MAT 101

Supplementary Mathematics

Integration

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Introduction

Integration is the reverse of differentiation. Suppose that we are given the derivative of a function, $\frac{dy}{dx}$. The process by which we obtain y as a function of x is known as **Integration.**

Example

(1) *Let* $\frac{dy}{dx} = c$ *where* c *is a constant.*
Then $y = cx$ *is a solution. But notice that* $y = cx + d$ *is also a solution, for different values of* x*, are parallel lines with the same gradient,* c*.*
For example the lines $y = 2x + 3, y = 2x$ *and* $y = 2x - 5$ *as shown in the figure below.*

Figure: Graph show the plots of $y = 2x + 3, y = 2x$ and $y = 2x - 5$

Notation: If $y = \frac{dy}{dx} = c$, then 'y is equal to the integral of c with respect to x', and the statement is written :

$$\int c\, dx = cx + d$$

Integration of x^n, $n \neq -1$

We have that

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\therefore \; \frac{d}{dx}\left(\frac{x^{n-1}}{n+1}\right) = x^n, \quad n \neq -1$$

Hence

$$\int x^n \,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C, \quad \text{where } C \text{ is constant}$$

the case $n = -1$ being excluded.

Remark

We exclude the case $n = -1$ because if we input the value $n = -1$ into the value of the integral of x^n the we will get the value $\frac{x^{-1+1}}{-1+1} = \frac{x^0}{0} = \frac{1}{0}$ which makes no sense.

Example

(2) *It has been proved that*

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log x) = \frac{1}{x}$$

then it follows that

$$\int \frac{1}{x}\, dx = \log x + c, \quad x > 0$$

(3) *Let* $\frac{dy}{dx} = 1 - \frac{1}{x^2}$. *Find* y.
$dy = \left(1 - \frac{1}{x^2}\right) dx \implies \int dy = \int \left(1 - \frac{1}{x^2}\right) dx$

Example (Cont'd)

$$y = \int \left(1 - \frac{1}{x^2}\right)\, dx = \int \left(1 - x^{-2}\right)\, dx$$

$$= \left[x - \frac{x^{-1}}{-1} + C\right] = x + x^{-1} + C$$

$$= x + \frac{1}{x} + C$$

Arithmetic of Indefinite Integrals

Suppose $f(x)$ and $g(x)$ are integrable functions of x. Then the following rules hold:

1. $$\int \{f(x) \pm g(x)\}\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

2. $$\int \alpha f(x)\, dx = \alpha \int f(x)\, dx$$

   where α is a constant.

Example

Evaluate the following integrals:

(1)

$$\int (2x^3 - 3x + 4)\, dx$$

(2)

$$\int \left(x^{\frac{3}{2}} + \sqrt{x}\right)\, dx$$

(3)

$$\int \frac{1}{\sqrt{x}}\, dx$$

Solution

(1)

$$\int (2x^3 - 3x + 4)\,dx = 2\int x^3\,dx - 3\int x\,dx + 4\int dx$$
$$= 2\frac{x^4}{4} - 3\frac{x^2}{2} + 4x + C$$
$$= \frac{x^4}{2} - \frac{3x^2}{2} 4x + C$$

(2)

$$\int \left(x^{\frac{3}{2}} + \sqrt{x}\right)\,dx = \int \left(x^{\frac{3}{2}} + x^{\frac{1}{2}}\right)\,dx$$
$$= \frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}} + C$$

(3)

$$\int \frac{1}{\sqrt{x}}\,dx = \int x^{\frac{-1}{2}}\,dx$$
$$= 2x^{\frac{1}{2}}$$

Standard Integrals

(1)

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad n \neq 1$$

(2)

$$\int \frac{1}{x}\,dx = \ln + C|x|$$

(3)

$$\int \sin x\,dx = -\cos x + C$$

(4)

$$\int \cos x \, dx = \sin x + C$$

(5)

$$\int \tan x \, dx = \ln|\sec x| + C = -\ln|\cos x| + C$$

(6)

$$\int \cot x \, dx = \ln|\sin x| + C$$

(7)

