speed in rpm, $L = 2r$ be the length of the stroke or cylinder, D be the diameter of the cylinder, $A = (\pi/4)\,D^2$ be the area of the piston or cylinder, h_s be the suction head or height of the axis of the cylinder from water surface in sump and h_d be the delivery head or height of delivery outlet above the cylinder axis.

$$\text{Volume of water sucked in during suction stroke: } v = AL$$

$$\text{Numbers of crank revolution per second} = \frac{N}{60}$$

Theoretical discharge of the pump per second is given by,

$$\boxed{Q_{th} = \frac{ALN}{60}} \tag{26.1}$$

Weight of water discharged per second is given by,

$$W = \rho_w g Q_{th} = \frac{\rho_w g ALN}{60} \tag{26.2}$$

Work done per second (w) is given by the product of weight of water per second and the total height through which water is lifted.

$$\therefore w = \frac{\rho_w g ALN(h_s + h_d)}{60} = \rho_w g Q_{th} H \tag{26.3}$$

Here $$H = h_s + h_d$$

Theoretical power required for driving the pump is given by,

$$P_{th} = \frac{\rho_w g ALN(h_s + h_d)}{60 \times 1000} = \frac{\rho_w g Q_{th} H}{1000} \text{ kW} \tag{26.4}$$

26.3.4 Working of a Double Acting Reciprocating Pump

Figure 26.3(a) shows a schematic diagram of a double acting reciprocating pump in which both sides of the piston are used for suction and delivery of the water (or liquid). Therefore, separate suction and delivery valves are provided for the front as well as back sides of the piston.

In a double acting reciprocating pump, when there is a suction stroke on one side of the piston, at the same time there is delivery stroke on the other side of the piston. When crank rotates from I.D.C. (inner dead centre) in the clockwise direction (i.e., piston moves towards crank side also known as forward stroke), a vacuum is created on the left side of the piston and the liquid is sucked in from the sump through the suction valve SV_1 fitted to the suction pipe SP_1. Simultaneously, the liquid taken in towards the backside of the piston is pressed and a high pressure causes the delivery valve DV_2 to open and the liquid is discharged through the discharge pipe DP_2. This operation continues till the piston reaches the head side, i.e., the crank moves to O.D.C. (outer dead centre).

With further rotation, the crankshaft moves towards I.D.C. (i.e., piston moves away from the crank also known as backward stroke) which presses the liquid and discharges it through the delivery valve DV_1 fitted to the delivery pipe DP_1. Simultaneously, a vacuum is created on the right side of the piston and the liquid is sucked in from the sump through the suction valve SV_2 fitted to the suction pipe SP_2. When the crank reaches I.D.C., the piston reaches to its initial position and thus, the cycle is completed. As the crank rotates, the processes towards front and back sides of the piston are repeated and the pump works continuously.

There are continuous delivery strokes and thus, a double acting reciprocating pump gives more uniform discharge than a single acting reciprocating pump. The variation of discharge (Q) through delivery pipe with crank angle (α) is shown in Figure 26.3(b).

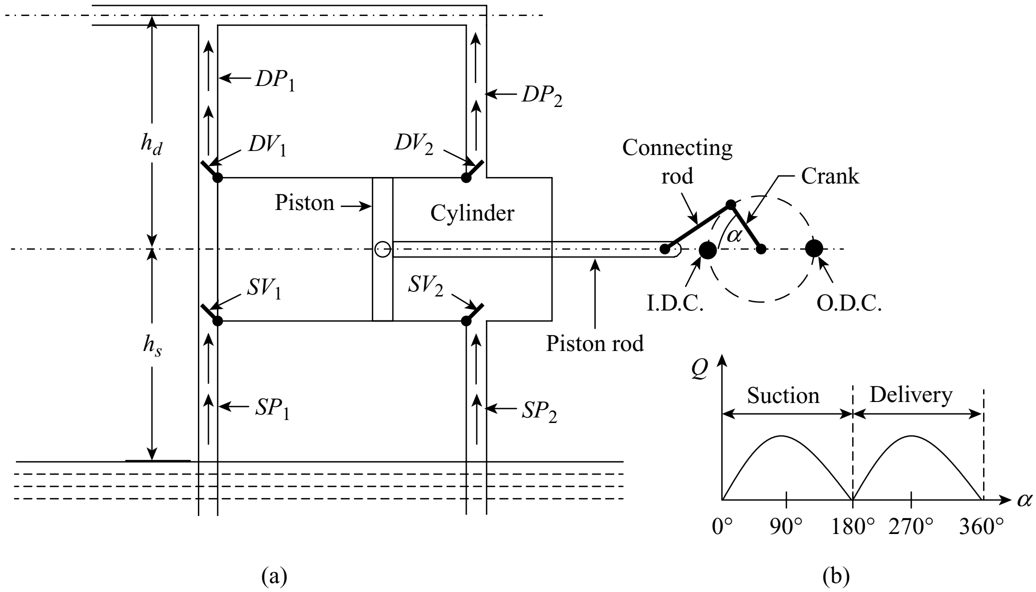


Figure 26.3 *(a) Double acting reciprocating pump (b) Variation of discharge*

26.3.5 Discharge, Work Done and Power Required for Driving a Double Acting Reciprocating Pump

Consider a double acting reciprocating pump as shown in Figure 26.3. Let D be the diameter of the piston, d be the diameter of the piston rod, $A = (\pi/4)\,D^2$ be the area of one side of the piston, $A_1 = (\pi/4)(D^2 - d^2) = A - A_p$ be the area on the other side of the piston where piston rod is connected to the piston and A_p is the area of cross section of the piston rod.

Theoretical volume of water sucked in during suction stroke is given by,

$$v = AL + (A - A_p)L = (2A - A_p)L$$

$$\text{Numbers of crank revolution per second} = \frac{N}{60}$$

Theoretical discharge of the pump per second is given by,

$$Q_{th} = \frac{(2A - A_p)LN}{60}$$

If area of the piston rod is neglected, then theoretical discharge of the pump per second is given below.

$$\boxed{Q_{th} = \frac{2ALN}{60}} \tag{26.5}$$

From Equation (26.5), it can be seen that the theoretical discharge of a double acting pump is twice that of a single acting pump.

Weight of water discharged per second is given by,

$$W = \rho_w g Q_{th} = \frac{2\rho_w g ALN}{60}$$

Work done per second (w) is given by the product of weight of liquid per second and the total height through which liquid is lifted.

$$\therefore w = \frac{2\rho_w g ALN(h_s + h_d)}{60} = \rho_w g Q_{th} H \tag{26.6}$$

Theoretical power required for driving the pump is given by,

$$P_{th} = \frac{2\rho_w g ALN(h_s + h_d)}{60 \times 1000} = \frac{\rho_w g Q_{th} H}{1000} \text{ kW} \tag{26.7}$$

26.4 ❐ COEFFICIENT OF DISCHARGE AND SLIP OF RECIPROCATING PUMP

26.4.1 Coefficient of Discharge

Due to leakage and imperfect operation of the valves, the actual discharge (Q_{act}) of a pump is less than the theoretical discharge (Q_{th}). The ratio of actual discharge to theoretical discharge is known as coefficient of discharge which is denoted by C_d.

Thus
$$C_d = \frac{Q_{\text{act}}}{Q_{\text{th}}} \tag{26.8}$$

When the coefficient of discharge is expressed in percentage, it is known as volumetric efficiency of the pump which is denoted by η_v. The volumetric efficiency depends on the dimensions of the pump and generally, its value varies from 85% to 98%.

26.4.2 Slip of the Reciprocating Pump

In most of the reciprocating pumps, actual discharge is less than theoretical discharge. The difference between the theoretical discharge (Q_{th}) and actual discharge (Q_{act}) is known as slip (S).

Thus
$$\boxed{S = Q_{\text{th}} - Q_{\text{act}}} \tag{26.9}$$

Generally, the slip is expressed as percentage slip and its expression is given below.

$$\%S = \frac{Q_{\text{th}} - Q_{\text{act}}}{Q_{\text{th}}} \times 100 = \left(1 - \frac{Q_{\text{act}}}{Q_{\text{th}}}\right) \times 100 = (1 - C_d) \times 100 \tag{26.10}$$

Generally, for pumps maintained in good condition, percentage slip is of the order of about 2% or even less.

26.4.3 Negative Slip of the Reciprocating Pump

In some reciprocating pumps, actual discharge may be more than the theoretical discharge. Thus, the slip of the pump will become negative. A reciprocating pump with a long suction pipe and a short delivery pipe operating at high speeds would be operating with a negative slip. In such cases, inertia of the liquid in the suction pipe would be large which opens the delivery valve before completion of the suction stroke. So, some of the liquid displaces to the delivery pipe even before the commencement of the delivery stroke. This results in more actual discharge than the theoretical discharge.

26.5 ❐ COMPARISONS OF RECIPROCATING AND CENTRIFUGAL PUMPS

Centrifugal pumps	Reciprocating pumps
The centrifugal pumps run at higher speeds and thus, the discharge capacity is high.	The speed is less and thus, the discharge capacity is low.
Suitable for large volumes of discharge at moderate pressures in a single stage.	Suitable for fairly low volumes of flow at high pressures.
As there are no valves, these pumps can handle highly viscous liquids, grit, slurry, oils, paper pulp, etc.	These pumps can handle only pure water and less viscous liquids free from impurities.
The wear and tear is less because of less moving parts.	The wear and tear is more because of more moving parts.
Simple in construction and its initial cost is low (about 4 to 5 times low).	More complex due to several moving parts and its initial cost is high.
Discharge is continuous. Its operation is smooth and without much noise.	Discharge is fluctuating. Its operation is complicated and with much noise.
Requires less maintenance cost and occupies less space.	Requires high maintenance cost and occupies more space.
They are compact and occupy less space.	They occupy large space (about 5 to 8 times more than a centrifugal pump).
The efficiency is high and torque is uniform.	The efficiency is low and torque is not uniform.
It needs priming.	It does not need priming.
The delivery valve should be closed before switching on the pump.	The delivery valve should not be closed before switching on the pump.
It can be operated at very high speeds without any danger of separation and cavitation.	The maximum speed is limited from the considerations of separation and cavitation.
No air vessel is required.	Air vessel is essential.
Its weight is less for same discharge.	Its weight is more for same discharge.

Example 26.1 A single acting reciprocating pump delivers 9 litres per second of water against a suction head of 4 m and a delivery head of 16 m while running at a speed of 60 rpm. The diameter and stroke of the piston are 200 mm and 300 mm, respectively. Determine (i) the theoretical discharge, (ii) coefficient of discharge, (iii) slip, (iv) percentage slip and (v) power required to drive the pump.

Solution

Let $Q_{\text{act}} = 9\ l/\text{s} = 0.009\ \text{m}^3/\text{s},\ h_s = 4\ \text{m},\ h_d = 16\ \text{m},\ N = 60\ \text{rpm},\ D = 200\ \text{mm} = 0.2\ \text{m}$ and $L = 300\ \text{mm} = 0.3\ \text{m}$.

(i) $$Q_{th} = \frac{ALN}{60} = \frac{\pi D^2 \times LN}{4\times 60} = \frac{\pi \times 0.2^2 \times 0.3\times 60}{4\times 60} = \mathbf{0.009425\ m^3/s}$$

(ii) $$C_d = \frac{Q_{act}}{Q_{th}} = \frac{0.009}{0.009425} = \mathbf{0.955}$$

(iii) $$S = Q_{th} - Q_{act} = 0.009425 - 0.009 = \mathbf{0.000425\ m^3/s}$$

(iv) $$\%S = \frac{Q_{th} - Q_{act}}{Q_{th}}\times 100 = \frac{0.0425}{0.009425}\times 100 = \mathbf{4.51\%}$$

(v) $H = h_s + h_d = 4 + 16 = 20$ m

$$P_{th} = \frac{\rho_w g Q_{th} H}{1000} = \frac{1000 \times 9.81 \times 0.009425 \times 20}{1000} = \mathbf{1.8492\ kW}$$

Example 26.2 A double acting reciprocating pump operating at 55 rpm has a piston diameter of 0.2 m and piston rod of diameter 40 mm which is on one side only. The stroke of the piston is 0.3 m. The suction and delivery heads are 5 m and 20 m, respectively. Determine (i) the theoretical discharge and (ii) power required to drive the pump.

Solution

Let $N = 55$ rpm, $D = 0.2$ m, $d = 40$ mm $= 0.04$ m, $L = 0.3$ m, $h_s = 5$ m and $h_d = 20$ m.

(i) $A = \frac{\pi}{4} D^2 = \frac{\pi}{4} \times 0.2^2 = 0.031416\ \text{m}^2$

$$A_p = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 0.04^2 = 0.001257\ \text{m}^2$$

$$Q_{th} = \frac{(2A - A_p)LN}{60} = \frac{(2 \times 0.031416 - 0.001257) \times 0.3 \times 55}{60} = \mathbf{0.01693\ m^3/s}$$

(ii) $H = h_s + h_d = 5 + 20 = 25$ m

$$P_{th} = \frac{\rho_w g Q_{th} H}{1000} = \frac{1000 \times 9.81 \times 0.01693 \times 25}{1000} = \mathbf{4.1521\ kW}$$

26.6 ❒ EFFECT OF ACCELERATION OF PISTON ON VELOCITY AND PRESSURE IN THE SUCTION AND DELIVERY PIPES

When the crank rotates, the piston reciprocates in the cylinder. During a stroke (suction or delivery stroke), the velocity of the piston does not remain same. At the start of each stroke, the velocity of piston is zero. The velocity increases during the first half of each stroke and reaches to its maximum value at the centre of the cylinder. Thus, it decreases during the latter half of stroke and again becomes zero at the end of each stroke. Therefore, the reciprocating motion of the piston causes acceleration during the beginning of each stroke and retardation at the end of each stroke. Since the water flowing through the pump remains in contact with the piston, the acceleration and retardation effects transmit to the water flowing through the suction and delivery pipes. It means the velocity of water flowing through the suction and delivery pipes is not uniform due to the action of accelerative or retarding head. These variations in the velocities of water in the suction and delivery pipes give rise to inertia pressures which causes a variation of pressure in the cylinder.

If the ratio of the length of connecting rod to the radius of crank is very large, then the piston is assumed to move with simple harmonic motion. A schematic view of a single cylinder and single acting reciprocating pump is shown in Figure 26.4.

Let $\omega = 2\pi N/60$ be the angular speed of the crank in radian per second, A be the area of the cylinder, a be the area of the suction or delivery pipe, l be the length of the suction or delivery pipe and r be the radius of the crank, V be the velocity of water in the cylinder and V_p be the velocity of water in the pipe.

Let the crank turn through an angle α in time t seconds from its I.D.C. (inner dead centre) during suction stroke.

Then $$\alpha = \omega t = (2\pi N/60)t$$

If x is the distance moved by the piston as shown in Figure 26.4, then we get the below expression.

$$x = r - r\cos\alpha = r - r\cos\omega t \qquad [\because \alpha = \omega t]$$

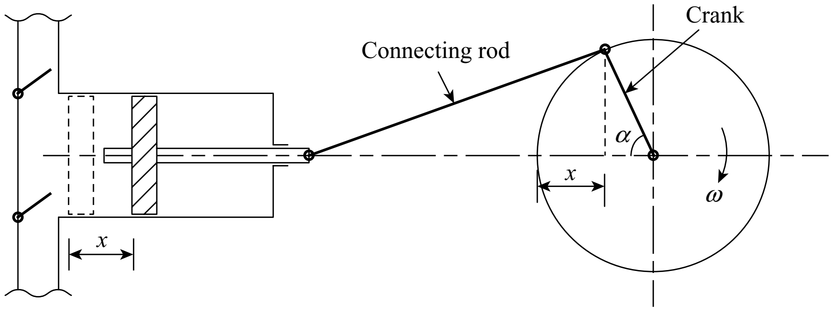


Figure 26.4 *Schematics of single cylinder single acting reciprocating pump*

$$V = \frac{dx}{dt} = \omega r \sin \omega t \tag{26.11}$$

From continuity equation, we get:

$$V_p = \frac{A}{a} \times V = \frac{A}{a} \omega r \sin \omega t \tag{26.11a}$$

The acceleration of water in pipe is given by,

$$a_w = \frac{dV_p}{dt} = \frac{A}{a} \omega^2 r \cos \omega t \tag{26.12}$$

The mass of water in pipe is given by,

$$m_w = \text{Density} \times \text{Volume} = \rho_w \times al$$

Force required to accelerate the water in the pipe is given by,

$$F = m_w a_w = \rho_w al \times \frac{A}{a} \omega^2 r \cos \omega t$$

Thus $$F = \rho_w lA\omega^2 r \cos \omega t$$

The intensity of pressure due to acceleration is given by,

$$p_a = \frac{F}{a} = \rho_w l \frac{A}{a} \omega^2 r \cos \alpha \qquad [\because \alpha = \omega t]$$

The pressure head due to acceleration is given by,

$$\boxed{h_a = \frac{p_a}{\rho_w g} = \frac{l}{g} \frac{A}{a} \omega^2 r \cos \alpha} \tag{26.13}$$

The pressure head due to acceleration in the suction and delivery pipes is given by using subscripts '*s*' and '*d*', respectively in Equation (26.13) as given below.

$$\boxed{h_{as} = \frac{l_s}{g} \frac{A}{a_s} \omega^2 r \cos \alpha} \tag{26.14}$$

$$\boxed{h_{ad} = \frac{l_d}{g} \frac{A}{a_d} \omega^2 r \cos \alpha} \tag{26.15}$$

It can be seen from Equation (26.13), that the pressure head developed due to acceleration acting on the piston varies with crank angle (α). The values of pressure head due to acceleration (h_a) for different values of α can be obtained from Equation (26.13) as given below.

(a) When $\alpha = 0°$ (beginning of stroke), $\cos 0° = 1$ and therefore, $h_a = \frac{l}{g}\frac{A}{a}\omega^2 r$. (26.16)

(b) When $\alpha = 90°$ (mid of stroke), $\cos 90° = 0$ and therefore, $h_a = 0$. (26.17)

(c) When $\alpha = 180°$ (end of stroke), $\cos 180° = -1$ and therefore, $h_a = -\frac{l}{g}\frac{A}{a}\omega^2 r$. (26.18)

Thus, the maximum accelerative pressure head $(h_a)_{\max} = \frac{l}{g}\frac{A}{a}\omega^2 r$. (26.19)

The Equations (26.16) to (26.19) are equally applicable to both the suction and delivery strokes.

It has been observed that for $0° < \alpha < 90°$, h_a has positive values and for $90° < \alpha < 180°$, h_a has negative values, which indicate that for the first half of the stroke, there is accelerative head and for the latter half of the stroke, there is retardation head.

Now if the ratio of the length of connecting rod to the radius of crank (i.e., n) is not very large, then the assumption of simple harmonic motion is not valid and in that case the expression for pressure head is given below.

$$h_a = \frac{l}{g}\frac{A}{a}\omega^2 r \cos\alpha\left(\cos\alpha + \frac{\cos 2\alpha}{n}\right) \tag{26.20}$$

From Equation (26.20), the following expressions can be worked out.

(a) When $\alpha = 0°$ (beginning of stroke): $h_a = \frac{l}{g}\frac{A}{a}\omega^2 r\left(1 + \frac{1}{n}\right)$ (26.21)

(b) When $\alpha = 90°$ (mid of stroke): $h_a = 0$ (26.22)

(c) When $\alpha = 180°$ (end of stroke): $h_a = \frac{l}{g}\frac{A}{a}\omega^2 r\left(1 - \frac{1}{n}\right)$ (26.23)

26.7 ❐ EFFECT OF VARIATION OF VELOCITY IN THE SUCTION AND DELIVERY PIPES

The frictional resistance offered to the water flowing through suction and delivery pipes causes the head loss. This loss of head due to friction is given by Darcy-Weisbach equation as expressed below.

$$h_f = \frac{4 f l V_p^2}{2gd} \tag{i}$$

Here, f is the coefficient of friction, l is the length of pipe, d is the diameter of pipe and V_p is the velocity of water in pipe.

By substituting $V_p = (A/a)\omega r \sin\alpha$ in expression (i), we get:

$$h_f = \frac{4 f l}{2gd}\left(\frac{A}{a}\omega r \sin\alpha\right)^2 \tag{26.24}$$

The variation of h_f with α is parabolic. The values of h_f for suction and delivery pipes are given by using subscripts '*s*' and '*d*', respectively in Equation (26.24) as given below.

$$h_{fs} = \frac{4 f l_s}{2 g d_s}\left(\frac{A}{a_s}\omega r \sin\alpha\right)^2 \tag{26.25}$$

$$h_{fd} = \frac{4 f l_d}{2 g d_d}\left(\frac{A}{a_d}\omega r \sin\alpha\right)^2 \tag{26.26}$$

The variation of h_f with α from Equation (26.24) is given below.

(a) When $\alpha = 0°$, $\sin 0° = 0$ and therefore, $h_f = 0.$ (26.27)

(b) When $\alpha = 90°$, $\sin 90° = 1$ and therefore, $h_f = \frac{4 f l}{2gd}\left(\frac{A}{a}\omega r\right)^2.$ (26.28)

(c) When $\alpha = 180°$, $\sin 180° = 0$ and therefore, $h_f = 0.$ (26.29)

Thus, the maximum friction head loss $(h_f)_{\max}$ is $\frac{4 f l}{2gd}\left(\frac{A}{a}\omega r\right)^2.$ (26.30)

The Equations (26.27) to (26.30) are equally applicable to both the suction and delivery strokes.

Example 26.3 A single acting reciprocating pump having a cylinder diameter of 0.15 m and stroke of 0.3 m is used to raise the water through a height of 20 m. Its crank rotates at 60 rpm. Find the theoretical power required to run the pump and theoretical discharge. If the actual discharge is 5 litres per second, then find the percentage slip. If the delivery pipe is 0.1 m in diameter and is 15 m long, then find the acceleration head at the beginning and mid of the stroke.

Solution

Let $D = 0.15$ m, $L = 0.3$ m, $H = 20$ m, $N = 60$ rpm, $Q_{act} = 5\ l/\text{s} = 0.005\ \text{m}^3/\text{s}$, $d_d = 0.1$ m and $l_d = 15$ m.

$$A = \frac{\pi}{4}D^2 = \frac{\pi}{4}\times 0.15^2 = 0.01767\ \text{m}^2$$

$$Q_{th} = \frac{ALN}{60} = \frac{0.01767\times 0.3\times 60}{60} = \mathbf{0.005301\ m^3/s}$$

$$P_{th} = \frac{\rho_w g Q_{th} H}{1000} = \frac{1000\times 9.81\times 0.005301\times 20}{1000} = \mathbf{1.04\ kW}$$

$$\%S = \frac{Q_{\text{th}} - Q_{\text{act}}}{Q_{\text{th}}}\times 100 = \frac{0.005301 - 0.005}{0.005301}\times 100 = \mathbf{5.68\%}$$

$$a_d = \frac{\pi}{4}{d_d}^2 = \frac{\pi}{4}\times 0.1^2 = 0.007854\ \text{m}^2$$

$$\omega = \frac{2\pi N}{60} = \frac{2\times\pi\times 60}{60} = 2\pi\ \text{rad/s}$$

$$r = \frac{L}{2} = \frac{0.3}{2} = 0.15\ \text{m}$$

The pressure head due to acceleration in the delivery pipe is given by,

$$h_{ad} = \frac{l_d}{g}\frac{A}{a_d}\omega^2 r\cos\alpha = \frac{15}{9.81}\times\frac{0.01767}{0.007854}\times(2\pi)^2\times 0.15\cos\alpha = 20.371\cos\alpha$$

At the beginning of delivery stroke: $\alpha = 0°$ and $h_{\text{ad}} = 20.371\cos 0° = \mathbf{20.371\ m}$

At the mid of delivery stroke: $\alpha = 90°$ and $h_{\text{ad}} = 20.371\cos 90° = \mathbf{0}$

Example 26.4 Both the diameter and stroke of a single acting reciprocating pump are 0.35 m and both the suction and delivery pipes are 0.2 m in diameter. The vertical lengths of the suction and delivery pipes are 5 m and 25 m, respectively. If the pump runs at 30 rpm and the coefficient of friction in the pipe is 0.02, then find the power required to run the pump.

Solution

Let $L = D = 0.35$ m, $d_s = d_d = 0.2$ m, $l_s = 5$ m, $l_d = 25$ m, $N = 30$ rpm and $f = 0.02$.

$$Q_{th} = \frac{ALN}{60} = \frac{\pi D^2 \times LN}{4\times 60} = \frac{\pi\times 0.35^2\times 0.35\times 30}{4\times 60} = 0.016837\ \text{m}^3/\text{s}$$

Since
$$h_{fs} = \frac{4fl_s}{2gd_s}\left(\frac{A}{a_s}\omega r\right)^2 = \frac{4fl_s}{2gd_s}\left[\frac{D^2}{{d_s}^2}\times\frac{2\pi N}{60}\times\frac{L}{2}\right]^2$$

$$\therefore h_{fs} = \frac{4\times 0.02\times 5}{2\times 9.81\times 0.2}\times\left[\frac{0.35^2}{0.2^2}\times\frac{2\pi\times 30}{60}\times\frac{0.35}{2}\right]^2 = 0.289\ \text{m}$$

Since
$$h_{fd} = \frac{4fl_d}{2gd_d}\left(\frac{A}{a_d}\omega r\right)^2 = \frac{4fl_d}{2gd_d}\left[\frac{D^2}{{d_d}^2}\times\frac{2\pi N}{60}\times\frac{L}{2}\right]^2$$

$$\therefore h_{fd} = \frac{4\times 0.02\times 25}{2\times 9.81\times 0.2}\times\left[\frac{0.35^2}{0.2^2}\times\frac{2\pi\times 30}{60}\times\frac{0.35}{2}\right]^2 = 1.445\ \text{m}$$

$$H = l_s + l_d + h_{fs} + h_{fd} = 5 + 25 + 0.289 + 1.445 = 31.73\ \text{m}$$

$$P_{th} = \frac{\rho_w g Q_{th} H}{1000} = \frac{1000\times 9.81\times 0.016837\times 31.73}{1000} = \mathbf{5.241\ kW}$$

26.8 ❐ INDICATOR DIAGRAMS

The indicator diagram is obtained by using an indicator fitted on the cylinder and thus, it is called an indicator diagram. In this diagram, the pressure head on the piston is plotted along the ordinate and the stroke length (L) along the abscissa. The work input to pump as given by Equation (26.3) can be calculated directly from the indicator diagram which shows the pressure of water in the cylinder corresponding to any crank position during the suction and delivery strokes.

26.8.1 Theoretical Indicator Diagram

The Figure 26.5 shows the theoretical indicator diagram of a single cylinder single acting reciprocating pump which has been drawn under ideal conditions. Therefore, this diagram is also known as ideal indicator diagram. It represents the work done by the pump during one complete cycle. The indicator diagram is obtained by neglecting the loss of head due to friction and the pressure head due to acceleration of water in the suction and delivery pipes.

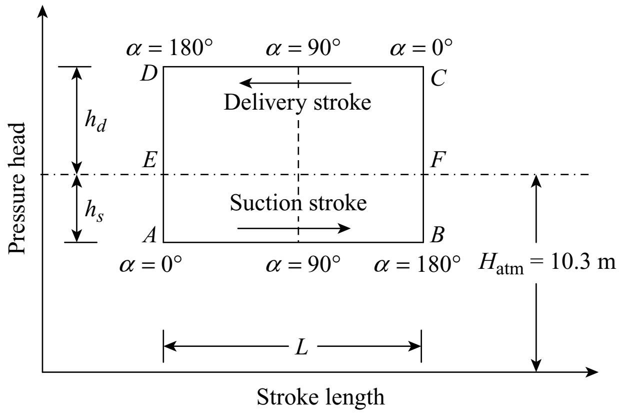


Figure 26.5 *Theoretical (or ideal) indicator diagram*

Let h_s be the suction head, h_d be the delivery head, H_{atm} be the atmospheric pressure head which is equal to 10.3 m of water, L be the stroke length and a_i be the area of indicator diagram.

In Figure 26.5, the horizontal line EF represents the atmospheric pressure head (H_{atm}). The pressure head in the cylinder during suction stroke is constant and equal to suction head (h_s). The line AB represents the pressure head in the cylinder during the suction stroke, which is below the atmospheric pressure by an amount equal to suction head (h_s). The pressure head in the cylinder during delivery stroke is constant and equal to delivery head (h_d). The line CD represents the pressure head in the cylinder during the delivery stroke, which is above the atmospheric pressure by an amount equal to delivery head (h_d).

The area $ABFE$ represents the work done by the piston during the suction stroke and the area $CDEF$ represents the work done by the piston during the delivery stroke. The total work done by the piston during complete revolution of the crank is then represented by the area $ABCD$ which is equal to $(h_s + h_d)L$.

The work input to drive a single cylinder single acting reciprocating pump is given by Equation (26.3) as follows.

$$w = \frac{\rho_w gAN}{60}[(h_s + h_d)L] = \left(\frac{\rho_w gAN}{60}\right)a_i = ka_i \qquad (26.31)$$

Here, $k = (\rho_w gAN)/60$ and a_i is the area of indicator diagram.

Thus, the work done by the pump is proportional to the area of indicator diagram.

For double acting pump, this diagram represents the pressure head on one side of the piston only. Therefore, the work done per revolution is represented by twice the area of this diagram when the area of piston rod is not considered.

