DISTANCE LEARNING CENTRE

MAT 141
ANALYTICAL GEOMETRY AND MECHANICS

by

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Introduction and Course Objectives

, The book consists of three main areas, namely: Coordinate Geometry, Vectors and Mechanics. Coordinate Geometry provides a bridge between algebra and geometry that makes it possible for geometric problems to be solved algebraically (or analytically). It also allows us to solve algebraic problems geometrically, but the former capability is far more important especially when numbers are assigned to essentially geometric concepts. The association between algebra and geometry is made by assigning numbers to points.

The idea of vector is introduced which focus on displacement, addition and subtraction of vectors. Other areas covered include differentiation, integration, scalar and triple product of vectors which leads to expression for calculating area and volume of shapes.

The book also describes behaviour of objets in circular motion, relate it to the projectile and its motion. It discusses the Simple Harmonic Motion (SHM), Elastic String, Simple and Conical pendulum. The book describe the dynamics of objects on smooth and rough inclined plane, connected particles, momentum and impulse, and principle of conservation of momentum. Other areas include effect of forces on rigid body and the resultant of such forces in equilibrium, center of mass and moment of inertia up to three dimension.

Lecture 1
Coordinate Geometry I

1.0 Introduction

A plane is divided into four equal parts formed by two perpendicular lines denoted as x and$y$ axes. Each of the four divisions of the plane is called a quadrant. The point where the lines intersect is the origin.

On the x-axis, any point to the right of the origin is positive, and negative to the left. On the y-axis, any point above the origin is positive and negative below.

Figure 1.0

1.1 Objectives

After this lecture, you should be able to:

(i) Have the notion of a point in a plane

(ii) Determine the distance betweenany two points in a plane.

(iii) Divide a straight line in the ratio say $r : s$

(iv) Determine mid-point of the line joining two points.

(v) Calculate the slope of a straight line.

(vi) Write equation of a straight line

1.2 A point in a plane

A point in a plane is an ordered pair (coordinate pair) from a number on x-axis and a number on y-axis.

Figure 1.2

The point assigned to the pair is the point where the perpendicular to the x-axis, at a intersect the perpendicular to the y-axis at b. Hence, every point has a pair and every pair has a point. The distance Oa and Ob is known as abscissa and ordinate of the point $p(a, b)$ respectively.

1.3 Pre-Test 1

Figure 1.3

Find the co-ordinate of the point
(a) A (b) H
(c) $\bar{U}$ (d) R

1.4 Distance between two points

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points on the plane. The distance between the points using Pythagora's Theorem is

Figure 1.4

$$AB = \sqrt{(x_2 - x_1) + (y_2 - y_1)} \text{ units} \tag{1.4}$$

1.5 Dividing a straight line in the Ratio $r : s$

Suppose point $H(a, b)$ divides the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $r : s$
Then, using the figure below:

Figure 1.5

Triangles AMR ad RPB are similar.

$$\therefore \quad AR : RB = r : s$$

and

$$\frac{AM}{RP} = \frac{AR}{RB} = \frac{r}{s} \tag{.5a}$$

Similarly,

$$\frac{RM}{BP} = \frac{AR}{RB} = \frac{r}{s} \tag{1.5b}$$

From Figure 1.5,

$$\frac{AM}{RP} = \frac{a - x_1}{x_2 - a} = \frac{r}{s}$$

and solving for 'a' gives

$$a = \frac{rx_2 + sx_1}{r + s} \tag{1.5c}$$

From

$$\frac{AR}{RB} = \frac{b - y_1}{y_2 - b} = \frac{r}{s}$$

Solving for b you obtain

$$b = \frac{ry_2 + sy_1}{r + s} \tag{1.5d}$$

1.6 Mid-Point of a Straight Line

Suppose $M(a, b)$is the mid-point of the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$. Then, the ratio $r : s = 1 : 1$ and the ordered pair of M is

$$a = \frac{x_2 + x_1}{2}; \quad b = \frac{y_2 - y_1}{2}$$

1.6.1 Example

Given the points $N(4, 7)$ and $B(2, -1)$. Find the:

(i) distance of line NB

(ii) mid-point of the of the line NB

(iii) Coordinates of the point that divides the line NB in ratio $1 : 3$.

Solution

(i) Let $x_1 = 2$, $x_2 = 4$, $y_1 = -1$, $y_2 = 7$. Then, the distance,

$$NB = \sqrt{(4 - 2)^2 + (7 + 1)^2} = 2\sqrt{3} \text{ units}$$

(ii) Suppose $M(a, b)$ is the mid-point of the distance NB, then,

$$a = \frac{472}{2} = 3, \ \ b = 3$$

$\therefore$ the mid-point is $(3, 3)$.

(iii) Suppose $H(a, b)$ divides the line in the ratio $r : 3 = 1 : 3$, then,

$$a = \frac{5}{2}; \ \ b = 1$$

and the co-ordinate of H is $\left(\frac{5}{2}, 1\right)$.

1.7 The slope of a straight line

The slope of a straight line is the measure of the tangent of the angle ψ makes with the x-axis. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points on a straight line. Then, the slope, is

$$\text{slope} \;\; s = \tan\psi = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Figure 1.7

1.7.1 Example

Find the slope of the line through $H_1(0, 4)$ and $H_2(5, 1)$.

Solution

Let $x_1 = 0$, $x_2 = 5$, $y_1 = 4$, $y_2 = 1$.
Slope, $s = -\dfrac{3}{5}$.

Remark

Hence, y decreases 3 units whenever x increases 5 units.

1.8 Equation of a straight line

1.8.1 Definition

An equation for a line is an equation that is satisfied by the co-ordinates of the points that lie only on the line.

1.8.2 Remarks

Equation of a straight line is uniquely defined if either:

(i) Two points are given on the straight line or

(ii) One point and the slope of the straight line are given.

In (i), suppose you are given points $A(x_1, y_1)$ and $B(x_2, y_2)$ on a straight line, the slope is

$$s = \frac{y_2 - y_1}{x_2 - x_1} \tag{1.8.2a}$$

and point $P(x, y)$ is an arbitrary point on the line. Then, the slope, s is

$$s = \frac{y - y_1}{x - x_1} \tag{1.8.2b}$$

Equating (1.8.2a) and (1.8.2b), you have

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} = s$$

$$\Rightarrow \quad y = s(x - x_1) + y_1$$

In (ii), if point $A(x_1, y_1)$ on a straight line with slope s are known, the equation of the line is obtained by

$$y = s(x - x_1) + y_1$$

where $P(x, y)$ is an arbitrary point on the line.

1.8.3 Forms of equation of a straight line

Let $P(x, y)$ be an arbitrary point on the line, then, different forms of equation of a straight line are:

(i) General linear form:

$$ax + by = c$$

where a, b, c are arbitrary constants, and a and b are not both zero.

(ii) Slope intercept form:

$$y = sx + c$$

where $s =$ slope, and c is the y-intercept.

(iii) Intercepts form:

$$\frac{x}{a} + \frac{y}{b} = 1$$

where a, b are respectively the x and y intercepts.

1.8.4 Example

(a) Find the equation of the straight line through the points $(-2, 1)$ and $(2, -2)$.

(b) Express the equation in the three different forms.

(c) Find the equation of the straight line whose slope is 1 and y-intercept is $\sqrt{2}$.

Solution

(a) Let $x_1 = -2$, $x_2 = 2$, $y_1 = 1$, $y_2 = -2$ and $P(x, y)$ be arbitrary point on the line.
Then,

$$\begin{aligned} \frac{y - y_1}{x - x_1} &= \frac{y_2 - y_1}{x_2 - x_1} \\ y &= \frac{-3}{4}(x + 2) + 1 \\ y &= -\frac{3}{4}x - \frac{1}{2} \text{ (slope-intercept equation)} \\ y + \frac{3}{4}x &= -\frac{1}{2} \text{ (General linear equation)} \end{aligned}$$

$$\frac{y}{-1/2} + \frac{x}{-2/3} = 1 \text{ (intercept equation)}$$

(b) Using $y = 5x + c$ and $y = x + \sqrt{2}$

1.90 Summary

In this lecture, we wrote equations:
(i) $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ units, to calculate shortest distance between $A(x_1, y_1)$ and $B(x_2, y_2)$.
(ii) Divide a straight line between two points in a given ratio, $r : s$ at point $H(a, b)$ where

$$a = \frac{rx_2 + sx_1}{r + s}$$
$$b = \frac{ry_2 + sy_1}{r + s}$$

(iii) $ax + by = c$ - General linear form
$y = sx + c$ - slope-intercept form.
$\frac{x}{a} + \frac{y}{b} = 1$ - Intercept form to describe the equations of a straight line.

1.10 Post-Test 1

(1) Find the slope and the intercepts of the line

(a) $y = 3x + 5$

(b) $2x - y = 6$

(2) Write an equation for the line through the points:

(a) $(-2, 0), (-2, -2)$ in general linear form.

(b) $(1, 3), (3, 1)$ in intercepts form.

(c) $(x_0, y_0), (x_1, y_1)$ in slope-intercept form.

(3) Write an equation for the line that passes through point $H(x, y)$ with slope, r

(a) $H(x, b), r = 2$

(b) $H(-1, 1), 4 = -1$ in slope-intercept form.

(4) If one end of a line segment is the point $(6, -2)$ and the mid-point is $(-1, 5)$, find the co-ordinates of the other end of the line segment.

(5) Find the point two-fifth of the way from a point $A(5, -1)$ to $B(-4, -5)$.

(6) Find the co-ordinates of the mid-points of the line segment joining each of the following points:

   (a) $(2, 2), (-3, 5)$

   (b) $(4, -1), (3, 3)$

   (c) $(2, 2), (4, 4)$

1.11 Solution to Pre-Test 1

(a) $A(-h, -9)$

(b) $H(-4, -4\frac{1}{2})$

(c) $\bar{U}(3, h)$

(d) $R(-h, 0)$

Contemporary Reading

1. O.O. Ugbebor and N.I. Akinwande: Analytical Geometry and Mechanics.

2. Godwin A. Odili: Calculus with Co-ordinate Geometry and Trigonometry.

3. R.O. Ayeni et.al.: Introductory University Mathematics.

Lecture 2
Co-ordinate Geometry of Straight Line II

2.0 Introduction

This lecture look at the concluding aspects of Equations of lines and planes. The gradient (slope) of a line is a vital tool for determining whether two lines are parallel or perpendicular, and the angle of inclination of two lines.

2.1 Objectives

After this lecture, you should be able to:

(i) Determine which lines are parallel or perpendicular.

(ii) Determine distance from the origin to the straight line.

(iii) Determine distance from a point to the straight line.

(iv) Determine the angle between two straight lines.

2.2 Pre-Test 2

Determine the gradient of each of the following:

(a) $8x + 5y = 20$

(b) $3x - 6y + 4 = 0$

(c) $y = 4x + 15$

(d) $4x + 5y - 3 = 0$

2.3 Parallel and Perpendicular Lines

Suppose we have two straight lines with equations:

$$y_1 = s_1x + d_1 \text{ and } y_2 = s_2x + d_2$$

Then, the equations are:

(i) Parallel if $s_1 - s_2 = 0$

(ii) Perpendicular if $s_1, x_2 + 1 = 0$

2.3.1 Illustrations

L_1 and L_2 are parallel since $s_1 - s_2 = 0$

L_1 and L_2 are perpendicular since $s_1s_2 + 1 = 0$

Figure 2.3.1

2.4 Distance from the origin to the straight line $ax + by + c = 0$

Let OH represent the perpendicular distance from the straight line $ax + by + c = 0$ to the origin. Suppose line OH makes angle ψ with the positive x-axis. Then,

.

Figure 2.4

$$\langle OAB = \psi \quad \text{(Why?)}$$

From the right-angled triangle OHB,

$$OH = OB \cos\psi \tag{2.4a}$$

Also, from right-angled triangle OBA,

$$\cos\psi = \frac{OA}{AB} \tag{2.4b}$$

From the equation $ax + by + c = 0$; when $x = 0$, $y = -c/b$ and when $y = 0$, $x = -c/a$.
Hence, the co-ordinates of point:
A is $(0, -c/b)$
B is $(-c/a, 0)$
and the distance

$$\begin{aligned} AB = \pm\left(\frac{c^2}{a^2} + \frac{c^2}{b^2}\right)^{\frac{1}{2}} &= \pm\left[\frac{c^2(a^2+b^2)}{a^2b^2}\right]^{\frac{1}{2}} \\ &= \pm\frac{c\sqrt{a^2+b^2}}{ab} \qquad (2.4c) \end{aligned}$$

Using (2.4c) in (2.4b) and (2.4b) in (2.4.a) we have

$$\begin{aligned} OH &= \frac{c}{a}\cdot\frac{c}{b}\cdot\frac{ab}{\pm c\sqrt{a^2+b^2}} \\ OH &= \frac{\pm c}{\sqrt{a^2+b^2}} \end{aligned}$$

Remark

The sign of OH is taken as that of c.