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

(8)

$$\int \mathrm{cosec} x \, dx = \ln|\mathrm{cosec} x - \cot x| + C$$

(9)

$$\int \sec^2 x\, dx = \tan x + C$$

(10)

$$\int \mathrm{cosec}^2 x\, dx = -\cot x$$

(11)

$$\int a^x\, dx = \frac{a^x}{\ln a} + C \quad a > 0, a \neq 0$$

(12)

$$\int \mathrm{e}^x\, dx = \mathrm{e}^x + C$$

(13)

$$\int \frac{dx}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\frac{x}{a} + C \text{ or } -\cos^{-1}\frac{x}{a} + C$$

(14)

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}}\, dx = \ln|x + \sqrt{x^2 \pm a^2}| + C$$

(15)

$$\int \frac{dx}{x^2 + a^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

(16)

$$\int \frac{dx}{x^2 - a^2}\, dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

(17)

$$\int \sqrt{x^2 \pm a^2}\, dx = \frac{x}{2}\sqrt{x^2 \pm a^2} + C$$

(18)

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{2} + C$$

Definite Integrals

Definite integration is the process of measuring the area under a function plotted on a graph.

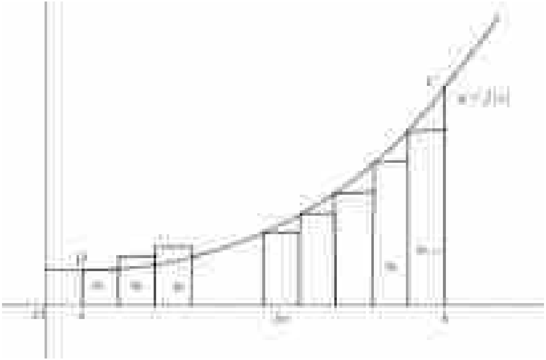


Figure: The graph of the function $y = f(x)$ with the area beneath divided into series

Let the area beneath the curve $y = f(x)$ from $x = a$ to $x = b$ be divided by ordinates into strip of equal width δx and the lengths of the ordinates from the left be $y_1, y_2, y_3, \ldots, y_n, y_{n+1}$ as shown in the figure above. Then the area under the steps from D to C formed by the rectangles is given by

$$y_1 \cdot \delta x + y_2 \cdot \delta x + y_3 \cdot \delta x + \ldots + y_m \cdot \delta x$$

which maybe written as

$$\sum_{x=a}^{b} y \cdot \delta x$$

As $\delta \to 0$, the area under the steps approaches the area under the curve DC, which is

$$\int_a^b y\,dx = \lim_{\delta x \to 0} \sum_{x=a}^{b} y \cdot \delta x$$

Then

$$\int_a^b y\,dx$$

is called the `definite integral` of y with respect to x from $x = a$ to $x = b$.

Remark

1. *The numbers a and b are called the lower and upper limit of the integral respectively.*
2. *The value of a definite integral is a constant.*

The evaluation of definite integral by the method of limits is usually difficult in most cases so as a result of this definite integral are evaluated using the **Fundamental theorem of Calculus**. We stated without proof the fundamental theorem of Calculus below.