26.8.2 Effect of Acceleration in Suction and Delivery Pipes on Indicator Diagram

The Figure 26.6 shows the indicator diagram that includes the effect of acceleration.

Effect of acceleration during suction stroke The acceleration head in the pipe during suction stroke (h_{as}) is given by Equation (26.14) as follows.

$$h_{as} = \frac{l_s}{g}\frac{A}{a_s}\omega^2 r \cos\alpha$$

At the beginning of the suction stroke: $\alpha = 0°$ and $h_{as} = \dfrac{l_s}{g}\dfrac{A}{a_s}\omega^2 r.$ (26.32)

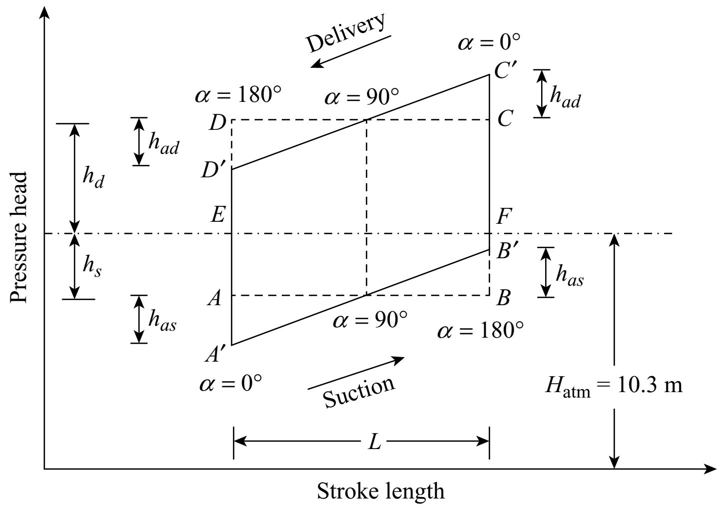


Figure 26.6 *Effect of acceleration on indicator diagram*

Thus, at the beginning of the suction stroke, the acceleration head h_{as} is positive. As at the beginning of the stroke, the liquid in the suction pipe is to be accelerated which requires an additional pressure drop in the cylinder. Therefore, the acceleration effect results in an increased suction from EA to EA'. Thus, the relation for pressure head in the cylinder at the beginning of suction stroke is given below,

$$= (h_s + h_{as}) \text{ Below atmospheric head (vac)} = [H_{atm} - (h_s + h_{as})] \text{ Absolute}$$

At the middle of the suction stroke, $\alpha = 90°$ and thus, $\cos 90° = 0$. Therefore, the acceleration head $h_{as} = 0$. At the middle of the suction stroke, the pressure head will be only h_s below the atmospheric pressure head.

At the end of the suction stroke: $\alpha = 180°$ and $h_{as} = -\dfrac{l_s}{g}\dfrac{A}{a_s}\omega^2 r.$ (26.33)

Thus, at the end of the suction stroke, the acceleration head h_{as} is negative. As at the end of the suction stroke, the liquid in the suction pipe is to be retarded. For this, there should be pressure rise in the cylinder. Therefore, the acceleration effect results in a reduced vacuum from FB to FB'. Thus, the relation for pressure head in the cylinder at the end of suction stroke is given below.

$$= (h_s - h_{as}) \text{ Below atmospheric head (vac)} = [H_{\text{atm}} - (h_s - h_{as})] \text{ Absolute}$$

The base of the indicator diagram for the suction stroke is changed from AB to $A'B'$ and the work done during suction stroke is represented by $A'B'FE$. However, area $ABFE$ is equal to the area $A'B'FE$. It means the net work done during the suction stroke is not changed on account of accelerating effects in the suction pipe.

Effect of acceleration during delivery stroke The acceleration head in the pipe during delivery stroke (h_{ad}) is given by Equation (26.15) as follows.

$$h_{ad} = \frac{l_d}{g}\frac{A}{a_d}\omega^2 r \cos\alpha$$

At the beginning of the delivery stroke: $\alpha = 0°$ and $h_{ad} = \dfrac{l_d}{g}\dfrac{A}{a_d}\omega^2 r.$ (26.34)

At the beginning of the delivery stroke, the acceleration head h_{ad} is positive. As at the beginning of the delivery stroke, the liquid in the delivery pipe is to be accelerated. For this, an additional pressure head should be developed in the cylinder. Therefore, the acceleration effect results in an increased pressure from FC to FC'. Thus, pressure head in the cylinder at the beginning of delivery stroke is given below.

$$= (h_d + h_{ad}) \text{ Above atmospheric head (gauge)} = [H_{\text{atm}} + (h_d + h_{ad})] \text{ Absolute}$$

At the middle of the delivery stroke, $\alpha = 90°$ and thus, $\cos 90° = 0$. Therefore, the acceleration head $h_{ad} = 0$. Thus, at the middle of the delivery stroke, the pressure head will be only h_d above the atmospheric pressure head.

At the end of the delivery stroke: $\alpha = 180°$ and $h_{ad} = -\frac{l_d}{g}\frac{A}{a_d}\omega^2 r.$ (26.35)

Thus, at the end of the delivery stroke, the acceleration head h_{ad} is negative. As at the end of the delivery stroke, the liquid in the delivery pipe is to be retarded. For this, there should be a pressure drop in the cylinder. Therefore, the acceleration effect causes drop in pressure head from DE to $D'E$. Thus, pressure head in the cylinder at the end of delivery stroke is given below.

$$= (h_d - h_{ad}) \text{ Above atmospheric head (gauge)} = [H_{\text{atm}} + (h_d - h_{ad})] \text{ Absolute}$$

The base of the indicator diagram for the delivery stroke is changed from *CD* to $C'D'$ and the work done during delivery stroke is represented by $C'D'EF$. However, area *CDEF* is equal to the area $C'D'EF$. It means the net work done during the delivery stroke is not changed on account of accelerating effects in the delivery pipe.

From Figure 26.6, it can be seen that due to acceleration in suction and delivery pipes, the indicator diagram has changed from *ABCD* to $A'B'C'D'$. However, the area of indicator diagram remains unaltered and hence, the work done remains same. Thus, it is inferred that the inertia pressure developed in the suction and delivery pipes does not affect the work input but it causes only a variation of pressure in the cylinder. The straight lines $A'B'$ and $C'D'$ in the indicator diagram will be slightly curved when the piston does not move with simple harmonic motion.

The acceleration head limits the performance of a reciprocating pump as (i) it limits the suction height and increases the frictional losses and (ii) to get higher flow rate, the speed cannot be increased due to separation.

26.8.3 Maximum Speed of a Reciprocating Pump

The maximum speed of a reciprocating pump depends on the phenomenon of cavitation which occurs when the pressure in the cylinder during suction and delivery strokes falls below the vapour pressure of the liquid flowing through the suction and delivery pipes. Cavitation causes discontinuity of the flow, i.e., the separation of the liquid takes place. The pressure at which the separation takes place is called separation pressure and the head is called separation pressure head which is denoted by h_{sep}. For water, the separation pressure head is about 2.5 m of water absolute or (10.3 – 2.5) = 7.8 m of water below atmospheric pressure head.

Maximum speed during suction stroke It can be seen from Figure 26.6 that the minimum pressure head during suction stroke occurs at the beginning of the stroke which corresponds to the point A'. During suction stroke, when the suction pressure falls below the separation pressure, it can separate at point A'. Thus, the separation of flow can take place at the beginning of the stroke only. Therefore, to avoid separation, the absolute pressure head at the point A' given as $[H_{atm} - (h_s + h_{as})]$ must not fall below the separation pressure head (h_{sep}).

$$H_{atm} - h_s - h_{as} > h_{sep} \tag{26.36}$$

In the limiting condition, we get:

$$H_{atm} - h_s - h_{as} = h_{sep}$$

$$H_{atm} - h_s - \frac{l_s}{g}\frac{A}{a_s}\omega^2 r = h_{sep} \qquad \text{[Substitute Equation (26.32)]}$$

Thus
$$\boxed{\frac{l_s}{g}\frac{A}{a_s}\left(\frac{2\pi N}{60}\right)^2 r = H_{atm} - h_s - h_{sep}} \qquad (26.37)$$

The maximum speed of the reciprocating pump without separation during suction stroke can be calculated from Equation (26.37).

Maximum speed during delivery stroke It can be seen from Figure 26.6 that the minimum pressure head during delivery stroke occurs at the end of the stroke which corresponds to the point D'. Therefore, to avoid separation, the absolute pressure head at the point D' given as $[H_{atm} + (h_d - h_{ad})]$ must not fall below the absolute separation pressure head (h_{sep}).

$$H_{atm} + h_d - h_{ad} > h_{sep} \qquad (26.38)$$

Thus
$$H_{atm} + h_d - h_{ad} = h_{sep} \qquad \text{[Limiting condition]}$$

$$H_{atm} + h_d - \frac{l_d}{g}\frac{A}{a_d}\omega^2 r = h_{sep} \qquad \text{[Substitute Equation (26.34)]}$$

Thus
$$\boxed{\frac{l_d}{g}\frac{A}{a_d}\left(\frac{2\pi N}{60}\right)^2 r = H_{atm} + h_d - h_{sep}} \qquad (26.39)$$

The maximum speed of the reciprocating pump without separation during delivery stroke can be calculated from Equation (26.39).

The minimum of the two speeds given by Equations (26.37) and (26.39) is the maximum speed of the reciprocating pump without separation during suction and delivery strokes.

The delivery pipe of a reciprocating pump may have two arrangements with possible regions of separation as shown in Figure 26.7.

(i) In the arrangement shown in Figure 26.7(a), the delivery pipe is first vertical and then horizontal.

Absolute pressure head at the inlet of the delivery pipe $= H_{atm} + h_d - h_{ad}$

Pressure head at the bend $= (H_{atm} + h_d - h_{ad}) - h_d = H_{atm} - h_{ad}$ (i)

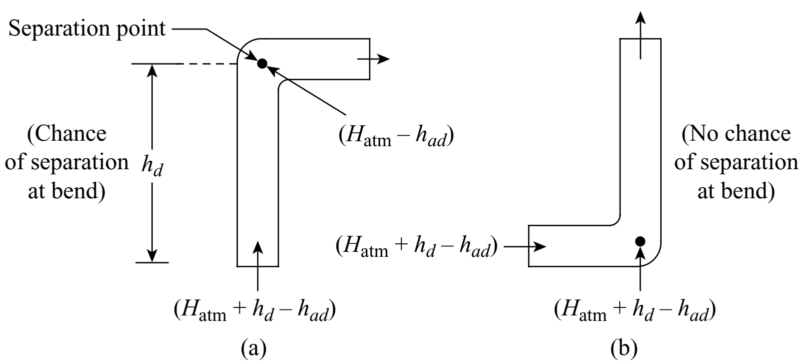


Figure 26.7 *Arrangements of delivery pipes*

(ii) In the arrangement shown in Figure 26.7(b), the delivery pipe is first horizontal and then vertical.

$$\text{Pressure head at the bend} = H_{atm} + h_d - h_{ad} \quad \text{(ii)}$$

From the expressions (i) and (ii), it is clear that the pressure head at the bend is lower in the first arrangement. Therefore, there are more chances of separation to occur in the first arrangement.

Example 26.5 The diameter and stroke of single acting reciprocating pump are 0.15 m and 0.35 m, respectively. Both the suction and delivery pipes are 0.1 m in diameter. The lengths of the suction and delivery pipes are 5 m and 30 m, respectively. The centre of the pump is 4 m above the water surface in the sump and 25 m below the delivery water level. If the pump is working at 30 rpm and atmospheric pressure is 76 cm of mercury, then find (i) the pressure heads on the piston at the beginning, middle and end of the suction stroke, (ii) pressure heads on the piston at the beginning, middle and end of the delivery stroke and (iii) power required to run the pump.

Solution

Let $D = 0.15$ m, $L = 0.35$ m, $d_s = d_d = 0.1$ m, $l_s = 5$ m, $l_d = 30$ m, $h_s = 4$ m, $h_d = 25$ m, $N = 30$ rpm and $p_{\text{atm}} = 76$ cm of Hg.

(i) $H_{\text{atm}} = \dfrac{76 \times 10 \times 13.6}{1000} = 10.336 \text{ m of water}$

Acceleration head in the pipe during suction stroke (h_{as}) is given by,

$$h_{as} = \frac{l_s}{g}\frac{A}{a_s}\omega^2 r \cos\alpha = \frac{l_s}{g}\frac{D^2}{{d_s}^2}\left(\frac{2\pi N}{60}\right)^2 \frac{L}{2}\cos\alpha$$

At the beginning of the delivery stroke: $\alpha = 0°$

$$\therefore h_{as} = \frac{5}{9.81} \times \frac{0.15^2}{0.1^2} \times \left(\frac{2\pi \times 30}{60}\right)^2 \times \frac{0.35}{2}\cos 0° = 1.981 \text{ m}$$

Thus, pressure head in the cylinder at the beginning of suction stroke is given by,

$$= (h_s + h_{as}) \text{ (vac)} = [H_{atm} - (h_s + h_{as})] \text{ m of water (abs)}$$

$$= 10.336 - (4 + 1.981) = \mathbf{4.355 \text{ m of water (abs)}}$$

At the middle of the delivery stroke: $\alpha = 90°$ and $h_{as} = 0$.

Thus, the pressure head at the middle of the suction stroke is given by,

$$= (h_s + h_{as}) = h_s = 4 \text{ m (vac)} = 10.336 - 4 = \mathbf{6.336 \text{ m of water (abs)}}$$

At the end of the suction stroke: $\alpha = 180°$

$$\therefore h_{as} = \frac{5}{9.81} \times \frac{0.15^2}{0.1^2} \times \left(\frac{2\pi \times 30}{60}\right)^2 \times \frac{0.35}{2}\cos 180° = -1.981 \text{ m of water}$$

Thus, pressure head in the cylinder at the end of suction stroke is given by,

$$= (h_s - h_{as}) \text{ below atmospheric head (vac)} = [H_{atm} - (h_s - h_{as})] \text{ absolute}$$

$$= 10.336 - (4 - 1.981) = \mathbf{8.317 \text{ m of water (abs)}}$$

(ii) Acceleration head in the pipe during delivery stroke (h_{ad}) is given by,

$$h_{ad} = \frac{l_d}{g}\frac{A}{a_d}\omega^2 r\cos\alpha = \frac{l_d}{g}\frac{D^2}{d_d^{\,2}}\left(\frac{2\pi N}{60}\right)^2\frac{L}{2}\cos\alpha$$

At the beginning of the suction stroke: $\alpha = 0°$

$$\therefore h_{ad} = \frac{30}{9.81}\times\frac{0.15^2}{0.1^2}\times\left(\frac{2\pi\times 30}{60}\right)^2\times\frac{0.35}{2}\cos 0° = 11.884 \text{ m}$$

Thus, pressure head in the cylinder at the beginning of delivery stroke is given by,

$$= (h_d + h_{ad})\text{ (gauge)} = [H_{atm} + (h_d + h_{ad})]\text{ absolute}$$

$$= 10.336 + (25 + 11.884) = \mathbf{47.22 \text{ m of water (abs)}}$$

At the middle of the suction stroke: $\alpha = 90°$ and $h_{ad} = 0$.

Thus, the pressure head at the middle of the delivery stroke is given by,

$$= (h_d + h_{ad}) = h_d = 25 \text{ m (gauge)} = 10.336 + 25 = \mathbf{35.336 \text{ m of water (abs)}}$$

At the end of the delivery stroke: $\alpha = 180°$

$$\therefore h_{ad} = \frac{30}{9.81}\times\frac{0.15^2}{0.1^2}\times\left(\frac{2\pi\times 30}{60}\right)^2\times\frac{0.35}{2}\cos 180° = -11.884 \text{ m of water}$$

Thus, pressure head in the cylinder at the end of delivery stroke is given by,

$$= (h_d + h_{ad})\text{ (gauge)} = [H_{atm} + (h_d + h_{ad})]\text{ abs}$$

$$= 10.336 + (25 - 11.884) = \mathbf{23.452 \text{ m of water (abs)}}$$

(iii) $Q_{th} = \dfrac{ALN}{60} = \dfrac{\pi D^2 \times LN}{4\times 60} = \dfrac{\pi\times 0.15^2\times 0.35\times 30}{4\times 60} = 3.0925\times 10^{-3}\ \text{m}^3/\text{s}$

$H = h_s + h_d = 4 + 25 = 29$ m

$$P_{th} = \frac{\rho_w g Q_{th} H}{1000} = \frac{1000\times 9.81\times 3.0925\times 10^{-3}\times 29}{1000} = \mathbf{0.8798 \text{ kW}}$$

Example 26.6 For a single acting reciprocating pump, the diameter and the length of the suction pipe are 5 cm and 6 m and that of delivery pipe is 4 cm and 18 m, respectively. The diameter of the piston and stroke length is 0.124 m and 0.224 m, respectively. The centre of the pump is 4 m above the water level in the sump and the delivery tank is 16 m above the centre line of the pump. The separation of water occurs at 7.8 m below the atmospheric pressure head. Determine the maximum speed at which the pump can run without separation.

Solution

Let $d_s = 5\text{ cm} = 0.05\text{ m}$, $l_s = 6$ m, $d_d = 4\text{ cm} = 0.04$ m, $l_d = 18$ m, $D = 0.124$ m, $L = 0.224$ m, $h_s = 4$ m, $h_d = 16$ m and $h_{sep} = (H_{atm} - 7.8)$ m (abs).

During suction stroke, the absolute pressure head is minimum at the beginning of the stroke and thus, separation can takes place at the beginning of the stroke only.

Thus $$\frac{l_s}{g}\frac{A}{a_s}\omega^2 r = H_{atm} - h_s - h_{sep}$$

or $$\frac{l_s}{g}\frac{D^2}{{d_s}^2}\left(\frac{2\pi N}{60}\right)^2\frac{L}{2} = H_{atm} - h_s - h_{sep}$$

$$\frac{6}{9.81}\times\frac{0.124^2}{0.05^2}\times\left(\frac{2\pi N}{60}\right)^2\times\frac{0.224}{2} = H_{atm} - 4 - (H_{atm} - 7.8)$$

$$4.62\times10^{-3}N^2 = 3.8$$

$$\therefore N = \sqrt{\frac{3.8}{4.62\times10^{-3}}} = 28.68 \text{ rpm}$$

During delivery stroke, the absolute pressure head is minimum at the end of the stroke and thus, separation can takes place at the end of the stroke only.

Thus $$\frac{l_d}{g}\frac{A}{a_d}\omega^2 r = H_{atm} + h_d - h_{sep}$$

or $$\frac{l_d}{g}\frac{D^2}{{d_d}^2}\left(\frac{2\pi N}{60}\right)^2\frac{L}{2} = H_{atm} + h_d - h_{sep}$$

$$\frac{18}{9.81}\times\frac{0.124^2}{0.04^2}\times\left(\frac{2\pi N}{60}\right)^2\times\frac{0.224}{2} = H_{atm} + 16 - (H_{atm} - 7.8)$$

$$0.022N^2 = 23.8$$

$$\therefore N = \sqrt{\frac{23.8}{0.022}} = 32.89 \text{ rpm}$$

The maximum speed of the pump without separation during suction and delivery stroke will be a minimum of 28.68 rpm and 32.89 rpm.

$$\therefore \text{Maximum speed} = \mathbf{28.68\ rpm}$$

Example 26.7 A single acting reciprocating pump delivers water at a height of 20 m through a delivery pipe 30 m long and 0.125 m in diameter. The diameter of the piston and stroke length is 0.225 m and 0.42 m, respectively. The atmospheric pressure head is 10.3 m of water and the cavitation occurs at 2.5 m of water absolute. Determine the maximum speed at which the pump can run without separation on the delivery side if (i) pipe runs first horizontally and then vertically upwards and (ii) pipe raise first vertically and then runs horizontally.

Solution

Let $h_d = 20$ m, $l_d = 30$ m, $d_d = 0.125$ m, $D = 0.225$ m, $L = 0.42$ m, $H_{atm} = 10.3$ m and $h_{sep} = 2.5$ m (abs).

(i) Pressure head at the bend = $H_{atm} + h_d - h_{ad}$

To avoid separation, the pressure head at the bend must be equal to or greater than separation head. At this limit, we get the following expression.

$$H_{atm} + h_d - h_{ad} = h_{sep} \Rightarrow h_{ad} = H_{atm} + h_d - h_{sep}$$

or $$\frac{l_d}{g}\frac{A}{a_d}\omega^2 r = H_{atm} + h_d - h_{sep}$$

or
$$\frac{l_d}{g}\frac{D^2}{{d_d}^2}\left(\frac{2\pi N}{60}\right)^2\frac{L}{2}=H_{atm}+h_d-h_{sep}$$

$$\frac{30}{9.81}\times\frac{0.225^2}{0.125^2}\times\left(\frac{2\pi N}{60}\right)^2\times\frac{0.42}{2}=10.3+20-2.5$$

$$0.02282N^2=27.8$$

$$\therefore N=\sqrt{\frac{27.8}{0.02282}}=\mathbf{34.9\ rpm}$$

(ii) Pressure head at the bend $= H_{atm}-h_{ad}$

To avoid separation, the pressure head at the bend must be equal to or greater than separation head. At this limit, we get the following expression.

$$H_{atm}-h_{ad}=h_{sep}\Rightarrow h_{ad}=H_{atm}-h_{sep}$$

or
$$\frac{l_d}{g}\frac{A}{a_d}\omega^2 r=H_{atm}-h_{sep}$$

or
$$\frac{l_d}{g}\frac{D^2}{{d_d}^2}\left(\frac{2\pi N}{60}\right)^2\frac{L}{2}=H_{atm}-h_{sep}$$

$$\frac{30}{9.81}\times\frac{0.225^2}{0.125^2}\times\left(\frac{2\pi N}{60}\right)^2\times\frac{0.42}{2}=10.3-2.5$$

$$0.02282N^2=7.8$$

$$\therefore N=\sqrt{\frac{7.8}{0.02282}}=\mathbf{18.49\ rpm}$$

26.8.4 Effect of Friction in Suction and Delivery Pipes on Indicator Diagram

The head loss due to friction in suction and delivery pipes is given by Equations (26.25) and (26.26), respectively as follows.

$$h_{fs}=\frac{4fl_s}{2gd_s}\left(\frac{A}{a_s}\omega r\sin\alpha\right)^2 \text{ and } h_{fd}=\frac{4fl_d}{2gd_d}\left(\frac{A}{a_d}\omega r\sin\alpha\right)^2$$

From these equations, it can be seen that the variation of h_{fs} or h_{fd} with α is parabolic. The change in pressure head during suction and delivery stroke inside the cylinder is given below.

(a) When $\alpha=0°$, $\sin 0°=0$ and therefore, $h_{fs}=0$ and $h_{fd}=0$.

(b) When $\alpha=90°$, $\sin 90°=1$ and thus, $h_{fs}=\frac{4fl_s}{2gd_s}\left(\frac{A}{a_s}\omega r\right)^2$ and $h_{fd}=\frac{4fl_d}{2gd_d}\left(\frac{A}{a_d}\omega r\right)^2$.

(c) When $\alpha=180°$, $\sin 180°=0$ and therefore, $h_{fs}=0$ and $h_{fd}=0$.

Therefore, it can be seen that the frictional losses are zero at the beginning and end of the suction as well as delivery strokes. The effect of h_{fs} and h_{fd} on the indicator diagram is shown in Figure 26.8. It can be seen that the area of the diagram increases in comparison to the ideal indicator diagram by the amount *AGBA* and *CIDC*.

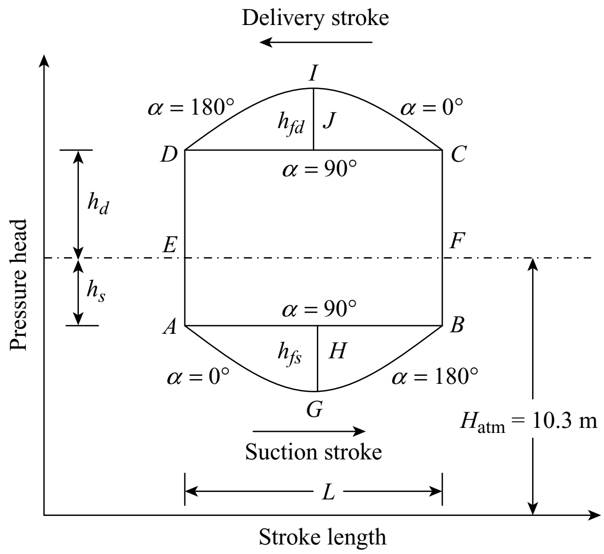


Figure 26.8 *Effect of friction on indicator diagram*

The area of the parabola *AGB* represents the work done against the friction in suction pipe.

$$\text{Area } AGBA = AB \times \frac{2}{3} GH = L \times \frac{2}{3} h_{fs} = L \times \frac{2}{3}\left[\frac{4 f l_s}{2 g d_s}\left(\frac{A}{a_s}\omega r\right)^2\right]$$

Similarly, the area of the parabola CID represents the work done against the friction in delivery pipe.

$$\text{Area } CIDC = CD \times \frac{2}{3} IJ = L \times \frac{2}{3} h_{fd} = L \times \frac{2}{3}\left[\frac{4 f l_d}{2 g d_d}\left(\frac{A}{a_d}\omega r\right)^2\right]$$

26.8.5 Effect of Acceleration and Friction in Suction and Delivery Pipes on Indicator Diagram

The acceleration head (h_a) and friction head (h_f) at any moment of flow in suction and delivery pipes is given by Equations (26.13) and (26.24), respectively as follows.

$$h_a = \frac{l}{g}\frac{A}{a}\omega^2 r \cos\alpha \text{ and } h_f = \frac{4 f l}{2 g d}\left(\frac{A}{a}\omega r \sin\alpha\right)^2$$

(a) Change in pressure head during suction stroke The pressure head on the piston or plunger during suction stroke for any angle of crank is equal to $h_s + h_{as} + h_{fs}$.

(i) At the beginning of the suction stroke, $\alpha = 0°$ and thus, from Equations (26.13) and (26.24), we get the below expressions.

$$h_{as} = \frac{l_s}{g}\frac{A}{a_s}\omega^2 r \text{ and } h_{fs} = 0$$

Thus, pressure head in the cylinder is given by,

$$= (h_s + h_{as}) \text{ below atmospheric pressure head} = [H_{atm} - (h_s + h_{as})] \text{ absolute}$$

(ii) At the middle of the suction stroke, $\alpha = 90°$ and thus, from Equations (26.13) and (26.24), we get the below expressions.

$$h_{as} = 0 \text{ and } h_{fs} = \frac{4 f l_s}{2 g d_s}\left(\frac{A}{a_s}\omega r\right)^2$$

Thus, pressure head in the cylinder is given by,

$$= (h_s + h_{fs}) \text{ below atmospheric pressure head} = [H_{atm} - (h_s + h_{fs})] \text{ absolute}$$

(iii) At the end of the suction stroke, $\alpha = 180°$ and thus, from Equations (26.13) and (26.24), we get the below expressions.

$$h_{as} = -\frac{l_s}{g}\frac{A}{a_s}\omega^2 r \text{ and } h_{fs} = 0$$

Thus, pressure head in the cylinder is given by,

$$= (h_s - h_{as}) \text{ below atmospheric pressure head} = [H_{atm} - (h_s - h_{as})] \text{ absolute}$$

(b) Change in pressure head during delivery stroke The pressure head on the piston or plunger during delivery stroke for any angle of crank is equal to $h_d + h_{ad} + h_{fd}$.

(i) At the beginning of the delivery stroke $\alpha = 0°$ and thus, from Equations (26.13) and (26.24), we get the below expressions.

$$h_{ad} = \frac{l_d}{g}\frac{A}{a_d}\omega^2 r \text{ and } h_{fd} = 0$$

Thus, pressure head in the cylinder is given by,

$$= (h_d + h_{ad}) \text{ above the atmospheric pressure head} = [H_{\text{atm}} + (h_d + h_{ad})] \text{ absolute}$$

(ii) At the middle of the delivery stroke, $\alpha = 90°$ and thus, from Equations (26.13) and (26.24), we get the below expressions.

$$h_{ad} = 0 \text{ and } h_{fd} = \frac{4 f l_d}{2 g d_d}\left(\frac{A}{a_d}\omega r\right)^2$$

Thus, pressure head in the cylinder is given by,

$$= (h_d + h_{fd}) \text{ above the atmospheric pressure head} = [H_{atm} + (h_d + h_{fd})] \text{ absolute}$$

(iii) At the end of the delivery stroke, $\alpha = 180°$ and thus, from Equations (26.13) and (26.24), we get the below expressions.

$$h_{ad} = -\frac{l_d}{g}\frac{A}{a_d}\omega^2 r \text{ and } h_{fd} = 0$$

Thus, pressure head in the cylinder is given by,

$$= (h_d - h_{ad}) \text{ above the atmospheric pressure head} = [H_{atm} - (h_d - h_{ad})] \text{ absolute}$$

The Figure 26.9 shows the combined effect of acceleration and friction in suction and delivery pipes.