2.5 Distance from a point (x_1, y_1) to the straight line $ax + by + c = 0$

Figure 2.5a

To obtain the perpendicular distance RP from a point $R(x_1, y_1)$ to the straight line $ax + by + c = 0$, we use the following sketch:

Figure 2.5b

where

- OH is the perpendicular distance from origin to the straight line $ax + by + c = 0$
- TE, DR are perpendicular to OH
- OH makes angle ψ with the positive x-axis.
- E is the foot of the perpendicular distance from R to the x-axis.

2.6 The angle between two straight lines

To obtain the angle ψ between two straight lines L_1 given by $y = s_1 x + c$ and L_2 given by $y_2 = s_2 x + d$, we consider the sketch below:

Figure 2.6

where

- ψ_1 is the angle line L_1 makes with the positive x-axis.
- ψ_2 is the angle line L_2 makes with the positive x-axis.
- ψ is the angle between L_1 and L_2

Let $\tan\psi_1 = s_1$, $\tau\psi_2 = s_2$.
Then,

$$\psi = \psi_2 - \psi_1 \quad \text{(Why?)}$$

and

$$\tau\psi = \tan(\psi_2 - \psi_1)$$

From the sketch,

$$\langle REN = \psi \quad \text{(Why?)}$$

In $\triangle REN$,

$$RN = RE\sin\psi = y_1\sin\psi \tag{2.6a}$$

In $\triangle ETO$,

$$OT = OE\cos\psi = x_1\cos\psi \tag{2.6b}$$

$RP = DH$ (Opposite sides of rectangle $RPHD$)

$$\begin{aligned} RP &= OH - (OT + TD) \\ &= OH - (OT + RN) \quad \text{(Why?)} \\ &= OH - x_1\cos\psi - y_1\sin\psi \end{aligned} \tag{2.6c}$$

But from $\triangle OBA$,

$$\cos\psi = \frac{-a}{\pm\sqrt{a^2+b^2}} \tag{2.6d}$$

$$\sin\psi = \frac{-b}{\pm\sqrt{a^2+b^2}} \tag{2.6e}$$

Substituting (2.6e) and (2.6d) in (2.6c), we have

$$\begin{aligned} RP &= OH - x_1\left(\frac{-a}{\pm\sqrt{a^2+b^2}}\right) - y_1\left(\frac{-b}{\pm\sqrt{a^2+b^2}}\right) \\ &= \frac{ax_1 + by_1 + c}{\pm\sqrt{a^2+b^2}} \end{aligned}$$

$\therefore$ the equation needed to obtain the perpendicular distance from a point say $R(x_1, y_1)$ to the straight line $ax + by + c = 0$ is

$$\frac{ax_1 + by_1 + c}{\pm\sqrt{a^2+b^2}} \tag{2.6f}$$

2.7 Summary

In this lecture, we wrote equations:
(i) $= \dfrac{\pm c}{\sqrt{a^2+b^2}}$ to determine the perpendicular distance from the straight line $ax+by+c=0$ to the origin.
(ii) $\dfrac{ax_1+by_1+c}{\pm\sqrt{a^2+b^2}}$ to determine the perpendicular distance from a point $R(x_1,y_1)$ to the straight line $ax+by+c=0$.
(iii) $\tan\psi = \dfrac{\psi_2-\psi_1}{1+\psi_2\psi_1}$ to determine the tangent of the angle between two straight lines meeting at a point.
And discovered that two lines are:
(i) Parallel if their gradient are sum up to zero ($s_1+s_2=0$)
(ii) Perpendicular if the product of their gradient plus one equals zero ($1+s_1s_2=0$)

$$= \frac{\tan\psi_2-\tan\psi_1}{1+\tan\psi_2\tan\psi_1}$$

$$\therefore \quad \tan\psi = \frac{s_2-s_1}{1+s_2s_1} \qquad (2.6g)$$

2.8 Examples

(a) Given the points $A(4,4)$, $B(3,2)$ and $C(9,5)$, find the:

(1) Equation of the line AC

(2) Perpendicular distance from B to the line AC.

(3) Equation of the straight line passing through the point B and is

(i) parallel to the line AC.

(ii) perpendicular to the line AC.

(4) Find the tangent of the angle between the lines AB and AC.

(b) The points $A(18,-8)$ and $B(-7,-8)$ are two vertices of an isosceles triangle ABC with $AB=AC$. If the altitude AN has the equation $4x+3y=48$, find the co-ordinate of N and C.

Solutions

(a) Let $P(x, y)$ be arbitrary point on the line passing through $A(4, 4)$ and $C(9, 5)$.

(1) Thus, the equation is

$$\frac{y-4}{x-4} = \frac{5-4}{9-4} = \frac{1}{5}$$

and $5y - x - 16 = 0$ is the equation of line AC.

(2) Perpendicular distance from point $B(3, 2)$ to the line AC given by $5y - x - 16 = 0$ is obtain using

$$\frac{ax_1 + by_1 + c}{\pm\sqrt{a^2 + b^2}}$$

where $x_1 = 3$, $y_1 = 2$, $b = 5$, $a = -1$, $c = -16$.
Substituting, we have

$$= \frac{9}{\sqrt{26}} = \frac{9\sqrt{26}}{26} \text{ units.}$$

(3) Gradient of line AC is $\frac{1}{5}$.
Let $P(x, y)$ be arbitrary point. Then, equation of the straight line passing through the point B and parallel to AC is obtain by

$$\begin{aligned} \frac{y-2}{x-3} &= \frac{1}{5} \\ \therefore \quad 5y - x &= 7 \end{aligned}$$

The equation of the line passing through point B and perpendicular to line AC is given by

$$\frac{y-2}{x-3} = -5$$

$$\therefore \quad y = 13 - 5x$$

(4) Slope of:

$$\begin{aligned} AB &= 2 \\ AC &= \frac{1}{5} \quad \text{(By 1 above)} \end{aligned}$$

Using equation (2.6g), the tangent angle ψ between AB and AC is

$$\begin{aligned} \tan\psi &= \frac{9}{4} \\ \therefore \quad \psi &= \tan^{-1}\left(\frac{9}{7}\right) \\ \psi &= 52.12^0 \end{aligned}$$

(b) Sketch

From the sketch,
Line $AN \perp$ line B, and from

$$\begin{aligned} 4x + 3y &= 48 \qquad (i) \\ y &= -\frac{4}{3}x + 16 \end{aligned}$$

Thus, gradient of line AN is $-4/3$ and gradient of line BC, $s_2 = \frac{3}{4}$ (Why?).
Let $P(x, y)$ be arbitrary point on the straight line through points $B(-7, -8)$. Then, equation of line BC is obtain by

$$\frac{y+8}{x+7} = \frac{3}{4}$$

$$4y + 32 = 3x + 21 \qquad (ii)$$

$$4y - 3x = -11$$

Solving (i) and (ii) simultaneously, we have

$$y = 4, \quad x = 9$$

and the distance,

$$\begin{aligned} BN &= \sqrt{(9+7)^2 + (4+11)^2} \\ &= 20 \text{ units} \end{aligned}$$

But

$$\begin{aligned} |BC| &= 2|BN| \quad \text{(Why?)} \\ &= 40 \text{ units} \\ |BC| &= \sqrt{(x+7)^2 + (y+8)^2} \\ 40 &= x^2 + y^2 + 14x + 16y + 113 \end{aligned}$$

$$x^2 + y^2 + 143x + 16y = 1487 \qquad (iii)$$

Distance of line NC, through $C(x_1, y_1)$ and gradient $\dfrac{3}{4}$ is

$$\begin{aligned} |NC| = \frac{y-4}{x-9} &= \frac{3}{4} \\ 4y - 3x &= -11 \end{aligned}$$

and

$$x = \frac{1}{3}(4y + 11) \qquad (iv)$$

Substituting (iv) in (iii), we have

$$y^2 + 16y - 512 = 0$$

$$y = 16 \text{ or } y = -32$$

$$\text{and } x = 25 \text{ or } x = -39$$
$$\therefore \quad C'(25, 16) \text{ or } C(-39, -32)$$

Hence $C(-39, -32)$ is the required point.

2.9 Post-Test 2

(1) Find the angle between the lines $x + 4y = 12$ and $y - 2x + 6 = 0$.

(2) Given the points $A(2, 3)$ and $B(0, -1)$, find the coordinate of the point C such that the angle BAC is a right angle and the length BC is 5 units.

(3) Find the distance from the point $P(2, 1)$ to the line $y = x + 2$.

(4) A particle starts at $A(-2, 3)$ and its coordinate change by $-6/5$. Write an equation to predict the position of the particle at any point $P(x, y)$.

(5) Find the acute angle between the lines:

  (i) $2y = 3x - 8$ and $5y = x + 7$

  (ii) $2x + y = 4$ and $y - 3x + 7 = 0$

2.10 Solution to the Pre-Test 2

(a) Gradient $= -\frac{8}{5}$

(b) Gradient $= \frac{1}{2}$

(c) Gradient $= 4$

(d) Gradient $= -\frac{|}{4}5$

Contemporary Reading

1. O.O. Ugbebor and N.I. Akinwande: Analytical Geometry and Mechanics.

2. Godwin A. Odili: Calculus with Co-ordinate Geometry and Trigonometry.

Lecture 3
Coordinate Geometry of Conic Sections I

3.0 Introduction

A cone is a figure with circular base and a pointing end called vertex.

Conic sections are the curves in which a plane cuts a double cone. The curves (conics) are formed when a plane (section) cuts through the double cones.

We need the curves to describe the paths of planets, comets, moons, asteroids, satellites or anybody that is moved through space by gravitational forces.

3.1 Objectives

After this lecture, you should be able to:

(i) Define conic sections and give example.

(ii) Classify conic sections based on eccentricity.

(iii) Write equation of a circle.

(iv) Write equation of the tangent at the point (x_1, y_1) on the circle.

(v) Write equation of a parabola.

(vi) Write equation of tangent to the parabola

(vii) Write equation of normal to the parabola.

3.2 Definition (conic section)

A conic section is the locus of a point which moves in a plane such that the ratio of its distance from a fixed line (focus) to its distance from a fixed line (directrix) is a constant (Eccentricity).

3.2.1 Illustration

Figure 3.2.1

where
$F =$ Focus
$P(x, y) =$ Arbitrary point on the plane
$LL' =$ Directrix
$H =$ Foot of the perpendicular to the fixed line (Directric) LL'.
Eccentricity, $e = \dfrac{PF}{PH}$.

3.3 Pre-Test

Figure 3.3

Write an equation for line (i) CD (ii) MN.

3.4 Examples

(i)

A plane section which is perpendicular to the generator describes a circle.

(ii) A plane section cutting one of the double cones and which is neither parallel to a generator nor perpendicular to the common axis describes an Ellipse.

(iii) A plane section which is parallel to a generator describes a Parabola.

(iv) A plane section which cuts both cones but does not pass through the vertex describes a Hyperbola

3.5 Remarks

(i) The locus is a pair of straight lines if $e = \infty$.

(ii) The locus is a circle if $e = 0$

(iii) The locus is a parabola if $e = 1$.

(iv) The locus is an ellipse if $e < 1$.

(v) The locus is a hyperbola if $e > 1$.

3.6 Equation of a circle

A circle is the locus of a point P which moves such that it is at a constant distance r from a focus. Hence, we have

$$PF = r$$

and

$$PC^2 = r^2 \quad \text{(Why?)}$$

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \tag{3.6a}$$

and expanding (3.6a), we have

$$x^2 + y^2 - 2xx_0 - 2yy_0 + c = 0 \tag{3.6b}$$

where

$$\left.\begin{aligned} & c = x_0^2 + y_0^2 - r^2 \\ \text{and} \quad & r = \sqrt{x_0 + y_0 - c} \end{aligned}\right\} \tag{3.6c}$$

3.7 Example

Find the equation of a circle centre $(-2, 3)$, radius 7.

