MTS 105 (ALGEBRA AND TRIGONOMETRY FOR BIOLOGICAL SCIENCES)

(SEQUENCES, SERIES AND BINOMIAL EXPANSION)

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INRODUCTION

A set of numbers such as $a_1, a_2, a_3, ..., a_{n-1}, a_n$ which appear in a definite order is called a sequence or a progression. Here, a_1 is called the first term, a_2 is called the second term,...,a_n is called the n^{th} term of the sequence.

EXAMPLE

(1). $5, 8, 11, 14, ...$

(2). $8, 4, 2, 1, \frac{1}{2}, ...$

(3). $1, 3, 5, 7, ..., 2n - 1$

If the number of terms in a sequence terminates, then we have a finite sequence. Otherwise, the sequence is infinite.

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SEQUENCE

DEFINITION

The expression $a_1 + a_2 + a_3+, ... + a_{n-1} + a_n$ which is formed from the sequence $a_1, a_2, a_3, ..., a_{n-1}, a_n$ is called a series. If the expression terminates, then it is called a finite series. Otherwise, it is an infinite series.

We see therefore, that sequences lead to series.

REMARK

In a sequence, two consecutive terms are seperated by a comma, while in a series, two consecutive terms are seperated by either $+$ or $-$.

We now consider two types of sequences, namely; arithmetic progression and geometric progression.

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ARITHMETIC PROGRESSION

ARITHMETIC PROGRESSION

A sequence is said to be an arithmetic progression (A.P) if the difference between any of its two consecutive terms is a constant. That is, a sequence of the form: $a, a+d, a+2d, a+3d, \ldots$; where a is the first term and d is called the common differnce. Here, $a_{k+1} - a_k = d$ for all k such that $1 \leq k \leq n$. The k^{th} term (T_k) of an A.P. fromm this pattern is given by $T_k = a + (k-1)d$. That is, (2^{nd} term -1^{st} term) =(3^{rd} term -2^{nd} term)= ... =(n^{th} term $-(n-1)^{th}$ term)= d. Or, $T_2 - T_1 = T_3 - T_2 = \ldots = T_n - T_{n-1} = d$. The n^{th} term a finite A.P. is called the last term, denoted by L. So, $T_n = a + (n-1)d$.

ARITHMETIC SERIES

DEFINITION

The sum (addition) of an A.P. is called an Arithmetic series. The sum of the first n terms of an A.P. is denoted by S_n. That is,

$$S_n = a + (a + d) + (a + 2d) + ... + [a + (n - 2)d] + [a + (n - 1)d] \quad (1)$$

Now, reversing the series, we have

$$S_n = [a + (n - 1)d] + [a + (n - 2)d] + ... + (a + 2d) + (a + d) + a \quad (2)$$

Adding expressions (1) and (2), we have

$2S_n = [2a+(n-1)d]+[2a+(n-1)d]+...+[2a+(n-1)d]+[2a+(n-1)d] = n[2a+(n-1)d]$.

So, $S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}[a + a + (n - 1)d] = \frac{n}{2}(a + L)$ (since $L = a + (n - 1)d$).

So, $S_n = \frac{n}{2}$(first term+ last term).

EXAMPLE

EXAMPLE

Given the arithmetic sequence $2, 5, 8, ..., 443$, find:

(i). The number of terms in the sequence

(ii). The sum of the sequence

SOLUTION

(i). Now, the first term of the sequence is 2 and the last term is 443. The common difference d is obtained from $2 + d = 5$. So, $d = 3$. The n^{th} term is given by $T_n = a + (n-1)d$. But the last term is T_n since the sequeence is finite. So, $443 = 2 + (n-1)3$. So, $n = 148$.

(ii). The sum of the sequence is given by $S_n = \frac{n}{2}[2a + (n-1)]d = \frac{n}{2}(a + L) = 32,930$.

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EXAMPLE

EXAMPLE

In a fund raising dinner, 64% of the money collected is given as cash prizes. There are eight categoties of cash prizes. The first prize winner gets 39,000, the second prize winner gets 35,000, the third prize winner gets 31,000, and so on. How much money:

(i). does the 8^{th} prize winner get?

(ii). is given as cash prizes altogether?

(iii). was raised at the dinner?

SOLUTION

SOLUTION

For the prizes given, we have the follwing sequence:

$39,000, 35,000, 31,000, ...$

So, $a = 39,000, d = -4,000$.

(i). To find how much the eight prize winner gets means that we should find the 8^{th} term T_8. But $T_n = [a + (n-1)d]$. So, $T_8 = [39,000 + (8-1)(-4000)] = 11,000$. So, the eight prize winner gets $11,000$.