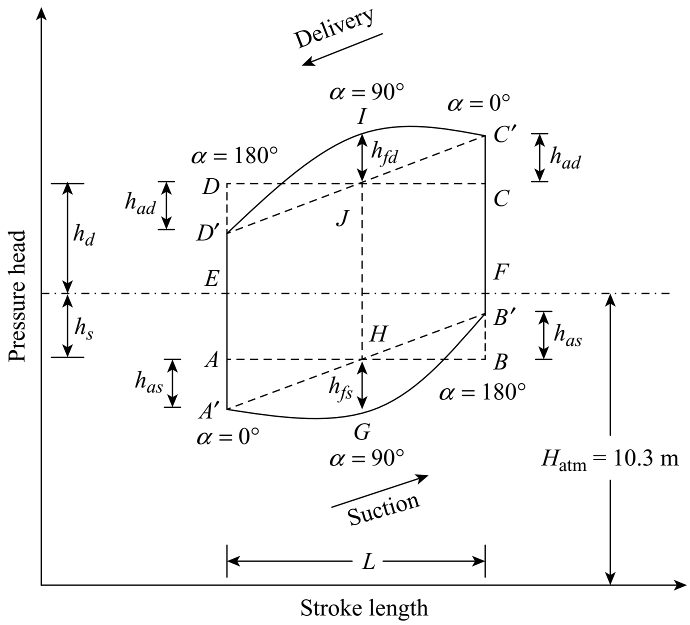


Figure 26.9 *Effect of acceleration and friction on indicator diagram*

The parabola $A'GB'A'$ represents the work done against friction in suction pipe and the parabola $C'ID'C'$ represents the work done against friction in delivery pipe. The total work done is represented by the area $A'GB'C'ID'A'$.

Area of indicator diagram during suction stroke is given by,

$$= \text{Area } A'GB'FEA' = \text{Area } A'B'FEA' + \text{Area } A'GB'A'$$

$$= \text{Area } ABFEA + \text{Area } A'GB'A' = h_s L + \frac{2}{3} h_{fs} L = \left(h_s + \frac{2}{3} h_{fs} \right) L \qquad \text{(i)}$$

Area of indicator diagram during delivery stroke is given by,

$$= \text{Area } FC'ID'EF = \text{Area } EFC'D' + \text{Area } C'ID'C'$$

$$= \text{Area } EFCDE + \text{Area } C'ID'C' = h_d L + \frac{2}{3} h_{fd} L = \left(h_d + \frac{2}{3} h_{fd} \right) L \qquad \text{(ii)}$$

$$\text{Area of indicator diagram } = \left(h_s + h_d + \frac{2}{3} h_{fs} + \frac{2}{3} h_{fd} \right) L \qquad \text{[Add (i) and (ii)]}$$

The work done by the pump is proportional to the area of the indicator diagram.

Thus

$$w \propto \left(h_s + h_d + \frac{2}{3} h_{fs} + \frac{2}{3} h_{fd} \right) L$$

$$\therefore w = k \left(h_s + h_d + \frac{2}{3} h_{fs} + \frac{2}{3} h_{fd} \right) L \qquad [k = \text{Constant}]$$

For a single acting reciprocating pump, we get:

$$k = \frac{\rho_w g A N}{60}$$

For a double acting reciprocating pump, we get:

$$k = \frac{2\rho_w gAN}{60}$$

Therefore, work done per second by a single acting pump is given by,

$$w = \frac{\rho_w gAN}{60} \times \left(h_s + h_d + \frac{2}{3} h_{fs} + \frac{2}{3} h_{fd} \right) L \tag{26.40}$$

Work done per second by a double acting pump is given by,

$$w = \frac{2\rho_w gAN}{60} \times \left(h_s + h_d + \frac{2}{3} h_{fs} + \frac{2}{3} h_{fd} \right) L \tag{26.41}$$

Example 26.8 A single acting reciprocating pump has a stroke length of 0.15 m. The suction pipe is 7 m long and the ratio of suction pipe diameter to the piston diameter is 3 : 4. The water level in the sump is 2.5 m below the axis of the pump cylinder and the pipe connecting the sump and pump cylinder is 7.5 cm in diameter. If the crank is running at 75 rpm, then find the pressure head on the piston at the beginning, middle and end of the suction stroke. Take friction coefficient as $f = 0.01$ and atmospheric pressure head as 10.3 m of water.

Solution

Let $L = 0.15$ m, $l_s = 7$ m, $d_s/D = 3/4$, $h_s = 2.5$ m, $d_s = 7.5$ cm $= 0.075$ m, $N = 75$ rpm, $f = 0.01$ and $H_{atm} = 10.3$ m.

$$r = \frac{L}{2} = \frac{0.15}{2} = 0.075 \text{ m}$$

$$\frac{A}{a_s} = \frac{(\pi/4)D^2}{(\pi/4){d_s}^2} = \frac{4^2}{3^2} = \frac{16}{9}$$

$$\omega = \frac{2\pi N}{60} = \frac{2\times\pi\times 75}{60} = 7.854 \text{ rad/s}$$

$$h_{as} = \frac{l_s}{g}\frac{A}{a_s}\omega^2 r\cos\alpha = \frac{7}{9.81}\times\frac{16}{9}\times 7.854^2 \times 0.075\cos\alpha = 5.87\cos\alpha \tag{i}$$

Since $$h_{fs} = \frac{4 f l_s}{2 g d_s}\left(\frac{A}{a_s}\omega r\sin\alpha\right)^2$$

$$\therefore h_{fs} = \frac{4\times 0.01\times 7}{0.075\times 2\times 9.81}\times\left(\frac{16}{9}\times 7.854\times 0.075\sin\alpha\right)^2 = 0.209\sin^2\alpha \tag{ii}$$

(i) At the beginning of the suction stroke, $\alpha = 0°$ and thus, from expressions (i) and (ii), we obtain the following result.

$$h_{as} = 5.87\cos 0° = 5.87 \text{ and } h_{fs} = 0.209\sin^2 0° = 0$$

Pressure head on the piston in the beginning of suction stroke is given by,

$= (h_s + h_{as})$ below atmospheric pressure head $= [H_{atm} - (h_s + h_{as})]$ absolute

$= 10.3 - (2.5 + 5.87) =$ **1.93 m (abs)**

(ii) At the middle of the suction stroke, $\alpha = 90°$ and thus, from expressions (i) and (ii), we obtain the following data.

$$h_{as} = 5.87\cos 90° = 0 \text{ and } h_{fs} = 0.209\sin^2 90° = 0.209$$

Thus, pressure head in the cylinder is given by,

$$= (h_s + h_{fs}) \text{ below atmospheric pressure head} = [H_{atm} - (h_s + h_{fs})] \text{ absolute}$$

$$= 10.3 - (2.5 + 0.209) = \mathbf{7.591\ m\ (abs)}$$

(iii) At the end of the suction stroke, $\alpha = 180°$ and thus, from expressions (i) and (ii), we obtain the following results.

$$h_{as} = 5.87\cos 180° = -5.87 \text{ and } h_{fs} = 0.209\sin^2 180° = 0$$

Thus, pressure head in the cylinder is given by,

$$= (h_s - h_{as}) \text{ below atmospheric pressure head} = [H_{atm} - (h_s - h_{as})] \text{ absolute}$$

$$= 10.3 - (2.5 - 5.87) = \mathbf{13.67\ m\ (abs)}$$

26.9 ❐ AIR VESSELS

An air vessel is a closed chamber which contains compressed air at its top portion and liquid (water) being pumped at the bottom portion of the chamber. There is an opening at the base of the chamber through which water may flow into the vessel or it may flow out from the vessel. The air is compressed when the water enters the vessel and it expands when the water flows out from the vessel. One air vessel is fitted to the suction pipe and another is fitted to the delivery pipe at the points near the cylinder of a reciprocating pump as shown in Figure 26.10. The following are the functions of an air vessel.

1. It reduces the possibility of separation in suction pipe.
2. Pump can run at higher speeds and gives higher discharge.

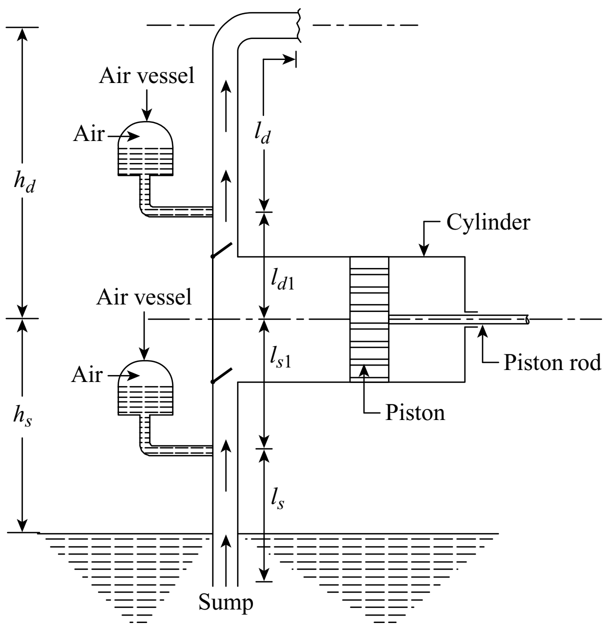


Figure 26.10 *A reciprocating pump with air vessels*

3. Length of suction pipe below the air vessel can be increased.
4. A large amount of work in overcoming the frictional resistance in suction and delivery pipes can be saved.
5. It gives uniform discharge from the pump.

An air vessel in a reciprocating engine acts like an intermediate reservoir to absorb pressure fluctuations and performs like a flywheel of an engine.

During the first half of the suction stroke, the piston moves with acceleration. This forces the water in the suction pipe to move with a velocity greater than the mean velocity of flow. Therefore, the flow rate of water entering the cylinder may be more than the mean discharge. This excess quantity of water required due to accelerating effects will be supplied from the air vessel. Thus, the velocity of flow of water in suction pipe below the point at which air vessel is fitted will be equal to the mean velocity of flow.

During the second half of the suction stroke, the piston moves with retardation. So, the water in the suction pipe also retards. Thus, the velocity of water flowing in the suction pipe is less than the mean velocity. Thereby, the water flow rate entering the cylinder is less than the mean discharge. Due to air vessel, the velocity of water in the suction pipe below the point at which air vessel is fitted is equal to the mean velocity of flow. As the flow required in the cylinder is less than the mean flow, the excess quantity of water flowing through the suction pipe will be stored in the air vessel which compresses the air inside the vessel. This stored water will be supplied during the first half of the next suction stroke and the same cycle will be repeated.

During the first half of the delivery stroke, the piston moves with acceleration and hence, the water is forced into the delivery pipe at a rate of flow greater than the mean discharge. The quantity of water in excess of average discharge flows into the air vessel and thereby, it compresses the air.

During the second half of the delivery stroke, the piston moves with retardation and hence, the piston velocity drops below than the mean velocity of water. Therefore, water forced in the delivery pipe from the cylinder is less than the mean discharge. The compressed air in the air vessel then forces excess stored water into the delivery pipe and thus, it maintains the constant rate of water flow in the delivery pipe.

It is inferred that an air vessel maintains a uniform velocity of flow in the portion of the suction pipe below the point at which the air vessel is fitted. However, the fluctuations in the velocity of flow due to accelerating effects occurs in the portion of the suction pipe between the cylinder and the point at which the air vessel is connected to the suction pipe, i.e., length l_{s1}. Similarly, an air vessel fitted to the delivery pipe also maintains the uniform rate of flow of water in the delivery pipe beyond the point at which the air vessel is fitted. However, the velocity of flow fluctuates due to accelerating effects in the portion of the delivery pipe between the cylinder and the point at which the air vessel is fitted to the delivery pipe, i.e., length l_{d1}. The acceleration pressure heads develop in the lengths l_{s1} and l_{d1} in suction and delivery pipes, respectively due to fluctuations in the velocity of flow. These acceleration pressure heads may be considerably reduced by fitting the air vessels to the suction and delivery pipes at a point very close to the cylinder.

26.10 ❐ THEORETICAL ANALYSIS OF AIR VESSELS

Let N be the crank speed in rpm and $\omega = 2\pi N/60$ be the angular speed,

$L = 2r$ be the length of the stroke or cylinder, here r be the radius of crank,

$A = (\pi/4)\,D^2$ be the area of cylinder, here D is the diameter of the cylinder,

$a = (\pi/4)\,d^2$ be the area of the suction or delivery pipe, here d is the diameter of the suction or delivery pipe,

h_s and h_d be the suction and delivery heads, respectively,

l_s be the length of suction pipe below air vessel,

l_{s1} be the length of suction pipe between cylinder and air vessel,

l_d be the length of delivery pipe beyond air vessel,

l_{d1} be the length of delivery pipe between cylinder and air vessel,

h_{as} and h_{ad} be the pressure head due to acceleration in the suction and delivery pipes, respectively,
h_{fs} and h_{fd} be the pressure head loss in the suction and delivery pipes due to friction, respectively,
h_{fs1} be the pressure head loss in the portion l_{s1} of suction pipe due to friction and
h_{fd1} be the pressure head loss in the portion l_{d1} of delivery pipe due to friction.

Theoretical discharge of a single acting pump per second is given by Equation (26.1) as follows.

$$Q_{th} = \frac{ALN}{60}$$

Mean velocity in the suction or delivery pipe is given by,

$$V_m = \frac{Q_{th}}{a} = \frac{ALN}{60a}$$

Since $$N = \frac{60\omega}{2\pi} \text{ and } L = 2r$$

$$\boxed{\therefore V_m = \frac{A \times 2r}{60 \times a} \times \frac{60\omega}{2\pi} = \frac{A\omega r}{\pi a}} \tag{26.42}$$

The velocity of liquid in the lengths portion l_{s1} and l_{d1} of suction and delivery pipes is given from Equation (26.11a) as follows.

$$V_p = \frac{A}{a}\omega r \sin\alpha \qquad [\alpha = \omega t]$$

26.10.1 Water Flow Rate In and Out of Air Vessel

(i) **For single acting reciprocating pump** The velocity of water in the cylinder of a single acting pump at any instant is given from Equation (26.11) as follows.

$$V = \omega r \sin\alpha \qquad [\alpha = \omega t]$$

The discharge at any instant to or from the cylinder is given by,

$$Q_i = VA = \omega r \sin\alpha \times A \tag{26.43}$$

The mean discharge in the suction or delivery pipe is given by,

$$Q_m = V_m a = \left(\frac{A\omega r}{\pi a}\right) a = \frac{A\omega r}{\pi} \tag{26.44}$$

The difference of the above two discharges given by Equations (26.43) and (26.44) will give the water flow rate in or out of the air vessel.

Therefore, the rate of flow of water into the air vessel is given by,

$$= \omega r \sin\alpha \times A - \frac{A\omega r}{\pi} = A\omega r\left(\sin\alpha - \frac{1}{\pi}\right) \tag{26.45}$$

Now considering that the air vessel is fitted to the delivery pipe. If Equation (26.45) is positive, then it means the water is flowing into the air vessel fitted to the delivery pipe. However, if Equation (26.45) is negative, then it means that the water is flowing from the air vessel.

When air vessel is considered to be fitted to the suction pipe, then the above condition will be reversed. If Equation (26.45) is positive, then it means the water is flowing from the air vessel. In case, if Equation (26.45) is negative, then it means that the water is flowing into the air vessel fitted to the suction pipe.

There is no flow of water into or from the air vessel when,

$$A\omega r\left(\sin\alpha - \frac{1}{\pi}\right) = 0$$

Thus $$\sin\alpha = \frac{1}{\pi} \Rightarrow \alpha = 18°34' \text{ or } 161°26'$$

Therefore, for crank angle (α) equal to $18°34'$ and $161°26'$, there will be no flow into or from the air vessel.

(ii) For double acting reciprocating pump In the case of double acting reciprocating pump, the water flow rate into the pipe at any instant remains same and is given by Equation (26.43). However, the mean discharge is double that of the single acting pump and it is given by two times of Equation (26.44) as follows.

$$Q_m = \frac{2A\omega r}{\pi}$$

Thus, the rate of flow of water into the air vessel is given by,

$$= \omega r \sin\alpha \times A - \frac{2A\omega r}{\pi} = A\omega r\left(\sin\alpha - \frac{2}{\pi}\right) \quad (26.46)$$

Considering that air vessel is fitted to the delivery pipe. Again if Equation (26.46) is positive, then it means the water is flowing into the air vessel fitted to the delivery pipe. If Equation (26.46) is negative, then it means that the water is flowing from the air vessel.

When air vessel is considered to be fitted to the suction pipe then the above condition will be reversed. If Equation (26.46) is positive, then it means the water is flowing from the air vessel. In case, if Equation (26.46) is negative, then it means that the water is flowing into the air vessel fitted to the suction pipe.

There is no flow of water into or from the air vessel when,

$$A\omega r\left(\sin\alpha - \frac{2}{\pi}\right) = 0$$

Thus $$\sin\alpha = \frac{2}{\pi} \Rightarrow \alpha = 39°32' \text{ or } 140°28'$$

Therefore, for crank angle (α) equal to $39°32'$ or $140°28'$, there will be no flow into or from the air vessel.

26.10.2 Pressure Heads in the Cylinder During Suction Stroke of a Reciprocating Pump with Air Vessel

The pressure head due to acceleration in the suction pipe for the length l_{s1} is given from Equation (26.14) as follows.

$$h_{as1} = \frac{l_{s1}}{g}\frac{A}{a_s}\omega^2 r\cos\alpha \quad (26.47)$$

Since $$V_s = \frac{A}{a_s}\omega r\sin\alpha$$

Thus, the loss of head due to friction in the suction pipe for the length l_{s1} is given by,

$$h_{fs1} = \frac{4f l_{s1} V_s^2}{2gd_s} = \frac{4f l_{s1}}{2gd_s}\left(\frac{A}{a_s}\omega r\sin\alpha\right)^2 \quad (26.48)$$

Since $$V_{ms} = \frac{A\omega r}{\pi a_s}$$

Thus, the loss of head due to friction in the suction pipe for the length l_s, i.e., below the air vessel is given by,

$$h_{fs} = \frac{4 f l_s V_{ms}^2}{2gd_s} = \frac{4 f l_s}{2gd_s}\left(\frac{A\omega r}{\pi a_s}\right)^2 \tag{26.49}$$

Thus, total suction pressure head developed during suction stroke of a reciprocating pump with air vessel for any crank angle α is given by the summation of the static suction head, accelerating head between cylinder and air vessel, friction head loss between cylinder and air vessel, uniform frictional head loss below air vessel, and mean velocity head.

$$\therefore H_s = h_s + h_{as1} + h_{fs1} + h_{fs} + \frac{V_{ms}^2}{2g} \tag{26.50}$$

When the length l_{s1} is negligible then the second and third terms in Equation (26.50) can be neglected. Sometimes, velocity head is also neglected. Thus, Equation (26.50) is written as follows.

$$H_s = h_s + h_{fs} \tag{26.51}$$

Now substituting Equations (26.47), (26.48), (26.49), and (26.42) in Equation (26.50), we get:

$$H_s = h_s + \frac{l_{s1}}{g}\frac{A}{a_s}\omega^2 r\cos\alpha + \frac{4 f l_{s1}}{2gd_s}\left(\frac{A}{a_s}\omega r\sin\alpha\right)^2 + \frac{4 f l_s}{2gd_s}\left(\frac{A\omega r}{\pi a_s}\right)^2 + \frac{1}{2g}\left(\frac{A\omega r}{\pi a_s}\right)^2 \tag{26.52}$$

Thus, the values of pressure head in the cylinder for different values of α are obtained from Equation (26.52) as follows.

(i) When $\alpha = 0°$ (beginning of stroke), $\cos 0° = 1$ and $\sin 0° = 0$.

$$\therefore H_s = h_s + \frac{l_{s1}}{g}\frac{A}{a_s}\omega^2 r + \frac{4 f l_s}{2gd_s}\left(\frac{A\omega r}{\pi a_s}\right)^2 + \frac{1}{2g}\left(\frac{A\omega r}{\pi a_s}\right)^2 \tag{26.53}$$

(ii) When $\alpha = 90°$ (mid of stroke), $\cos 90° = 0$ and $\sin 90° = 1$.

$$\therefore H_s = h_s + \frac{4 f l_{s1}}{2gd_s}\left(\frac{A}{a_s}\omega r\right)^2 + \frac{4 f l_s}{2gd_s}\left(\frac{A\omega r}{\pi a_s}\right)^2 + \frac{1}{2g}\left(\frac{A\omega r}{\pi a_s}\right)^2 \tag{26.54}$$

(iii) When $\alpha = 180°$ (end of stroke), $\cos 180° = -1$ and $\sin 180° = 0$.

$$\therefore H_s = h_s - \frac{l_{s1}}{g}\frac{A}{a_s}\omega^2 r + \frac{4 f l_s}{2gd_s}\left(\frac{A\omega r}{\pi a_s}\right)^2 + \frac{1}{2g}\left(\frac{A\omega r}{\pi a_s}\right)^2 \tag{26.55}$$

The terms $\frac{l_{s1}}{g}\frac{A}{a_s}\omega^2 r$ and $\frac{4 f l_{s1}}{d_s 2g}\left(\frac{A}{a_s}\omega r\right)^2$ can be neglected in Equations (26.53) to (26.55).

26.10.3 Pressure Heads in the Cylinder During Delivery Stroke of a Reciprocating Pump with Air Vessel

The same analysis applied for the suction stroke is used to find the pressure heads developed during the delivery stroke. The corresponding relations for delivery stroke may be given by replacing subscript ‘s’ by ‘d’ in Equations (26.47) to (26.55).

Thus, total delivery pressure head developed during delivery stroke of a reciprocating pump with air vessel for any crank angle α is given by the summation of the static delivery head, accelerating head between cylinder and air vessel, friction head loss between cylinder and air vessel, uniform frictional head loss beyond air vessel and mean velocity head.

$$\therefore H_d = h_d + h_{ad1} + h_{fd1} + h_{fd} + \frac{{V_{md}}^2}{2g} \tag{26.56}$$

When the length l_{d1} is negligible then the second and third terms in Equation (26.56) can be neglected. Sometimes, velocity head is also neglected. Thus, Equation (26.56) is written as follows.

$$H_d = h_d + h_{fd} \tag{26.57}$$

Now substituting various values in Equation (26.56), we get:

$$H_d = h_d + \frac{l_{d1}}{g}\frac{A}{a_d}\omega^2 r \cos\alpha + \frac{4 f l_{d1}}{2gd_d}\left(\frac{A}{a_d}\omega r \sin\alpha\right)^2 + \frac{4 f l_d}{2gd_d}\left(\frac{A\omega r}{\pi a_d}\right)^2 + \frac{1}{2g}\left(\frac{A\omega r}{\pi a_d}\right)^2 \tag{26.58}$$

Thus, the values of pressure head in the cylinder for different values of α are obtained from Equation (26.58) as follows.

(i) When $\alpha = 0°$, (beginning of stroke), $\cos 0° = 1$ and $\sin 0° = 0$.

$$\therefore H_d = h_d + \frac{l_{d1}}{g}\frac{A}{a_d}\omega^2 r + \frac{4 f l_d}{2gd_d}\left(\frac{A\omega r}{\pi a_d}\right)^2 + \frac{1}{2g}\left(\frac{A\omega r}{\pi a_d}\right)^2 \tag{26.59}$$

(ii) When $\alpha = 90°$ (mid of stroke), $\cos 90° = 0$ and $\sin 90° = 1$.

$$\therefore H_d = h_d + \frac{4 f l_{d1}}{2gd_d}\left(\frac{A}{a_d}\omega r\right)^2 + \frac{4 f l_d}{2gd_d}\left(\frac{A\omega r}{\pi a_d}\right)^2 + \frac{1}{2g}\left(\frac{A\omega r}{\pi a_d}\right)^2 \tag{26.60}$$

(iii) When $\alpha = 180°$ (end of stroke), $\cos 180° = -1$ and $\sin 180° = 0$.

$$\therefore H_d = h_d - \frac{l_{d1}}{g}\frac{A}{a_d}\omega^2 r + \frac{4 f l_d}{2gd_d}\left(\frac{A\omega r}{\pi a_d}\right)^2 + \frac{1}{2g}\left(\frac{A\omega r}{\pi a_d}\right)^2 \tag{26.61}$$

The terms $\frac{l_{d1}}{g}\frac{A}{a_d}\omega^2 r$ and $\frac{4 f l_{d1}}{d_d 2g}\left(\frac{A}{a_d}\omega r\right)^2$ can be neglected in Equations (26.59) to (26.61).

26.10.4 Work Done by a Reciprocating Pump with Air Vessel and Its Effect on Indicator Diagram

The theoretical work done per second by a pump fitted with air vessels to both the suction and delivery pipes is given by,

$$w = \frac{\rho_w g ALN}{60}\left[(h_s + h_d) + (h_{fs} + h_{fd}) + \frac{2}{3}(h_{fs1} + h_{fd1}) + \frac{{V_{ms}}^2 + {V_{md}}^2}{2g}\right] \tag{26.62}$$

When the small quantities are neglected, then we get,

$$w = \frac{\rho_w g\, ALN}{60}[(h_s + h_d) + (h_{fs} + h_{fd})] \tag{26.63}$$

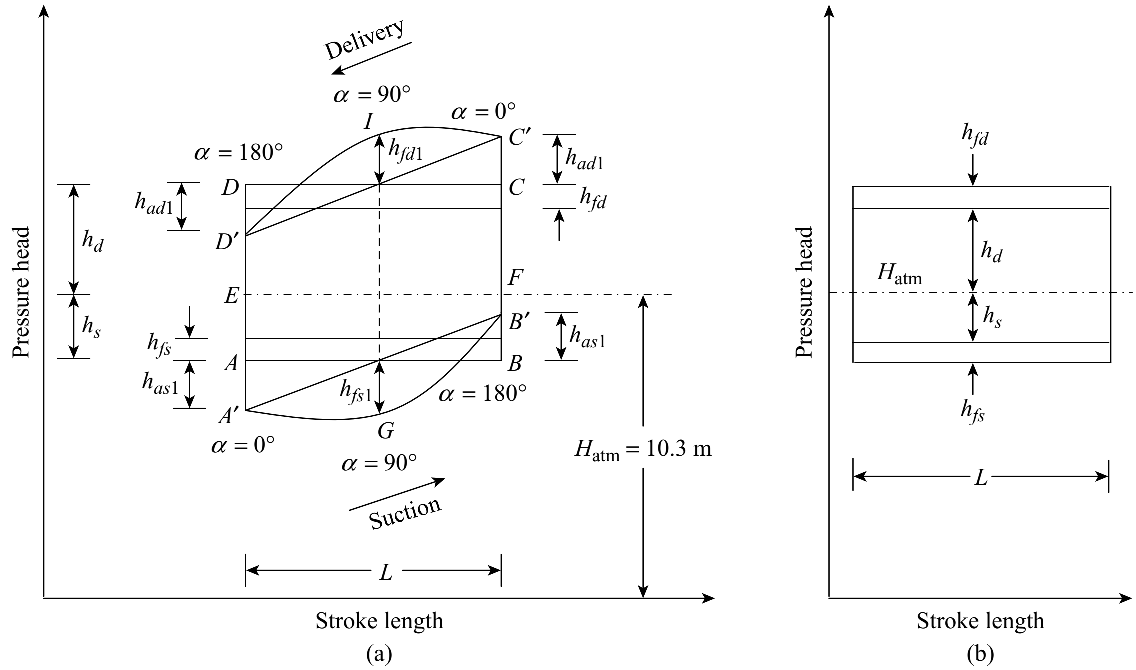


Figure 26.11 *Indicator diagram for a reciprocating pump with air vessels*

Thus, the theoretical power required to drive the pump fitted with air vessels is given by,

$$P = \frac{\rho_w g\, ALN}{60 \times 1000}[(h_s + h_d) + (h_{fs} + h_{fd})] \text{ kW} \tag{26.64}$$

The modified indicator diagram incorporating the effect of air vessels is illustrated in Figure 26.11(a). The shape of the indicator diagram will become rectangular when the small quantities are neglected as shown in Figure 26.11(b).

Thus, the Equation (26.63) can also be written as,

$$w = \frac{\rho_w gAN}{60}[(h_s + h_d + h_{fs} + h_{fd})L] = \frac{\rho_w gAN}{60} \times (\text{Area of indicator diagram})$$

26.10.5 Maximum Speed of a Reciprocating Pump with Air Vessel

The maximum permissible speed of a reciprocating pump is limited by the drop of pressure in the cylinder at the beginning of the suction stroke. Therefore, to avoid separation the absolute pressure head at the beginning of stroke (i.e., when $\alpha = 0°$) should not fall below the separation head.

$$[H_{atm} - H_s] > h_{sep}$$

Substituting Equation (26.53) in the above expression, we get:

$$\left[H_{atm} - \left\{h_s + \frac{l_{s1}}{g}\frac{A}{a_s}\omega^2 r + \frac{4fl_s}{2gd_s}\left(\frac{A\omega r}{\pi a_s}\right)^2 + \frac{1}{2g}\left(\frac{A\omega r}{\pi a_s}\right)^2\right\}\right] > h_{sep}$$

In the limiting condition, we get:

$$h_{sep} = H_{atm} - h_s - \frac{l_{s1}}{g}\frac{A}{a_s}\omega^2 r - \frac{4fl_s}{2gd_s}\left(\frac{A\omega r}{\pi a_s}\right)^2 - \frac{1}{2g}\left(\frac{A\omega r}{\pi a_s}\right)^2 \tag{26.65}$$

or

$$h_{sep} = H_{atm} - h_s - \frac{l_{s1}}{g}\frac{A}{a_s}\left(\frac{2\pi N}{60}\right)^2 r - \frac{4fl_s}{2gd_s}\left(\frac{2A\pi Nr}{60\pi a_s}\right)^2 - \frac{1}{2g}\left(\frac{2A\pi Nr}{60\pi a_s}\right)^2 \tag{26.65a}$$

If the small quantities are neglected, then the limiting speed may be calculated from the following expression as given below.

$$h_{sep} = H_{atm} - h_s - \frac{4fl_s}{2gd_s}\left(\frac{2ANr}{60\,a_s}\right)^2 \tag{26.66}$$

During the delivery stroke, the value of accelerating head between the cylinder and air vessel, i.e., h_{ad1} is very small. Thus, there is least possibility of falling pressure below the separation pressure.