Solution

Let $P(x, y)$ be arbitrary point using equation (3.6b), we have

$$x^2 + y^2 - 2(-2)x - 2(3)y + c = 0$$

where $c = (-2)^2 + (3)^2 - (7)^2 = -39$ and, the equation is

$$x^2 + y^2 + 4x - 6y - 39 = 0$$

3.8 Equation of the tangent at the point (x_1, y_1) on the circle $x^2 + y^2 - 2xx_0 - 2yy_0 + c = 0$

Figure 3.8.1

$$\text{Slope of } \ EF = \frac{y_1 - y_0}{x_1 - x_0}$$

and

$$\text{Slope of } \ AB = -\frac{(x_1 - x_0)}{y_1 - y_0} \quad \text{(Why?)}$$

Equation of tangent AB

$$\frac{y - y_1}{x - x_1} = -\frac{x_1 - x_0}{y_1 - y_0}$$

$$yy_1 + xx_1 + y_1y_0 - yy_0 + x_1x_0 - xx_0 - (y_1^2 + x_1^2) = 0$$

But

$$x_1^2 + y_1^2 = 2x_1x_0 + 2y_1y_0 - c$$

and substituting, we have

$$yy_1 + xx_1 - y_0(y_1 + y) - x_0(x_1 + x) + c = 0$$

$$xx_1 + yy_1 - x_0(x_1 + x) - y_0(y_1 + y) + c = 0 \qquad (3.8a)$$

is the equation of the tangent to the circle.

3.8.1 Example

Find the equation of the tangent to the circle

$$x^2 + y^2 + 10x - 12y + 11 = 0 \text{ at } (2, 7)$$

Solution

Using equation (3.8a).
Let $P(x, y)$ be arbitrary point. Then the required equation is

$$2x + 7y + 5(2 + x) - 6(7 + y) + 11 = 0$$

where centre of the circle is $(-5, 6)$.

3.9 Equation of a parabola open to the right

A parabola is the set of points in a plane which are equidistant from a given focus and a directrix in the plane

Figure 3.9

where

LL'	=	Directrix
V	=	Vertex coinciding with the origin
H	=	Foot of the perpendicular from $P(x,y)$ to the directrix
CD	=	Chord parallel to the directrix (Latus rectum)

From figure 3.9,

$$\begin{aligned} PF^2 &= (x-\hat{x})^2 + y^2 \\ &= (x+\hat{x})^2 \quad \text{(Why?)} \end{aligned}$$

But

$$e = \frac{PF}{PM} = 1$$

$$\Rightarrow \quad PF^2 = PM^2$$

$$\therefore \quad (x-x)^2 + y^2 = (x+\hat{x})^2$$

and

$$y^2 = 4\bar{x}x \tag{3.9}$$

is the equation of a parabola.

3.10 Remarks

(i) The equation $y^2 = 4\hat{x}x$ is the simplest (canonical) form of the origin as its vertex and the point $(\bar{x}, 0)$ as its focus.

(ii) Since $y^2 > 0$, the curve of the parabola lies entirely on the positive x-axis positive it $x > 0$ and on the negative x-axis if $x < 0$.

(iii) As $x \to +\infty$, $y \to \pm\infty$.

(iv) For any value of x, there are two equal and opposite values of y. Hence, the curve is symmetrical about the x-axis.

(v) The point V is called the vertex of the parabola.

(vi) The length of the latus rectum is $4\bar{x}$.

(vii) If the vertex of the parabola is a point (h, k) other than the origin, the equation is of the form

$$(y-k)^2 = 4\bar{x}(x-k)$$

3.11 Example

(1) Find the focus and directrix of the parabola $y^2 = 8x$

(2) Find the equation of the parabola $y^2 = 4\bar{x}x$ with focus $(1, 2)$ and directrix $\bar{x} = 4$.

Solution

(1) Remember that $y^2 = 4\bar{x}x$ and equating, we have

$$\begin{aligned} 4\bar{x}x &= 8x \\ \therefore \quad \bar{x} &= 2 \end{aligned}$$

Thus, focus $= F(2, 0)$
Directrix $= -2$.

(2) Let $P(x, y)$ denote an arbitrary point. Then,

$$\begin{aligned} PF^2 &= (x-1)^2 + (y-2)^2 \\ PH^2 &= (x+4)^2 + (y-y)^2 = (x+4)^2 \end{aligned}$$

But

$$\begin{aligned} e &= \frac{PF}{PH} = 1 \\ \Rightarrow &= PF^2 \end{aligned}$$

$$\begin{aligned} & (x-1)^2 + (y-2)^2 = (x+4)^2 \\ \therefore \quad & x^2 - 2x + 1 + y^2 - 4y + 4 = x^2 + 8x + 16 \\ \therefore \quad & y^2 - 4y - 10x - 11 = 0 \end{aligned}$$

is the required equation.

3.12 Equation of Tangent to the Parabola

Differentiating the equation $y^2 = 4\bar{x}x$ with respect to x gives

$$\frac{dy}{dx} = \frac{2\bar{x}}{y}$$

$\therefore$ the slope of the tangent to the parabola at the point (x_1, y_1) is obtained by

$$s_1 = \frac{2\bar{x}}{y} \tag{3.12a}$$

and

$$y - y_1 = \frac{2\bar{x}}{y}(x - x_1) \tag{3.12b}$$

3.13 Equation and Normal to the Parabola

The slope of the normal at the point (x_1, y_1) is obtained by

$$s_2 = -\frac{y_1}{2\bar{x}} \quad \text{(Why?)} \tag{3.13a}$$

and the required equation is

$$y - y_1 = \frac{-y}{2\bar{x}}(x - x_1) \tag{3.13b}$$

3.13.1 Example

Find the point of intersection of the line $x - 2y + 6 = 0$ and the parabola $y^2 = 6x$, and the equation of the tangents and normals to the parabola at the point of intersection.

Solution From

$$x - 2y + 6 = 0$$

$$x = 2y - 6$$

$$\therefore \quad y^2 = 6(2y - 6)$$

$\Rightarrow \quad y^2 - 12y + 36 = 0$ and the roots are $y = y_1 = y_2 = 6$.
Hence, $x = 6$ and point of intersection is $(6, 6)$ the tangent at $(6, 6)$ is

$$y - 6 = \frac{2\bar{x}}{6}(x - 6) \ \text{ Using (3.12b)}$$

But

$$4\bar{x} = 6$$
$$\bar{x} = \frac{3}{2}$$

Substituting for $\bar{x}$, we have

$$y - 6 = \frac{1}{2}(x - 6)$$

$$\therefore \quad 2y - x - 6 = 0$$

is the required tangent equation.
The normal equation is obtained using equation (3.13b)

$$y - 6 = -2(x - 6)$$

$$\therefore \quad y + 2x - 18 = 0$$

is the required normal equation.

3.14 Summary

In this lecture, we:
(i) Classify conic section as:
(a) Circle if eccentricity, $e = 0$
(b) Parabola if $e = 1$
(c) Ellipe if $e < 1$
(d) Hyperbola if $e > 1$
(ii) Wrote equation of: (a) Circle as:
$x^2 + y^2 - 2xx_0 - 2yy_0 + c = 0$
(b) Tangent to the circle at (x_1, y_1) as:
$x_1x + y_1y - x_0(x_1 + x) - y_0(y_1 + y) + c = 0$
(c) Parabola as $y^2 = 4\bar{x}x$
(d) Tangent equation to the parabola at (x_1, y_1) as
$y - y_1 = \frac{2\bar{x}}{y_1}(x - x_1)$
(e) Normal equation to the parabola at (x_1, y_1) as
$y - y_1 = \frac{-y_1}{2\bar{x}}(x - x_1)$

3.15 Post-Test 3

(1) Find the equation of the circle which has the line joining the points $(4, -3)$ and $(-1, 7)$ as its diameter.
Hint: Centre of the circle is the mid-point of the diameter, and the radius is half the diameter length.

(2) Find the centre and radius of a circle $2x^2 + 2y^2 - 28x + 12y + 114 = 0$.

(3) Find an equation for the circle through the points (0,0) and (6,0) that is tangent to the line $y = 1$.

(4) If the tangents to the parabola $y^2 = 4ax$ at the points $(at^2, 2at)$ and $(as^2, 2as)$ meet at the point (p, q), show that $a^2(t - s)^2 = q^2 - 4ap$.

(5) Find the points of intersection of the parabola $y^2 = 4ax$ and the circle $x^2 + y^2 = 64a^2$.

(6) Find the equation of the circle which passes through the points (4,1), (5,4) and (0,1). Check whether the point (3,5) lies on the circle.

(7) Find the equation of the tangent to the parabola $y^2 = 4ax$ which is parallel to the line $y + 2x = 0$.

3.16 Solution to Pre-Test 3

(i) Slope, $CD = 1$.
Let $P(x, y)$ be arbitrary point in the plane, then, line equation,

$$\begin{aligned} y - y_1 &= x - x_1 \\ \Rightarrow \quad y &= x - x_1 + y_1 \\ y &= x - 1 \\ \text{and} \qquad y - x + 1 &= 0 \end{aligned}$$

(ii) Equation of line MN,
Gradient of $MN = -1$
Equation, $y + 1 = -x - 2$

$$\therefore \quad y + x + 3 = 0$$

Contemporary Reading

1. O.O. Ugbebor and N.I. Akinwande: Analytical Geometry and Mechanics.

2. R.O. Ayeni et.al.: Introductory University Mathematics.

Lecture 4
Coordinate Geometry of Conic Sections II

4.0 Introduction

An ellipse is the set of points in a plane whose distance from two fixed points in the plane have a constant sum. A hyperbola is the set of points in a plane whose distances from two fixed points in the plane have a constant difference

Ellipse
Figure 4.0

where:
$F_1, F_2 :=$ Foci
$V_1, V_2 :=$ Vertices
$KK', LL' :=$ Directrices
$H_1, H_2 :=$ Feet of the perpendiculars from arbitrary point $P(x, y)$ to the directrices.

4.1 Objectives

After this lecture, you should be able to:

(i) State properties of ellipse

(ii) Write equation of an ellipse

(iii) Write equation for tangent at a point on the ellipse

(iv) Write equation for normals at a point on the ellipse

(v) State properties of hyperbola

(vi) Write equation of a hyperbola

(vii) Write equation for tangent at a point on the hyperbola

(viii) Write equation for normal at a point on the hyperbola.

(ix) Write a general equation for quadratic curve, and use it to classify conic section.

4.2 Pre-Test 4

An ellipse has and axis.

4.3 Equation of an ellipse

Applying definition of ellipse on figure 4.0, we have

$$e = \frac{F_2V_2}{V_2M_2} = \frac{F_2V_1}{V_1M_2} = \frac{PF_2}{PH_2} < 1 \tag{4.3a}$$

$$F_2V_2 = eV_2M_2, F_2V_1 = eV_1M_2 \tag{4.3b}$$

and

$$F_2V_2 + F_2V_1 = e(V_1M_2 + V_2M_2) \tag{4.3c}$$

$$F_2V_1 - F_2V_2 = e(V_1M_2 - V_2M_2) \tag{4.3d}$$

But

$$\begin{aligned}
F_2V_2 + F_2V_1 = V_1V_2 &= e[(CM_2 - CV_2) + (CV_1 + CM_2)] \text{ (using 4.3c)} \\
2a &= e[2CM_2 - a + a] \\
2a &= 2eCM_2 \\
\therefore \quad CM_2 &= \frac{a}{e}
\end{aligned}$$

and co-ordinate of point $M_2 = (\frac{a}{e},\ 0)$
$\therefore \quad CM_1 = \left(-\frac{a}{e}, 0\right)$.
From (4.3d),

$$\begin{aligned}
CF_2 + CV_1 - (CV_2 - CF_2) &= eV_1V_2 \\
2CF_2 &= 2ea \\
\therefore \quad CF_2 &= ae
\end{aligned}$$

Hence, co-ordinate of F_2 are $(ae, 0)$ and F_1 is $(-ae, 0)$.
Also,

$$\begin{aligned}
PF_2 &= ePH_2 \\
PF_2^2 &= e^2PH_2^2
\end{aligned}$$

But

$$PF_2^2 = (x - ae)^2 + y^2$$

and

$$PH_2^2 = \left(\frac{a}{e} - x\right)^2 + (y - y)^2 = \left(\frac{a}{e} - x\right)^2$$

$$\begin{aligned}
\therefore \quad (x - ae)^2 + y^2 &= e^2\left(\frac{a}{e} - x\right)^2 \\
x^2(1 - e^2) + y^2 &= a^2(1 - e^2)
\end{aligned}$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1 \tag{4.3e}$$

and

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{4.3f}$$

where $b^2 = a^2(1-e^2)$.