(ii). To find the total amount of momey given as cash prizes means to find the sum of S_n. But $S_n = \frac{n}{2}[2a + (n-1)]d = \frac{n}{2}(a+L) = \frac{8}{2}(39000 + 11000) = 200000$. So, the total amount given is $200,000$.

SOLUTION

(iii). Remember that it was 64% of the money raised at the dinner that was given as cash prizes. Now total amount given as cash prizes is 200, 000. So, if the total money raised was P, then 64% of P is 200, 000. So, we find P.

$\frac{64}{100}P = 200000$. So, $P = 312500$.

Hence, the money realised at the dinner is 312, 500.

ARITHMETIC MEAN

ARITHMETIC MEAN

If x, y, z are three consecutive terms of an A.P., then $y = \frac{1}{2}(x + y)$ is the arithmetic mean of x and z. This is easily seen from the folwinf argument:

Since the common differnce is d, then $d = y - x = z - y$. So, $2y = x + z$. Hence, $y = \frac{1}{2}(x + y)$.

EXAMPLE

Find six arithmetic means between -3 and 18.

SOLUTION

We require eight terms in arithmetic progression such that -3 is the first term and 18 is the 8^{th} term. If d is the common differnce, then $T_n = y = \frac{n}{2}[2 + (n - 1)d]$, we have $18 = T_8 = -3 + 7d$. Hence, $d = 3$. The arithmetic means are therfore, $0, 3, 6, 9, 12, 15$ obtained by adding d to each preceeding term.

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EXAMPLE

EXAMPLE

Find three numbers in an A.P. whose sum is 21 and whose product is 315.

SOLUTION

Let the numbers be $a-d, a, a+d$. Then $a-d+a+a+d=21$. Therefore, $3a=21$. Hence, $a=7$.

Now, $a(a-d)(a+d)=a(a^2-d^2)=315$

$a^2-d^2=45$ (since $a=7$). So, $d^2=4$. Then $d\pm 2$. Hence, the required numbers are $5, 7, 9$.

GEOMETRIC PROGRESSION

GEOMETRIC PROGRESSION

A sequence of the form $a, ar, ar^2, ar^3, ...$ is called a geometric progression (G.P.); where $a_1 = a$ is the first term and r is the common ratio. That is, if a_k is the k^{th} term and a_{k+1} is the $(k+1)^{th}$ term, then $a_k(r) = a_{k+1}$. So, $r = \frac{a_{k+1}}{a_k}$; showing that the ratio of any two consecutive terms is constant.

From the pattern of a G.P., we see that the n^{th} term $T_n = ar^{n-1}$

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GEOMETRIC SERIES

DEFINITION

The sum of the first n terms of a G.P. is called a geometric series. That is,

$$S_n = a + ar + ar^2 + ar^3 + ... + ar^{n-1} \tag{3}$$

Now, multiplying both sides of expression (3) by r gives

$$rS_n = ar + ar^2 + ar^4 + ... + ar^{n-1} + ar^n \tag{4}$$

Subtracting expression (4) from expression (3), gives $S_n - rS_n = a - ar^n$. So, $S_n(1-r) = a(1-r^n)$.

Hence,

$$S_n = a\frac{(1-r^n)}{1-r} \tag{5}$$

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SUM TO INFINITY

SUM TO INFINITY

From expression (5), we see that if $r > 1$, then S_n becomes very large so that LtS_n as $n \to \infty$ is ∞

But if $-1 < r < 1$, then as n increases, we see that $r^n \to 0$; so that LtS_n as $n \to \infty$ is $\frac{a}{1-r}$.

We therefore say that the series converges to the sum $\frac{a}{1-r}$. The condition for converrence of the series is that $\mid r \mid < 1$

EXAMPLE

To what sum does the series $\frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \frac{7}{10000} + \ldots$ converge?

SOLUTION

Now, this is a geometric series with first term $\frac{7}{10}$ and common ratio $\frac{1}{10}$

Now, $\mid \frac{1}{10} \mid < 1$. So, the series converges to $S_n = \frac{\frac{7}{10}}{1-\frac{1}{10}} = \frac{7}{9}$.

EXAMPLE

EXAMPLE

If the sum of the first n terms of a sequence is given by $S_n = 9(1 - \frac{1}{3^n})$.

(i). Find the first and second terms of the sequence

(ii). Find the n^{th} term of the sequence

(iii). Show that the sequence is a G.P. and find the common ratio.