26.10.6 Work Saved Against Friction by Fitting Air Vessel

Air vessels fitted in a reciprocating pump eliminates the fluctuations in the velocity of flow in suction and delivery pipes. This reduces the head loss due to friction in suction and delivery pipes and thus, it saves certain amount of energy. This can be determined by finding the difference between the work done against friction without air vessel and with air vessel.

(i) For single acting reciprocating pump The following analysis is applicable to both the suction as well as delivery stroke of the reciprocating pump. The velocity of flow in suction and delivery pipes is given from Equation (26.11a) as follows.

$$V = \frac{A}{a}\omega r \sin\alpha$$

The loss of head due to friction is given by,

$$h_f = \frac{4flV^2}{2gd} = \frac{4fl}{2gd}\left(\frac{A}{a}\omega r \sin\alpha\right)^2$$

As variation of h_f with α is parabolic in nature, the indicator diagram for the loss of head due to friction is a parabola. The work done by the pump against friction per stroke is given by the area of the indicator diagram due to friction. Therefore, the work done by pump against friction without air vessel is given by,

$$w_1 = \frac{2}{3}Lh_f \qquad \text{[i.e., area of the parabola]}$$

Here, h_f is the height at $\alpha = 90°$ and thus, we have,

$$w_1 = \frac{2}{3}L \times \frac{4fl}{2gd}\left(\frac{A}{a}\omega r\right)^2 \tag{26.67}$$

When the air vessel is fitted to the reciprocating pump, the velocity of flow through the pipes is the mean velocity of flow which is given by Equation (26.42) as follows.

$$V_m = \frac{A\omega r}{\pi a}$$

The loss of head due to friction is given by,

$$h_f = \frac{4flV_m^2}{2gd} = \frac{4fl}{2gd}\left(\frac{A\omega r}{\pi a}\right)^2 \quad (26.68)$$

As the h_f is independent of α, the work done against friction with air vessel will be the area of rectangle (refer Figure 26.11b).

$$w_2 = L \times h_f = L \times \frac{4fl}{2gd}\left(\frac{A\omega r}{\pi a}\right)^2$$

Thus, work saved is given by,

$$w_s = w_1 - w_2 = \frac{2}{3}L \times \frac{4fl}{2gd}\left(\frac{A}{a}\omega r\right)^2 - L \times \frac{4fl}{2gd}\left(\frac{A\omega r}{\pi a}\right)^2$$

$$\therefore w_s = \frac{4fl}{2gd} \times L\left(\frac{A}{a}\omega r\right)^2\left(\frac{2}{3} - \frac{1}{\pi^2}\right)$$

Now the percentage of work saved is given by,

$$\%w_s = \frac{w_s}{w_1} \times 100 = \left[\frac{4fl}{2gd} \times L\left(\frac{A}{a}\omega r\right)^2\left(\frac{2}{3} - \frac{1}{\pi^2}\right) \div \left\{\frac{2}{3}L \times \frac{4fl}{2gd}\left(\frac{A}{a}\omega r\right)^2\right\}\right] \times 100$$

$$\boxed{\therefore \%w_s = \left[\left(\frac{2}{3} - \frac{1}{\pi^2}\right) \div \frac{2}{3}\right] \times 100 = 84.8\%}$$

Therefore, by using air vessels in single acting reciprocating pump, 84.8% of the work done against friction can be saved.

(ii) For double acting reciprocating pump In case of double acting reciprocating pump without air vessel, the work lost in friction per stroke remains same as given in case of single acting reciprocating pump which is given by Equation (26.67) as follows.

$$w_1 = \frac{2}{3}L \times \frac{4fl}{2gd}\left(\frac{A}{a}\omega r\right)^2$$

When the air vessel is fitted to the double acting reciprocating pump, the velocity of flow through the pipes is the mean velocity of flow as given below.

$$V_m = \frac{Q_{th}}{a} = \frac{2ALN}{60a}$$

Since $N = \dfrac{60\omega}{2\pi}$ and $L = 2r$

$$\therefore V_m = \frac{2A \times 2r}{60a} \times \frac{60\omega}{2\pi} = \frac{2A\omega r}{\pi a} \quad (26.69)$$

The loss of head due to friction for double acting pump is given by,

$$h_f = \frac{4flV_m^2}{2gd} = \frac{4fl}{2gd}\left(\frac{2A\omega r}{\pi a}\right)^2 \quad (26.70)$$

As h_f is independent of α, the work done against friction with air vessel will be the area of rectangle (refer Figure 26.11b).

$$w_2 = L \times h_f = L \times \frac{4fl}{2gd}\left(\frac{2A\omega r}{\pi a}\right)^2$$

Thus, work saved is given by,

$$w_s = w_1 - w_2 = \frac{2}{3}L \times \frac{4fl}{2gd}\left(\frac{A}{a}\omega r\right)^2 - L \times \frac{4fl}{2gd}\left(\frac{2A\omega r}{\pi a}\right)^2$$

$$\therefore w_s = \frac{4fl}{2gd} \times L\left(\frac{A}{a}\omega r\right)^2\left(\frac{2}{3} - \frac{4}{\pi^2}\right)$$

Now percentage of work saved is given by,

$$\%w_s = \frac{w_s}{w_1} \times 100 = \left[\frac{4fl}{2gd} \times L\left(\frac{A}{a}\omega r\right)^2\left(\frac{2}{3} - \frac{4}{\pi^2}\right) \div \left\{\frac{2}{3}L \times \frac{4fl}{2gd}\left(\frac{A}{a}\omega r\right)^2\right\}\right] \times 100$$

$$\boxed{\therefore \%w_s = \left[\left(\frac{2}{3} - \frac{4}{\pi^2}\right) \div \frac{2}{3}\right] \times 100 = 39.2\%}$$

Therefore, by using air vessels in the double acting reciprocating pump, 39.2% of the work done against friction can be saved.

Example 26.9 The diameter and stroke of a single acting reciprocating pump are 0.15 m and 0.3 m, respectively. When the pump runs at 35 rpm, it lifts water through a head of 14 m above the centre of pump. The length and diameter of delivery pipe are 20 m and 0.1 m, respectively. When an air vessel is fitted on the delivery side 1.5 m above the centre of the pump and $f = 0.009$, determine the total pressure in the cylinder at the beginning and mid of the delivery stroke.

Solution

Let $D = 0.15$ m, $L = 0.3$ m, $N = 35$ rpm, $h_d = 14$ m, $l_d = 20$ m, $d_d = 0.1$ m, $l_{d1} = 1.5$ m and $f = 0.009$.

$$r = \frac{L}{2} = \frac{0.3}{2} = 0.15 \text{ m}$$

$$A = \frac{\pi}{4}D^2 = \frac{\pi}{4} \times 0.15^2 = 0.01767 \text{ m}^2$$

$$a_d = \frac{\pi}{4}d_d{}^2 = \frac{\pi}{4} \times 0.1^2 = 0.007854 \text{ m}^2$$

$$\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 35}{60} = 3.6652 \text{ rad/s}$$

$$\frac{l_{d1}}{g}\frac{A}{a_d}\omega^2 r = \frac{1.5}{9.81} \times \frac{0.01767}{0.007854} \times 3.6652^2 \times 0.15 = 0.6932$$

$$\frac{4fl_d}{2gd_d}\left(\frac{A\omega r}{\pi a_d}\right)^2 = \frac{4 \times 0.009 \times 20}{2 \times 9.81 \times 0.1} \times \left(\frac{0.01767 \times 3.6652 \times 0.15}{\pi \times 0.007854}\right)^2 = 0.0569$$

$$\frac{1}{2g}\left(\frac{A\omega r}{\pi a_d}\right)^2 = \frac{1}{2\times 9.81}\times\left(\frac{0.01767\times 3.6652\times 0.15}{\pi\times 0.007854}\right)^2 = 0.0079$$

The total pressure at the beginning of delivery stroke is given by,

$$H_d = h_d + \frac{l_{d1}}{g}\frac{A}{a_d}\omega^2 r + \frac{4fl_d}{2gd_d}\left(\frac{A\omega r}{\pi a_d}\right)^2 + \frac{1}{2g}\left(\frac{A\omega r}{\pi a_d}\right)^2$$

$$\therefore H_d = 14 + 0.6932 + 0.0569 + 0.0079 = \mathbf{14.758\ m}$$

Now $$\frac{4fl_{d1}}{2gd_d}\left(\frac{A}{a_d}\omega r\right)^2 = \frac{4\times 0.009\times 1.5}{0.1\times 2\times 9.81}\times\left(\frac{0.01767}{0.007854}\times 3.6652\times 0.15\right)^2 = 0.0421$$

The total pressure at the mid of delivery stroke is given by,

$$H_d = h_d + \frac{4fl_{d1}}{2gd_d}\left(\frac{A}{a_d}\omega r\right)^2 + \frac{4fl_d}{2gd_d}\left(\frac{A\omega r}{\pi a_d}\right)^2 + \frac{1}{2g}\left(\frac{A\omega r}{\pi a_d}\right)^2$$

$$\therefore H_d = 14 + 0.0421 + 0.0569 + 0.0079 = \mathbf{14.1069\ m}$$

Example 26.10 A single acting reciprocating pump of diameter and stroke as 0.25 m and 0.46 m, respectively runs at 55 rpm. The length and diameter of delivery pipe are 50 m and 0.12 m, respectively. The pump is fitted with an air vessel on the delivery side at the centre line of the pump. Determine the power saved in overcoming friction in the delivery pipe if the piston executes a simple harmonic motion and $f = 0.009$.

Solution

Let $D = 0.25$ m, $L = 0.46$ m, $N = 55$ rpm, $l_d = 50$ m, $d_d = 0.12$ m and $f = 0.009$.

$$r = \frac{L}{2} = \frac{0.46}{2} = 0.23\ \text{m}$$

$$A = \frac{\pi}{4}D^2 = \frac{\pi}{4}\times 0.25^2 = 0.0491\ \text{m}^2$$

$$a_d = \frac{\pi}{4}d_d{}^2 = \frac{\pi}{4}\times 0.12^2 = 0.01131\ \text{m}^2$$

$$\omega = \frac{2\pi N}{60} = \frac{2\times\pi\times 55}{60} = 5.76\ \text{rad/s}$$

Loss of head due to friction without air vessel is given by,

$$h_{fd} = \frac{4fl_d}{2gd_d}\left(\frac{A\omega r}{a_d}\right)^2 = \frac{4\times 0.009\times 50}{0.12\times 2\times 9.81}\times\left(\frac{0.0491\times 5.76\times 0.23}{0.01131}\right)^2 = 25.29\ \text{m}$$

Power required to overcome friction without air vessel is given by,

$$P_1 = \frac{\rho_w gALN}{60\times 1000}\times\frac{2}{3}h_{fd} = \frac{1000\times 9.81\times 0.0491\times 0.46\times 55}{60\times 1000}\times\frac{2}{3}\times 25.29 = 3.424\ \text{kW}$$

Loss of head due to friction with air vessel is given by,

$$h_{fd} = \frac{4fl_d}{2gd_d}\left(\frac{A\omega r}{\pi a_d}\right)^2 = \frac{4\times 0.009\times 50}{0.12\times 2\times 9.81}\times\left(\frac{0.0491\times 5.76\times 0.23}{\pi\times 0.01131}\right)^2 = 2.562 \text{ m}$$

Power required to overcome friction with air vessel is given by,

$$P_2 = \frac{\rho_w gALNh_{fd}}{60\times 1000} = \frac{1000\times 9.81\times 0.0491\times 0.46\times 55\times 2.562}{60\times 1000} = 0.52 \text{ kW}$$

$$\text{Power saved} = P_1 - P_2 = 3.424 - 0.52 = \mathbf{2.904\ kW}$$

Example 26.11 A double acting reciprocating pump has a bore of 0.15 m and stroke of 0.46 m. The diameter and length of suction pipe are 10 cm and 6.5 m, respectively and the suction lift is 4 m. Assume that the pump has a simple harmonic motion, atmospheric pressure head is 10.3 m of water and separation takes place at 2.5 m of water absolute. If $f = 0.02$, then find the maximum speed at which the said pump can be operated (i) without any air vessel on the suction side and (ii) with a large air vessel on the suction side close to the pump.

Solution

Let $D = 0.15$ m, $L = 0.46$ m, $d_s = 10$ cm $= 0.1$ m, $l_s = 6.5$ m, $h_s = 4$ m, $H_{atm} = 10.3$ m, $h_{sep} = 2.5$ m and $f = 0.02$.

$$r = \frac{L}{2} = \frac{0.46}{2} = 0.23 \text{ m}$$

(i) The maximum speed of the reciprocating pump without separation during suction stroke can be calculated from Equation (26.37) as follows.

$$\frac{l_s}{g}\frac{A}{a_s}\left(\frac{2\pi N}{60}\right)^2 r = H_{atm} - h_s - h_{sep}$$

Thus

$$\frac{6.5}{9.81}\times\frac{0.15^2}{0.1^2}\times\left(\frac{2\pi N}{60}\right)^2\times 0.23 = 10.3 - 4 - 2.5$$

$$0.00376N^2 = 3.8 \Rightarrow N = \sqrt{\frac{3.8}{0.00376}} = \mathbf{31.79\ rpm}$$

(ii) The maximum speed of the reciprocating pump without separation during suction stroke when air vessel is used can be calculated from Equation (26.65) as follows.

$$h_{sep} = H_{atm} - h_s - \frac{l_{s1}}{g}\frac{A}{a_s}\omega^2 r - \frac{4fl_s}{d_s 2g}\left(\frac{A\omega r}{\pi a_s}\right)^2 - \frac{1}{2g}\left(\frac{A\omega r}{\pi a_s}\right)^2$$

$$h_{sep} = H_{atm} - h_s - 0 - \frac{4fl_s}{d_s 2g}\left(\frac{A\omega r}{\pi a_s}\right)^2 - \frac{1}{2g}\left(\frac{A\omega r}{\pi a_s}\right)^2$$

$$h_{sep} = H_{atm} - h_s - \frac{1}{2g}\left(\frac{A\omega r}{\pi a_s}\right)^2\left(\frac{4fl_s}{d_s} + 1\right)$$

Thus

$$2.5 = 10.3 - 4 - \frac{1}{2\times 9.81}\times\left[\frac{(\pi/4)\times 0.15^2\times\omega\times 0.23}{\pi\times(\pi/4)\times 0.1^2}\right]^2\times\left(\frac{4\times 0.02\times 6.5}{0.1} + 1\right)$$

$$8.5746\times10^{-3}\,\omega^2 = 10.3-4-2.5 = 3.8$$

$$\therefore \omega = \sqrt{\frac{3.8}{8.5746\times10^{-3}}} = 21.05 \text{ rad/s}$$

Thus
$$\frac{2\pi N}{60} = 21.05$$

$$\therefore N = \frac{21.05\times60}{2\times\pi} = \mathbf{201.01\ rpm}$$

Example 26.12 A single acting reciprocating pump is to raise a liquid of density 1200 kg/m^3 through a vertical height of 11.5 m from 2.5 m below pump axis to 9 m above it. The plunger moves with simple harmonic motion has diameter 0.125 m and stroke 0.225 m. The suction and delivery pipes are of 75 mm diameter and 3.5 m and 13.5 m long, respectively. A large size air vessel is fitted to the delivery pipe only near the pump axis. If separation takes place at 0.88 bar below atmospheric pressure, then find (i) the maximum speed with which the pump can run without separation taking place and (ii) power required to drive the pump, if $f = 0.02$. Neglect slip for the pump.

Solution

Let $\rho = 1200$ kg/m^3, $(h_s + h_d) = 11.5$ m, $h_s = 2.5$ m, $h_d = 9$ m, $D = 0.125$ m, $L = 0.225$ m, $d_s = d_d = 75$ mm $= 0.075$ m, $l_s = 3.5$ m, $l_d = 13.5$ m, $h_{sep} = 0.88$ bar and $f = 0.02$.

$$h_{sep} = \frac{0.88\times10^5}{1200\times9.81} = 7.47 \text{ m}$$

$$r = \frac{L}{2} = \frac{0.225}{2} = 0.1125 \text{ m}$$

(i) $A = \frac{\pi}{4}D^2 = \frac{\pi}{4}\times0.125^2 = 0.0123 \text{ m}^2$

$$a_s = a_d = \frac{\pi}{4}d_s{}^2 = \frac{\pi}{4}\times0.075^2 = 0.00442 \text{ m}^2$$

Limiting condition for no separation is given by,

$$h_s + h_{as} = h_{sep} \Rightarrow 2.5 + h_{as} = 7.47$$

$$\therefore h_{as} = 7.47 - 2.5 = 4.97 \text{ m}$$

Maximum speed of pump without separation during suction stroke can be determined by,

$$h_{as} = \frac{l_s}{g}\frac{A}{a_s}\omega^2 r = \frac{l_s}{g}\frac{A}{a_s}\left(\frac{2\pi N}{60}\right)^2 r$$

Thus
$$\frac{3.5}{9.81}\times\frac{0.0123}{0.00442}\times\left(\frac{2\pi N}{60}\right)^2\times0.1125 = 4.97$$

$$\therefore N = \sqrt{\frac{4.97\times9.81\times0.00442\times60^2}{3.5\times0.0123\times(2\pi)^2\times0.1125}} = \mathbf{63.7\ rpm}$$

(ii) $Q_{th} = \dfrac{ALN}{60} = \dfrac{0.0123 \times 0.225 \times 63.7}{60} = 2.938 \times 10^{-3}\ \text{m}^3/\text{s}$

Thus $$V_{md} = \frac{Q_{th}}{a_d} = \frac{2.938 \times 10^{-3}}{0.00442} = 0.665\ \text{m/s}$$

The loss of head due to friction in delivery pipe is given by,

$$h_{fd} = \frac{4 f l_d {V_{md}}^2}{2 g d_d} = \frac{4 \times 0.02 \times 13.5 \times 0.665^2}{2 \times 9.81 \times 0.075} = 0.324\ \text{m}$$

Maximum value of h_{fs} during suction stroke is given by,

$$h_{fs} = \frac{4 f l_s}{2 g d_s}\left(\frac{A\omega r}{a_s}\right)^2 = \frac{4 f l_s}{2 g d_s}\left(\frac{A \times 2\pi N \times r}{a_s \times 60}\right)^2$$

$$\therefore h_{fs} = \frac{4 \times 0.02 \times 3.5}{0.075 \times 2 \times 9.81} \times \left(\frac{0.0123 \times 2 \times \pi \times 63.7 \times 0.1125}{0.00442 \times 60}\right)^2 = 0.83\ \text{m}$$

Power required to drive the pump is given by,

$$P = \frac{\rho g Q_{th}}{1000}\left(h_s + h_d + \frac{2}{3} h_{fs} + h_{fd}\right)\text{kW}$$

$$\therefore P = \frac{1200 \times 9.81 \times 2.938 \times 10^{-3}}{1000} \times \left(2.5 + 9 + \frac{2}{3} \times 0.83 + 0.324\right) = \mathbf{0.4281\ kW}$$

26.11 ❐ CHARACTERISTIC CURVES OF A RECIPROCATING PUMP

The constant speed characteristic curves for a reciprocating pump are obtained by plotting its discharge (Q), power input (P) and overall efficiency (η) against the head (H) developed by keeping the speed (N) constant. For obtaining the variable speed characteristic curves, the pump is operated at different speeds and its discharge is plotted against the speed by keeping the head constant. The characteristic curves of a reciprocating pump are shown in Figure 26.12.

1. ***Q* versus *H* curve:** It can be seen that the discharge of a reciprocating pump operating at constant speed slightly decreases as the head developed by the pump increases as shown in Figure 26.12(a). However, under ideal conditions, it is found to be independent of the head developed by the pump.

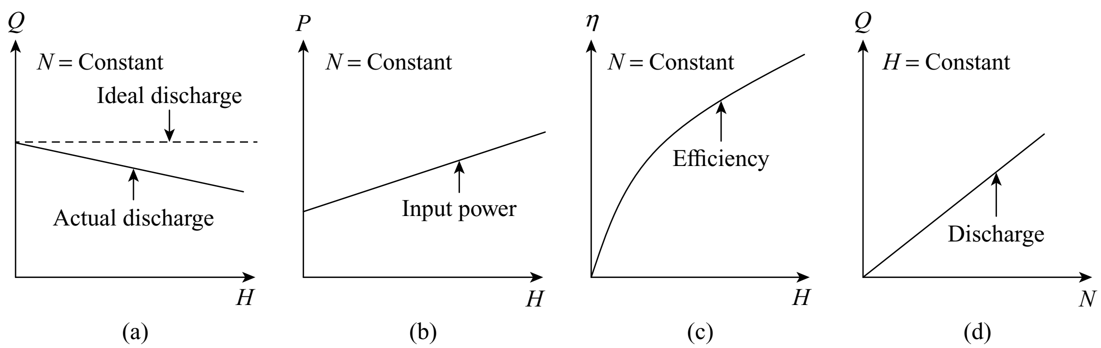


Figure 26.12 *Operating characteristic curves of a reciprocating pump*

2. ***P* versus *H* curve:** The input power curve starts away from the origin because even at zero discharge, some power is required to overcome the mechanical losses. The input power curve is observed to increase almost linearly with the increase in head developed by the pump as shown in Figure 26.12(b).
3. **η versus *H* curve:** The overall efficiency curve of a reciprocating pump is also found to increase with the increase in head developed by the pump as shown in Figure 26.12(c).
4. ***Q* versus *N* curve:** The discharge is observed to increase almost linearly with the increase in speed of the pump as shown in Figure 26.12(d).

26.12 ❒ ROTARY POSITIVE DISPLACEMENT PUMPS

Rotary positive displacement pumps were developed to avoid the complexity of construction and restriction on speeds of the reciprocating pumps. These may be constant delivery pump or variable delivery pump. Due to complex construction, the variable delivery pump are expensive than constant delivery pumps. The rotary positive displacement pumps have a stationary housing in which a power driven unit rotates and displaces the liquid and it also controls the opening and closing of suction and delivery ports. These pumps have the advantage of both the reciprocating and centrifugal pumps and it can produce moderately high pressure while running at higher speeds. These pumps are mainly used for pumping lubricant to the motors, engines, turbines and various machine tools. These are also used in oil hydraulic control systems but these pumps are not suitable for pumping of water.

Some of the rotary positive displacement pumps, namely vane pump, lobe pump, axial piston pump, gear pump, screw pump and radial piston pump are briefly discussed in this section. These rotary positive displacement pumps give continuous discharge of liquid at a uniform rate of flow and they are also known as constant delivery pumps. These pumps can also be used to give variable delivery by using variable speed control mechanism or by regulating the flow of liquid by valves.

26.12.1 Vane Pump

A vane pump consists of a hollow rotor which is eccentrically mounted in the casing as shown schematically in Figure 26.13. The hollow rotor disc has equispaced radial slots each fitted with a sliding vane. These vanes are free to slide radially. The vanes are kept pressed on the casing by means of springs and thus, it provides proper sealing between the suction and discharge connections. The liquid is trapped between the vanes and the casing. When the rotor rotates, the trapped liquid is forced to the delivery side (or pressure side). The theoretical volume displaced by these pump per second is given as follows.

$$Q_{th} = 2eb[2\pi(R-e) - n\,t] \times \frac{N}{60} \tag{26.71}$$

Here, e is the eccentricity between the rotor and the casing, b is the width of vane, R is the inner radius of the casing, n is the number of vanes, t is the thickness of vane and N is the rotor speed in rpm.

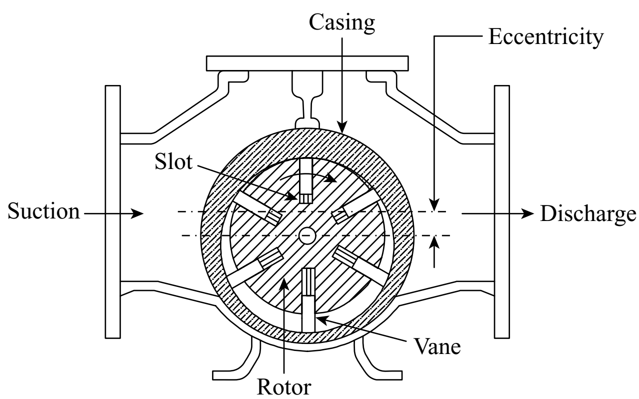


Figure 26.13 *Vane pump*

A single stage pump is able to develop pressures varying from 1.75 MPa to 7 MPa. For obtaining higher pressures, more than one stage can also be employed. The vane pump units find wide range of applications in machine tools and other industrial applications. Its volumetric efficiency varies from 82% to 92%, mechanical efficiency typically varies from 90% to 95% and overall efficiency varies from 80% to 95%.

26.12.2 Lobe Pump

The schematic view of two lobed and three lobed pumps are shown in Figure 26.14(a) and 26.14(b), respectively. The lobes form a liquid tight joint at each meshing point with the pump casing. Both the rotors do not contact each other and are driven externally. The liquid is to be filled before starting the pump. The liquid continuously traps in the pockets formed between the lobes and the pump casing. When the rotor rotates, the trapped oil is forced to the delivery side. Generally, these pump units are used with oil. These pumps have higher capacity and are less noisy than the gear pumps. These units give smooth but non-uniform flow rate of liquid. Due to smaller number of mating elements, these pump units have a greater amount of pulsation in its output.

26.12.3 Axial Piston Pump

It consists of a cylinder block which contains a number of cylinders bored in it along a circle. Each cylinder houses a piston as schematically shown in Figure 26.15.

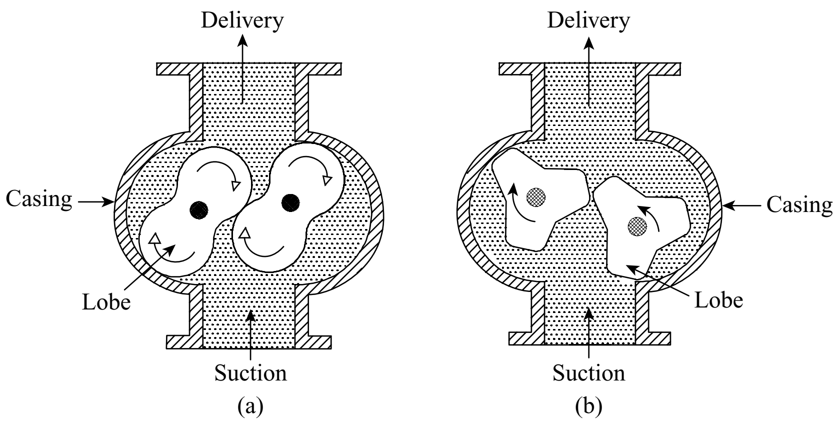


Figure 26.14 *Lobe pumps*

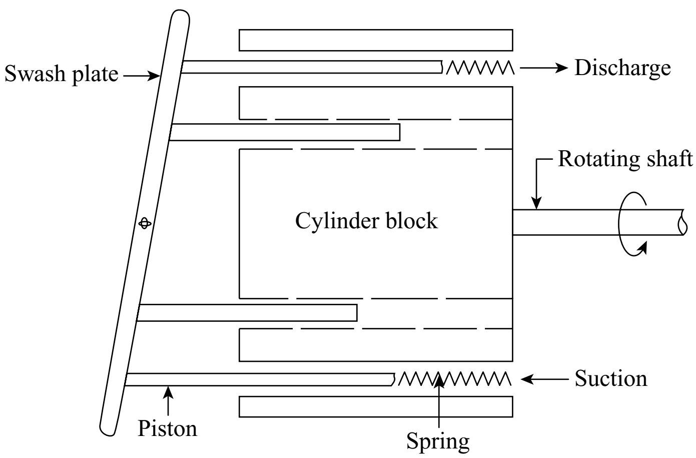


Figure 26.15 *Axial piston pump*

The cylinder block is fixed to the driving shaft. The pistons are spring loaded and are butt against the swash plate. When the cylinder block rotates, the pistons move in and out according to the distance between the swash plate and the cylinder block. There are two ports separated from each other towards the head side of the block. One port is connected to the suction side and the other to the discharge side. The sliding movements of the pistons produce the suction and compression in the cylinder. The tilting of swash plate is used to adjust the displacement of the pump. These pumps are reversible and can develop pressures up to 200 bar. Its overall efficiency varies from 90% to 98%.

26.12.4 Gear Pump

The gear pumps can be external gear pump or internal gear pump. The external gear pump is discussed in the next chapter. Here, internal gear pump is explained whose schematic view is shown in Figure 26.16.