Equation (4.3f) is the simplest form of equation for an ellipse.

4.4 Tangent Equation at point (x_1, y_1) on the Ellipse

Differentiating equation (4.3f) with respect to x gives

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

and, the gradient at (x_1, y_1),

$$\frac{dy}{dx} = -\frac{b^2x_1}{a^2y_1}$$

Hence, tangent equation at (x_1, y_1) is

$$y - y_1 = \frac{-b^2x_1}{a^2y_1}(x - x_1)$$

$$\tfrac{y_1y}{b^2} + \tfrac{x_1x}{a^2} = \tfrac{y_1^2}{b^2} + \tfrac{x_1^2}{a^2} = 1 \quad \text{(Why?)} \tag{4.4a}$$

$$\therefore \quad \tfrac{y_1y}{b^2} + \tfrac{x_1x}{a^2} = 1$$

is the required tangent equation.

4.5 Normal equation at (x_1, y_1) on the Ellipse

The normal at (x_1, y_1) has the gradient $\dfrac{a^2y_1}{b^2x_1}$ (Why?)

Thus, the required normal equation is

$$y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1) \tag{4.5}$$

4.6 Example

Find the eccentricity of the ellipse $2x^2 + 3y^2 = 1$, and the tangent, normal to the ellipse at the point $x = \frac{1}{4}$ in the positive quadrant.

Solution

$2x^2 + 3y^2 = 1$ in canonical form is

$$\frac{x^2}{1/2} + \frac{y^2}{1/3} = 1$$

$\therefore \quad a = \dfrac{1}{2}, \quad b = \dfrac{1}{3}.$

From $b^2 = a^2(1 - e^2)$, we have

$$e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{5}}{3}$$

$\therefore$ eccentricity is $\sqrt{5}/3$

At $x = \frac{1}{4}$, $y = \pm\frac{4}{2}\sqrt{\frac{7}{6}}$. using the given equation. The point $(x_1, y_1) = \left(\dfrac{1}{4}, \sqrt{\dfrac{7}{24}}\right)$ and, the tangent equation is

$$\begin{aligned} y - \sqrt{\frac{7}{24}} &= \frac{-\frac{1}{9} \cdot \frac{1}{4}}{\frac{1}{4} \cdot \sqrt{\frac{7}{24}}} \left(x - \frac{1}{4}\right) \\ y - 0.54 &= -0.20(x - 0.25) \\ y &= -0.20x + 0.59 \end{aligned}$$

is the required tangent equation, and the normal equation is

$$\begin{aligned} y - \sqrt{\frac{7}{24}} &= 5(x - 0.25) \\ y - 0.54 &= 5x - 1.25 \\ y &= 5x - 0.71 \end{aligned}$$

is the required normal equation. **4.7 Remarks**

(i) If $\dfrac{x^2}{a^2} > 1$ in $\dfrac{y^2}{b^2} = 1 - \dfrac{x^2}{a^2}$, the value of y will be imaginary. Thus, we have $-a \leq x \leq a$ and $-b \leq y \leq b$, and the ellipse is a closed curve.

(ii) The ellipse is symmetrical about the x and y axes.

(iii) Line V_1V_2 and B_1B_2 are the major and minor axis respectively.

(iv) When $x = 0$, $y = \pm b$ and $B_1B_2 = 2b$. Similarly, when $y = 0$, $x = \pm a$ and $V_1V_2 = 2a$.

(v) The latus rectum (a chord through F parallel to the directrix), $x = ae$.

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) = b^2(1-e^2) = \frac{b^4}{a^2} \ \text{(since } b^2 = a^2(1-e^2))$$

$\therefore \ y = \pm\dfrac{b^2}{a^2}$ on the latus rectum, and length of the latus rectum is $2\dfrac{b^2}{a}$.

(vi) If points (x_1, y_1) and (x_2, y_2) lie on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (4.7a)$$

Then,

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \qquad (4.7b)$$

$$\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1 \qquad (4.7c)$$

and equation of the line joining the points is obtained by

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} \qquad (4.7d)$$

Substituting (4.7c) from (4.7b) gives

$$\frac{(x_1 - x_2)(x_1 + x_2)}{a^2} + \frac{(y_1 - y_2)(y_1 + y_2)}{b^2} = 0 \qquad (4.7e)$$

Verify.

Combining (4.7d) and (4.7e) gives

$$\frac{x(x_1 + x_2)}{a^2} + \frac{y(y_1 + y_2)}{b^2} = 1 + \frac{x_1 x_2}{a^2} + \frac{y_1 y_2}{b^2} \tag{4.7f}$$

Verify.
But if the points coincide, the line becomes the tangent, and equation (4.7f) becomes

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \tag{4.7g}$$

and in slope intercept form,

$$y = -\frac{xx_1}{a^2 y_1} + \frac{b^2}{y_1} \tag{4.7h}$$

Normal equation is obtained by

$$y = \frac{a^2 y_1}{x_1}(x - x_1) \tag{4.7i}$$

where $\frac{a^2 y_1}{x}$ is the gradient of the normal at (x_1, y_1) (Why?).

4.8 Equation of a Hyperbola

Let $F_1, F_2 :=$ Foci
$V_1, V_2 :=$ Vertices
$KK', LL' =$ Directrices
$H_1, H_2 :=$ Feet of the perpendicular from arbitrary point $p(x, y)$

Figure 4.8

Applying definition of hyperbola on figure (4.8), we have

$$e = \frac{F_2V_2}{V_2R_2} = \frac{F_2V_1}{V_2R_2} = \frac{PF_2}{PH_2} > 1 \tag{4.8a}$$

$$\left.\begin{aligned} F_2V_2 &= eV_2R_2 \\ F_2V_1 &= eV_1R_2 \end{aligned}\right\} \tag{4.8b}$$

Let $NV_2 = NV_1 = a$, then the coordinate of V_2 are $(a, 0)$ and V_1 are $(-a, 0)$. Then,

$$F_2V_2 + F_2V_1 = e(V_2R_2 + V_1R_2) \tag{4.8c}$$

$$F_2V_1 - F_2V_2 = e(V_1R_2 - V_2R_2) \tag{4.8d}$$

Using (4.8c), we obtain

$$F_2V_2 + F_1V_1 = V_1V_2 = e[(NV_2 - NR_2) + (NV_1 + NR_2)]$$

which gives $2a = 2eNR_2$.
$\therefore \quad NR_2 = \frac{a}{e}$ and the coordinate of point R_2 are $(\frac{a}{e},\ 0)$ and R_1 are $(-a/e,\ 0)$. Using equation (4.8d), we can obtain coordinate of F. Thus,

$$F_2V_1 - F_2V_2 = (NV_1 + NF_2) - (NV_2 - NF_2) = eV_1V_2$$

and $NF_2 = ae$ so that the coordinates of F_2 are $(ae, 0)$ and F_1 are $(-ae, 0)$. Furthermore, we have

$$PF_2 = ePH_2$$

But

$$PF_2 = (x - ae)^2 + y^2$$

and

$$PH_2 = (x - \frac{a}{e})^2$$

Hence,

$$PF_2^2 = e^2PH_2^2$$

and substituting gives,

$$(x - ae)^2 + y^2 = e^2(x - \frac{a}{e})^2$$

Solving, we obtain

$$x^2(e^2-1) - y^2 = a^2(e^2-1)$$

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1 \qquad (4.8e)$$

Let $b^2 = a^2(e^2-1)$ and substituting in (4.8e), you have

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (4.8f)$$

is the canonical form of the equation of a hyperbola with centre, foci $(\pm ae, 0)$ and directrices, $x = \pm\frac{a}{e}$.

4.9 Equation for tangent at a point (x_1, y_1) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Differentiating the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

with respect to x, we obtain

$$\text{Slope, } \frac{dy}{dx} = \frac{b^2 x}{a^2 y} \qquad (4.9a)$$

and the tangent equation at $)x_1, y_1)$ is

$$\frac{y - y_1}{x - x_1} = \frac{b^2 x_1}{a^2 y_1}$$

$$\therefore \quad y = \frac{b^2 x_1}{a^2 y_1}(x - x_1) + y_1 \qquad (4.9b)$$

4.10 Equation for normal at a point (x_1, y_1) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

From (4.9a), slope for normal is

$$m_2 = -\frac{a^2 y}{b^2 x} \quad \text{(Why?)}$$

$\therefore$ the normal equation at (x_1, y_1) to the hyperbola is

$$\begin{aligned} \frac{y-y_1}{x-x_1} &= \frac{-a^2y}{b^2x} \\ \therefore \quad y &= \frac{-a^2y_1}{b^2x_1}(x-x_1)+y_1 \end{aligned} \tag{4.10}$$

4.11 Remark

An asymptotic to a hyperbola is a tangent at infinity, and it is obtained by $y = \pm\frac{a}{b}$.

4.12 Example

Find the eccentricity, foci, vertices, directrices and the asymptotes, tangent and normal at (3,4) of the hyperbola

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

Solution

$$a = 3, \quad b = 4$$

From $b^2 = a^2(e^2 - 1)$, we have

$$e = \pm\frac{5}{3}$$

$\therefore$ eccentricity $= \pm\frac{5}{3}$
Foci $= (\pm ae, 0) = (\pm 3, 0)$
Directrices, $x = \pm\frac{b}{a} = \pm\frac{4}{3}$
By equation (4.9a) and (4.9b) respectively,
Tangent equation at (3,4) is

$$y = \frac{4}{2}x \quad \text{(Verify)}$$

Normal equation at (3,4) is

$$y = -\frac{3}{4}x + \frac{13}{16} = \frac{1}{4}\left(-3x + \frac{13}{4}\right)$$

$$16y + 12x - 13 = 0 \quad \text{(Verify)}$$

Asymptotes $\pm\frac{b}{a} = \pm\frac{4}{3}$.

4.13 The Graphs of Quadratic Equations

The Cartesian graph of any quadratic equation is of the form

$$A_1x^2 + A_2xy + A_3y^2 + A_4x + A_5y + A_6 = 0 \tag{4.13}$$

in which $A_2 \neq 0, \quad \forall\, i = 1, 2, 3$ is approximately always a conic section.

4.13.1 Examples of quadratic curves

$A_1x^2 + A_2xy + A_3y^2 + A_4x + A_5y + A_6 = 0$.

Conic Section	A_1	A_2	A_3	A_4	A_5	A_6	Equation	Remarks
Circle	1		1	-16		-9	$x^2 + y^2 = 9$	$A_1 = A_3$
Parabola			1	-16			$y^2 = 16x$	Quadratic in y linear in x
Ellipse	9		16			-144	$9x^2 + 16y^2$ =144	A_1, A_2 have same sign $A_1 \neq A_3$
Hyperbola	1		-1			-1	$x^2 - y^2 = 1$	A_1, A_3 have opposite sign

4.14 Summary

In this lecture:

(a) We derive equation for:

(i) An ellipse as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(ii) Tangent equation at a point (x_1, y_1) on the Ellipse as

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1$$

(iii) Normal equation at (x_1, y_1) on the Ellipse as

$$y = \frac{a^2 y_1}{b^2 x_1}(x - x_1) + y_1$$

(iv) A hyperbola as

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(v) Tangent at a point (x_1, y_1) on the hyperbola as

$$y = \frac{b^2 x_1}{a^2 y_1}(x - x_1) + y_1$$

(vi) Normal at a point (x_1, y_1) on the hyperbola as

$$y = \frac{a^2 y_1}{b^2 x_1}(x - x_1) + y_1$$

(vii) Asymptote to a hyperbola as $y = \pm\frac{b}{a}$

(b) We derive the directrices of

(i) Ellipse as $d = \pm\frac{a}{e}$

(ii) Hyperbola as $e = \sqrt{a^2 + b^2/a}$

(c) We described the directices of

(i) Ellipse as $d = \pm\frac{a}{e}$

(ii) Hyperbola as, $d = \pm\frac{a}{e}$

(d) We described a conic section using quadratic curve,
$A_1x^2 + A_2xy + A_3y^2 + A_4x + A_5y + A_6 = 0$
where A_1, A_2, A_3 are not all equal to zero.