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SOLUTION

SOLUTION

Now, $S_n = 9(1 - \frac{1}{3^n})$.

(i). First term $= a_1 = S_1 = 9(1 - \frac{1}{3^1}) = 6$

Second term $= a_2 = S_2 - S_1 = 9(1 - \frac{1}{3^2}) = 8 - 6 = 2$

(ii). The n^{th} term $= a_n = S_n - S_{n-1} = 9(1 - \frac{1}{3^n}) - 9(1 - \frac{1}{3^{n-1}}) = 9(\frac{1}{3^{n-1}} - \frac{1}{3^n}) = (\frac{2}{3^{n-2}})$

(iii). From (ii) above, it follows that

$a_{n-1} = \frac{2}{3^{n-3}}$. But $a_n = \frac{2}{3^{n-2}}$. So

$\frac{a_n}{an-1} = \frac{3^{n-3}}{3^{n-2}} = \frac{1}{3}$

Hence, the sequence is a G.P. with common ratio $r = \frac{1}{3}$.

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GEOMETRIC MEAN

GEOMETRIC MEAN

If a, b, c are three consecutive terms of a G.P., then $b = \sqrt{ac}$ is called the geometric mean of a and c.

To show this, recall that the ratio of any two consecutive terms of a G.P. is a constant.

That is, $\frac{b}{a} = \frac{c}{b} = r$

So, $b^2 = ac$; giving us $b = \sqrt{ac}$

REMARK

REMARK

The arithmetic mean (A.M) $\geq$ the geometric mean (G.M). This is shown as follws:

Now, let a, b be two numbers such that $a \geq 0$ and $b \geq 0$. Then $\sqrt{a}$ and $\sqrt{b}$ exist. Consider

$$(\sqrt{a} - \sqrt{b})^2 \geq 0 \tag{6}$$

Expanding expression (1) gives $(\sqrt{a})^2 - 2\sqrt{a}\sqrt{b} + (-\sqrt{b})^2 \geq 0$.

Hence, $a - 2\sqrt{ab} + b \geq 0$

So, $a + b \geq 2\sqrt{ab}$

Therfore, $\frac{1}{2}(a + b) \geq \sqrt{ab}$ as required.

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EXAMPLE

EXAMPLE

Find four geometric means between 2 and 486

SOLUTION

We require six numbers in geometric progression such that the first is 2 and the sixth is 486.

Let r be the common ratio. Then since $T_n = ar^{n-1}$, we have

$2r^5 = 486$

So, $r^5 = 243$

Therefore $r = \sqrt[5]{243} = 3$.

Hence, the required geometric means are $2(3), 2(3)^2, 2(3)^3, 2(3)^4$. That is, $6, 18, 54, 162$.

BINOMIAL EXPANSION

BINOMIAL EXPANSION

We consider the expansion of the sum (or difference) of two quantities. For example, if we are to expand an expression of the form $(a+b)^n$, where n is a positive integer, what would be the result?

Now, by arithmetic multiplication, we have the following:

$(1+x)^1 = 1+x$

$(1+x)^2 = (1+x)(1+x) = 1+2x+x^2$

$(1+x)^3 = (1+2x+x^2)(1+x) = 1+3x+3x^2+x^3$

$(1+x)^4 = (1+x)^3(1+x) = 1+4x+6x^2+4x^3+x^4$

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BINOMIAL EXPANSION

We observe that in each of the results above, the coefficients of the first and last terms are 1. Also, each of the other coefficients in $(1+x)^{n+1}$ is the sum of the corresponding coefficient and the preceding one in the expansion of $(1+x)^n$. Thus, we can lay out the coefficients of the successive powers in the form of a triangle, known as Pascal's triangle, using these rules as follows:

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

The above triangle represents the coefficients of powers of a and b in the expansion of $(a+b)^n$ with respect to the positive integer n.

REMARK

REMARK

(i) Upon expansion, the powers of a decreases from n to 0, while the powers of b increases from 0 to n.

(ii) Each term of the expansion is of degree n.

(ii) The expansion of $(a+b)^n$ contains $n+1$ terms.

EXAMPLE

Expand the following:

(i) $(a+b)^3$ (ii) $(a+b)^4$

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SOLUTION

SOLUTION

(i) Now, there are two things to do here. First, we write out the ascending (or descending) powers of a and b. Then we write out their corresponding coefficients using the Pasccal's triangle.