Theorem (Fundamental Theorem of Calculus)

'For a general continuous function $f(x)$, we have

$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b = [F(x)]_a^b = F(b) - F(a)$$

Where

$$\frac{dF(x)}{dx} = f(x)$$

Example

Evaluate the following definite integrals

1. $$\int_{-2}^{4} x^2 + 2x - 8 \, dx$$

2. $$\int_{0}^{\frac{\pi}{2}} \cos x \, dx$$

3. $$\int_{3}^{4} \frac{6}{x^3} \, dx$$

Solution

(1)

$$\int_0^1 x^2 + 2x - 8\,dx = \left[\frac{x^{2+1}}{3} + \frac{2x^{1+1}}{2} - 8x\right]_0^1$$
$$= \left[\frac{x^3}{3} + x^2 - 8x\right]_0^1$$
$$= \left[\frac{(1)^3}{3} + (1)^2 - 8(1)\right] - \left[\frac{(0)^3}{3} + (0)^2 - 8(0)\right]$$
$$= \frac{1}{3} + 1 - 8 - 0 = \frac{1+3-24}{3}$$
$$= \frac{-20}{3}$$

(2)

$$\int_0^{\frac{\pi}{2}} \cos x \, dx = [\sin x]_0^{\frac{\pi}{2}}$$
$$= \sin \frac{\pi}{2} - \sin 0 = 1 - 0$$
$$= 0$$

(3)

$$\int_3^4 \frac{6}{x^3} \, dx = \int_3^4 6x^{-3} \, dx = \left[\frac{6x^{-3+1}}{-3+1} \right]_3^4$$
$$= \left[\frac{-3}{x^2} \right]_3^4$$
$$= \frac{-3}{4^2} - \left[\frac{-3}{3^2} \right] = \frac{-3}{16} + \frac{3}{9}$$
$$= \frac{-27+48}{144}$$
$$= \frac{21}{48}$$

Integration by Substitution

Consider the indefinite integral

$$\int f(x)\,dx$$

of any arbitrary function $y = f(x)$. suppose that $f(x)$ can be written as $f(x) = g(u)\frac{du}{dx}$ for some function of a suitably chosen variable $u = u(x)$ then

$$\int f(x)\,dx = \int g(u)\,du\Big|_{u=u(x)}$$

Note: This can also be applied to definite integrals.

Example

Evaluate the following integrals

(1)

$$\int x^2 \sqrt{x^3+1}\, dx$$

(2)

$$\int \frac{dx}{ax+b}$$

(3)

$$\int_0^1 \frac{dx}{\sqrt[3]{1+5x}}$$

(1) Given $\int x^2\sqrt{x^3+1}\,dx$ put $u = x^3 + 1$, then we have $\frac{du}{dx} = 3x^2$.
Substituting this into the given integral we have

$$\begin{aligned}
\int x^2\sqrt{x^3+1}\,dx &= \int x^2 \cdot u^{\frac{1}{2}} \cdot \frac{1}{3x^2}\,du \\
&= \frac{1}{3}\int u^{\frac{1}{2}}\,du \\
&= \frac{1}{3}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C \\
&= \frac{2}{9}(x^2+1)^{\frac{3}{2}} + C \\
&= \frac{2}{9}(\sqrt{x^2+1})^3 + C
\end{aligned}$$

(2) Given $\int \frac{dx}{ax+b}$, put $u = ax + b$ then $\frac{du}{dx} = a$ $\implies adx = du$ then $dx = \frac{du}{a}$ substituting these into the integral we have

$$\int \frac{dx}{ax+b} = \int \frac{1}{u} \cdot \frac{du}{a}$$
$$= \frac{1}{a} \int \frac{1}{u}\, du$$
$$= \frac{1}{a} \ln(u) + C$$
$$= \frac{1}{a} \ln(ax+b) + C$$

(3) Given $\int_0^1 \frac{dx}{\sqrt[3]{1+5x}}$ set $u = 1 + 5x$ then we have $\frac{du}{dx} = 5$
$\implies 5dx = du$ then $dx = \frac{du}{5}$
when $x = 0, u = 1$ and when $x = 1, u = 6$
substituting these into the integral we have