4.15 Post-Test 4

1(a) Find the points of contact of the line $y + 2a - 3 = 0$ with the ellipse $4x^2 + y^2 = 5$.

(b) Find the equations of the tangent and normal at these points.

2. Sketch the hyperbola

$$9x^2 - 9y^2 = 16$$

3. Find the center, vertices, foci, eccentricity and asymptote of the hyperbola, $4y^2 = x^2 - 4x$.

4.16 Solution to Pre-Test 4

Major and Minor.

Contemporary Reading

1. O.O. Ugbebor and N.I. Akinwande: Analytical Geometry and Mechanics.

2. Godwin Alo. Odili: Calculus with Co-ordinate Geometry and Trigonometry.

Lecture 5
Vectors in $\mathbb{R}^3$ (I)

5.0 Introduction

A vector is a straight line of a given length or size (magnitude) in a specified direction. Suppose you walk $10km$ due north and $8km$ due east. Then, your displacement is as follows:

Figure 5.0

From the figure, the vectors are: (i) $\overrightarrow{SG}$, (ii) $\overrightarrow{GP}$, (iii) $\overrightarrow{SP}$.
We begin this lecture by showing how to perform algebraic expressive on vectors, define unit vectors, directional cosine of vectors, state and identify conditions necessary for three collinear points, write vector equation of a straight line and write vector equation of a plane.

5.1 Objectives

After this lecture, you should be able to:

(i) Define vectors, in $\mathbb{R}^2, \mathbb{R}^3$.

(ii) Perform algebraic expression on vectors in $\mathbb{R}^3$.

(iii) Define unit vectors in $\mathbb{R}^3$

(iv) Define directional cosines of vectors in $\mathbb{R}^3$.

(v) State and identity conditions necessary for three collinear points

(vi) Write vector equation of a straight line.

(vii) Write vector equation of a plane.

5.2 Pre-Test 5

Using the figure 5.0, what is the magnitude of:
(a) SG, (b) GP and (c) SP.

5.3 Definitions

5.3.1: A vector is a quantity that has both magnitude and direction.
For example,
Force = Distance in the direction of force.
Other examples of vectors are displacement, velocity, acceleration, momentum, electric and magnetic field.

5.3.2: A scalar quantity has only size but no direction. Examples of scalar quantity are mass, length, time, temperature, volume, density, work, quantity of heat.

5.4 Note Notations

Let $A, B, C, D, \ldots$ denote points in space with P representing an arbitrary point, and 0 an arbitrary origin. Then:

5.4.1: The vectors, $\overrightarrow{OA} = a$

5.4.2: The arbitrary vector $\overrightarrow{OP} = I$

5.4.3: Scalar values shall be denoted by $a, b, \ldots,$ or α, β.

5.5 Vectors in $\mathbb{R}^2$

A vector in $\mathbb{R}^2$ is a 2-dimensional coordinate system in the direction of x, y-axes respectively.

Figure 5.5

5.6 Vector in $\mathbb{R}^3$

A vector in $\mathbb{R}^3$ is a 3-dimensional coordinate system in the direction x, y, z-axes respectively.

Figure 5.6

5.7 Vector Algebra

(i) A vector say ($\underline{a}$) and another vector say '$\underline{b}$' are equal if they both have the same magnitude and direction, and we write $\underline{a} = \underline{b}$.

(ii) The vector $-\underline{b}$ has the same magnitude as the vector $\underline{b}$ but opposite direction.

(iii) The vector $\underline{c} = \underline{a} + \underline{b}$ is obtained using the parallelogram law by placing the initial point of $\underline{b}$ on the terminating point of $\underline{a}$ and then joining the initial point of $\underline{a}$ to the terminal point of $\underline{b}$ as shown below.

Figure 5.7

(iv) $\underline{a} + \underline{b} = \underline{b} + \underline{a}$ (Commutative law of addition)

(v) $\underline{a} + (\underline{b} + \underline{c}) = (\underline{a} + \underline{b}) + \underline{c}$ (associativity law of addition)

(vi) $\psi\underline{a} = \underline{a}\psi$ (commutative law of scalar multiplication)

(vii) $(\psi + \beta)\underline{a} = \psi\underline{a} + \beta\underline{a}$ (Distributive law)

(viii) $\psi(\beta\underline{a}) = (\psi\beta)\underline{a} =$ (associative law of scalar multiplication)

(ix) $\psi(\underline{a} + \underline{b}) = \psi\underline{a} + \psi\underline{b}$ (Distributive law)

5.8 Definitions (Unit vector)

(i) A unit vector is any vector whose length is equal to the unit of length along the coordinate axes.
Let $\underline{a} = x_i + y_j$, then, the magnitude of $\underline{a}$, is

$$|\underline{a}| = |x_i + y_j| = \sqrt{x^2 + y^2}$$

(ii) The direction of a unit vector, $\underline{a}$ is obtained by

$$\frac{\underline{a}}{|\underline{a}|} = \frac{x_i + y_j}{\sqrt{x^2 + y^2}} = \frac{x}{\sqrt{x^2 + y^2}}i + \frac{y}{\sqrt{x^2 + y^2}}j$$

(iii) Suppose $P(x, y, z)$ is an arbitrary point in space, its position vector, is $\overrightarrow{OP} = \underline{r} = x_i + y_j + z_k$ and its magnitude is

$$|\overrightarrow{OP}| = |\underline{r}| = \sqrt{x^2 + y^2 + z^2}$$

Similarly, if $\overrightarrow{OA} = \underline{a} = a_1i + a_2j + a_3k$ is the position vector of a point A. Then, its magnitude,

$$|\overrightarrow{OA}| = |\underline{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

5.8.1 Example

Given that $\underline{a} = 3i - 4j$, find the direction of $\underline{a}$.

Solution

Magnitude of $\underline{a}$ is $|\underline{a}| = \sqrt{3^2 + 4^2} = 5$.
Direction $= \frac{3}{5}i - \frac{4}{5}j$

5.8.2 Example

Given that $\underline{a} = i - 2j + 4k$. Find the length of $\underline{a}$.

Solution

Length of $\underline{a}$ is $|\underline{a}| = 21$ units (verify).

5.8.3 Remark

The vectors i, j and k are units vectors because:

$$
\begin{aligned}
|i| = 1i + 0j + 0k &= \sqrt{1^2 + 0^2 + 0^2} = 1 \\
|j| = 0i + 1j + 0k &= \sqrt{0^2 + 1^2 + 0^2} = 1 \\
|k| = 0i + 0j + 1k &= \sqrt{0^2 + 0^2 + 1^2} = 1
\end{aligned}
$$

5.9 Direction Cosines

If the line OP makes angle α_1, α_2 and α_3 respectively with x, y and z axes. Then, we define cosines of these angles as:

(i) $l = \frac{x}{r} = \cos\alpha_1$

(ii) $m = \frac{y}{r} = \cos\alpha_2$

(iii) $n = \frac{z}{r} = \cos\alpha_3$
where l, m, n are the direction cosines of the line OP.

(iv) $l^2 + m^2 + n^2 = 1$ (Verify).

5.10 Point Dividing a Line in a Given Ratio

Figure 5.10

Suppose P divides the line AC in the ratio $m : n$ then,

$$\begin{aligned} \frac{AP}{PC} &= \frac{m}{n} \\ \text{and} \quad n\overrightarrow{AP} &= m\overrightarrow{PC} \\ n(\underline{r} - \underline{a}) &= m(\underline{b} - \underline{r}) \\ \therefore \quad \underline{r} &= \frac{m\underline{b} + n\underline{a}}{m+n} \end{aligned}$$

5.10.1 Remarks

(i) If $m : n$ is positive, then the point P lies in-between the points A and B.

(ii) If P is the mid-point of A and B, then $m = n = 1$ and

$$\underline{r} = \frac{1}{2}(\underline{a} + \underline{b})$$

(iii) If $m : n$ is negative and lies between 0 and -1, then P is outside AC and is closer to A than C.

(iv) If $m : n$ is negative and lies between -1 and $-\infty$, then P is outside AC and is closer to C than A.

5.11 Three collinear points

If the three points A, B, C are collinear and distinct, then there exist l, m, n such that $l\underline{a} + m\underline{b} + n\underline{c} = 0$, and $l + m + n = 0$. Conversely, if the equations hold then the points are collinear.

5.12 Example

Let L, M, N be the mid-point of the sides BC, CA, AB respectively of a triangle ABC. show that

(i) $\overrightarrow{NM} = \frac{1}{2}\overrightarrow{BC}$

(ii) $\overrightarrow{AL} + \overrightarrow{BM} + \overrightarrow{CN} = 0$

(iii) The median AL, BM and CN have a common point of trisection.

Solution

Figure 5.12

From figure 5.12,

$$\begin{aligned}
\text{(i)}\quad \overrightarrow{NM} &= \overrightarrow{BM} - \overrightarrow{BN} \ \text{(because } \overrightarrow{NM} + \overrightarrow{BN} = \overrightarrow{BM}) \\
&= (\overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC}) - \frac{1}{2}\overrightarrow{BA} \ \text{(since } BN = \frac{1}{2}BA) \\
&= \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{AC}) \\
\therefore\quad \overrightarrow{NM} &= \frac{1}{2}\overrightarrow{BC} \\
\text{(ii)}\quad \overrightarrow{AL} &= \overrightarrow{BL} - \overrightarrow{BA} = \frac{1}{2}\overrightarrow{BC} - \overrightarrow{BA} = \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{AC}) - \overrightarrow{BA} \\
\overrightarrow{BM} &= \overrightarrow{MC} - \overrightarrow{BC} = \frac{1}{2}\overrightarrow{AC} - \overrightarrow{BC} = \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{BC}) - \overrightarrow{BC} \\
\overrightarrow{CN} &= \overrightarrow{BN} - \overrightarrow{AC} = \frac{1}{2}\overrightarrow{BA} - \overrightarrow{AC} = \frac{1}{2}(\overrightarrow{AC} + \overrightarrow{BC}) - \overrightarrow{AC} \\
\therefore\quad \overrightarrow{AL} + \overrightarrow{BB} + \overrightarrow{CN} &= \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{AC}) - \overrightarrow{BA} + \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{BC}) - \overrightarrow{BC} \\
&\quad + \frac{1}{2}(\overrightarrow{AC} + \overrightarrow{BC}) - \overrightarrow{AC} = 0 \ \text{(Verify)}
\end{aligned}$$

(iii) From $\frac{1}{2}(\overrightarrow{BA} + \overrightarrow{AC})$ and $\overrightarrow{MC} = \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{AC})$. Eliminate $\overrightarrow{AC}$ to obtain:

$$\frac{2\overrightarrow{BL} + \overrightarrow{AC}}{3} = \frac{2\overrightarrow{MC} + \overrightarrow{BA}}{3}$$

Thus, the common point of intersection of the median AL, BM is a point of trisection to each of the lines.

5.13 Vector Equation of Straight Line

5.13.1 Remark

Equation of a line is uniquely determined of either:

(i) A point on the line and a vector parallel to the line are given.
Or

(ii) Two points on the line are given.
Let assume case (i).
Suppose point P lies on a straight line, L which is parallel to a vector V.

Figure 5.13.1

Then, $\overrightarrow{P_0P} = \psi V$ for some ψ (Why?).
For any arbitrary point, $P(x, y, z)$ on the line L

$$(x - x_0)i + (y - y_0)j + (z - z_0)k = \psi(a_i + b_j + c_k)$$

and we have,

$$P(x, y, z) = P_0(x_0, y_0, z_0) + \psi(a, b, c) \qquad (5.13.1a)$$

or

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \qquad (5.13.1b)$$

Now assuming case (ii).
Suppose A_1 and A_2 are points on a straight line, then the vector $\underline{b} - \underline{a}$ is parallel to the line and its equation is

$$(x, y, z) = (x_0, y_0, z_0) + \psi(b_1 - a_1, b_2 - a_2, b_3 - a_3)$$

or

$$\psi = \frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3}$$

5.13.2 Example

(i) Write a vector equation for a line through $(-2, 0, 4)$ parallel to the vector $\underline{V} = 2i + 4j - 2k$.