So, we have $a^3b^0 + a^2b^1 + a^1b^2 + a^0b^3 =$

$a^3 + a^2b + ab^2 + b^3$

Now, putting the corresponding coefficients obtained from the Pascal's triangle, we have

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

(ii) By following the same steps in (i), we have

$a^4b^0 + a^3b + a^2b^2 + ab^3 + a^0b^4 =$

$a^4 + a^3b + a^2b^2 + ab^3 + b^4$ Now, putting the corresponding coefficients, we have

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

EXAMPLE

EXAMPLE

Expand $(2x+3y)^3$ in descending powers of x

SOLUTION

The expansion of $(2x+3y)^3$ is similar to the expansion of $(a+b)^3$:

Here, $a = 2x$, $b = 3y$.

But $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

So, putting $2x$ for a and $3y$ for b, we have

$(2x+3y)^3 = (2x)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2 + (3y)^3 =$

$8x^3 + 3(4x^2)(3y) + 3(2x)(9y^2) + 27y^3 =$

$8x^3 + 36x^2y + 54xy^2 + 27y^3$

EXAMPLE

EXAMPLE

Obtain the expansion of $(2x - \frac{1}{2})^4$

SOLUTION

Now, $(2x - \frac{1}{2}) = (2x + (\frac{-1}{2}))^4$ which is similar to $(a+b)^4$; where $a = 2x$ and $b = \frac{-1}{2}$.

But $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

Now, putting $2x$ for a and $\frac{-1}{2}$ for b, we have

$(2x + (\frac{-1}{2}))^4 = (2x)^4 + 4(2x)^3(\frac{-1}{2}) + 6(2^2x^2(\frac{1}{4})) + 8x(\frac{-1}{8}) + (\frac{1}{16}) =$

$16x^4 - 16x^3 + 6x^2 - x + \frac{-1}{16}$

EXAMPLE

10

EXAMPLE

Expand $(1.01)^3$

SOLUTION

Now, $1.01 = 1 + 0.01 = 1 + \frac{1}{100}$

So, $(1.01)^3 = (1 + \frac{1}{100})^3$. This is similar to $(a+b)^3$; where $a = 1$ and $b = \frac{1}{100}$.

But $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

Putting 1 for a and $\frac{1}{100}$ for b, we have

$(1.03)^3 = (1 + \frac{1}{100})^3 =$

$1^3 + 3(1)^2(\frac{1}{100}) + 3(1)(\frac{1}{100})^2 + (\frac{1}{100})^3 =$

$1 + 3(\frac{1}{100}) + 3(\frac{1}{10000}) + \frac{1}{1000000} =$

1.030301.

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BINOMIAL THEOREM

REMARK

Instead of relying on the Pascal's triangle to perform our expansion, there is a formula we can use. This is given by the Binomial Theorem.

BINOMIAL THEOREM

Suppose n is a positive integer. Then for any real numbers a and b, we have $(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}a^0 b^n$;

where $\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r(r-1)(r-2)\ldots(2)(1)}$ are called binomial coefficients.

REMARK

REMARK

(i) The expression $\binom{n}{r}$ gives us the coefficients of the terms in the expansion just the way the Pascal's triangle does.

(ii) As expected, the number a in the expansion decreases from n to 0 while the number b increases from 0 to n.

(iii) $\binom{n}{0} = 1, \quad \binom{n}{n} = 1$

(iv) Since $\binom{n}{r} = \binom{n}{n-r}$, it follows that the coefficients of the binomial expansion are symmetrical about the middle. There is one middle term (that is, the $\frac{n}{2}th$ term) if n is even, and two middle terms (that is, the $\frac{n-1}{2}th$ and $\frac{n+1}{2}th$ terms) if n is odd.

EXAMPLE

EXAMPLE

Expand $(2x+3y)^3$ using the Binomial Theorem.

SOLUTION

Let $2x = a$ and $3y = b$ so that $(2x+3y)^3 = (a+b)^3$. Here, $n = 3$.

Now,

$$(a+b) = \binom{3}{0}a^3b^0 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}a^0b^3 \quad (6)$$

Let us now find the binomial coefficients:

$\binom{3}{0} = 1, \quad \binom{3}{1} = 3, \quad \binom{3}{2} = 3, \quad \binom{3}{3} = 1$

Putting the binomial coefficients into expression (6), we have:

$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$

But $a = 2x$ $b = 3y$.

So, $(2x+3y)^3 = (2x)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2 + (3y)^3 =$
$8x^3 + 36x^2y + 54xy^2 + 27y^3$

REMARK

REMARK

We can obtain the coefficient of any term in the expansion without necessarily carrying out the expansion.

The coefficient of a^{n-r} in the expansion of $(a+b)^n$ is $\binom{n}{r}b^r$.