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt[3]{1+5x}} &= \frac{1}{5}\int_1^6 u^{\frac{-1}{3}} \cdot \frac{1}{5}\, du \\ &= \frac{1}{5}\left[\frac{3}{2}u^{\frac{2}{3}}\right]_1^6 \\ &= \frac{3}{10}(6)^{\frac{2}{3}} - \frac{3}{10} \\ &= \frac{3}{10}(6^2)^{\frac{1}{3}} - \frac{3}{10} \\ &= \frac{3}{10}(\sqrt[3]{36} - 1)\end{aligned}$$

Exercise

Evaluate the following integrals

(1)

$$\int (2\cos\theta - 3\sin\theta)\, d\theta$$

(2)

$$\int_{-1}^{1} x\sqrt{1+x^2}\, dx$$

(3)

$$\int \frac{4}{(1+2x)^3}\, dx$$

(4)

$$\int (ax+b)^n\, dx$$

Exercise

(5)

$$\int e^{ax+b}\ dx$$

(6)

$$\int_1^2 \frac{dx}{(3-5x)^2}\,dx$$

(7)

$$\int_0^{\pi} \sin(4\pi x+7)\,dx$$

(8)

$$\int \frac{1}{x}\sin(\ln x)\,dx$$

Applications of the Definite Integral as the area under a curve

Suppose we want to find the area between the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ as shown in the shaded region below.
$\frac{dA}{dx} = y = f(x)$ then we have

$$\int_a^b dA = \int_a^b f(x)\,dx$$

$$A = [F(x) + C]_a^b = F(b) + C - [F(a) + C]$$

$$A = F(b) - F(a)$$

Example

Find the area between the curve $y = 20 - 3x^2$ and the x-axis and the the lines $x = 1$ and $x = 2$

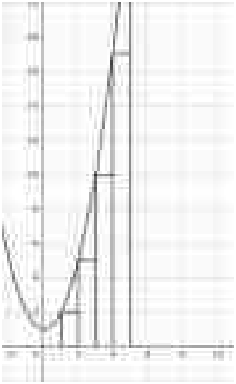
Solution

$\frac{dA}{dx} = y = 20 - 3x^2$ $\quad f(x) = 20 - 3x^2$ $\quad$ then we have the following

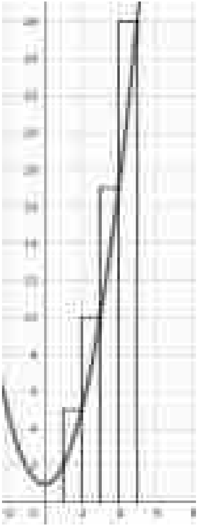
$$A = \int_1^2 f(x)\,dx = \int_1^2 (20 - 3x^2)\,dx = \left[20x - \frac{3x^3}{3}\right]_1^2$$
$$= \left[20x - x^3\right]_1^2$$
$$= (40 - 8) - (20 - 1)$$
$$= 32 - 19$$
$$= 13 \text{ square units}$$

Area as the limit of a sum

Suppose we want to find the area between the curve $y = x^2 + 1$, the x-axis and the lines $x = 1$ and $x = 5$ as shown in the shaded region below



Calculating the area of each rectangles in the above figures we have:
Estimated value of $A = 2 + 5 + 10 + 17 = 34$ square units.
Note: the value of A above is called an underestimate since the rectangles that we use their values are below the curve.
If we try and make the rectangles to be above the curve as shown in the figure below



Now using the fundamental theorem of Calculus we have

$$\int_1^5 x^2 + 1\,dx = \left[\frac{x^3}{3} + x\right]_1^3$$
$$= \left(\frac{125}{3} + 5\right) - \left(\frac{1}{3} + 1\right) = \frac{125 + 15}{3} - \frac{1 + 3}{3}$$
$$= \frac{140}{3} - \frac{4}{3} = \frac{140 - 4}{3} = \frac{136}{3}$$
$$= 45\frac{1}{3} \text{ square units}$$