It consists of an internal gear (or an idler), an outer driving spur gear and an external casing. The internal gear is fitted eccentrically to the outer spur gear. The space between the outside diameter of driving gear and the inside diameter of idler is sealed by a crescent shape projection which entraps the liquid to be pumped. When the driving gear gets its motion from the motor, the teeth come out of the mesh and there is an increase in volume. This creates a vacuum and thus, it draws liquid from the reservoir. The liquid fills in the space between the teeth of the driving gear and idler. As the driving gear keeps on rotating, the teeth mesh and the entrapped liquid is forced out to the delivery side. Typically its volumetric efficiency varies from 80% to 90%.

26.12.5 Screw Pumps

A screw pump is an axial flow type pump. It consists of a screw which rotates in a closely fitted cylindrical housing. The casing has suction and delivery ports. There may be one, two or three screws. In a two screw pump, one will be the rotor and the other will be an idler, whereas in a three screw pump there will be two idlers on either side of the rotor. Two idlers act as seal to the power rotor and are driven by fluid pressure. One screw and two screws pumps are shown in Figure 26.17(a) and 26.17(b), respectively.

The screw may be single helical or double helical. The double helical screws are more balanced and give more discharge than single helical screws. The liquid is carried forward to the discharge side along the rotor in pockets formed between teeth and the casing, like a nut on the power screw. These pumps deliver non-pulsating continuous flow. Also large discharge and high pressure are possible with these units.

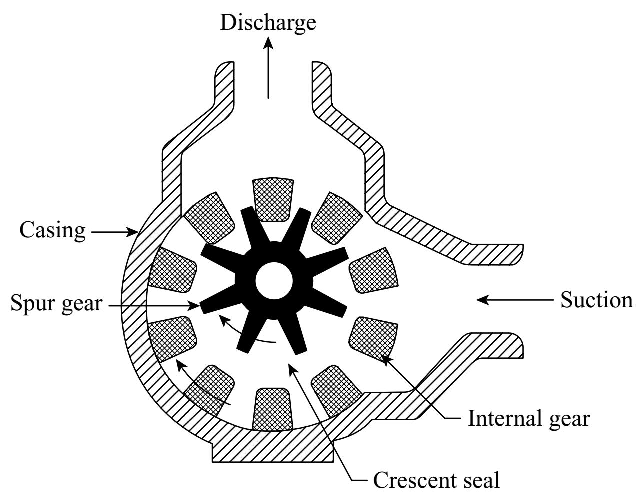


Figure 26.16 *Internal gear pump*

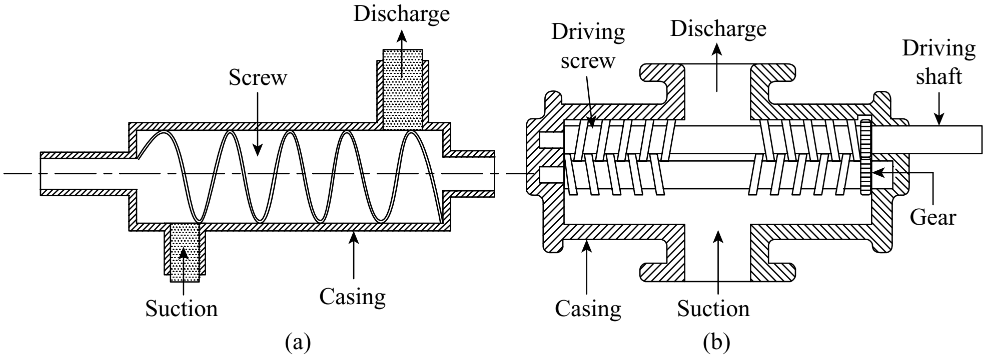


Figure 26.17 *Screw pumps*

26.12.6 Radial Piston Pump

The schematic view of a radial piston pump is illustrated in Figure 26.18.

It consists of a cylindrical rotating block which carries the pistons radially. It rotates about an axis offset from the fixed ring (or reaction ring) fixed to the casing. The pump gets its power from motor through the shaft. It rotates the cylinder block due to which the pistons move in and out. The pistons always remain in contact with the fixed ring due to centrifugal force. The inlet and outlet ports are at the centre of the cylinder block. The suction and discharge lines connect to the side of the casing.

As the shaft rotates, the pistons on the suction side move away and suck liquid while diametrically opposite pistons move inside and increase pressure which forces out the liquid. The theoretical discharge by this pump is given below.

$$Q_{th} = \pi end^2 \times \frac{N}{60} \tag{26.72}$$

Here, e is the eccentricity of the fixed ring and the casing, n is the number of pistons, d is the diameter of piston and N is the rotating block speed in rpm.

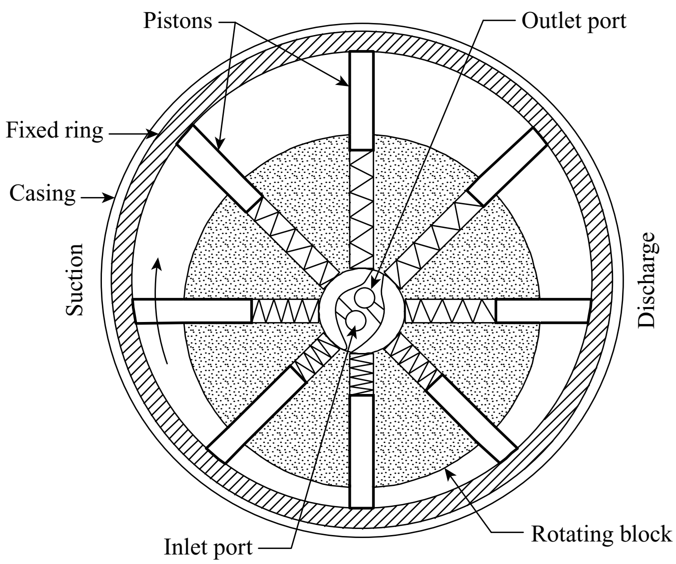


Figure 26.18 *Radial piston pump*

These pumps are considered as an ideal pump for use in heavy duty machines, such as control systems of aircraft, governors of hydraulic turbines, gas turbines and steam turbines, heavy duty earth moving equipments. These units are capable of developing pressure as high as 400 MPa. Its overall efficiency varies from 85% to 95%.

Summary

1. Theoretical discharge of the reciprocating pump per second is given by,

   (i) $Q_{th} = (ALN)/60$ (Single acting)

   (ii) $Q_{th} = (2ALN)/60$ (Double acting)

   Here, A is the area of the piston or cylinder, L is the length of the stroke or cylinder and N is the crank speed in rpm.

2. **Work done per second:**

   (i) $w = [\rho_w g ALN \times (h_s + h_d)]/60$ (Single acting pump)

   (ii) $w = [2\rho_w gALN \times (h_s + h_d)]/60$ (Double acting pump)

   Here, h_s is the suction head and h_d is the delivery head.

3. The difference between the theoretical discharge (Q_{th}) and actual discharge (Q_{act}) is known as slip (S).

4. The pressure head due to acceleration in the suction and delivery pipes is given by,

$$h_{as} = \frac{l_s}{g}\frac{A}{a_s}\omega^2 r\cos\alpha \quad \text{(Suction)}$$

$$h_{ad} = \frac{l_d}{g}\frac{A}{a_d}\omega^2 r\cos\alpha \quad \text{(Delivery)}$$

   Here, l is the length of the suction or delivery pipe, a is the area of the suction or delivery pipe, r is the radius of the crank, α is the crank angle and ω is the angular speed of crank.

5. **Maximum pressure head due to acceleration:**

$$(h_a)_{\max} = \frac{l}{g}\frac{A}{a}\omega^2 r$$

6. The loss of head due to friction in suction and delivery pipe is given by,

$$h_{fs} = \frac{4fl_s}{2gd_s}\left(\frac{A}{a_s}\omega r\sin\alpha\right)^2 \text{ and } h_{fd} = \frac{4fl_d}{2gd_d}\left(\frac{A}{a_d}\omega r\sin\alpha\right)^2$$

7. **Maximum value of head loss due to friction:**

$$(h_f)_{\max} = \frac{4fl}{2gd}\left(\frac{A}{a}\omega r\right)^2$$

8. In the indicator diagram, the pressure head on the piston is plotted along the ordinate and the stroke length (L) along the abscissa. The work done by the pump is proportional to the area of indicator diagram.

9. Maximum speed of the reciprocating pump without separation during suction and delivery stroke can be calculated from the following equations.

$$\frac{l_s}{g}\frac{A}{a_s}\left(\frac{2\pi N}{60}\right)^2 r = H_{atm} - h_s - h_{sep} \quad \text{(During suction stroke)}$$

$$\frac{l_d}{g}\frac{A}{a_d}\left(\frac{2\pi N}{60}\right)^2 r = H_{atm} + h_d - h_{sep} \quad \text{(During delivery stroke)}$$

   Here, H_{atm} is the atmospheric pressure head and h_{sep} is the separation pressure head.

10. Work done per second due to acceleration and friction in suction and delivery pipes is given by,

    (i) $w = \dfrac{\rho_w gAN}{60}\left(h_s + h_d + \dfrac{2}{3}h_{fs} + \dfrac{2}{3}h_{fd}\right)L$

    (Single acting pump)

    (ii) $w = \dfrac{2\rho_w gAN}{60}\left(h_s + h_d + \dfrac{2}{3}h_{fs} + \dfrac{2}{3}h_{fd}\right)L$

    (Double acting pump)

    Here, h_{fs} and h_{fd} is the head loss due to friction in suction and delivery pipes, respectively.

11. An air vessel is a closed chamber which contains compressed air at its top portion and liquid (water) being pumped at the bottom portion of the chamber. A large amount of work in overcoming the frictional resistance in suction and delivery pipes can be saved by using an air vessel.

12. **The rate of flow of water into the air vessel:** $= A\omega r[\sin\alpha - (1/\pi)]$ (single acting pump) and $= A\omega r[\sin\alpha - (2/\pi)]$ (double acting pump)

13. **Mean velocity in the suction or delivery pipe:** $V_m = A\omega r/\pi a$

14. **The theoretical work done per second by a pump fitted with air vessels to both the suction and delivery pipes:**

$$w = \frac{\rho_w g ALN}{60}[(h_s + h_d) + (h_{fs} + h_{fd})]$$

15. **Maximum speed of the reciprocating pump with air vessel without separation during suction stroke:**

$$h_{sep} = H_{atm} - h_s - \frac{4fl_s}{2gd_s}\left(\frac{2ANr}{60\,a_s}\right)^2$$

16. In single acting reciprocating pump, 84.8% of the work done against friction can be saved by using air vessels while in double acting reciprocating pump 39.2% work can be saved.

17. Rotary positive displacement pumps were developed to avoid the complexity of construction and restriction on speeds of the reciprocating pumps. Constant delivery pumps are vane pump, lobe pump, axial piston pump, gear pump, screw pump and radial piston pump.

Multiple-choice Questions

1. Generally, reciprocating pumps are best suited for
(a) Where constant heads are required despite fluctuation in discharge.
(b) Where constant supplies are required despite fluctuation in pressure.
(c) For pumping large liquid flows for medium heads at high speeds.
(d) None of the above.

2. The limiting value of separation pressure head for water in absolute unit is
(a) 10.3 m.
(b) 7.3 m.
(c) 3.5 m.
(d) 2.5 m.

3. In a reciprocating pump, the minimum absolute pressure during suction and delivery strokes occurs, respectively at the
(a) End of suction stroke and end of delivery stroke.
(b) Mid of suction stroke and beginning of delivery stroke.
(c) Beginning of suction stroke and end of delivery stroke.
(d) None of the above.

4. The reciprocating pump cannot run at high speed due to
(a) High rate of pulsation in flow.
(b) Increased acceleration head.
(c) Increased possibility of cavitation.
(d) All the above.

5. The slip for reciprocating pump may be
(a) –ve.
(b) +ve.
(c) +ve or –ve.
(d) Zero.

6. Air vessels in a reciprocating pump are used
(a) To increase the pump head.
(b) To smoothen the flow.
(c) To increase the efficiency.
(d) To reduce the acceleration heads to minimum.

7. In suction and delivery pipes, maximum head loss due to friction occurs at
(a) The beginning of the stroke.
(b) The mid of the stroke.
(c) At the end of the stroke.
(d) None of the above.

8. The work saved by fitting an air vessel to a single and double acting reciprocating pump is respectively
(a) 92.3% and 48.8%.
(b) 84.8% and 39.2%.
(c) 48.8% and 92.3%.
(d) 39.2% and 84.8%.

9. In negative slip of a reciprocating pump, the actual discharge as compared to theoretical discharge is
(a) Equal.
(b) Less.
(c) More.
(d) None of the above.

10. For pumping highly viscous liquids, which of the following pump is to be used?
(a) Turbine pump.
(b) Centrifugal pump.
(c) Plunger pump.
(d) Screw pump.

11. Which one of the following statement is false?
(a) Generally, axial piston pump have odd number of cylinders to give uniform flow.
(b) Axial piston pumps have better performance than gear and vane pumps.
(c) A gear pump can operate at higher pressures than an axial piston pump.
(d) None of the above.

12. Which of the following is the rotary positive displacement pump?
(a) Screw pump.
(b) Vane pump.
(c) Gear pump.
(d) All the above.

13. Which one of the following can generate the highest pressure?
(a) Lobe pump.
(b) Vane pump.
(c) Screw pump.
(d) Gear pump.

Review Questions

1. Describe the principle, constructional and working details of a reciprocating pump. Also determine the discharge, work and power input for it.
2. Differentiate between the single acting and double acting reciprocating pumps. Also discuss the working principle, discharge, work and power required for a double acting reciprocating pump.
3. Define slip, coefficient of discharge and negative slip of a reciprocating pump.
4. Give comparisons between centrifugal and reciprocating pumps.
5. How the acceleration of the piston of a reciprocating pump affects the velocity and acceleration of the water in suction and delivery pipes? Also derive an expression for the pressure head developed due to acceleration of the piston. Assume that the piston has simple harmonic motion.
6. Define indicator diagram. Show that work done by the reciprocating pump is proportional to the area of the indicator diagram.
7. Discuss the effects of acceleration on the indicator diagram with a neat graph during both suction and delivery strokes.
8. Derive expressions for maximum speed of reciprocating pump without separation during both suction and delivery strokes.
9. What is an air vessel? Also give its function for reciprocating pump.
10. Show from the first principle that the work saved in a single acting and double acting reciprocating pump by fitting an air vessel is 84.8% and 39.2%, respectively.
11. Derive expressions for water flow rate in and out of air vessel for single and double acting reciprocating pumps. Also determine the crank angle for which there will be no flow into or from the air vessel.
12. Discuss the pressure heads in cylinder during delivery stroke of a reciprocating pump with air vessel for different values of crank angles.
13. Derive an expression for the work done by a reciprocating pump with air vessel and also discuss its effect in indicator diagram.
14. Derive an expression for the maximum speed of a reciprocating pump with air vessel during suction stroke.
15. Explain the characteristic curves of reciprocating pump with neat sketches.
16. What do you mean by rotary positive displacement pumps? Explain the constructional and working of a vane and internal gear pumps.
17. Briefly discuss (i) radial piston pump, (ii) lobe pump and (iii) axial piston pump.

Problems

1. A single acting reciprocating pump running at 100 rpm delivers 15 *l*/s water. The diameter and stroke of the cylinder are 200 mm and 300 mm, respectively. Determine (i) the coefficient of discharge and (ii) percentage slip.
   [**Ans.** 0.9554, 4.46%]
2. A double acting reciprocating pump running at 50 rpm takes water from 3 m and delivers at 40 m. The diameter and stroke of the piston are 0.18 m and 0.36 m, respectively. Determine the power required to drive the pump if mechanical efficiency is 0.86. Also determine the discharge of the pump neglecting the area of the piston rod.
   [**Ans.** 7.505 kW, 0.0153 m^3/s]
3. The stroke and piston area of a single acting reciprocating pump are 0.3 m and 0.15 m^2, respectively. The water is lifted through a total head of 12 m while the pump is running at 50 rpm. If the actual discharge of the pump is 36.5 litres per second and its mechanical efficiency is 88%, then find (i) the coefficient of discharge, (ii) percentage of slip and (iii) power required to drive the pump.
   [**Ans.** 0.9733, 2.67%, 4.883 kW]
4. The diameter of piston and piston rod of a double acting reciprocating pump are 250 mm and 50 mm, respectively. The suction and delivery heads are 5 m and 17.5 m, respectively. If the mean piston speed is 0.5 m/s, then find (i) the discharge and (ii) force required to push the piston in and out strokes.
   [**Ans.** 0.024 m^3/s, 10.48 kN, 10.72 kN]
5. A single acting reciprocating pump having a cylinder diameter of 170 mm and stroke of 250 mm is used to raise the water through a total height of 30 m. Its crank rotates at 60 rpm. Find the theoretical power required to run the pump and theoretical discharge. If the actual discharge is 5.35 litres per second, then find the percentage slip. If the delivery pipe is 100 mm in diameter and is 20 m long, then find the acceleration head at the beginning and middle of the delivery stroke.
   [**Ans.** 1.669 kW, 0.00567 m^3/s, 5.64%, 29.05 m, 0]
6. A single acting reciprocating pump operating at 50 rpm has a piston diameter of 120 mm and stroke of 200 mm. The suction and delivery heads are 3 m and 12 m, respectively. Determine (i) the theoretical discharge, (ii) force required

for working the piston during the suction stroke, (iii) force required for working the piston during the delivery strokes if the efficiencies of suction and delivery strokes are 70% and 75%, respectively and (iv) the power required to drive the pump.

[**Ans.** 0.00188 m^3/s, 0.4755 kN, 1.775 kN, 0.374 kW]

7. The diameter and stroke of a single acting reciprocating pump are 150 mm and 300 mm, respectively. The pump is 4 m above the water surface. The diameter and length of a suction pipe are 100 mm and 5 m, respectively. If the pump is running at 40 rpm and atmospheric pressure head is 10.3 m of water, then find (i) the pressure head due to acceleration at the beginning of the suction stroke, (ii) maximum pressure head due to acceleration, and (iii) pressure head in the cylinder at the beginning and end of the suction stroke.

[**Ans.** (i) 3.02 m, 3.02 m, 3.28 m of water abs, 9.32 m of water abs]

8. If in the problem 7, the delivery pipe diameter is 100 mm and length is 30 m and the water is delivered in the tank which is 25 m above the centre of the pump, then determine (i) the pressure head due to acceleration at the beginning and end of delivery stroke and (ii) pressure head in the cylinder at the beginning and end of delivery stroke.

[**Ans.** (i) 18.12 m, 53.42 m of water abs, –18.12 m, 17.18 m of water abs]

9. The diameter and stroke of single acting reciprocating pump are 150 mm and 300 mm, respectively. Both the suction and delivery pipes are 100 mm in diameter. The lengths of the suction and delivery pipes are 5 m and 30 m, respectively. The centre of the pump is 3 m above the water surface in the sump and 20 m below the delivery water level. If the pump is working at 35 rpm and atmospheric pressure is given as 10.3 m of water, then find (i) the pressure heads on the piston at the beginning, middle and end of the suction stroke, (ii) pressure heads on the piston at the beginning, middle, and end of the delivery stroke and (iii) power required to run the pump.

[**Ans.** (i) 4.99, 7.3, 9.61 m of water abs (ii) 44.16, 30.3, 16.44 m of water abs, (iii) 0.6972 kW]

10. The bore and stroke of a single acting reciprocating pump are 150 mm and 300 mm, respectively. The total head is 20 m. The diameter and length of delivery pipe are 100 mm and 22 m, respectively. If the pump is working at 50 rpm, then determine (i) the theoretical power required to run the pump and (ii) acceleration head at the beginning, middle and end of the delivery stroke in the cylinder.

[**Ans.** 0.8668 kW, (ii) 20.75 m, 0 m, –20.75 m]

11. For a single acting reciprocating pump, the diameter and the length of the suction pipe are 4.5 cm and 6.5 m and that of delivery pipe are 3.5 cm and 18.5 m, respectively. The diameter of the piston and stroke length is 10.5 cm and 20.5 cm, respectively. The centre of the pump is 4.5 m above the water level in the sump and the delivery tank is 14.5 m above the centre line of the pump. The separation of water occurs at 8 N/cm^2 below the atmospheric pressure head. Determine the maximum speed at which the pump can run without separation when atmospheric pressure head is 10.3 m of water.

[**Ans.** 19 rpm]

12. The diameter and stroke of a single acting reciprocating pump are 120 mm and 300 mm, respectively. The water is lifted by a pump through a total head of 30 m. The diameter and the length of delivery pipe are 100 mm and 20 m, respectively. Determine (i) the theoretical discharge and theoretical power required to run the pump if its speed is 40 rpm, (ii) percentage slip if the actual discharge is 2.21 litres per second and (iii) acceleration head at the beginning and middle of the delivery stroke.

[**Ans.** 2.262 litres/s, 0.6657 kW, 2.29%, 7.73 m, 0 m]

13. A single acting reciprocating pump has a diameter of 10 cm and stroke length 20 cm. The diameter and length of the suction pipe are 5 cm and 6.5 m, respectively. The suction lift of the pump is 3.2 m and the separation occurs when the pressure in the pump falls below 2.5 m of water absolute. If the atmospheric pressure head is 10.3 m of water, then determine the maximum speed at which pump can run without separation in the suction pipe.

[**Ans.** 39.82 rpm]

14. A single acting reciprocating pump drawing water from a sump and delivering to a tank has a cylinder of diameter of 100 mm and a stroke length of 200 mm. The water level in the sump is 4.5 m below the centre line of the pump and in the tank water level is 13 m above the pump centre. The diameter and length of the suction pump are 40 mm and 5 m while that of delivery pipe the diameter and length are 30 mm and 20 m, respectively. If atmospheric pressure head is 762 mm of mercury and the separation occurs at 2.5 m of water absolute, then find the maximum speed of the pump.

[**Ans.** 28.98 rpm]

15. A three throw pump has cylinder of 22.5 cm diameter and stroke of 45 cm each. The pump delivers water at rate of 0.075 cubic metres per second at a head of 81 m. Friction losses in the suction and delivery pipes are 1 m and 18 m, respectively. If the velocity of water in the delivery pipe is 0.9 m/s, overall efficiency is 80% and slip is 3%, then find the speed of the pump and power required to drive the pump.

[**Ans.** 86.4 rpm, 73.575 kW]

16. A single acting reciprocating pump delivers water at a height of 25 m through a delivery pipe 40 m long and 140 mm in diameter. The diameter of the piston and stroke length is 250 mm and 440 mm, respectively. The atmospheric pressure head is 10.3 m of water and the cavitation occurs at 2.5 m of water absolute. Determine the maximum speed at which the pump can run without separation on the delivery side if (i) the pipe runs first horizontally and then vertically

upwards and (ii) the pipe raise first vertically and then runs horizontally.

[**Ans.** 32.34 rpm, 15.77 rpm]

17. The bore and stroke of a reciprocating pump are 25 cm and 50 cm, respectively. The pump delivers water through a 10 cm delivery pipe to a tank located at 15 m above it and 28 m horizontally from it. The atmospheric pressure head is 10.3 m of water and connecting rod - crank ratio is 5. If separation occurs at a pressure of 22.4 kN/m^2 absolute, then determine the safe speed at which pump should run for the following arrangements of delivery pipe, such as (a) the delivery pipe is horizontal from the pump and then vertical up to the tank and (b) the delivery pipe is vertical from the pump and then horizontal up to the tank.

[**Ans.** 19.57 rpm, 11.54 rpm, case (i)]

18. A single acting reciprocating pump has a stroke length of 0.2 m. The suction pipe is 8.5 m long and its diameter is 7.5 cm. The ratio of piston diameter to the suction pipe diameter is 4 : 3. The pump is 3.5 m above the water level in the sump. If the crank is running at 40 rpm, then find the pressure head on the piston at the beginning, middle and end of the suction stroke. Take friction coefficient as $f = 0.009$.

[**Ans.** 6.2 m, 3.615 m, 9.5 m]

19. A double acting reciprocating pump running at 30 rpm has a stroke length and diameter as 0.4 m and 0.2 m, respectively. The pump sucks water from a sump 1.2 m below through a suction pipe of length and diameter as 2.6 m and 0.1 m, respectively. The water is delivered 25 m above the pump through a delivery pipe 0.1 m in diameter and 35 m long. When the piston has moved through a distance of 0.1 m from the inner dead centre, then find (a) the force on the piston from suction side, (b) force on the piston from delivery side and (c) net force due to fluid pressure on the piston. Assume simple harmonic motion, friction coefficient as $f = 0.009$ and atmospheric pressure head = 10.3 m of water.

[**Ans.** 2410.05 N, 7475.77 N, 5065.72 N]

20. A single acting reciprocating pump running at 30 rpm has a stroke length and diameter as 0.21 m and 0.105 m, respectively. The lengths of suction and delivery pipes are 9 m and 27 m, respectively. The diameters of both the suction and delivery pipes are 85 mm. The suction and delivery heads are 5 m and 16 m, respectively. Find (i) the pressure head in the cylinder at the beginning, middle and end of the suction stroke, (ii) pressure head in the cylinder at the beginning, middle and end of the delivery strokes and (iii) also determine the power required for driving the pump. Take friction coefficient $f = 0.01$ and atmospheric pressure head = 10.3 m of water.

[**Ans.** 3.849 m, 5.245 m, 6.751 m, 30.652 m, 26.4641 m, 21.948 m, 0.18861 kW]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (b)	**2.** (d)	**3.** (c)	**4.** (d)	**5.** (c)
6. (d)	**7.** (b)	**8.** (b)	**9.** (c)	**10.** (d)
11. (c)	**12.** (d)	**13.** (c)		

Chapter **27**

Hydraulic Systems

27.1 ❐ INTRODUCTION

There are various hydraulic systems and devices which are used to transmit power with the help of an incompressible fluid under pressure. In these machines, hydraulic energy is transmitted through the liquid medium for which water or oil is used. Generally, these devices are based on the principles of hydrostatics and hydrokinetics (or rotodynamic). These devices either store hydraulic energy and then transmit this energy when required or magnify the hydraulic energy many times and then transmit the same. Some of these devices are (i) hydraulic press, (ii) hydraulic accumulator, (iii) hydraulic intensifier, (iv) hydraulic ram, (v) hydraulic lift, (vi) hydraulic crane, (vii) hydraulic coupling, (viii) hydraulic torque converter, (ix) air lift pump, (x) jet pump and (xi) gear pump. In this chapter, the function, construction and working details of the above mentioned hydraulic devices are described.

27.2 ❐ HYDRAULIC PRESS

The hydraulic press is a device used for lifting heavy loads by the application of much smaller force. The first hydraulic press was built in 1795 by Joseph Bramah and it is still in use. It is based on Pascal's law which states that intensity of pressure is transmitted equally in all directions through a mass of fluid at rest.

27.2.1 Working Principle

The working principle of a hydraulic press may be explained with the help of Figure 27.1(a). The hydraulic press system consists of two cylinders say C_1 and C_2 of different diameters. The larger diameter cylinder C_1 has a ram while the smaller diameter cylinder C_2 has a plunger. These two cylinders are connected by a chamber that contains fluid in it through which pressure is transmitted.

When a small force F is applied on the plunger in the downward direction, a pressure is produced on the liquid in contact with the plunger. This pressure is transmitted equally in all directions and it acts on the ram in the upward direction. Thus, the heavier load placed on the ram is lifted up.

Let W be the weight to be lifted, F be the force applied on the plunger, $A = (\pi/4)D^2$ be the area of ram, $a = (\pi/4)d^2$ be the area of plunger and $p = (F/a)$ be the pressure intensity produced by force.

According to Pascal's law, the pressure intensity (p) will be equally transmitted in all directions and it is given below.

$$p = \frac{F}{a} \text{ also } p = \frac{W}{A}$$

Thus

$$\frac{W}{A} = \frac{F}{a} \tag{i}$$

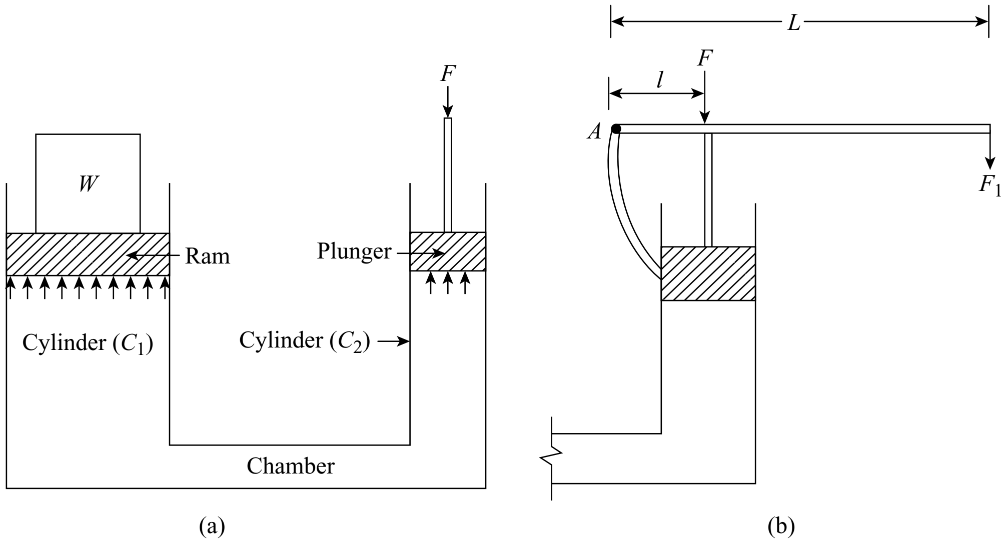


Figure 27.1 *Working of a hydraulic press*

$$\therefore \boxed{W = F \times \frac{A}{a}} \tag{27.1}$$

The mechanical advantage is given by the following relation.

$$\boxed{\text{M.A.} = \frac{\text{Load lifted}}{\text{Force applied}} = \frac{W}{F} = \frac{A}{a}} \tag{27.2}$$

If a force is applied by the lever on a plunger, then a force F_1 smaller than F can lift the load W as shown in Figure 27.1(b).