(ii) Find its parametric equations.

Solution

Using equation (5.13.1a)

$$P_0(x_0, y_0, z_0) = (-2, 0, 4)$$

Thus,

$$(x, y, z) = (-2, 0, 4) + \psi(2, 4, -2)$$

(ii) $x = -2 + 2\psi$
$y = 4\psi$
$z = 4 - 2\psi$
where ψ is the parameter.

5.14 Vector Equation of a Plane

Suppose the points A, B, C lie on a plane, then the vectors $\overrightarrow{AB} = \underline{b} - \underline{a}$, $\overrightarrow{AC} = \underline{c} - \underline{a}$ lie on the plane. Therefore, the position vector of an arbitrary point on the plane is obtained by

$$\underline{r} = \underline{a} + \psi(\underline{b} - \underline{a}) + \beta(\underline{c} - \underline{a}) \tag{5.14.1}$$

If the point A is the origin, the equation (5.14.1‘) becomes

$$\underline{r} = \alpha\underline{b} + \beta\underline{c}$$

But $(1 - \alpha - \beta)\underline{a} + \alpha\underline{b} + \beta\underline{c} - \underline{r} = 0$.
From (5.14.1) and the sum of the coefficients is zero.
Hence, if the points A, B, C, D are coplanar, then there exist scalar numbers $\psi_1, \psi_2, \psi_3, \psi_4$ such that

$$\psi_1\underline{a} + \psi_2\underline{b} + \psi_3\underline{c} + \psi_4\underline{d} = 0$$

and

$$\psi_1 + \psi_2 + \psi_3 + \psi_4 = 0$$

5.14.2 Example

Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $N = 5i + 2j - k$.

Solution

Let $P(x, y, z)$ be arbitrary point on the plane, then, we have:

$$5x + 2y - z = -22$$

and

$$5x + 2y - z + 22 = 0$$

is the required equation.

5.15 Summary

(a) Define:
- (i) A vector as any quantity having magnitude and specified direction
- (ii) Vector in 2, 3 coordinate points as $\underline{v} = x_i + y_j$ and $\underline{v} = x_i + y_j + z_k$
- (iii) Unit vector as $\dfrac{\underline{v}}{|\underline{v}|}$.

(b) Wrote
- (i) Vector equation of a straight line as
  $(x, y, z) = (x_0, y_0, z_0) + \psi(v_1, v_2, v_3)$
  where ψ is the parameter of the equation.
- (ii) Vector equation of a plane through
  $P(x_0, y_0, z_0)$ perpendicular to $\underline{v} = a_i + b_j + c_k$ as
  $ax + by + cz = ax_0 + by_0 + cz_0$

5.16 Post-Test 5

(1) If $\underline{r} = 3i + 4j + 3k$; $\underline{u} = 6j + 7k$; $\underline{t} = 2i - 5j$
Evaluate:

- (a) $2\underline{r} - 3\underline{u} - 5\underline{t}$
- (b) $\underline{r} + \underline{u} - 2\underline{t}$
- (c) $\underline{r} - 3\underline{u} + \underline{t}$
- (d) Find the magnitude of $4\underline{r}$
- (e) Find the direction of $\underline{t}$

(2) Find the cosine of the angles ψ_1, ψ_2, ψ_3 which the vector $\underline{b} = 4i - 5j + 3k$ makes with the x, y and z axis respectively.

(3) If the vectors $\underline{v}, \underline{v}, \underline{r}$ form the sides of a triangle, prove that $r^2 = \underline{u}^2 + \underline{v}^2 - 2uv\cos\mathbb{R}$.

(4) If $\underline{a} = 2i - 4k$; $b = 3i - 4j + k$, find a unit vector perpendicular to the plane $\underline{a}$ and $\underline{b}$.

(5) Find an equation for the line through $A_0(3, 5, 1)$ perpendicular to $\underline{v} = 4i + 3j + k$.

(6) Find the parametric equation for the line segment through $M(-3, 2, 4)$ and $N(2, -2, 5)$.

(7) Find the point in which the line $x = 4 + 2t$; $y = 4t$; $z = 2 + t$ meets the plane $3x + 4y + 8z = 8$.

Solution to Pre-Test 5

(a) $10km$, (b) $8km$ (c) $2\sqrt{41}km$

Contemporary Reading

1. O.O. Ugbebor and N.I. Akinwande: Analytical Geometry and Mechanics.

2. C.J. Tranter and C.G. Lambi: Advanced Level Mathematics.

Lecture 6
Vectors in $\mathbb{R}^3$ (II)

6.0 Introduction

In this lecture, you will learn how to perform dot and cross vectors product, discover how you can use this knowledge to classify vectors as parallel or perpendicular. Moreover, you will learn how dot and vectors product leads to calculation of areas and volume of some shapes.

6.1 Objectives

After this lecture, you should be able to:

(i) Perform scalar (DOT) multiplication on vectors

(ii) Calculate angle of inclination between vectors

(iii) Perform cross multiplication on vectors

(iv) Perform triple scalar product on vectors

(v) Perform triple vector product

(vi) Write vector equation of a plane

(vii) Write equation for perpendicular distance from a point to a line.

6.2 Pre-Test 6

(1) Find the area of the triangle ABC

(2)

Find the volume of the figure.

6.3 Dot (Scalar) Product

The scalar product of two vectors, $\underline{a}$ and $\underline{b}$ inclined at an angle ψ as display on figure 6.3 is the number.

Figure 6.3

$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos \psi \tag{6.3}$$

6.4 Properties of scalar product

Let $\underline{a}, \underline{b}, \underline{c}$ be vectors and α be a scalar. Then:

(i) $i \cdot i = j \cdot j = k \cdot k = 1$

(ii) $i \cdot j = j \cdot k = i \cdot k = 0$

(iii) $\underline{a} \cdot \underline{a} = a_1^2 + a_2^2 + a_3^2$ (Verify)

(iv) $\underline{a} \cdot \underline{b} = a_1b_1 + a_2b_2 + a_3b_3$ (Verify)

(v) $\underline{a} \cdot \underline{b} = \underline{a} \cdot \underline{a}$ (Verify)

(vi) $\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} + \underline{c}$

(vii) $\alpha(\underline{a} \cdot \underline{b}) = (\alpha\underline{a}) \cdot \underline{b} = \underline{a} \cdot (\alpha\underline{b})$

(viii) $\cos\psi = \dfrac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{a}|} = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{|\underline{a}||\underline{b}|}$

(ix) If $\underline{a} \cdot \underline{b} = 0$ and $\underline{a} \neq 0 \neq \underline{b}$, then $\psi = \frac{\pi}{2}$ and $\underline{a}$ and $\underline{b}$ are perpendicular.

6.5 Example

Given that the vectors $\underline{a} = i - 2j + 3k$ and $\underline{b} = 6i + 3j + 2k$:

(i) evaluate $\underline{a} \cdot \underline{b}$

(ii) Find the cosine of the angle between the vectors $\underline{a}$ and $\underline{b}$.

Solution

(i) $\underline{a} \cdot \underline{b} = i \cdot 6i + (-2j)(3j) + (3k) \cdot (2k)$

(ii) Let the angle between the vectors $\underline{a}$, $\underline{b}$ be β, then,

$$\cos\beta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}$$

But $|\underline{a}| = 14, \quad |\underline{b}| = 49$
$\therefore \quad \cos\beta = \dfrac{3}{343}$.
and $\cos\beta = 0.0087$.

6.6 Cross Products

The vector product of two vectors $\underline{a}$ and $\underline{b}$ inclined at an angle ψ is obtained by
$\underline{a} \times \underline{b} = \underline{nab} \sin\psi$

where $\underline{n}$ is the perpendicular unit vector to the plane of the vector $\underline{a}$ and $\underline{b}$, and $0 \leq \psi \leq \frac{\pi}{2}$.

6.7 Properties of vector product

Let

$$\begin{aligned} \underline{a} &= a_1 i + a_2 j + a_3 k \\ \underline{b} &= b_1 i + b_2 j + b_3 k \\ \underline{c} &= c_1 i + c_2 j + c_3 k \end{aligned}$$

and β be a scalar.
Then,

(i) $i \times i = j \times j = k \times k = 0$

(ii) $i \times j = -(j \times i) = k,\ j \times k = -(k \times j) = i,\ \ k \times i = -(i \times k) = j$

(iii) $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$

(iv) $\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$

(v) $\beta(\underline{a} \times \underline{b}) = (\beta\underline{a}) \times \underline{b} = \underline{a} \times (\beta\underline{b})$

6.8 Area of a parallelogram

Area of a parallelogram = base × perpendicular height

$$
\begin{aligned}
|\underline{a} \times \underline{b}| = |n\underline{a} \times \underline{h}| &= |\underline{n}||\underline{a}||\sin\psi| \\
\therefore \quad |\underline{a} \times \underline{b}| &= |\underline{n}||\underline{a}||\sin\psi| \qquad (6.8)
\end{aligned}
$$

is the required formular to calculate the area of a parallelogram whose sides are vectors.

6.9 The determinant formula for $|\underline{a} \times \underline{b}|$

Let

$$
\begin{aligned}
\underline{a} &= a_1 i + a_2 j + a_3 k \\
\underline{b} &= b_1 i + b_2 j + b_3 k
\end{aligned}
$$

Then,

$$
\begin{aligned}
\underline{a} \times \underline{b} &= (a_1 i + a_2 j + a_3 k) \times (b_1 i + b_2 j + b_3 k) \\
&= a_1 i \times b_1 i + a_2 i \times b_2 j + a_1 i \times b_3 k \\
&\quad + a_2 j \times b_1 i + a_2 j \times b_2 j + a_2 j \times b_3 k \\
&\quad + a_3 k \times b_1 i + a_3 k \times b_2 j + a_3 k \times b_3 k \\
&= (a_1 b_2 - a_2 b_1)k + (a_3 b_1 - a_1 b_3)j + (a_2 b_3 - a_3 b_2)i \\
&= (a_2 b_3 - a_3 b_2)i + (a_3 b_1 - a_1 b_3)j + (a_1 b_2 - a_2 b_1)k \\
\therefore \quad \underline{a} \times \underline{b}| &= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}
\end{aligned}
$$

6.10 Example

If the vectors $\underline{a} = 3i - 2j + 3k$, $\underline{b} = 4i + j + 2k$, find the area of the parallelogram enclosed by $\underline{a}$ and $\underline{b}$.

Solution

$$\begin{aligned} \underline{a} \times \underline{b}| &= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= -7i + 6j + 11k \\ \therefore \quad \text{area} &= |\underline{a} \times \underline{b}| = 206 \text{ sq. units} \end{aligned}$$

6.11 Tripe Scalar Product

The product $\underline{a} \cdot \underline{b} \times \underline{c}$ is called the triple scalar product of $\underline{a}, \underline{b}, \underline{c}$. It gives the volume of a parallelepiped (a parallelogram pipe) whose sides are $\underline{a}, \underline{b}, \underline{c}$

Figure 6.11

The product,

$$\underline{a} \cdot \underline{b} \times \underline{c} = \underline{c} \cdot \underline{a} \times \underline{b} = \underline{b} \cdot \underline{c} \times \underline{a}$$

Suppose $\underline{a}, \underline{b}, \underline{c} \neq 0$ and $\underline{a} \times \underline{b} \times \underline{c} = 0$, then, the vectors $\underline{a}, \underline{b}, \underline{c}$ lie on a common plane (i.e. the three vectors are coplanar).
Let

$$\begin{aligned} \underline{a} &= a_1 i + a_2 j + a_3 k \\ \underline{b} &= b_1 i + b_2 j + b_3 k \\ \underline{c} &= c_1 i + c_2 j + c_3 k \end{aligned}$$

then,

$$\underline{a} \cdot \underline{b} \times \underline{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

6.12 Example

Find the volume of the parallelepiped determined by $\underline{a} = i + 2j + k$, $\underline{b} = 3i + 3k$, and $\underline{c} = 7j + 4k$.

Solution

$$\begin{aligned} \underline{a} \cdot \underline{b} \times \underline{c} &= \begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 0 & 7 & 4 \end{vmatrix} \\ &= -31 \end{aligned}$$

6.13 Triple Vector Product

The product $(\underline{a} \times \underline{b}) \times \underline{c}$ is defined by:

$$(\underline{a} \times \underline{b}) \times \underline{c} = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{b} \cdot \underline{c})\underline{b} \qquad (6.13)$$

and it lies in the plane of $\underline{b}$ and $\underline{c}$.