This implies that the true value of satisfies the inequality

$$34 < A < 58$$

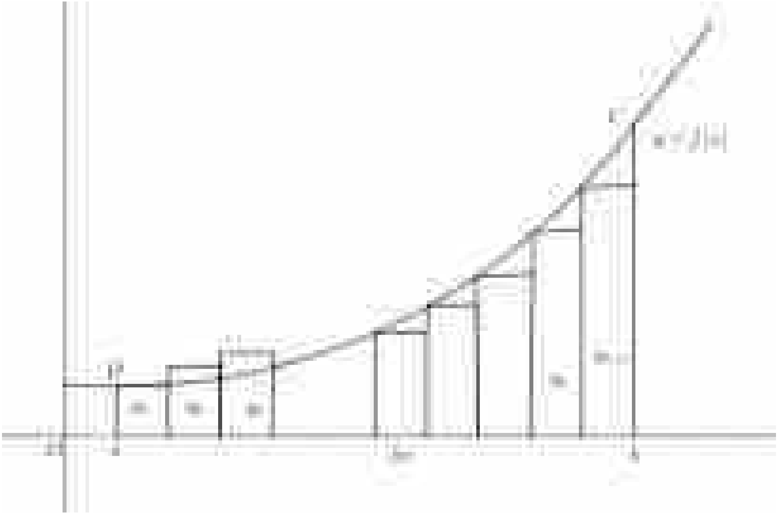
If we increase the number of rectangles say to $8, 16$ rectangles our bounds/ranges for A become closer.
Note: As we have in the figure below, each of the rectangle have length of y_i where $i = 1, 2, 3, \dots, n$ and the have the same breadth δx Let the area of each rectangles be δA_i which is equivalent to saying

$$\delta A_i = y_i \delta x$$

Then approximately we have

$$A \approx \sum_{i=1}^{n} \implies A \sum_{i=1}^{n} y_i \delta x$$

In the limit, as $n \to \infty$, and $\delta x \to 0$, the result is no longer an approximation; it is exact.



At this point,

$$A = \sum_{i=1}^{n} y_i \delta x$$

is written

$$A = \int y \, dx$$

Which is the integral of $y = f(x)$ with respect to x

Exercise

Given the curve $y = 12x(x-1)(x-3)$

(a) *calculate the area bounded by this curve , the x-axis and the ordinate* $x = 3$

(b) *Evaluate the integral*

$$\int_0^3 12x(x-1)(x-3)\, dx$$

and compare your result

Example

Find the area bounded by by $y = x^2$, the x-axis and ordinates $x = 2$ and $x = 5$.

Always sketch your curve to see where the area lie. The required area, A is shown below.

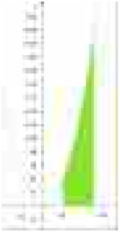


Figure: Area of $y = x^2$ between $x = 2$ and $x = 5$ and the x-axis

The area of the region bounded by the curves $y = f(x),\ y = g(x)$, and the lines $x = a,\ x = b$ where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$ is given by

$$\begin{aligned} A &= [\text{ area under } y = f(x)] - [\text{ area under } y = g(x)] \\ &= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \\ &= \int_a^b [f(x) - g(x)]\,dx \end{aligned}$$

Example

Find the area of the region bounded above by $y = \mathrm{e}^x$, bounded below by $y = x$ and bounded on the sides by $x = 0$ and $x = 1$.

The upper boundary curve is $y = \mathrm{e}^x$ and the lower boundary curve is $y = x$. So we use the area formula with $f(x) = \mathrm{e}^x$, $g(x) = x$, $a = 0$, and $b = 1$:
Then

$$\begin{aligned} A &= \int_0^1 (\mathrm{e}^x - x)\, dx \\ &= \left[\mathrm{e}^x - \frac{1}{2}x^2\right]_0^1 \\ &= \mathrm{e} - \frac{1}{2} - 1 \\ &= \mathrm{e} - 1.5 \end{aligned}$$

Example

Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$

We first find the points of intersection of the parabolas by solving the equations simultaneously. This gives

$x^2 = 2x - x^2 \implies 2x^2 - 2x = 0$

$2x(x-1) = 0 \implies x = 0 \text{ or } 1$

when $x = 0 \quad y = 0^2 = 0$

when $x = 1 \quad y = 1^2 = 1$

The points of intersection are $(0, 0)$ and $(1, 1)$.