Taking moments about A, we get:

$$F_1 \times L = F \times l$$

$$\therefore F = F_1 \times \frac{L}{l} \tag{27.3}$$

Substituting the value of F in Equation (27.1), we get:

$$\boxed{W = F_1 \times \frac{L}{l} \times \frac{A}{a}} \tag{27.4}$$

$$\therefore \boxed{MA = \frac{W}{F_1} = \frac{L}{l} \times \frac{A}{a}} \tag{27.5}$$

Here, (L/l) is called the leverage of the hydraulic press.

The effective force transmitted to generate pressure on the liquid is reduced by the amount lost in friction. The pressure intensity in a static mass of fluid is also same throughout. If the percentage packing friction for each ram and plunger is denoted by k, then expression (i) is written as follows.

$$\frac{F}{a} \times \left[1 - \frac{k}{100}\right] = \frac{W}{A \times [1 - (k/100)]} \tag{27.6}$$

Thus $$W = F \times \frac{A}{a} \times \left[1 - \frac{k}{100}\right]^2 = F_1 \times \frac{L}{l} \times \frac{A}{a} \times \left[1 - \frac{k}{100}\right]^2 \quad \text{[Substitute Equation (27.3)]}$$

$$\therefore \text{M.A.} = \frac{W}{F_1} = \frac{L}{l} \times \frac{A}{a} \times \left[1 - \frac{k}{100}\right]^2$$

The volume of liquid displaced by the plunger (Q) is equal to that reaches to the ram area. If h is the stroke of the plunger, n is the number of strokes made per second and H is the lift of the ram in a second then the expression is given below.

$$Q = a \times h \times n = A \times H$$

$$\boxed{\therefore n = \frac{A \times H}{a \times h} = \frac{(\pi/4)D^2 \times H}{(\pi/4)d^2 \times h} = \frac{D^2 \times H}{d^2 \times h}} \quad (27.7)$$

If n numbers of strokes are completed in t seconds, then work supplied by the plunger is given below.

$$\boxed{w = \frac{F \times h \times n}{t} = \frac{W \times H}{t}} \quad (27.8)$$

Thus, the power required to drive the plunger is $P = (w/1000)$ kW.

27.2.2 Actual Hydraulic Press

The simplest form of an actual hydraulic press is shown in Figure 27.2 and it consists of a ram sliding in a fixed cylinder. A movable plate is attached to the lower end of the ram which moves up and down with the ram between two fixed plates. The upper and lower fixed plates are joined by columns. The ram is operated by liquid under pressure which is supplied by a pump. Usually, a hydraulic accumulator is provided between the press and the pump, where it stores high pressure liquid while the press is at rest. When liquid under pressure is supplied to the cylinder, then the ram moves in the downward direction. It exerts a force on any material placed between the lower fixed plate and the movable plate. Thus, the material gets pressed or any desired mechanical operation is performed. To bring back the ram in its initial position, the liquid from the cylinder is removed. As a result, the ram along with the movable plate moves up by the action of return weights.

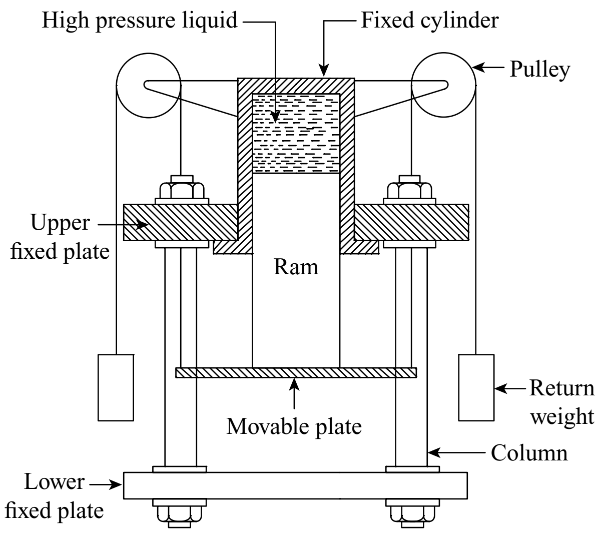


Figure 27.2 *Actual hydraulic press*

27.2.3 Applications

Hydraulic press is used to complete the jobs which require tremendous pressure. Generally, electrically driven pumps are used to supply oil under pressure to these devices. In hydraulic presses, a total thrust ranging from about 50 MN to 100 MN can be produced. Commonly, the hydraulic presses are used in (i) sheet metal press work, (ii) die sinking, (iii) cotton press, (iv) forging press, (v) bakelite press, (vi) plate press, (vii) metal pushing press, (viii) packing press, (ix) filter press, (x) drawing and pushing rods and (xi) punching, bending, drawing, straightening operations of any metal piece.

Example 27.1 A hydraulic press has a ram of 250 mm diameter and a plunger of 25 mm diameter. (i) If the applied force on the plunger is 40 N, then determine the weight lifted. (ii) If a lever with a leverage of 10 is used for applying force on the plunger, then determine the force needed at the end of the lever.

Solution

Let $D = 250\text{ mm} = 0.25\text{ m}$, $d = 25\text{ mm} = 0.025\text{ m}$, $F = 40\text{ N}$ and $L/l = 10$.

Let W be the weight lifted and F_1 be the force needed at the end of the lever.

(i) $$W = \frac{F\times A}{a} = \frac{F\times D^2}{d^2} = \frac{40\times 0.25^2}{0.025^2} = \mathbf{4000\ N}$$

(ii) $$F_1 = W\times\frac{l}{L}\times\frac{a}{A} = W\times\frac{l}{L}\times\frac{d^2}{D^2} = 4000\times\frac{1}{10}\times\frac{0.025^2}{0.25^2} = \mathbf{4\ N}$$

Example 27.2 A hydraulic press has a ram of 10 cm diameter and a plunger of 1.25 cm. What force will be required on the plunger to lift a load of 25 kN? If the plunger has a stroke of 20 cm, then how many strokes will be required to lift the load by 55 cm? Also determine the volume of additional liquid required. Further, if the time taken to lift the load is 8 minutes and the frictional effects are neglected, then what would be the power required by the motor to drive the plunger?

Solution

Let $D = 10\text{ cm} = 0.1\text{ m}$, $d = 1.25\text{ cm} = 0.0125\text{ m}$, $W = 25\text{ kN} = 25\times10^3\text{ N}$, $h = 20\text{ cm} = 0.2\text{ m}$, $H = 55\text{ cm} = 0.55\text{ m}$ and $t = 8\text{ min} = 480\text{ s}$.

Let F be the force required on the plunger, n be the number of strokes of the plunger, v be the volume of additional liquid and P be the power of the motor.

$$F = \frac{W\times a}{A} = \frac{W\times d^2}{D^2} = \frac{25\times10^3\times0.0125^2}{0.1^2} = \mathbf{390.625\ N}$$

$$n = \frac{D^2\times H}{d^2\times h} = \frac{0.1^2\times0.55}{0.0125^2\times0.2} = \mathbf{176}$$

$$v = \frac{\pi}{4}D^2\times H = \frac{\pi}{4}\times0.1^2\times0.55 = \mathbf{0.00432\ m^3}$$

$$P = \frac{W\times H}{1000\times t} = \frac{25\times10^3\times0.55}{1000\times480} = \mathbf{0.0286\ kW}$$

Example 27.3 The diameters of ram and plunger of a hydraulic press are 0.25 m and 30 mm, respectively, and the leverage of the handle is 10 : 1. The press can lift a load of 200 kN through 1.25 m in 2 minutes with a plunger stroke of 250 mm. Determine (i) the force applied at the end of lever, (ii) the number of strokes completed by the plunger per second and (iii) the power required to drive the plunger of the press. Assume the packing friction of the plunger as well as the ram equal to 4% of the load.

Solution

Let $D = 0.25$ m, $d = 30$ mm $= 0.03$ m, $L/l = 10:1$, $W = 200$ kN $= 200 \times 10^3$ N, $H = 1.25$ m, $t = 2$ min $= 120$ s, $h = 250$ mm $= 0.25$ m and $k = 4\%$.

Let F be the force on the plunger, F_1 be the force needed at the end of the lever, n be the number of strokes of the plunger and P be the power required to drive the plunger.

(i) $\frac{F}{a} \times [1-(k/100)] = \frac{W}{A \times [1-(k/100)]}$

$$F = \frac{W \times a}{A \times [1-(k/100)]^2} = \frac{W \times d^2}{D^2 \times [1-(k/100)]^2}$$

$$\therefore F = \frac{200 \times 10^3 \times 0.03^2}{0.25^2 \times [1-(4/100)]^2} = 3125 \text{ N}$$

$$F_1 = F \times \frac{l}{L} = 3125 \times \frac{1}{10} = \mathbf{312.5\ N}$$

(ii) $n = \frac{D^2 \times H}{d^2 \times h} = \frac{0.25^2 \times 1.25}{0.03^2 \times 0.25} = 347.22 \approx \mathbf{348}$

(iii) $P = \frac{F \times h \times n}{1000 \times t} = \frac{3125 \times 0.25 \times 348}{1000 \times 120} = \mathbf{2.266\ kW}$

27.3 ❐ HYDRAULIC ACCUMULATOR

The hydraulic accumulator temporarily stores the energy of liquid under pressure and supplies it for any sudden or intermittent requirement. The hydraulic machines such as lifts or cranes require large amount of energy in the form of liquid under pressure during upward motion of the load only. This energy is supplied from the hydraulic accumulator. When these devices move in the downward direction, no energy is practically used. At that time, the energy supplied by the pump is stored in the accumulator. Therefore, an accumulator stores energy during the idle period of the machine and supplies it along with the uniform supply from the pump to the machine during its working stroke when large quantity of liquid under pressure is required. Thus, it acts as a pressure regulator, which means it damps out pressure surges and shocks in the hydraulic system. This function is analogous to that of an electric storage battery and the flywheel of a reciprocating engine.

27.3.1 Simple Hydraulic Accumulator

A simple hydraulic accumulator is illustrated in Figure 27.3. It consists of a fixed vertical cylinder containing a sliding ram or plunger. The bottom end of the cylinder has two openings, namely inlet and outlet of the cylinder. A heavy load is placed on the top of the ram to generate pressure inside the cylinder chamber. The inlet of the cylinder is connected to the pump which continuously supplies liquid under pressure to the cylinder. The outlet of the cylinder is connected to the machine which may be lift, crane, press, etc.

Initially, the ram is at its lowermost position. When liquid is not required by the machine, the pump delivers the liquid under pressure to the cylinder. It raises the loaded ram till it reaches its uppermost position in the cylinder. This constitutes the upward stroke of the ram. Now at this

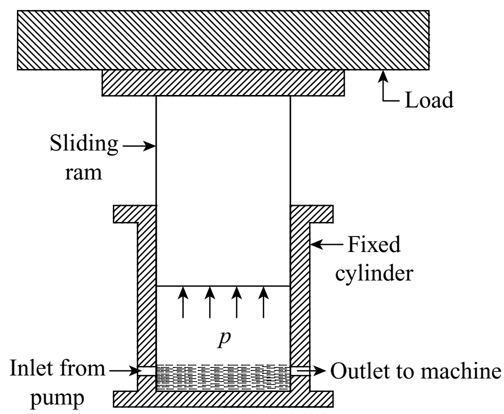


Figure 27.3 *Simple hydraulic accumulator*

position, the cylinder is full of liquid under pressure and hence, the accumulator has stored maximum amount of energy. Later on, when the machine requires a large amount of energy during its working stroke, then the hydraulic accumulator supplies the stored energy. Thus, the ram gradually moves in the downward direction. It constitutes the downward stroke of the ram during which the liquid under pressure is delivered to the machine.

27.3.2 Capacity of Accumulator

The maximum amount of hydraulic energy that the accumulator can store is known as the capacity of the accumulator.

Let L be the stroke or lift of the ram, D be the diameter of the sliding ram, $A=(\pi/4)D^2$ be the area of the sliding ram, (AL) be the volume of accumulator, p be the pressure intensity of liquid supplied by the pump and W be the total weight of the loaded ram (including the weight of the ram) as given in the following expression.

$$W = p \times A$$

The work done in lifting the ram is given by,

$$w = W \times \text{Stroke of the ram} = WL = pA \times L \tag{27.9}$$

The work done in lifting the ram is equal to the energy stored in the accumulator. Thus, the capacity of accumulator is given by the following expression.

$$\boxed{C = W \times L = pA \times L = p \times \text{Volume of accumulator}} \tag{27.10}$$

Example 27.4 An accumulator has a ram of 0.2 m diameter and lift of 6.5 m. If the liquid is supplied at a pressure of 5000 kN/m^2, then determine (i) the load on the ram and (ii) capacity of the accumulator in kWh.

Solution

Let $D = 0.2$ m, $L = 6.5$ m and $p = 5000$ kN/m^2. Let W be the load on the ram and C be the capacity of the accumulator.

(i) $W = p \times A = p \times \frac{\pi}{4} D^2 = 5000 \times \frac{\pi}{4} \times 0.2^2 = \mathbf{157.08\ kN}$

(ii) $1\text{ kWh} = 1000 \times 60 \times 60 = 3.6 \times 10^6$ Nm

$$C = \frac{W \times L}{3.6 \times 10^6} = \frac{157.08 \times 10^3 \times 6.5}{3.6 \times 10^6} = \mathbf{0.28362\ kWh}$$

Example 27.5 The ram of an accumulator is 0.35 m diameter and it weighs 50 kN. Determine the additional weight to be placed over it to develop a pressure of 4 MPa.

Solution

Let $D = 0.35$ m, $W = 50$ kN and $p = 4$ MPa $= 4 \times 10^6$ N/m^2.

$$A = \frac{\pi}{4} D^2 = \frac{\pi}{4} \times 0.35^2 = 0.09621\text{ m}^2$$

$$\text{Pressure} = \frac{W}{A} = \frac{50 \times 10^3}{0.09621} = 519696.5\text{ N/m}^2$$

$$\text{Required pressure} = 4 \times 10^6 - 519696.5 = 3480303.5\text{ N/m}^2$$

$$\text{Required extra load} = \text{Required pressure} \times \text{Area}$$

$$\therefore W_{\text{extra}} = \frac{3480303.5 \times 0.09621}{10^3} = \mathbf{334.84\ kN}$$

Example 27.6 The ram of an accumulator is 0.5 m diameter and it is loaded with 30 kN including its own weight. If the frictional resistance against the movement of the ram is 4% of the total weight, then determine the intensity of liquid pressure when the ram is moving up and down with uniform velocity.

Solution

Let $D = 0.5$ m, $W = 30\text{ kN} = 30\times10^3$ N and $F_f = 4\%$ of W.

(i) When the ram is moving up with a uniform velocity, we get:

$$A = \frac{\pi}{4}D^2 = \frac{\pi}{4}\times 0.5^2 = 0.19635\text{ m}^2$$

$$F_f = \frac{4}{100}\times 30\times10^3 = 1200\text{ N}$$

Thus, total force on the ram is given by,

$$F_t = 30000 + 1200 = 31200\text{ N}$$

$$p = \frac{F_t}{A} = \frac{31200}{0.19635\times10^3} = \mathbf{158.9\ kN/m^2}$$

(ii) When the ram is moving down with a uniform velocity, we get:

$$F_t = 30000 - 1200 = 28800\ N$$

$$p = \frac{F_t}{A} = \frac{28800}{0.19635\times10^3} = \mathbf{146.68\ kN/m^2}$$

Example 27.7 An accumulator is loaded with 40 kN weight. The ram has a diameter of 0.3 m and stroke of 6 m. Its friction may be assumed as 5% of the total load. It takes two minutes to fall through its full stroke. Find the total work supplied and power delivered to the hydraulic machine by the accumulator, when 0.0075 m^3/s water is being delivered by a pump, while accumulator descends with the stated velocity.

Solution

Let $W = 40$ kN, $D = 0.3$ m, $L = 6$ m, $F_f = 5\%$ of W, $t = 2\text{ min} = 120$ s and $Q = 0.0075\text{ m}^3/\text{s}$.

$$A = \frac{\pi}{4}D^2 = \frac{\pi}{4}\times 0.3^2 = 0.070686\text{ m}^2$$

$$F_f = \frac{5}{100}\times 40\times10^3 = 2000\text{ N}$$

The net force is given by,

$$F_{\text{net}} = 40000 - 2000 = 38000\text{ N}$$

Work supplied by the accumulator per second is given by,

$$w_{acc} = \frac{F_{\text{net}}\times L}{t} = \frac{38000\times 6}{120} = 1900\text{ Nm/s}$$

$$p = \frac{F_{\text{net}}}{A} = \frac{38000}{0.070686\times10^3} = 537.59\text{ kN/m}^2$$

$$H = \frac{p}{\rho_w g} = \frac{537.59\times10^3}{1000\times 9.81} = 54.8\text{ m}$$

Work supplied by the pump per second is given by,

$$w_p = \rho_w gQH = 1000 \times 9.81 \times 0.0075 \times 54.8 = 4031.91 \text{ Nm/s}$$

Total work supplied per second to the machine is given by,

$$w_{\text{total}} = w_{acc} + w_p = 1900 + 4031.91 = \mathbf{5931.91\ Nm/s}$$

$$\text{Power} = \frac{w_{\text{total}}}{1000} = \frac{5931.91}{1000} = \mathbf{5.932\ kW}$$

27.3.3 Differential Hydraulic Accumulator

The differential hydraulic accumulator is another form of accumulator in which the liquid is stored at a high pressure by a comparatively small load on the ram. It is also known as Tweddell's differential accumulator and it is shown in Figure 27.4. It consists of a fixed vertical ram which has central liquid passage of small diameter through which the liquid supplied from the pump enters the cylinder. The fixed vertical ram is surrounded by a closely fitting brass bush (or sleeve). The bush is surrounded by an inverted sliding cylinder having a circular collar projecting outwards at the base. The weights are placed on the collar to load the cylinder. The passages for liquid to enter and leave the sliding cylinder are provided in the fixed ram and are connected to the inlet and outlet pipes. The liquid supplied from the pump enters the inverted cylinder through the central vertical passage provided in the fixed ram. The liquid exerts an upward force on the internal annular area of the inverted sliding cylinder which is equal to the horizontal cross-sectional area of the brass bush. This causes the loaded cylinder to move upwards and thus, the hydraulic energy is stored in the accumulator.

Let L be the vertical lift of the moving cylinder, D be the external diameter of the brass bush, d be the diameter of fixed ram, $a = (\pi/4)(D^2 - d^2)$ be the annular area of the cylinder or area of the bush, (aL) be the volume of the accumulator, p be the pressure intensity of liquid supplied by the pump, W be the total weight of the sliding cylinder (including the weight placed on the cylinder) and t be the time in seconds.

$$p = \frac{W}{a} \tag{27.11}$$

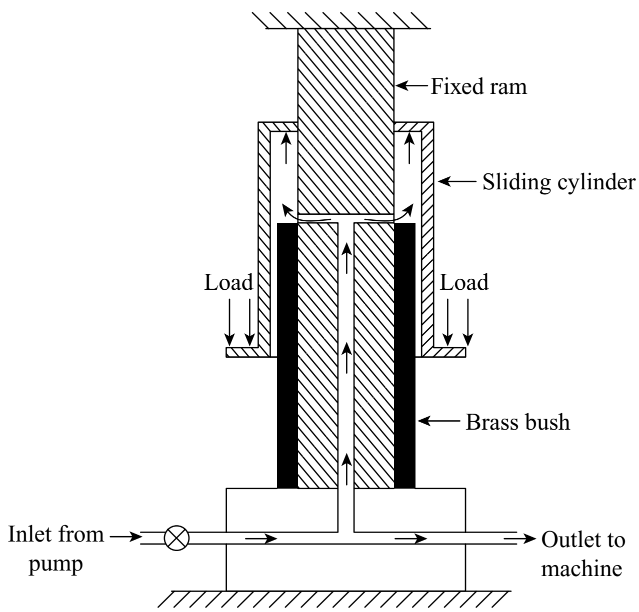


Figure 27.4 *Differential hydraulic accumulator*

From Equation (27.11), it is observed that pressure intensity can be increased with a small load 'W' by making the area of the bush 'a' small.

Energy stored in the accumulator = Capacity of accumulator = Total weight × Vertical lift

Therefore, the capacity of accumulator is given by,

$$C = W \times L = pa \times L = p \times \text{Volume of accumulator} \tag{27.12}$$

The work done by the accumulator is given by,

$$w = \frac{\text{Total weight} \times \text{Stroke of the ram}}{\text{Time}}$$

$$\boxed{\therefore w = \frac{W \times L}{t} = \frac{p(\pi/4)(D^2 - d^2) \times L}{t}} \tag{27.13}$$

Example 27.8 The diameters of two portions of the ram of a differential accumulator are 0.15 m and 0.14 m, respectively. The stroke of the accumulator is 1 m and it is supplied with water at a pressure of 1000 m of water. Evaluate the load on the ram and the capacity of the accumulator.

Solution

Let $D = 0.15$ m, $d = 0.14$ m, $L = 1$ m and $H = 1000$ m of water.

$$p = \rho_w g H = 1000 \times 9.81 \times 1000 = 9.81 \times 10^6 \text{ N/m}^2$$

$$W = p \times \frac{\pi}{4}(D^2 - d^2) = 9.81 \times 10^6 \times \frac{\pi}{4} \times (0.15^2 - 0.14^2) = \mathbf{22343.79\ N}$$

$$C = \frac{WL}{3.6 \times 10^6} = \frac{22343.79 \times 1}{3.6 \times 10^6} = \mathbf{6.207 \times 10^{-3}\ kWh}$$

27.4 ❐ HYDRAULIC INTENSIFIER

The hydraulic intensifier is a device which is used to increase the intensity of pressure of the liquid or water. It is accomplished by utilizing the hydraulic energy of a larger quantity of liquid at low pressure. This device is required when the hydraulic machines, such as hydraulic press, hydraulic crane and hydraulic lift needs liquid at very high pressure which may not be directly available from a pump. It is possible by using an intensifier between the pump and the machine.

A hydraulic intensifier consists of a fixed ram through which the high pressure liquid is supplied to the machine. The fixed ram is surrounded by a sliding cylinder which contains high pressure liquid. The sliding cylinder is surrounded by a fixed inverted cylinder which contains the low pressure liquid from the main supply as shown in Figure 27.5. The valves V_1 and V_4 allow low pressure liquid from the supply, valve V_3 discharges low pressure liquid to exhaust and valve V_2 allows high pressure liquid to the machine.

Initially, when the sliding cylinder is at its bottom-most position, the fixed cylinder is full of low pressure liquid. Now the valves V_2 and V_4 are closed and the valve V_1 is opened. It permits low pressure liquid into the sliding cylinder and meanwhile valve V_3 is also opened. It allows low pressure liquid from the fixed cylinder to the exhaust and the sliding cylinder moves upwards. When the sliding cylinder reaches its topmost position, the inside of the sliding cylinder is full of low pressure liquid. Now the valves V_1 and V_3 are closed and the valves V_2 and V_4 are opened. Therefore, the low pressure liquid from the supply enters the fixed cylinder which forces the sliding cylinder to move downwards. As a result, the liquid in the sliding cylinder gets compressed and its pressure increases. Thus, the high pressure liquid is forced out of the sliding cylinder through the fixed ram to the machine. A hydraulic intensifier can raise the pressure intensity of liquid up to about 160 MN/m^2.

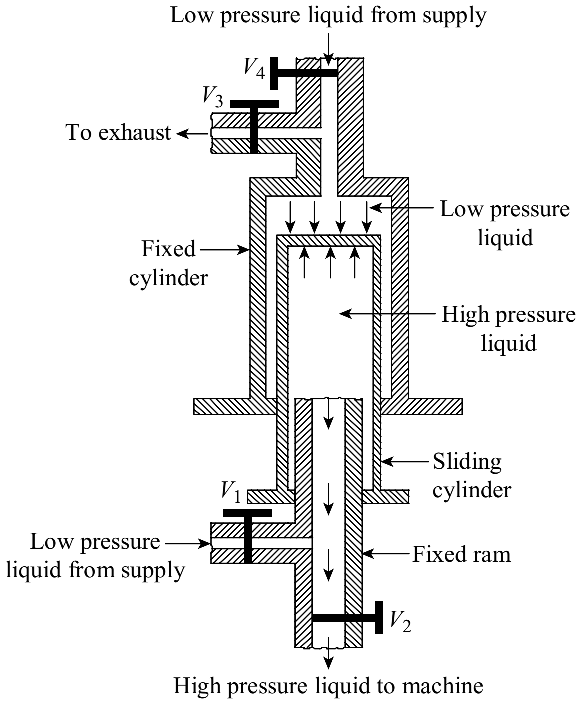


Figure 27.5 *Hydraulic intensifier*

Let p_1 be the intensity of pressure of low pressure liquid in the fixed cylinder, D be the external diameter of the sliding cylinder, $A = (\pi/4)D^2$ be the area of the sliding cylinder, p_2 be the intensity of pressure of high pressure liquid inside the sliding cylinder, d be the diameter of the fixed ram and $a = (\pi/4)d^2$ be the area of the fixed ram.

The total upward force equals total downward force and therefore, by neglecting friction effects, we derive the following expression.

$$p_1 \times A = p_2 \times a$$

$$\boxed{\therefore p_2 = p_1 \times \frac{A}{a} = p_1 \times \frac{(\pi/4)D^2}{(\pi/4)d^2} = \frac{p_1 \times D^2}{d^2}} \tag{27.14}$$

If the friction effects are considered and k is the percentage of friction loss at each packing of the intensifier, then we get the following expression.

$$p_1 A\left[1 - \frac{k}{100}\right] = \frac{p_2 a}{[1-(k/100)]}$$

$$\therefore p_2 = \frac{p_1 A}{a} \times \left[1 - \frac{k}{100}\right]^2 = \frac{p_1 D^2}{d^2} \times \left[1 - \frac{k}{100}\right]^2 \tag{27.14a}$$

The intensifier described above supplies high pressure liquid during the downward stroke only and thus, it is single acting. However, double acting intensifiers are also made which supply high pressure liquid continuously.

Example 27.9 The hydraulic intensifier receives low pressure water at a pressure of 4500 kN/m^2 and delivers it to the device at a pressure of 18000 kN/m^2. Determine the diameters of the fixed ram and the sliding cylinder of the intensifier if it has a capacity of 0.02 m^3 and stroke of 1.2 m.

Solution

Let $p_1 = 4500 \text{ kN/m}^2$, $p_2 = 18000 \text{ kN/m}^2$, $C = 0.02 \text{ m}^3$ and $L = 1.2 \text{ m}$. Let the diameters of the fixed ram and sliding cylinder be d and D, respectively.

$$\text{Capacity} = \text{Area of fixed ram} \times \text{Stroke length}$$

or
$$C = \frac{\pi}{4} d^2 \times L$$

Thus
$$0.02 = \frac{\pi}{4} d^2 \times 1.2$$

$$\therefore d = \sqrt{\frac{4 \times 0.02}{\pi \times 1.2}} = \mathbf{0.1457 \text{ m}}$$

From Equation (27.14) we get:

$$D = \sqrt{\frac{p_2 \times d^2}{p_1}} = \sqrt{\frac{18000 \times 0.1457^2}{4500}} = \mathbf{0.2914 \text{ m}}$$

Example 27.10 A hydraulic intensifier receives water into the fixed cylinder from an overhead tank through a pipeline 60 mm in diameter, 90 m long and with friction coefficient of 0.01 under a head of 15 m. The diameter of sliding cylinder is given 300 mm while that of fixed ram is 120 mm. The sliding cylinder moves 0.9 m in 30 seconds during the working stroke. Calculate the pressure head and the power delivered at the outlet of the fixed ram.

Solution

Let $d_1 = 60 \text{ mm} = 0.06 \text{ m}$, $l = 90 \text{ m}$, $f = 0.01$, $H = 15 \text{ m}$, $D = 300 \text{ mm} = 0.3 \text{ m}$, $d = 120 \text{ mm} = 0.12 \text{ m}$, $L = 0.9 \text{ m}$ and $t = 30 \text{ s}$.