Remark

The products, $\underline{a} \times (\underline{b} \times \underline{c})$ and $(\underline{a} \times \underline{b}) \times \underline{c}$ are not always equal.

6.13.1 Example

Given that $\underline{a} = i - j + 2k$; $\underline{b} = 2i + j + k$; $\underline{c} = i + 2j - k$, show that

$$(\underline{a} \times (\underline{b}) \times \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{b} \cdot \underline{c})\underline{a}$$

Solution

From the L.H.S. of the equation, we have

$$\underline{a} \times \underline{b} = -3i + 3j + 3k$$

so that $(\underline{a} \times \underline{b}) \times \underline{c} = -9i - 9k$ and from the R.H.S. of the equation, $(\underline{a} \cdot \underline{c}) = -3$, $\underline{b} - \underline{c} = 3$ so that

$$\begin{aligned} (\underline{a} \cdot \underline{c})\underline{b} - (\underline{b} - \underline{c})\underline{a} &= 3\underline{b} - 3\underline{a} = -9i - 9k \\ \therefore \quad (\underline{a} \times \underline{b}) \times \underline{c} &= (\underline{a} \cdot \underline{c})\underline{b} - (\underline{b} \cdot \underline{c})\underline{a} \end{aligned}$$

Remark

$$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$$

6.14 Perpendicular distance from a point to a line

Figure 6.14

Consider the straight AP passing through point A with position vector $\underline{a}$. Suppose P is an arbitrary point on the line with position vector $\underline{r}$, and $\underline{e}$ is the unit vector parallel to the straight line AP. Then, the vectors $\underline{r} - \underline{a}$ and $\underline{e}$ are parallel vectors, and

$$(\underline{r} - \underline{a}) \times \underline{e} = 0 \qquad (6.14a)$$

which is equation of the straight line AP.

Let the perpendicular distance from a point M with position vector $\underline{m}$ to the line AP be MN, N is the foot of the perpendicular line from M to AP. Hence, we have

$$MN = AM \sin\psi = |(\underline{r} - \underline{a}) \times \underline{e}|$$

where ψ is the angle line AM makes with line AP. The position vector of the point N is given by

$$\underline{n} = \underline{a} + \underline{e} \cdot (\underline{m} - \underline{a})\underline{e}$$

The vector $\overrightarrow{MN}$ is given by

$$\begin{aligned} \overrightarrow{MN} &= \overrightarrow{MA} + \overrightarrow{AN} \\ &= \underline{a} - \underline{m} + \underline{e} \cdot (\underline{m} - \underline{a})\underline{e} \qquad (6.14b) \end{aligned}$$

and perpendicular distance is $|\overrightarrow{MN}|$.

6.14.1 Example

Find the perpendicular distance from the point $4i - 2j + 3k$ to the straight line which passes through the points $2i + 3j - 4k$ and $8i + 6j - 8k$.

Solution

Let $\underline{a} = 2i + 3j - 4k$, $\underline{r} = 8i + 6j - 8k$, $\underline{m} = 4i - 2j + 3k$.
Then,

$$\underline{e} = \frac{\underline{r} - \underline{a}}{\underline{r} - \underline{a}|} = \frac{6i + 3j - 4k}{\sqrt{61}}$$

Using equation (6.14b), we have

$$\begin{aligned} \underline{m} - \underline{a} &= 2i - 5j + 7k \\ \underline{e}(\underline{m} - \underline{a}) &= -\frac{31}{\sqrt{61}} \\ \underline{e} \cdot (\underline{m} - \underline{a})\underline{e} &= -\frac{186i - 93j + 124k}{61} \end{aligned}$$

But $\underline{a} - \underline{m} = -(\underline{m} - \underline{a}) = -2i + 5j = 7k$.
$\therefore$ the perpendicular distance from point $4i - 2j + 3k$ is

$$|\underline{a} - \underline{m} + \underline{e} \cdot (\underline{m} - \underline{a})\underline{e}| = 7.9 \text{ units}$$

6.15 Vector equation of a plane

Let $\underline{a}$ be a vector on a plane perpendicular to a vector $\underline{n}$. Suppose $\underline{r}$ is the position vector of an arbitrary point P on the plane. Then, $\underline{r} - \underline{a}$ and $\underline{n}$ are perpendicular. Thus,

$$(\underline{r} - \underline{a}) \cdot \underline{n} = 0 \qquad (6.15a)$$

which is the vector equation of the plane.
If $\underline{n}$ is the unit vector perpendicular to the plane. The perpendicular distance from the origin to the plane is

$$\underline{r} = \underline{a} \cdot \underline{n} \qquad (6.15b)$$

which can be written as

$$\underline{r} - \underline{a} \cdot \underline{n} = 0 \qquad (6.15c)$$

If the unit vectors $\underline{u}_1$ and $\underline{u}_2$ are the unit normal to two planes; the angle of inclination ψ of the two planes is obtained by

$$\cos\psi = \underline{u}_1 \cdot \underline{u}_2 \tag{6.15d}$$

6.15.1 Example

Find the perpendicular vector to the vectors $\underline{a} = 2i - j + k$ and $\underline{b} = i + 2j + 3k$.

Solution

$$\begin{aligned} \underline{v} = \underline{a} \times \underline{b} &= \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} \\ &= -5i + 5j + 5k. \end{aligned}$$

6.16 Summary

In this lecture, we:

(a) Performed:

(1) dot multiplication on the vectors as $\underline{n} \cdot \underline{a} \cdot \underline{b}$

(2) cross multiplication on vector as $\underline{a} \times \underline{b}$

(3) triple scalar multiplication as $\underline{a} \cdot \underline{b} \times \underline{c}$

(b) Defined:

(1) angle between two vectors say $\underline{a}$ and $\underline{b}$ as $\cos\psi = \dfrac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}$

(2) cross product of two vectors as $\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

where $\underline{a} = a_1 i + a_2 j + a_3 k$; $\underline{b} = b_1 i + b_2 j + b_3 k$.

(3) dot product of two vectors as

$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

(4) triple scalar product as

$\underline{a} \cdot \underline{b} \times \underline{c} = \underline{c} \cdot \underline{a} \times \underline{b} = \underline{b} \cdot \underline{c} \times \underline{a}$

(5) triple vector product as

$(\underline{a} \times \underline{b}) \times \underline{c} = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{b} \cdot \underline{c})\underline{a}$

(c) Wrote:

(1) Equation for perpendicular distance from a point to a line.

(2) Vector equation of a plane.

6.17 Post-Test 6

(1) Find the cosine of the angle between $3x-2y-6z=5$ and $6x+2y-9z=4$.

(2) Find the area of a triangle whose vertices are $A(-1,-1)$, $B(3,3)$, $C(2,1)$

(3) Given vectors $\underline{a}, \underline{b}, \underline{c}$, use dot product and cross product to describe a vector in the plane of $\underline{b}$ and $\underline{c}$ perpendicular to $\underline{a}$.

(4) Find a vector that is perpendicular to both $\underline{a} = 2i+2j+2k$ and $\underline{b} = 4i+4j$

(5) Given that $\underline{a} = a_1 i + a_2 j + a_3 k$ and $\underline{b} = b_1 i + b_2 j + b_3 k$

   (a) Evaluate:
   (i) $\underline{a} \times \underline{b}$ (ii) $\underline{b} \times \underline{a}$

   (b) Is $\underline{a} \times \underline{b} = \underline{b} \times \underline{a}$

(6) Find the area of a parallelogram whose diagonals are the vectors $\underline{a} = i - 3j + 4k$ and $\underline{b} = 3i + j - 2k$.

(7) Find the volume of a parallelepiped $\underline{a} = 6i - 7k$; $\underline{b} = 3i + 4j + 2k$; $\underline{c} = 3i + 8k$.

(8) Let $\underline{a}, \underline{b}, \underline{c}$ be vectors and are collinear. Show that x, y, z which are not all zero exist such that

$$x + y + z = 0$$

and

$$x\underline{a} + y\underline{b} + z\underline{c} = 0$$

(9) Find the volume of a tetrahedron $\underline{a} = 3i + 2j + 6k$; $\underline{b} = 4i - 2j + 7k$ and $\underline{c} = 2i + 4j + 3k$
(**Hint:** Volume $= \left|\frac{1}{6}(\underline{a} \times \underline{b} \cdot \underline{c})\right|$)

6.18 Solution to Pre-Test 6

$$\begin{aligned}(1)\quad \text{Area} &= \frac{1}{2} \times 8 \times \sin 30^0 \\ &= 2 \text{ sq. units}\end{aligned}$$

$$\begin{aligned}(2)\quad \text{Volume} &= 5cm \times 7cm \times 30cm \\ &= 105cm^3\end{aligned}$$

Contemporary Reading

O.O. Ugbebor and N.I. Akinwande: Analytical Geometry and Mechanics.

Lecture 7
Dynamic I - (Velocity and Acceleration)

7.0 Introduction

In this lecture, we combine calculus and vectors to describe the motion of a body in space. You will learn how position of a body in space is use to describe its velocity, acceleration, relative velocity, energy, power, work, speed and direction.

7.1 Objectives

After this lecture, you should be able to:

(i) Define the velocity of a position vector.

(ii) Define the acceleration of a position vector.

(iii) Define speed of moving body.

(iv) Define the force acting on a body.

(v) Define momentum of a body.

(vi) Define kinetic energy of a given body.

(vii) Define potential energy of a given body.

(viii) Define work done on a given body.

(ix) Define power of a given system

(x) Define relative velocity.

(xi) Define relative acceleration.

(xii) Express direction of a moving body.

7.2 Pre-Test 7

(a) Given that:

(1) $y(x) = 2x^3 + 6x^2 + 3$

(2) $y(x) = \frac{x^4}{4} - \sqrt{x}$

Find $\frac{dy(x)}{dx}$ of questions (1) and (2)

(b) Evaluate

(1) $\int_0^2 \left(\frac{3x^2}{12} + 6x^3\right) dx$

(2) $\int_{-1}^3 (2x + 4)dx$

7.3 Velocity, Acceleration, Speed of a position vector

Figure 7.3

Let $\overrightarrow{OP} = \underline{r}(t) = x(t)i + y(t)j + z(t)k$ be the position vector of a body moving in space at time t. Then, the velocity of P is obtained by the derivative.

7.3.1 Example

The vector $\underline{r}(t) = (2\cos t)i + (2\sin t)j + t^2k$ gives the position of a moving body at time t. Find the body:

(i) Velocity

(ii) Acceleration

(iii) Speed at $t = 3$ minutes.

Solution

$$\begin{array}{lll}
\text{(i)} \quad \text{Velocity,}\underline{v}(t) & = & -2\sin ti + 2\cos tj + 2tk \\
 & = & 2(\cos tj - \sin ti + tk) \\
\text{(ii)} \quad \text{Acceleration, } \underline{a}(t) & = & -2\cos tj - 2\sin tj + 2k \\
\text{(iii)} \quad \text{Speed, } s = |\underline{v}| & = & 2\sqrt{2} \text{ units/minutes}
\end{array}$$

$$\underline{v}(t) = \frac{d\underline{r}(t)}{dt} = \frac{dx(t)}{dt}i + \frac{dy(t)}{dt}j + \frac{dz(t)}{dt}k$$

and the acceleration of P is

$$\begin{aligned}
\underline{a}(t) = \frac{d\underline{v}(t)}{dt} &= \frac{d}{dt}\left(\frac{dx(t)}{dt}\right)i + \frac{d}{dt}\left(\frac{dy(t)}{dt}\right)j + \frac{d}{dt}\left(\frac{dz(t)}{dt}\right)k \\
&= \frac{d^2x(t)}{dt^2}i + \frac{d^2y(t)}{dt^2}j + \frac{d^2z(t)}{dt^2}k
\end{aligned}$$

The speed of the body is the magnitude of the velocity. Thus, speed,

$$s = |\underline{v}| = (x^2 + y^2 + z^2)^{1/2}$$

7.4 Direction of a moving body

Let $\underline{r}(t) = x(t)i + y(t)j + z(t)k$ be the vector position of a body in space. Then, the direction of the body in space is obtained by

$$\underline{d}(t) = \frac{\underline{v}(t)}{|\underline{v}(t)|} = \frac{x(t)}{\sqrt{x^2 + y^2 + z^2}}i + \frac{y(t)}{\sqrt{x^2 + y^2 + z^2}}j + \frac{z(t)}{\sqrt{x^2 + y^2 + z^2}}k$$

7.4.1 Example

A body is located at $\underline{r}(t) = \cos ti + \sin tj + t^2k$ at time t.