EXAMPLE

Find the coefficient of x^{10} in the expansion of $(x-3)^{14}$.

SOLUTION

We know that the coefficient of a^{n-r} in the expansion of $(a+b)^n$ is $\binom{n}{r}b^r$.

So, the coefficient of $a^{10} = a^{14-4}$; where $n = 14, r = 4$, is $\binom{14}{4}b^4$.

But $(x-3)^{14}$ is to be expanded. Now, $(x-3)^{14} = [x + (-3)]^{14}$.

Let $a = x$ and $b = -3$. Then the coefficient of x^{10} is

$\binom{14}{4}(-3)^4 = 81081$.

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GENERAL TERM

GENERAL TERM OF A BINOMIAL EXPANSION

The general term of a binomial expansion $(a+b)^n$ is given by

$\binom{n}{r}a^{n-r}b^r$.

EXAMPLE

Find the term in x^9 and the term independent of x in the expansion of $(2x^2 - \frac{1}{x})^{12}$.

SOLUTION

The general term is $\binom{12}{r}(2x^2)^{12-r}(-x^{-1})^r =$

$\binom{12}{r}(-1)^r 2^{12-r}x^{24-3r}$.

If a term has x^9, then $24 - 3r = 9$. So, $r = 5$. This implies that the term having x^9 is the $6th$ term. The $6th$ term is therefore $\binom{12}{5}(-1)^5 2^7 x^9 = -101376x^9$

The term independent of x has x^0. That is, $24 - 3r = 0$. So, $r = 8$.

This implies that the term we require is the $9th$ term. Therefore, the term independent of x is $\binom{12}{8}(-1)^8 2^4 = 7920$.

GREATEST TERM

Let a and b be given specific values and suppose that $(a+b)^n =$
$u_0 + u_1 + u_2 + u_3 + \ldots + u_n$;
where $u_r = \binom{n}{r} a^{n-r} b^r$.
Solve for r the inequalities $\frac{u_{r+1}}{u_r} > 1$ and $\frac{u_{r+1}}{u_r} < 1$
(i) if $u_0 < u_1 < \ldots < u_q$ and $u_q > u_{q+1} > \ldots > u_n$, then u_q is the greatest term of the binomial expansion.
(ii) if $u_0 < u_1 < \ldots < u_{q-1}$ and $u_q > u_{q+1} > \ldots > u_n$, then there are two equal greatest terms, which are u_{q-1} and u_q.

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EXAMPLE

EXAMPLE

Find the value of the greatest term in the expansion of $(1+2x)^8$ when $x=\frac{1}{3}$

SOLUTION

Now, $(1+2x)^8 = \sum_{r=0}^{8} u_r$; where $u_r = \binom{8}{r}(2x)^r$

$\frac{u_{r+1}}{u_r} = \frac{8-r}{r+1}(\frac{2x}{1})$

Solve for r, $\frac{u_{r+1}}{u_r} > 1$ when $x=\frac{1}{3}$

$\frac{8-r}{r+1}(\frac{2}{3}) > 1$ implies that $r < \frac{13}{5}$,

which implies that $r = 0, 1, 2$. That is, $u_0 < u_1 < u_2 < u_3$

Also, solve for r, $\frac{u_{r+1}}{u_r} < 1$ when $x=\frac{1}{3}$

$\frac{8-r}{r+1}(\frac{2}{3}) < 1$ implies $r > \frac{13}{5}$,

which implies that $r = 3, 4, 5, 6, 7$. That is, $u_8 < u_7 < u_6 < u_5 < u_4 < u_3$.

Hence, u_3 is the greatest term of the expansion, and $u_3 = \binom{8}{3}(\frac{2}{3})^3 = \frac{448}{27} = 16.6$.

EXERCISES

EXERCISES

(1). Find the sum of the first n terms of the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + ...$

(2). The first and last terms of a geometric series are 2 and 2048 respectively. The sum of the series is 2730. Find the number of terms and the common ratio.

(3). The first two terms of an arithmetic series are -2 and 3. How many terms are needed for the sum to be 306?

(4). For what value of x does the series $x + \frac{x}{1+x} + \frac{x}{(1+x)^2} + \frac{x}{(1+x)^3} + ...$ converge? Also find the sum to which it converges.

(5). Find the value(s) of a if the coefficient of x^2 in the expansion of $(1 + ax)^4(2 - x)^3$ is 6.

(6). Find the middle term and the term independent of x in expansion of $(3x^3 - \frac{1}{2x})^{12}$