Area of sliding cylinder is given by,

$$A = \frac{\pi}{4} D^2 = \frac{\pi}{4} \times 0.3^2 = 0.0707 \text{ m}^2$$

Area of fixed ram is given by,

$$a = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 0.12^2 = 0.01131 \text{ m}^2$$

The discharge of low pressure liquid entering the fixed cylinder is given by,

$$Q_1 = \frac{A \times L}{t} = \frac{0.0707 \times 0.9}{30} = 2.121 \times 10^{-3} \text{ m}^3/\text{s}$$

$$V = \frac{Q_1}{(\pi/4) d_1^2} = \frac{2.121 \times 10^{-3}}{(\pi/4) \times 0.06^2} = 0.75 \text{ m/s}$$

The head loss due to friction in the supply pipe is given by,

$$h_f = \frac{4 f l V^2}{2 g d_1} = \frac{4 \times 0.01 \times 90 \times 0.75^2}{2 \times 9.81 \times 0.06} = 1.72 \text{ m}$$

The pressure head of the low pressure water in the fixed cylinder is given by,

$$H_1 = 15 - 1.72 = 13.28 \text{ m}$$

Therefore, the pressure head delivered at the outlet of the fixed ram is given by,

$$H_2 = \frac{H_1 \times A}{a} = \frac{13.28 \times 0.0707}{0.01131} = \mathbf{83.015\ m}$$

Discharge of the high pressure water delivered at the outlet of the fixed ram is given by,

$$Q_2 = \frac{Q_1 \times a}{A} = \frac{2.121 \times 10^{-3} \times 0.01131}{0.0707} = 3.393 \times 10^{-4}\ \text{m}^3$$

Power delivered by the intensifier at the outlet of the fixed ram is given by,

$$P = \frac{\rho_w g Q_2 H_2}{1000} = \frac{1000 \times 9.81 \times 3.393 \times 10^{-4} \times 83.015}{1000} = \mathbf{0.27632\ kW}$$

27.5 ❐ HYDRAULIC RAM

The hydraulic ram is a type of pump which lifts a small quantity of water to a greater height from large quantity of water available at a smaller height. The hydraulic ram lifts water without the use of any external power. The schematic diagram of a typical hydraulic ram and its main components are illustrated in Figure 27.6. It consists of a valve chamber connected to the supply tank by an inclined supply pipe. The valve chamber is provided with a waste valve V_1 and a delivery valve V_2. Both these valves are non-return valves that permit the flow in one direction only. The waste valve V_1 opens inwards while the delivery valve V_2 opens outwards. The delivery valve V_2 connects the valve chamber to the air vessel which is connected to the delivery tank through the delivery pipe.

It works on the principle of water hammer. When a flowing liquid is suddenly brought to rest, the change in momentum of liquid mass causes a sudden rise in pressure. This rise in pressure is utilized to raise a portion of the liquid to higher levels.

When the supply valve fitted to the supply pipe is opened, water starts flowing from the supply tank to the valve chamber. The level of water rises in the valve chamber and the waste valve starts moving upwards. The waste valve V_1 being open, the water flows through it to the waste water channel. As the rate of discharge after the waste valve increases, the flow of water in the supply pipe accelerates. With increase in the velocity of flow in the supply pipe, dynamic pressure on the underside of the waste valve reaches to a stage which suddenly closes the waste valve. Due to the sudden closure of waste valve, water in the supply pipe is suddenly brought to rest which creates high pressure inside the valve chamber. This high pressure lifts the delivery valve V_2 and a part of water from the valve chamber enters the air vessel and compresses the air inside it. This compressed air exerts force on the water in the air vessel and a small quantity of water is supplied through the delivery pipe to the delivery tank placed to a greater height.

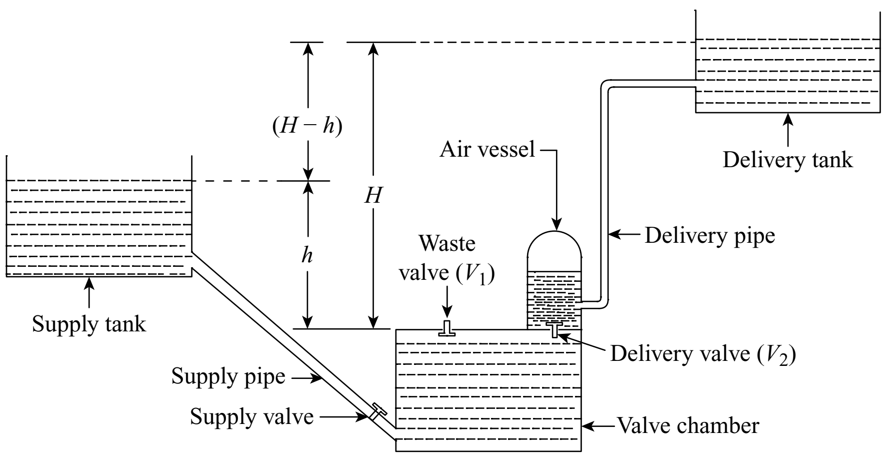


Figure 27.6 *Hydraulic ram*

When the water in the valve chamber loses its momentum, the waste valve V_1 opens. Now the flow of water from supply tank starts flowing to the valve chamber and this cycle is repeated.

The hydraulic ram works with water streams of 1 m^3/s to 40 m^3/s, fall heads of 1.5 m to 30 m, and lifts water up to heights about 300 m.

Let Q be the discharge through supply pipe, q be the discharge through delivery pipe, h be the height of water in supply tank above the valve chamber and H be the height of water raised from the valve chamber.

$$\text{Energy supplied to the ram} = \rho_w gQh$$

$$\text{Energy supplied by the ram} = \rho_w gqH$$

The efficiency of hydraulic ram is given by,

$$\boxed{\eta_D = \frac{\text{Energy supplied by the ram}}{\text{Energy supplied to the ram}} = \frac{\rho_w gqH}{\rho_w gQh} = \frac{q \times H}{Q \times h}} \tag{27.15}$$

The above expression for efficiency was suggested by D'Aubuisson and thus, it is known as D'Aubuisson's efficiency.

Rankine suggested another form of the above efficiency and hence, it is called Rankine efficiency. It is defined on the basis of the difference of water head in the discharge and the supply tank. The Rankine's efficiency is mathematically expressed as given below.

$$\boxed{\eta_R = \frac{q \times (H - h)}{(Q - q) \times h}} \tag{27.16}$$

The above two efficiencies, in terms of weight are given by,

$$\eta_D = \frac{w \times H}{W \times h} \tag{27.17}$$

$$\eta_R = \frac{w \times (H - h)}{(W - w) \times h} \tag{27.18}$$

Here, w is the weight of water delivered per second by the ram and W is the weight of water flowing from supply tank to the valve chamber per second.

Due to several energy losses, the maximum efficiency of hydraulic ram is usually limited to only about 75%. The main causes of energy losses are (i) friction and secondary losses which occur in the supply pipe, delivery pipe and in the valves and (ii) the velocity energy carried away by the water leaving the waste valve.

Some of the characteristic features of hydraulic ram are (i) it is suitable to pump water from streams for irrigation purposes and supplying water to houses in hilly and remote areas, (ii) it is quiet in operation and works automatically, (iii) it requires very little maintenance and running costs, (iv) it has no moving parts, so frequent oiling is not required, (v) it does not require any external source of energy to pump water but it needs large quantity of water at low heads.

Example 27.11 A hydraulic ram raises 0.005 m^3/s of water to a height 25 m above the ram through a 100 m long and 65 mm diameter delivery pipe. If the supply tank is 4 m above the ram and supplies 0.06 m^3/s of water, then determine the efficiency of the ram. Assume coefficient of friction as $f = 0.009$.

Solution

Let $q = 0.005 \text{ m}^3/\text{s}$, $H = 25$ m, $l = 100$ m, $d = 65 \text{ mm} = 0.065$ m, $h = 4$ m, $Q = 0.06 \text{ m}^3/\text{s}$ and $f = 0.009$.

Velocity of water in the delivery pipe is given by,

$$V = \frac{q}{a} = \frac{q}{(\pi/4)d^2} = \frac{0.005}{(\pi/4) \times 0.065^2} = 1.507 \text{ m/s}$$

The head loss due to friction in the delivery pipe is given by,

$$h_f = \frac{4flV^2}{2gd} = \frac{4\times 0.009\times 100\times 1.507^2}{2\times 9.81\times 0.065} = 6.411 \text{ m}$$

Therefore, the effective head developed by the ram is given by,

$$H_e = H + h_f = 25 + 6.411 = 31.411 \text{ m}$$

D'Aubuisson's efficiency is given by,

$$\eta_D = \frac{q\times H_e}{Q\times h} = \frac{0.005\times 31.411}{0.06\times 4}\times 100 = \mathbf{65.44\%}$$

Rankine's efficiency is given by,

$$\eta_R = \frac{q\times(H_e - h)}{(Q-q)\times h} = \frac{0.005\times(31.411-4)}{(0.06-0.005)\times 4}\times 100 = \mathbf{62.3\%}$$

Example 27.12 The parameters given for a hydraulic ram are supply head = 4 m, length of supply pipe = 6 m, diameter of supply pipe = 26 mm, delivery head = 8 m, length of delivery pipe = 15 m, diameter of delivery pipe = 13 mm, time taken to supply 14 N of water = 34 s and water wasted during the given time = 62 N. Determine the Rankine efficiency of the ram if coefficient of friction is $f = 0.009$.

Solution

Let $h = 4$ m, $l_s = 6$ m, $d_s = 26$ mm $= 0.026$ m, $H = 8$ m, $l_d = 15$ m, $d_d = 13$ mm $= 0.013$ m, $q_1 = 14$ N, $t = 34$ s, $q_2 = 62$ N and $f = 0.009$.

The total quantity of water supplied in 34 s is $Q_t = 62 + 14 = 76$ N.

$$Q = \frac{Q_t}{\rho_w g t} = \frac{76}{1000\times 9.81\times 34} = 2.2786\times 10^{-4} \text{ m}^3/\text{s}$$

Velocity of water flowing through supply pipe is given by,

$$V_s = \frac{Q}{(\pi/4)d_s^2} = \frac{2.2786\times 10^{-4}}{(\pi/4)\times 0.026^2} = 0.429 \text{ m/s}$$

The head loss due to friction in the supply pipe is given by,

$$h_{fs} = \frac{4fl_sV_s^2}{2gd_s} = \frac{4\times 0.009\times 6\times 0.429^2}{2\times 9.81\times 0.026} = 0.078 \text{ m}$$

Effective supply head is given by,

$$h_e = h - h_{fs} = 4 - 0.078 = 3.922 \text{ m}$$

$$q = \frac{q_1}{\rho_w g t} = \frac{14}{1000\times 9.81\times 34} = 4.1974\times 10^{-5} \text{ m}^3/\text{s}$$

Velocity of water flowing through delivery pipe is given by,

$$V_d = \frac{q}{(\pi/4)d_d^2} = \frac{4.1974\times 10^{-5}}{(\pi/4)\times 0.013^2} = 0.316 \text{ m/s}$$

The head loss due to friction in the delivery pipe is given by,

$$h_{fd} = \frac{4 f l_d V_d^2}{2 g d_d} = \frac{4 \times 0.009 \times 15 \times 0.316^2}{2 \times 9.81 \times 0.013} = 0.211 \text{ m}$$

Effective head developed is given by,

$$H_e = H + h_{fd} = 8 + 0.211 = 8.211 \text{ m}$$

$$\eta_R = \frac{q \times (H_e - h_e)}{(Q - q) \times h_e} = \frac{4.1974 \times 10^{-5} \times (8.211 - 3.922)}{(2.2786 \times 10^{-4} - 4.1974 \times 10^{-5}) \times 3.922} \times 100 = \mathbf{24.7\%}$$

27.6 ❐ HYDRAULIC LIFT

The hydraulic lift is a device which is used for carrying people or goods from one floor to another in a multi-storeyed building. The hydraulic lifts are of two types, namely direct acting hydraulic lift and suspended hydraulic lift.

27.6.1 Direct Acting Hydraulic Lift

It consists of a ram sliding in a fixed cylinder as shown in Figure 27.7. A cage or platform is fitted at the end of the ram where the passengers may stand or goods may be placed. When the fluid under pressure is admitted into the cylinder, the ram moves vertically up and the cage is lifted to any required height. The lift of the cage is equal to the stroke of the ram. The cage moves in the downward direction by removing the liquid from the fixed cylinder.

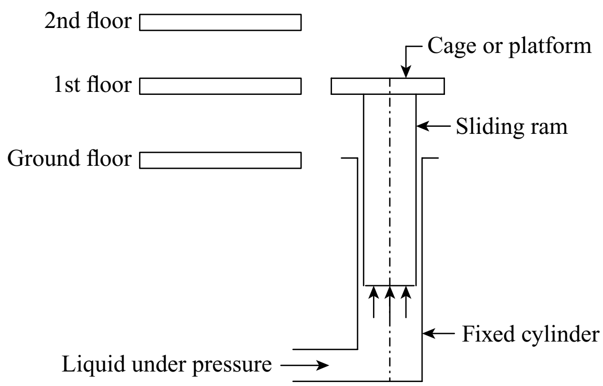


Figure 27.7 *Direct acting hydraulic lift*

27.6.2 Suspended Hydraulic Lift

The suspended hydraulic lift is a modified form of direct acting hydraulic lift. Generally, the modern lifts are suspended type and it has high velocity ratio. It consists of a cage which is suspended by wire ropes, a jigger consists of a fixed cylinder, a sliding ram and a set of two pulleys, namely fixed pulley block and movable pulley block as shown in Figure 27.8.

The lift cage runs between guides of round steel. It is suspended with four wire lifting ropes, each one with sufficient strength to support the load. When liquid under pressure is admitted into the fixed cylinder of the jigger, the sliding ram is forced to move towards left. The movable pulley block which is connected to the sliding ram also moves towards left. This increases the gap between the two pulley blocks. As a result, the wire rope connected to the cage is pulled and the cage is lifted up.

When the liquid from the fixed cylinder is removed, the cage is lowered. The removal of liquid causes the movement of the sliding ram towards right. This decreases the gap between the two blocks and thus, the cage is lowered due to increase in the length of the rope.

The lifting speeds of these lifts vary from 100 m/min to 200 m/min. The hydraulic lifts have been replaced by electric lifts which have become quite common these days. However, the hydraulic lifts are usually preferred when there is danger due to fire or explosion. These are used as standby units along with electric lifts.

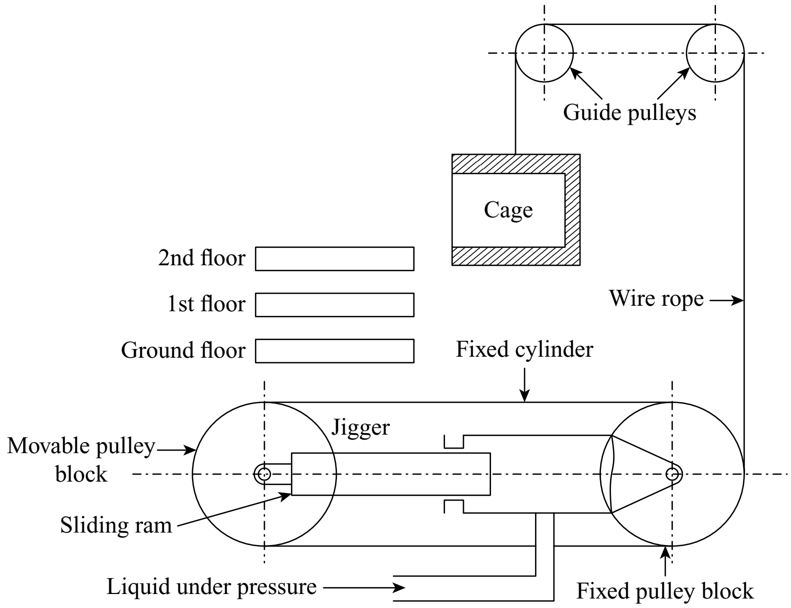


Figure 27.8 *Suspended hydraulic lift*

Example 27.13 A hydraulic lift is located at a distance of 725 m from the accumulator. It is supplied with water under pressure from the accumulator through a hydraulic main of diameter 90 mm. The accumulator maintains a steady pressure of 7 MPa. The lift ram is 200 mm in diameter. The load on the ram inclusive of its own weight is 120 kN. The friction of the ram is assumed equivalent to an addition of 5% of gross load on the ram. If the friction coefficient for the main is given as 0.009, then evaluate the speed at which the lift will ascend.

Solution

Let $l = 725$ m, $d = 90$ mm $= 0.09$ m, $p = 7$ MPa, $D = 200$ mm $= 0.2$ m, $W_1 = 120$ kN, $f_R = 5\%$ of W_1 and $f = 0.009$.

Let V be the velocity of flow in the main and V_{lift} be the speed at which the lift ascends.

The head loss due to friction in the main is given by,

$$h_f = \frac{4flV^2}{2gd} = \frac{4 \times 0.009 \times 725 \times V^2}{2 \times 9.81 \times 0.09} = 14.781\ \text{V}^2$$

The total load is given by,

$$W = W_1 + f_R = 120 + \frac{5}{100} \times 120 = 126 \text{ kN}$$

Pressure head due to total load is given by,

$$h = \frac{(W/A)}{\rho_w g} = \frac{4W}{\pi D^2 \rho_w g} = \frac{4 \times 126 \times 10^3}{\pi \times 0.2^2 \times 1000 \times 9.81} = 408.84 \text{ m}$$

Pressure head of water in the accumulator is given by,

$$H = \frac{p}{\rho_w g} = \frac{7 \times 10^6}{1000 \times 9.81} = 713.56 \text{ m}$$

Since $$h_f + h = H$$

$$14.781V^2 + 408.84 = 713.56$$

$$\therefore V = \sqrt{\frac{713.56 - 408.84}{14.781}} = 4.54 \text{ m/s}$$

Discharge from main = Discharge into the lift ram cylinder

$$\frac{\pi}{4}d^2 \times V = \frac{\pi}{4}D^2 \times V_{\text{lift}}$$

$$\therefore V_{\text{lift}} = \frac{d^2 \times V}{D^2} = \frac{0.09^2 \times 4.54}{0.2^2} = \mathbf{0.92\ m/s}$$

Example 27.14 A hydraulic lift is required to lift a load of 100 kN through a height of 18 m once in every 2 minutes. The lift travels up at the rate of 1.2 m per second. During working stroke of the lift, the water is supplied to it from the accumulator and from the pump at a pressure intensity of 3.5 MPa. If the efficiency of the lift is 80% and that of pump is 85%, then evaluate the power required to drive the pump and the minimum capacity of the accumulator. Neglect friction losses in the pipe.

Solution

Let $W = 100$ kN, $H = 18$ m, $t = 2$ min, $V = 1.2$ m/s, $p = 3.5$ MPa, $\eta_l = 0.8$ and $\eta_p = 0.85$.

The power supplied to the lift by the pump and accumulator is given by,

$$P = \frac{W \times V}{\eta_l} = \frac{100 \times 1.2}{0.8} = 150 \text{ kW}$$

Working period of the lift is given by,

$$t_w = \frac{H}{V} = \frac{18}{1.2} = 15 \text{ s}$$

Idle period of the lift is given by,

$$t_i = t - t_w = 120 - 15 = 105 \text{ s}$$

During idle period, the energy E_a is stored in the accumulator. If P_1 is the power output of the pump in kW, then the energy is given below.

$$E_a = P_1 \times 105 = 105P_1 \text{ kNm}$$

This energy is supplied to the lift during its working period of 15 seconds.

Therefore, power supplied by the accumulator is given by,

$$P_a = \frac{105P_1}{15} = 7P_1 \text{ kW}$$

Therefore, total power supplied by the accumulator and pump to the lift is given below.

$$P = 7P_1 + P_1 = 8P_1 \text{ kW}$$

Thus $$8P_1 = 150$$

$$\therefore P_1 = \frac{150}{8} = 18.75 \text{ kW}$$

Power required for driving the pump is given by,

$$P_p = \frac{P_1}{\eta_p} = \frac{18.75}{0.85} = \mathbf{22.06\ kW}$$

Power stored in the accumulator $= 3.5 \times 10^6 \times \text{Capacity}$

$$3.5 \times 10^6 \times \text{Capacity} = 105\,P_1$$

$$\therefore \text{Capacity} = \frac{105P_1}{3.5 \times 10^6} = \frac{105 \times 18.75 \times 10^3}{3.5 \times 10^6} = \mathbf{0.5625\ m^3}$$

27.7 ❐ HYDRAULIC CRANE

The hydraulic crane is a device which is used to lift heavy loads. It can lift loads as high as 50 tons to about 100 tons. It is widely used in warehouses, workshops, docks for loading and unloading ships, railways for lifting wagons and in heavy industries. Figure 27.9 shows a schematic view of a conventional hydraulic crane. It consists of a central pedestal, a mast, jib, tie, guide pulley and a jigger. The central pedestal supports a mast to which the jib (or arm) and tie is attached. The jib can be raised or lowered in order to increase or decrease the radius of action of the crane. The pedestal along with the mast can revolve about a vertical axis and the jib swings with the mast. By revolving the pedestal and lowering the jib, the load attached to the rope can be transferred to the desired place within the crane's area of action. The jigger that consists of a ram sliding in a fixed cylinder is used to lift or lower the load and it is attached to the mast. One end of the ram is fixed to a set of movable pulley block and the other is in contact with water. Another pulley block called fixed pulley block is attached to the fixed cylinder. A wire rope with one end fixed to a movable pulley (which is attached to the movable ram) is taken round all the pulleys of the two sets of the pulleys and finally, it is taken over the guide pulley attached to the jib. The other end of the rope is hooked to lift the load.

To lift the load with the help of crane, high pressure water is admitted in the cylinder of the jigger. The water forces the sliding ram to move vertically up. The movable pulley block attached with the ram also moves with it in the upward direction. This increases the distance between two pulley blocks which results in winding of the wire rope over the guide pulleys by the jigger. Thus, the load is lifted up.

To lower the load by the crane, the water from the cylinder of the jigger is removed through the outlet valve and this causes the ram to move down. Thereby, the distance between the two sets of pulley is reduced, which results in releasing more length of the wire rope. Thus, the load is lowered.

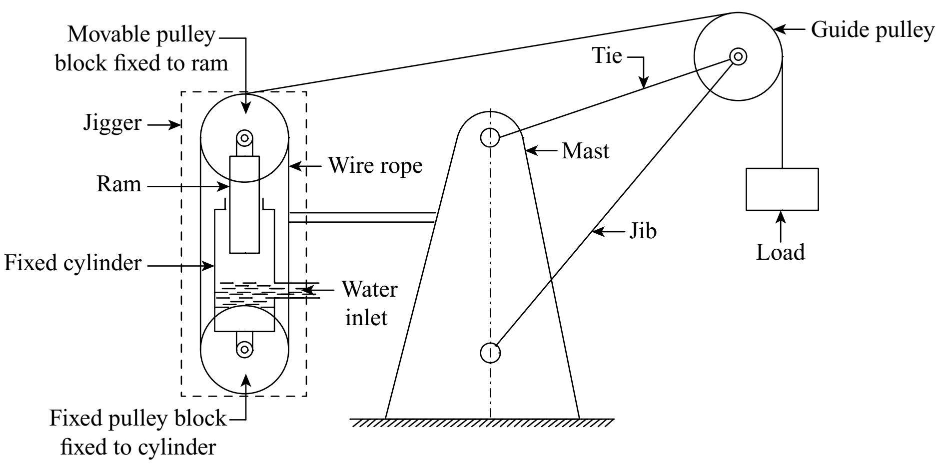


Figure 27.9 *Hydraulic crane*

Let p be the pressure force on the ram, W be the weight lifted, x be the distance moved by the weight which will be equal to the total height through which the load can be lifted, i.e., h and y be the distance moved by the force which will be equal to the stroke of the ram, i.e., L.

The efficiency of the crane is given by,

$$\eta = \frac{\text{Output}}{\text{Input}} = \frac{\text{Weight} \times \text{Distance moved by weight}}{\text{Force} \times \text{Distance moved by force}} = \frac{W \times x}{p \times y} \tag{27.19}$$

Velocity ratio is given by,

$$\text{V.R.} = \frac{\text{Distance moved by weight}}{\text{Distance moved by force}} = \frac{x}{y} = \frac{h}{L} \tag{27.20}$$

$$\boxed{\therefore \eta = \frac{W}{p} \times VR} \tag{27.21}$$

The velocity ratio of the crane hook to the ram of jigger depends on the number of pulleys in each set. If there are four pulleys in a set, then the velocity ratio will be 4 to one, which means that the load on the wire rope moves four times faster to the speed of the ram of the jigger. The lifting speed of a modern hydraulic crane may be about 75 m per minute. Nowadays hydraulic cranes have been replaced by electric cranes.

Example 27.15 A hydraulic crane utilizes 0.04 m^3 of water at 6 MPa to lift a load of 10 kN through a height of 12 m. Determine the efficiency of the crane.

Solution

Let $Q = 0.04\ \text{m}^3$, $p = 6\ \text{MPa} = 6 \times 10^6\ \text{Pa}$, $W = 10\ \text{kN} = 10 \times 10^3\ \text{N}$ and $x = 12$ m.

Power supplied to the crane is given by,

$$P_s = p \times Q = 6 \times 10^6 \times 0.04 = 240000\ \text{Nm}$$

Power produced by the crane is given by,

$$P_p = W \times x = 10 \times 10^3 \times 12 = 120000\ \text{Nm}$$

$$\eta = \frac{P_p}{P_s} = \frac{120000}{240000} \times 100 = \mathbf{50\%}$$

Example 27.16 A hydraulic crane lifts a load through a height of 12 m. The diameter of the ram is 0.16 m and the jigger has a velocity ratio of 6. The water is supplied by a pump at a pressure of 1 MPa to the cylinder. Calculate the weight lifted by the crane, stroke of the ram and the volume of water required to lift the weight if the efficiency is 55%.

Solution

Let $h = 12$ m, $D = 0.16$ m, $VR = 6$, $p_1 = 1\ \text{MPa} = 10^6$ Pa and $\eta = 0.55$.

$$A = \frac{\pi}{4} D^2 = \frac{\pi}{4} \times 0.16^2 = 0.0201062\ \text{m}^2$$

Pressure force on the ram is given by,

$$p = p_1 \times A = 10^6 \times 0.0201062 = 20106.2\ \text{N}$$

Since
$$\eta = \frac{W}{p} \times VR$$

Thus $$0.55 = \frac{W}{20106.2} \times 6$$

$$\therefore W = \frac{0.55 \times 20106.2}{6} = 1843.07 \text{ N}$$

$$L = \frac{h}{VR} = \frac{12}{6} = \mathbf{2\ m} \qquad [\because VR = h/L]$$

Volume of water is given by,

$$v = A \times L = 0.0201062 \times 2 = \mathbf{0.0402124\ m^3}$$

Example 27.17 An accumulator supplies water at a pressure of 5×10^6 Pa to a crane which lifts a weight of 12 kN through a height of 10 m. The weight moves with a speed of 20 m per minute once in every two minutes. The water is supplied to the accumulator by a pump. If the efficiency of crane is 75%, then determine (i) the capacity of jigger cylinder, (ii) capacity of the accumulator and (iii) minimum power required to run the pump.

Solution

Let $p_1 = 5 \times 10^6$ Pa, $W = 12$ kN $= 12 \times 10^3$ N, $x = 10$ m, $V = 20$ m/min, $t = 2$ min and $\eta = 0.75$.

(i) $$\text{Capacity} = \frac{W \times x}{p_1 \times \eta} = \frac{12 \times 10^3 \times 10}{5 \times 10^6 \times 0.75} = \mathbf{0.032\ m^3}$$

(ii) Total work input to the crane is given by,

$$w_{in} = 5 \times 10^6 \times 0.032 = 160000 \text{ Nm}$$

$$w_{in}/\text{min} = \frac{160000}{2} = 80000 \text{ Nm/min}$$

$$\text{Time to lift the weight by crane} = \frac{x}{V} = \frac{10}{20} = 0.5 \text{ min}$$

Work done by the pump to lift the weight is given by,

$$w_{\text{pump}} = w_{in}/\text{min} \times \text{Time to lift the weight} = 80000 \times 0.5 = 40000 \text{ Nm}$$

Energy supplied by the accumulator to the crane is given by,

$$E_s = w_{in} - w_{\text{pump}} = 160000 - 40000 = 120000 \text{ Nm}$$

$$E_s = \text{Pressure} \times \text{Capacity} = 5 \times 10^6 \times \text{Capacity}$$

Thus $$5 \times 10^6 \times \text{Capacity} = 120000$$

$$\therefore \text{Capacity} = \frac{120000}{5 \times 10^6} = \mathbf{0.024\ m^3}$$

(iii) Minimum power required to run the pump is given by,

$$P_{\min} = \frac{w_{in}/\text{sec}}{10^3} = \frac{80000}{60 \times 10^3} = \mathbf{1.3333\ kW}$$

27.8 ❐ HYDRAULIC COUPLING

The hydraulic coupling (or fluid coupling) is a device which is used to transmit power from driving shaft (input shaft) to driven shaft (output shaft) through a liquid medium (generally oil). There is no mechanical or rigid connection between the two shafts. The schematic view of a typical hydraulic coupling is shown in Figure 27.10. It consists of a radial pump impeller keyed to the driving shaft '*A*' and a radial flow turbine runner keyed to the driven shaft '*B*'. Both the impeller and runner are identical in shape and size. These two units are kept very close with their ends facing each other and are enclosed in a casing. The casing is completely filled with ordinary mineral lubricating oil. The oil in the casing transmits the torque from the pump impeller to the turbine runner.