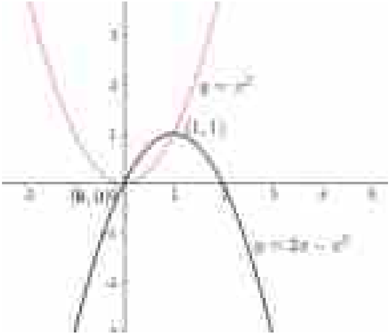


Figure: The graph showing the intersection of $y = x^2$ and $y = 2x - x^2$

We see from the graph that the top and bottom boundaries are

$$y_{\text{top}} = 2x - x^2 \quad \text{and } y_{\text{bottom}} = x^2$$

The area of a typical rectangle is

$$(y_T - y_B)\Delta x = (2x - x^2 - x^2)\Delta x$$

and the region lies between $x = 0$ and $x = 1$

So the total area is

$$
\begin{aligned}
A &= \int_0^1 (2x - 2x^2)\, dx \\
&= 2\int_0^1 (x - x^2)\, dx \\
&= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left[\frac{1}{2} - \frac{1}{3}\right] \\
&= \frac{1}{3}
\end{aligned}
$$

Example

Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution

By solving the two equations we find that the points of intersection are $(-1, 2)$ and $(5, 4)$.

We solve the equation of the parabola for x and notice that the graph at the left and right boundary curves are

$$x_{Left} = \frac{1}{2}y^2 - 3 \quad \text{and } x_{Right} = y + 1$$

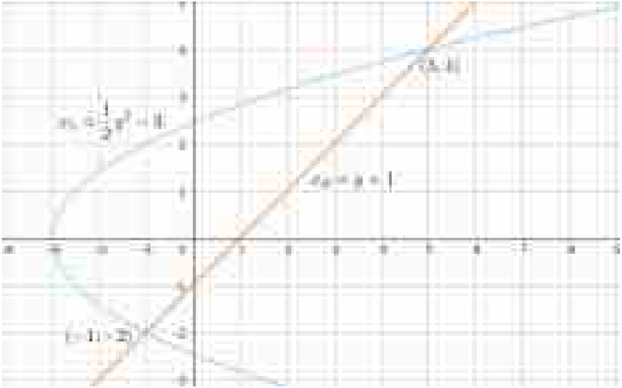


Figure: The graph showing the intersection of $x_{Left} = \frac{1}{2}y^2 - 3$ and$x_{Right} = y + 1$

We must integrate between the appropriate y-values, $y = -2$ and $y = 4$. Thus we have

$$\begin{aligned}
A &= \int_{-2}^{4} (x_R - x_L)\, dy \\
&= \int_{-2}^{4} \left[(y+1) - (\frac{1}{2}y^2 - 3) \right] dy \\
&= \int_{-2}^{4} \left[\frac{-1}{2}y^2 + y + 4 \right] dy \\
&= \left[\frac{-1}{2}\left(\frac{y^3}{3}\right) + \frac{y^2}{2} + 4y \right]_{-2}^{4} \\
&= \frac{-1}{6}(64) + 8 + 16 - \left(\frac{4}{3} + 2 - 8\right) \\
&= 18
\end{aligned}$$

Remark

If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$ and $y = d$, where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$, then its area is