Initially, both the shafts '*A*' and '*B*' are stationary. When the shaft '*A*' is rotated by the prime mover (engine or a motor), the pump impeller causes the oil to flow from its inner radius (eye) to the outer radius. This oil of increased energy enters the turbine runner vanes at its outer radius and flows inwardly to its inner radius and thus, it exerts a force on the runner vanes. As the speed of the driving shaft '*A*' increases, the torque on the turbine runner increases. Eventually, the magnitude of the torque overcomes the inertia of the driven unit. Thus, the turbine runner and the driven shaft '*B*' starts rotating. The oil from the runner flows back into the pump impeller and thus, it makes a continuous circulation.

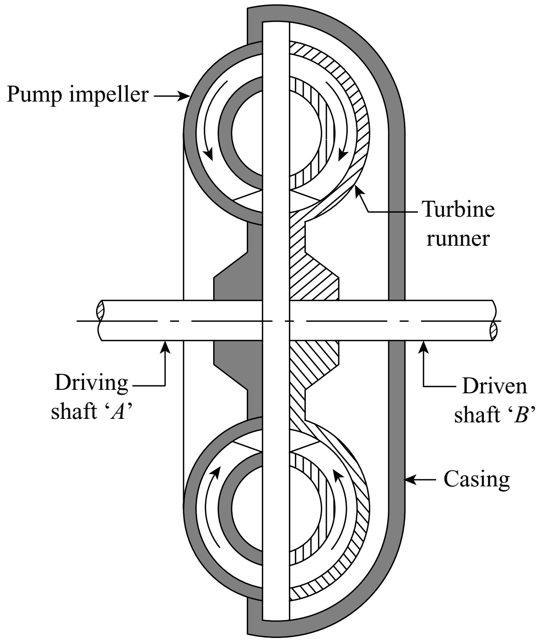


Figure 27.10 *Hydraulic coupling*

Let T be the torque which remains equal on driving shaft (impeller) and driven shaft (runner), ω_p be the angular speed of the pump impeller or driving shaft A, ω_t be the angular speed of the turbine runner or driven shaft B.

Efficiency of the coupling is given by,

$$\eta = \frac{\text{Power output}}{\text{Power input}} = \frac{T \times \omega_t}{T \times \omega_p} = \frac{\omega_t}{\omega_p} \tag{27.22}$$

Therefore, this ratio (ω_t/ω_p) is known as speed ratio.

Slip (s) of hydraulic coupling is defined by,

$$s = \frac{\omega_p - \omega_t}{\omega_p} = 1 - \frac{\omega_t}{\omega_p} = 1 - \eta \tag{27.23}$$

Generally, the efficiency of a hydraulic coupling is more than 94%. A typical efficiency versus speed ratio curve for a hydraulic coupling is shown in Figure 27.11(a). The efficiency starts at zero and increases uniformly with the speed ratio until it reaches to 95% and then, it reduces to zero.

The stall is the condition when the speeds of both the shafts '*A*' and '*B*' becomes equal due to which the slip becomes zero and efficiency becomes unity. Under such conditions, there is no flow of oil and hence, the coupling does not work.

The variation of driving shaft input torque with the driving shaft speed for different values of slip are parabolic curves which are illustrated in Figure 27.11(b). The input torque increases with the cube of driving speed for a given slip. Thus, the power transmitted also varies with the cube of speed for a given slip. In order to reduce the input torque, the slip must be of a smaller value.

Hydraulic couplings have low value of transmission efficiency in comparison to mechanical couplings. However, these are widely used in automobiles, marine engines, ropeway cable drive units, power driven excavators and agricultural machinery to transmit torque ranging from 0.7 kW to 26500 kW. The hydraulic couplings are very useful where smooth shock free operations are required and where large initial loads are involved.

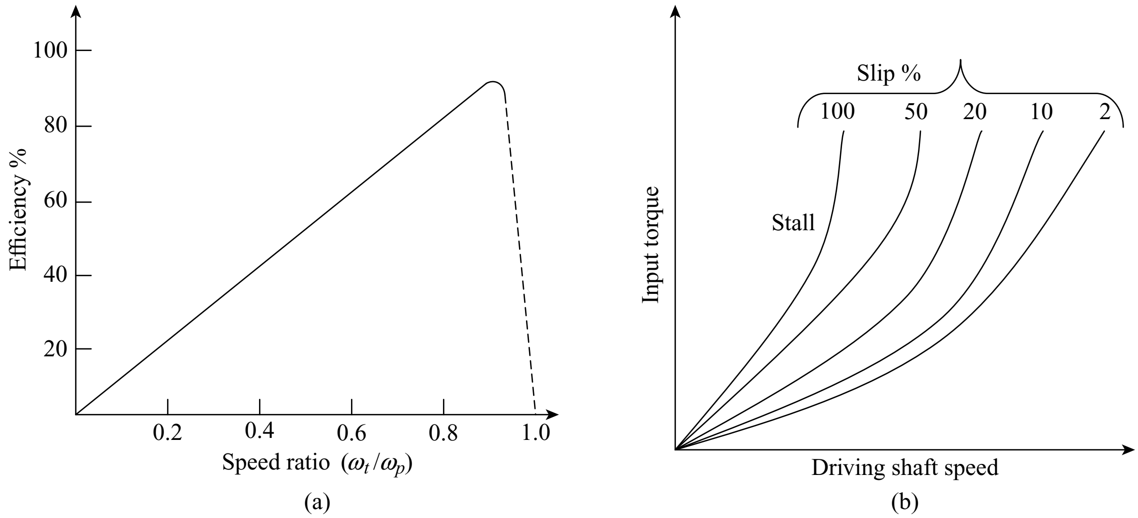


Figure 27.11 *Performance characteristics of a hydraulic coupling*

27.9 ❒ HYDRAULIC TORQUE CONVERTER

The hydraulic torque converter is a device which is used to transmit increased or decreased torque at the driven shaft. In case of hydraulic coupling, the torque output is equal to the torque input, but in case of hydraulic torque converter, the torque output can be increased or decreased. Usually, the torque converters are used to increase the torque at the driven shaft. The torque may be increased about five times the torque available at the driving shaft with an efficiency of about 90%. The hydraulic torque converters are used in automobile power transmission units, diesel locomotives and earth moving machinery.

It consists of a pump impeller mounted on the driving shaft (input shaft), a turbine runner fixed on the driven shaft (output shaft) and stationary guide vanes (also called stator or reaction member) fixed to the casing are provided between the impeller and the runner as shown in Figure 27.12(a). The construction of a hydraulic converter is similar to hydraulic coupling, except for a series of fixed guide vanes which are provided between the impeller and the runner.

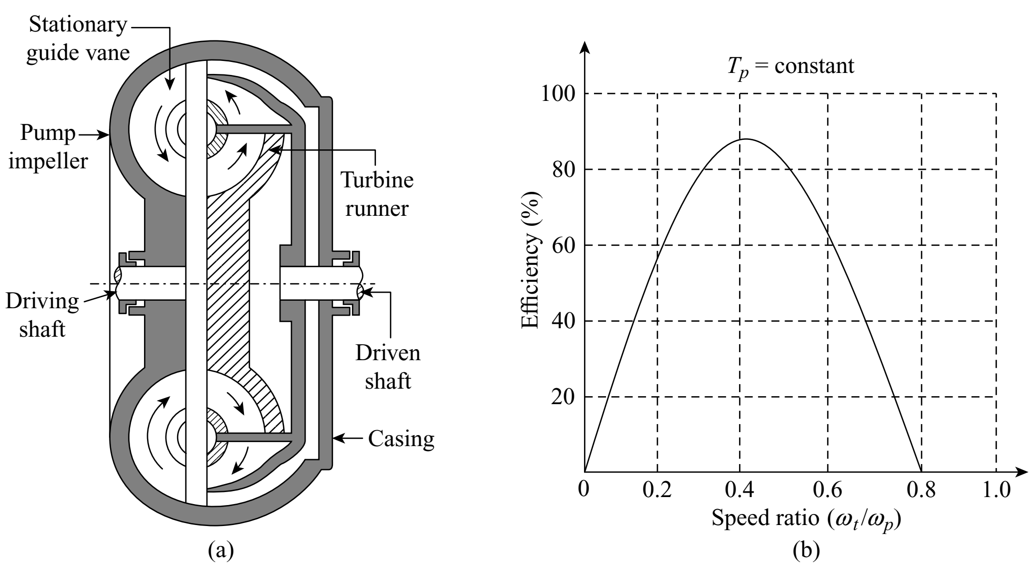


Figure 27.12 *Hydraulic torque converter and its characteristics*

The liquid flowing from the impeller goes to the runner and to a series of guide vanes. The guide vanes change the direction of liquid, as a result of which the torque delivered increases many times. By suitable design of the guide vanes, the torque transmitted to the driven shaft can be increased or decreased. Therefore, hydraulic torque converter is comparable to an electric transformer.

Let T_t be the torque transmitted to the turbine runner (driven shaft), T_p is the torque of the pump impeller (driving shaft), T_v is the variation of torque caused by guide vanes, ω_p is the angular speed of the pump impeller (driving shaft), ω_t is the angular speed of the turbine runner (driven shaft) and (ω_t/ω_p) is the speed ratio.

The torque relationship is given by,

$$T_t = T_p + T_v \tag{27.24}$$

The efficiency of the converter is given by,

$$\boxed{\eta = \frac{\text{Power output}}{\text{Power input}} = \frac{(T_p + T_v)\times\omega_t}{T_p \times \omega_p} = \frac{\omega_t}{\omega_p}\times\left(1+\frac{T_v}{T_p}\right)} \tag{27.25}$$

From Equation (27.25), it can be seen that if T_v is zero (i.e., when there is no stationary guide vanes), then the torque converter reduces to hydraulic coupling and we have $\eta = (\omega_t/\omega_p)$. If slip is considered, then $\eta = (1-s)$.

In Equation (27.25), if T_v is positive, then increased torque is obtained at the driven shaft. To achieve this, guide vanes are to be designed to receive the torque from the liquid in a direction opposite to that exerted on the driven shaft. Conversely, if T_v is negative, then reduced torque is obtained at the driven shaft. This is accomplished by designing the guide vanes to receive a torque from the liquid in the same sense as that of driven shaft.

In order to obtain a large reduction in speed and a large torque magnification, the hydraulic torque converters have two or more sets of turbine runners and fixed guide vanes.

The efficiency versus speed ratio curve for hydraulic torque converter at a given pump torque (T_p) is illustrated in Figure 27.12(b). It can be seen that efficiency increases with increase in speed ratio and it becomes maximum when speed ratio is approximately 0.5, but the efficiency drops at higher speed ratios. It is also observed that the efficiency of a torque converter is higher than a fluid coupling at lower speed ratios. Conversely, the coupling is more economical than a converter when the speed ratio approaches unity. Thus, the advantages of both the converter and the coupling can be obtained in a transmission system by designing it in such a way that it acts as a converter at low speed ratios and as a coupling at high speed ratios.

27.10 ❒ AIR LIFT PUMP

The air lift pump is a device which is used to lift water from a deep well or sump by using compressed air. It consists of a source of compressed air (air compressor), air supply pipe fitted with a set of air nozzles, and a rising main (or a delivery pipe) as shown in Figure 27.13. The compressed air is introduced through the nozzles fitted at the bottom of the air supply pipe. It mixes with water in the form of fine spray at the bottom of the rising main fixed in the well from which water is to be lifted. Thus, a mixture of air and water is formed within the rising main. The density of air water mixture is much less than that of pure water. Thus, a very small column of pure water can balance a very long column of air water mixture. Therefore, the flow of air water mixture begins through the rising main and it will be discharged at its top. The flow will continue as long as the supply of compressed air is maintained.

If h is the submergence (i.e., the height of static water level in the well above the tip of the nozzle) and H is the height of the point to which water is lifted above the tip of the nozzle, then $(H-h)$ is called useful lift. It has been observed that best results are obtained if the useful lift is less than submergence, i.e., $(H-h) < h$. Generally, the ratio $(H-h)/h$ varies from 0.25 to 1 for the values of h ranging from about 30 m to 90 m.

Some of the advantages of air lift pump are (i) it has no moving parts below the water level and therefore, no wear and tear of the pump due to suspended solid particles, (ii) it can raise more water through a bore hole of given diameter than any other pump, and (iii) it is suitable for draining water from the mines.

The efficiency of the air lift pump is low which varies from 20% to 40% and it may further impaired due to air leakage.

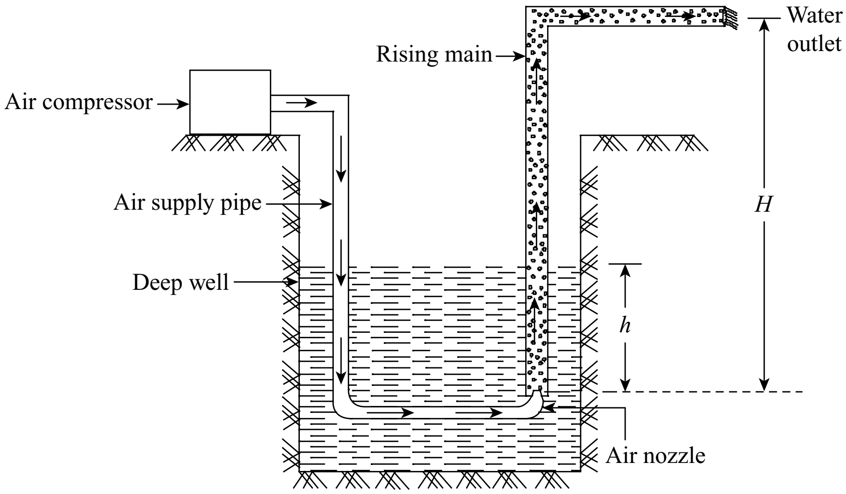


Figure 27.13 *Air lift pump*

27.11 ❐ JET PUMP

Jet pump is a device which is used for lifting water from deep well or sump by utilizing energy of water. It consists of a nozzle and a diffuser assembly mounted on a mixing chamber as shown in Figure 27.14. A stream of high pressure water from the delivery pipe of the pump is allowed to flow through the nozzle at the throat of the diffuser. The pressure energy of water converts into kinetic energy. This causes the pressure drop due to which suction is created and water is sucked in from the sump. A large supply of low pressure water is ensured. In the mixing zone, the streams of different velocities mix and there occurs some pressure rise. After mixing zone, the pressure of water mixture further recovers in the diffuser section due to decrease in velocity.

The jet pump can lift water up to a height of about 6 m. Its capacity ranges up to about 50 litres per second and they are used in mines. The jet pumps are also used for pumping oil or petroleum.

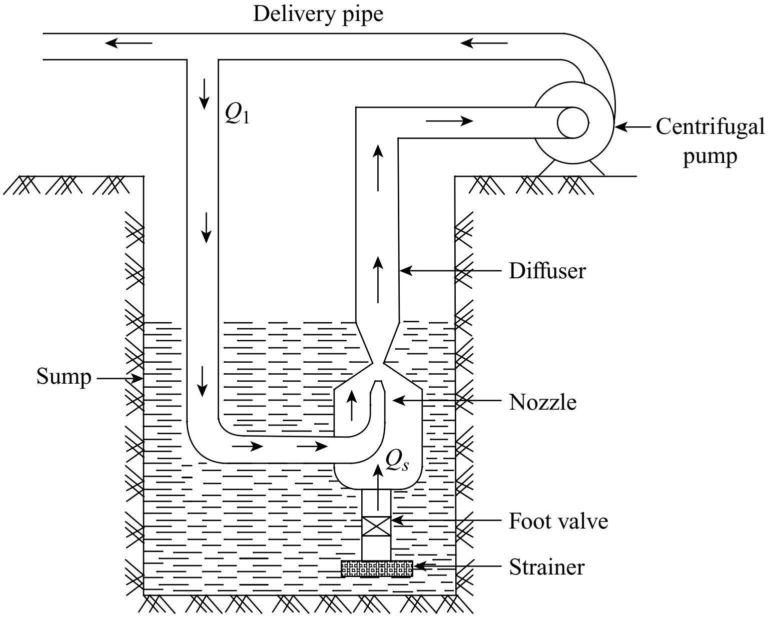


Figure 27.14 *Jet pump*

27.12 ❐ EXTERNAL GEAR PUMP

It is a constant delivery pump which gives a continuous discharge of oil at a uniform rate. An external gear pump unit consists of two identical intermeshing spur gears working with a fine clearance inside a closely fitting stationary casing as shown in Figure 27.15. One of the gears is keyed to the driving shaft of a motor. The other gear is idle and it revolves due to driving gear. Both gears rotate in the opposite direction. The oil is trapped between the teeth and the casing and carried round between the gears from the suction port to the discharge port. The oil pushed into the discharge port cannot slip back into the suction port due to the perfect meshing of the gears which acts as a seal.

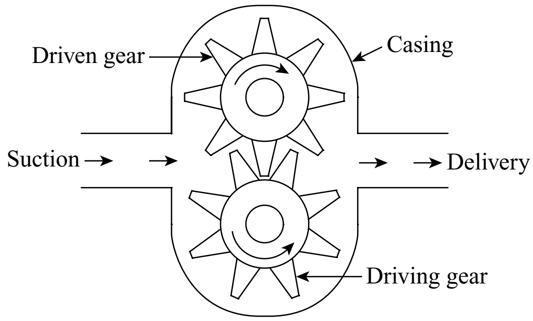


Figure 27.15 *External gear pump*

Let n be the number of teeth in each gear, N be the speed in revolutions per minute, l be the axial length of teeth and a be the area enclosed between two adjacent teeth and casing.

$$\text{Volume of oil pumped in one revolution} = 2aln$$

Therefore, the theoretical discharge per second is given by,

$$\boxed{Q_{th} = (2\,a\,l\,n)\times\frac{N}{60}} \tag{27.26}$$

The actual discharge will be less than the theoretical discharge. If η_{vol} is the volumetric efficiency, then the actual discharge is given below.

$$\boxed{Q_a = (2\,a\,l\,n)\times\frac{N}{60}\times\eta_{vol}} \tag{27.27}$$

The volumetric efficiency of gear pumps varies from 70% to 90%. Generally, these pumps are used for relatively low pressure applications up to 10 bar. These find applications in the supply of cooling water and supply of pressure oil for lubrication purposes of machine tools drives, turbines, etc. Gear pumps are noisy in operation and it requires more maintenance.

Summary

1. Hydraulic press is used for lifting heavy loads by the application of a much smaller force. It works on the basis of Pascal's law.
2. Hydraulic accumulator is used for storing the energy of liquid under pressure temporarily and supplies it for any sudden or intermittent requirement.
3. Capacity of the accumulator $= pAL$, here p is the liquid pressure supplied by the pump, A is the area of sliding ram and L is the stroke or lift of the ram.
4. Differential hydraulic accumulator is another form of accumulator in which the liquid is stored at a high pressure by a comparatively small load on the ram.
5. Hydraulic intensifier increases the intensity of pressure of the liquid by utilizing the energy of a larger quantity of liquid at low pressure.
6. Hydraulic ram is a pump which lifts a small quantity of water to a greater height from large quantity of water available at a smaller height. The efficiency of hydraulic ram is expressed in two ways as shown below.

   (i) D' Aubuisson's efficiency:

   $$\eta_D = \frac{qH}{Qh} \text{ or } \eta_D = \frac{w\times H}{W\times h}$$

   (ii) Rankine's efficiency:

   $$\eta_R = \frac{q\times(H-h)}{(Q-q)\times h} \text{ or } \eta_R = \frac{w\times(H-h)}{(W-w)\times h}$$

   Here, Q is the discharge through supply pipe, q is the discharge through delivery pipe, h is the height of water in supply tank above the valve chamber, H is the height of water raised from the valve chamber, w is the weight of water delivered per second by the ram and W is the weight of water flowing from supply tank to the valve chamber per second.

7. Hydraulic lift is used for carrying people or goods from one floor to another in a multi-storeyed building. These are of two types, namely direct acting and suspended hydraulic lift. Modern lifts are generally of suspended type.
8. Hydraulic crane is used to lift heavy loads as high as 50 tons to about 100 tons.
9. Hydraulic coupling (or fluid coupling) is a device used to transmit power from driving shaft to driven shaft through oil. There is no mechanical connection between the two shafts and torque remains equal on both shafts.
10. The hydraulic torque converter is a device used to transmit increased or decreased torque at the driven shaft. Usually, the torque converters are used to increase the torque at the driven shaft.
11. The air lift pump is used to lift water from a deep well by using compressed air.
12. Jet pump is used for lifting water from deep well by utilizing energy of water.
13. Gear pump is a constant delivery pump consisting of two identical intermeshing spur gears working with a fine clearance inside a casing.

Multiple-choice Questions

1. In a hydraulic ram,
   (a) Rankine efficiency < D'Aubuisson efficiency.
   (b) Rankine efficiency = D'Aubuisson efficiency.
   (c) Rankine efficiency > D'Aubuisson efficiency.
   (d) None of the above.
2. Hydraulic ram works on the
   (a) Principle of centrifugal action.
   (b) Principle of water hammer.
   (c) Principle of reciprocating action.
   (d) None of the above.
3. Which of the following device is used for transmitting increased torque to the driven shaft?
   (a) Hydraulic coupling.
   (b) Hydraulic ram.
   (c) Hydraulic intensifier.
   (d) Hydraulic torque converter.
4. Hydraulic coupling is used for:
   (a) Transmitting increased torque to the driven shaft.
   (b) Transmitting same torque to the driven shaft.
   (c) Transmitting decreased torque to the driven shaft.
   (d) None of the above.
5. Maximum efficiency of hydraulic ram is limited to about
   (a) 75%.
   (b) 50%.
   (c) 25%.
   (d) None of the above.
6. The lifting speed of suspended hydraulic lift varies from
   (a) 0 to100 m/min.
   (b) 100 to 200 m/min.
   (c) 200 to 300 m/min.
   (d) None of the above.
7. Hydraulic crane can lift loads up to about
   (a) 100 tons.
   (b) 200 tons.
   (c) 300 tons.
   (d) None of the above.
8. The lifting speed of modern hydraulic crane may be about
   (a) 25 m/min.
   (b) 50 m/min.
   (c) 75 m/min.
   (d) None of the above.
9. Maximum efficiency of a hydraulic torque converter will be when speed ratio is about
   (a) 0.3.
   (b) 0.5.
   (c) 0.7.
   (d) 0.9.
10. The efficiency of a hydraulic coupling will be zero when speed ratio is
    (a) 0.6.
    (b) 0.8.
    (c) 1.0.
    (d) None of the above.

Review Questions

1. Explain the constructional and working details of a hydraulic press with the help of a neat sketch. Also give the mechanical advantage and leverage of the press.
2. Describe with the aid of neat sketch the working of a hydraulic accumulator. Also obtain an expression for the capacity of a hydraulic accumulator.
3. Describe with a neat sketch the construction and working operation of a differential hydraulic accumulator. Also discuss how does it differ from a simple hydraulic accumulator?
4. Explain the constructional and working details of a hydraulic intensifier. Also mention some of the systems in which it is used.

5. Explain the construction and working of a hydraulic ram with a neat diagram. Also obtain the expressions for its efficiencies.
6. Explain with the help of a neat sketch the principle and working of a modern hydraulic lift.
7. Draw a neat diagram and explain the principle and working of a hydraulic crane.
8. Describe with the help of a neat sketch the constructional and working details of a hydraulic coupling. Also discuss its characteristics, merits and applications.
9. Explain the torque converter and its characteristics with neat sketches. Also give the difference between a fluid coupling and a fluid converter.
10. Write short note on (i) air lift pump and (ii) jet pump.
11. Explain the construction and working details of an external gear pump with the help of a diagram.

Problems

1. A hydraulic press has a ram of 200 mm diameter and plunger of 40 mm diameter with a stroke length of 275 mm. The weight exerted by press ram amounts to 10 kN and distance moved is 1 m in 3 minutes. Find (i) the force applied on the plunger, (ii) number of strokes performed by the plunger, (iii) work done by the press ram and (iv) power required to drive the plunger.

   [**Ans.** 0.4 kN, 91, 10 kNm, 0.055 kW]

2. A hydraulic press has a ram of 12 cm diameter and a plunger of 2 cm. What force will be required on the plunger to lift a load of 35 kN? If the plunger has a stroke of 35 cm, then how many strokes will be required to lift the load by 75 cm? Also determine the volume of additional liquid required. Further if the time taken to lift the load is 5 minutes and the frictional effects are neglected, then what would be the power of the motor required to drive the plunger?

   [**Ans.** 971.71 N, 78, 0.00848 m^3, 26250 Nm, 0.0875 kW]

3. A hydraulic press has a ram of 150 mm diameter and plunger of 30 mm diameter. The stroke length of the plunger is 200 mm and weight lifted is 900 N. If the distance moved by the weight is 1 m in 15 minutes, then determine (i) the force applied on the plunger, (ii) power required to drive the plunger and (iii) number of strokes performed by the plunger.

   [**Ans.** 36 N, 10^{-3} kW, 125]

4. The ram and plunger of a hydraulic press are 300 mm and 40 mm, respectively, and the leverage of the handle is 10 : 1. The press can lift a load of 200 kN through 1.5 m in 94 seconds with a plunger stroke of 300 mm. Determine (i) the force applied at the end of lever, (ii) the number of strokes completed by the plunger per second and (iii) power required to drive the plunger of the press. Assume the packing friction of the plunger as well as the ram as 5% of the load.

   [**Ans.** 0.3941 kN, 282, 3.547 kW]

5. If the diameter of the ram of an accumulator is 0.3 m and displacement is 105 litres, then find its stroke.

   [**Ans.** 1.485 m]

6. The liquid is supplied at a pressure of 150 kN/m^2 to an accumulator having a plunger of diameter 1.2 m. Determine the capacity of the accumulator and total weight placed on the ram (including the weight of ram) when total lift of the ram is given as 7.5 m.

   [**Ans.** 1272.375 kNm, 169.65 kN]

7. An accumulator has a ram diameter 300 mm and a lift of 6 m. The total weight on the accumulator is 80 kN. The packing friction is 5% of the load on the ram. Determine the power delivered to the machine if ram falls through the full height in 90 seconds and at the same time pump delivers 30 litres per second through the accumulator.

   [**Ans.** 37.325 kW]

8. A hydraulic accumulator has sliding ram of 460 mm diameter which slides through 8.5 m in 3.5 minutes during its working stroke, while weight on the ram including its self-weight is equivalent to 350 kN. The pump supplies water at a rate of 10 litres per second and packing friction amounts to 5% of total load. Determine (i) the pressure intensity of water, (ii) power delivered by accumulator to the machine and (iii) power required to drive the pump having efficiency 75%.

   [**Ans.** 2000.6 kN/m^2, 33.47 kW, 26.675 kW]

9. The diameters of two parts of the ram of a differential accumulator are 160 mm and 130 mm. The stroke is 1.25 m. If the pressure of water is 5000 kN/m^2 when the load is at rest at the upper end of the stroke or when the load is moving with uniform velocity, then what will be the weight of the loaded cylinder? How much energy can be stored in the accumulator? What will be the diameter of the ram of an ordinary accumulator to move the same load with the help of the same water pressure?

   [**Ans.** 34165 N, 42706.25 Nm, 0.0933 m]

10. The pressure intensity of liquid supplied to an intensifier is 250 kN/m^2 while the pressure intensity of water leaving the intensifier is 1250 kN/m^2. Determine the diameter of the fixed ram of the intensifier if the external diameter of the sliding cylinder is 0.25 m.

    [**Ans.** 0.1118 m]

11. A hydraulic intensifier gets low pressure liquid at a pressure of 5.2 MPa and delivers it to the machine at a pressure of 20.8 MPa. If the intensifier has a capacity of 0.035 m^3 and

stroke 1.5 m, then determine the diameters of the fixed ram and the sliding cylinder to be used for the intensifier.

[**Ans.** 0.1711 m, 0.3422 m]

12. A hydraulic intensifier has a ram diameter of 0.16 m and a sliding cylinder diameter of 0.8 m. Find the pressure on the low pressure side of the intensifier if the pressure at the outlet of the intensifier is 16 MPa. The loss due to friction at each packing of the intensifier is given 5% of the total force on each of the packing.

[**Ans.** 0.709 MPa]

13. The water is supplied at the rate of 2500 litres per minute from a height of 5 m to a hydraulic ram, which lifts 250 litres per minute to a height of 30 m from the ram. The length and diameter of the delivery pipe is 80 m and 60 mm, respectively. Determine the efficiency of the hydraulic ram if the coefficient of friction is 0.009.

[**Ans.** 70.64%, 67.38%]

14. The following particulars are given for a hydraulic ram, such as supply head = 5 m, delivery head = 25 m and ratio of water lifted to water wasted by the ram = 1 : 10. Determine the efficiency of the ram.

[**Ans.** 45.45%, 40%]

15. A hydraulic lift is required to lift a load of 90 kN through a height of 15 m once in every 1.75 minutes. The lift travels up at the rate of 1.2 m/s. During working stroke of the lift, the water is supplied to it from the accumulator and from the pump at a pressure intensity of 4000 kN/m^2. If the efficiency of the lift is 75% and that of pump is 80%, then calculate the power required to drive the pump and the minimum capacity of the accumulator. Neglect friction losses in the pipe.

[**Ans.** 21.43 kW, 0.3964 m^3]

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1. (a)	**2.** (b)	**3.** (d)	**4.** (b)	**5.** (a)
6. (b)	**7.** (a)	**8.** (c)	**9.** (b)	**10.** (c)

Index

I

J

K

L

M

N

O

P

Q

R

S

T

U

V

W

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