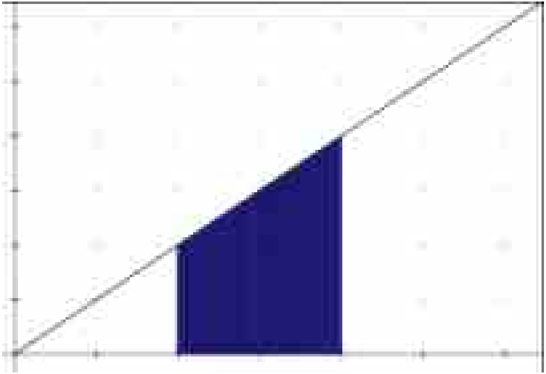
$$A \int_c^d [f(y) - g(y)]\, dy$$

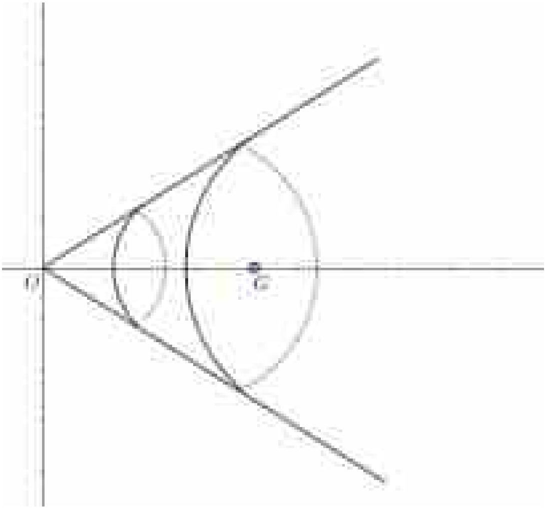
If we write x_R for the right boundary and x_L for the left boundary, then we have

$$A = \int_c^d (x_R - x_L)\, dy$$

Finding Volumes by Integration

When the shaded region in the figure below is rotated through 360^o about the x-axis, the solid obtained, illustrated in the second figure, is called a **solid of revolution**.





Thee volume of revolution is given by

$$V = \int_{x=a}^{x=b} \pi y^2 \, dx$$

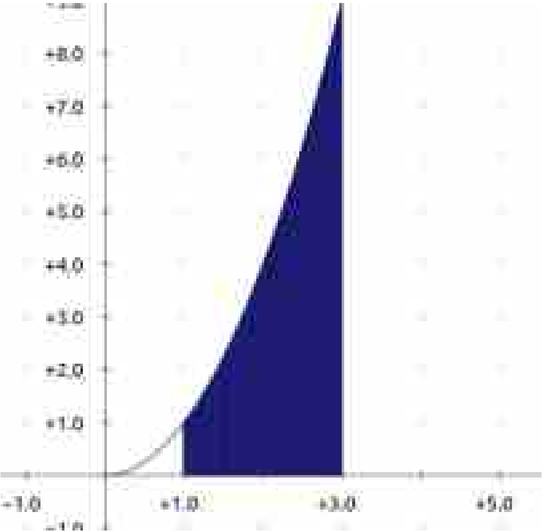
If the curve is rotated about the y-axis the volume of revolution is given by:

$$V = \int_{y=a}^{y=b} \pi x^2 \, dy$$

Example

The region between the curve $y = x^2$, the x-axis and the lines $x = 1$ and $x = 3$ is rotated through 360^o about the x-axis. Find the volume of revolution which is formed.

The region is shaded in the figure below


+8.0
+7.0
+6.0
+5.0
+4.0
+3.0
+2.0
+1.0
-1.0
+1.0
+3.0
+5.0

Using

$$V = \int_a^b \pi y^2 \, dx$$

then we have

$$\begin{aligned} Volume &= \int_1^3 \pi (x^2)^2 \, dx \\ &= \int_1^3 \pi x^4 \, dx \\ &= \left[\frac{\pi x^5}{5} \right]_1^3 \\ &= \frac{\pi}{5}(243 - 1) \\ &= \frac{242\pi}{5} \text{ cubic units} \end{aligned}$$