(i) Find the direction of the body at time $t = 2$

(ii) At what time, if any, are the body's velocity and acceleration orthogonal?

Solution

(i) $\underline{v}(t) = -\sin t i + \cos t j + 3t^2 k$.
Direction, $d(t) = -\frac{0.03}{\sqrt{10}} i + \frac{0.999}{\sqrt{10}} j + \frac{12}{\sqrt{10}} k$

(ii) $\underline{a}(t) = -\cos^2 t i + \sin^2 t j + 6tk$.
But for orthogonality,

$$\begin{aligned}\underline{v}(t) \cdot \underline{a}(t) &= \sin t \cos^2 t + \cos t \sin^2 t + 18t^3 = 0 \\ &= 18t^3 = 0\end{aligned}$$

$\therefore$ the only value is $t = 0$.

7.5 Relative Velocity and Acceleration

Suppose a boy and a girl position vectors from the origin are $\underline{r}_b$ and $\underline{r}_g$ respectively. Then, the relative positive vector denoted by $\underline{r}_{bg}$ of the girl relative to the boy at time t is

$$\underline{r}_{bg}(t) = \underline{r}_g(t) - \underline{r}_b(t)$$

Thus, the relative velocity, $\underline{v}_{bg}(t)$ of the girl relative to the boy at time t,

$$\underline{r}_{bg}(t) = \frac{d\underline{r}_{bg}(t)}{dt} = \frac{d(\underline{r}_g(t) - \underline{r}_b(t))}{dt}$$

and the relative acceleration, $\underline{a}_{bg}(t)$ of the girl relative to the boy at time t is

$$\underline{a}_{bg}(t) = \frac{d\underline{v}_{bg}(t)}{dt} = \frac{d^2\underline{r}_{bg}(t)}{dt^2}$$

7.5.1 Example

Two bodies A and B have position vectors:

$$\begin{aligned}h_1(t) &= 3x^2 i + y^3 j - 2z^2 k \\ h_2(t) &= x^2 i + 2y^3 j + 3z^2 k\end{aligned}$$

respectively relative to the origin.
Find:

(a) The position vector of the relative vector of A to vector B at time t.

(b) The relative velocity of A to B at time t.

(c) The relative acceleration of A to B at time t.

Solution

(a) The position vector,

$$\begin{aligned} \underline{r}(t) &= \underline{r}_A(t) - \underline{r}_B(t) \\ &= 2x_i^2 - y_j^3 - 5z^2k \end{aligned}$$

(b) Relative velocity

$$\underline{v}(t) = \frac{d\underline{r}(t)}{dt} = 4xi - 3y^2j - = 10zk$$

(c) Relative acceleration,

$$\underline{a}(t) = \frac{d\underline{v}(t)}{dt} = 4i - 6yj - 10k$$

7.6 Definitions

(i) **Mass** is the measure of a body inertia. This is the measure of a quantity of matter a body is made of.

(ii) Force is that which changes or tends to change the state of motion of a body.

(iii) The Newton's first law of motion states that, "Every body continues in its state of rest or uniform motion in a straight line except it is made to change that state by external forces.
Suppose m is the mass of a body in kg, a force F Newtons is applied to the body, and the rate of change of the body velocity is 'a'
Thus,

$$F = ma$$

(iv) Weight, W of a body is a force acting on a body of mass m at a point in space

$$W = mg$$

where g is the acceleration due to gravity.

(v) Momentum of a body is the product of its mass and velocity.
Hence,

$$\underline{P} = m\underline{v}$$

where $\underline{v}$ is the vector velocity of the body, m is the body mass.

(vi) Rate of change of momentum

$$= \frac{d\underline{P}}{dt} = m\frac{d\underline{v}}{dt} = m\underline{a} = F$$

$\therefore$ rate of change of momentum of a body is equivalent to the force acting on the body.

(vii) A force is said to do work when its point of application moves through a displacement $\underline{r}$.
Hence,

$$\text{Work done} = \underline{F} \cdot \underline{r} \ \text{ Joules}$$

(viii) Power is the rate at which work is done. It is measured in Joules per seconds.

(ix) Energy is the capacity for doing work. It is measured in Joules.

(x) Kinetic Energy is the energy a body has by virtue of its motion.
A body of mass m traveling with a velocity $\underline{v}$ has kinetic energy,

$$K.E. = \frac{1}{2}m\underline{v}^2$$

(xi) Potential Energy is the energy a body has by virtue of its position.
A body of mass m at a height h from the surface of the earth has potential energy,

$$P \cdot E = mgh$$

(xi) **Principle of conservation of Energy** states that energy can neither be created nor destroyed but can be transferred from one form to another. Thus,

$$K.E. + P.E = \text{constant}$$

7.7 Example

A body of mass m moving with velocity $a_1xi+a_2yj+a_3zk$ m/s is subjected to a force. Find:

(i) Force on the body

(ii) Momentum of the body

(iii) The body kinetic energy

(iv) The resistance force of an obstacle that brought the body to rest, after moving a distance of km.

Solution

(i) Force on the body,

$$\begin{aligned} \underline{f} = m\underline{a} &= m\frac{d\underline{v}}{dt} \\ &= m(a_1i + a_2j + a_3k)N \end{aligned}$$

(ii) Momentum

$$\begin{aligned} \underline{P} &= m\underline{v} \\ &= m(a_1xi + a_2yj + a_3zk) \end{aligned}$$

(iii) Kinetic energy of the body is

$$\begin{aligned} K.E. &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m(a_1x_i + a_2y_j + a_3z_k)^2 \text{ joules} \\ &= \frac{1}{2}m(a_1^2x^2 + a_2^2y^2 + a_3^2z^2) \end{aligned}$$

(iv) Note that work changes the kinetic energy from its initial value to zero. Therefore, work done against the obstacle is the kinetic energy of the body.

$$\underline{F} \times K = \frac{1}{2}m(a_1^2x^2 + a_2^2y^2 + a_3^2z^2)N$$

where $\underline{F}$ is the required resistance force, and

$$\underline{F} = \frac{1}{2}m(a_1^2x^2 + a_2^2y^2 + a_3^2z^2) - K$$

7.8 Summary

In this lecture, we learnt that given a position vector, $\underline{r}(t) = x(t)i + y(t)j + z(t)k$, its:

(i) Velocity, $\underline{v}(t) = \frac{d\underline{r}(t)}{dt} = \frac{dx(t)}{dt}i + \frac{dy(t)}{dt}j + \frac{dz(t)}{dt}k$

(ii) Acceleration, $\underline{a}(t) = \frac{d\underline{v}(t)}{dt} = \frac{d^2\underline{r}(t)}{dt^2}$

$$= \frac{d^2x(t)}{dt^2}i + \frac{d^2y(t)}{dt^2}j + \frac{d^2z(t)}{dt^2}k$$

(iii) Speed, $s = |v| = (x^2 + y^2 + z^2)^{1/2}$

(iv) Direction, $\underline{d}(t) = \frac{\underline{v}(t)}{|\underline{v}(t)|}$

$$= \frac{x(t)i + y(t)j + z(t)k}{\sqrt{x^2 + y^2 + z^2}}$$

(v) Relative velocity,

$$\underline{v}_{1,2}(t) = \frac{d(\underline{r}_1(t) - \underline{r}_2(t))}{dt}$$

(vi) Relative acceleration, $\underline{a}_{1,2}(t) = \frac{dv_{1,2}(t)}{dt}$

(vii) Force $= m\underline{a}$

(viii) Weight $W = mg$

(ix) Momentum, $\underline{P} = m\underline{v}$

(x) Rate of change of momentum,

$$\frac{d\underline{p}}{dt} = m\underline{a} = F$$

(xi) Work done, $W.D. = \underline{F} \cdot \underline{r}$ Joules

(xii) Kinetic Energy, $K.E. = \frac{1}{2}m\underline{v}^2$

(xiii) Potential Energy, $P.E. = mgh$

7.9 Post-Test 7

(1) The position vector of an object in space is

$$\underline{r}(t) = \frac{x^2}{2}(t)i + \frac{y^3(t)}{3}j - 2\sqrt{z}km.$$

Find its

(a) Velocity at $t = 0.5$sec.

(b) Acceleration at $t = 1.5$sec.

(c) Speed at $t = 2$sec.

(d) Direction at $(2, 1, 0)$ when $t = 1$sec.

(2) The position vector of a bird and a plane is $x^2i + y^3j + 2z^2k$ and $2x^2i + y^3j - z^2k$ respectively. Find the relative velocity and acceleration of the bird to the plane.

(3) The position vector of a moving train of mass $10kg$ is $4t^2i + \cos tj + \sin tk$. Find:

(a) Force acting on the moving train

(b) Weight of the train ($g = 9.8N$)

(c) Momentum of the train

(d) Rate of change of momentum at $t = 5$sec.

(e) Kinetic energy of the train

7.10 Solution to Pre-Test 7

(a) (1) $y'(x) = 6x(x + 2)$

(2) $y'(x) = x^3 - \dfrac{1}{2\sqrt{x}}$

(b) (1) $\left.\dfrac{x^3}{12} + \dfrac{3}{2}x^4\right|_0^2 = 24\dfrac{2}{3}$

(2) $\left. x^2 + 4x\right|_{-1}^2 = 15$

Contemporary Reading

1. O.O. Ugbebor and N.I. Akinwande: Analytical Geometry and Mechanics.

2. C.J. Tranter and C.G. Lambe: Advanced Level Mathematics, 4th Edition.

Lecture 8
Dynamics II

8.0 Introduction

This lecture is divided into two sections. Section A describes the behaviour of object in circular motion in terms of angular velocity, angular acceleration, speed, tangential and normal acceleration.

When an object is thrown, its travel in space to reach a maximum height. Then return to a point on the earth at a distance from its initial point. This is refer to as the projectile.

Section B describes the motion of a projectile.

8.1 Objectives

After this lecture, you should be able to calculate:

a(i) the angular velocity of a moving object is space.

(ii) angular acceleration of a moving object in space.

(iii) speed in a circular motion

(iv) the tangential acceleration of object in circular motion.

(v) the normal acceleration of object in circular motion.

b(i) Describe and calculate the position of a projectile 't' seconds after firing.

(ii) calculate height of a projectile in space.

(iii) calculate the orbit of a projectile in space.

(iv) calculate the flight time of a projective.

8.2 Pre-Test 8

Find the length of the arc s_1s_2.

Section A:

8.3 Circular Motion

Suppose an object H moves in the plane containing OX such that $\langle HOX = \psi$ radian at time t. Then, the angular velocity of the object is the time rate of change of ψ. Thus,

$$w = \frac{d\psi}{dt} = \dot{\psi} \tag{8.3}$$

Figure 8.3

8.4 Angular acceleration

The angular acceleration is time rate of change of angular velocity. Thus, Angular acceleration

$$\beta = \frac{dw}{dt} = \dot{w} = \frac{d^2w}{dt^2} = \dot{\psi}$$

Note that

$$\beta = \frac{d^2\psi}{dt^2} = \frac{d}{dt}\left(\frac{1}{2}\dot{\psi}^2\right)$$

8.5 Speed in a circular motion

Suppose an object moves in a circular motion described by the circle center 0 of Figure 8.3

Figure 8.5

Let's assume at time t, the distance P_1P_2 is $arcP_1P_2 = l$ and $\langle P_1OP_2 = \psi$. Then,

$$l = r\psi \tag{8.5}$$

Differentiating with respect to (w.r.t.) t given the speed of the object as:

$$v = \frac{dl}{dt} = r\frac{d\psi}{dt} = r\dot{\psi}$$

$$\therefore \quad \text{Speed,} \quad v = r\dot{\psi} = rw$$

8.6 Tangential acceleration in circular motion

The tangential acceleration of an object in circular motion at time t is the time rate of change of speed v.
Thus,

$$\text{Tangential acceleration,} \quad \frac{dv}{dt} = r\ddot{\psi} = r\dot{w} = r\beta.$$

8.7 Normal acceleration in circular motion

Normal acceleration in circular motion is the centripetal acceleration towards the centre of the circle.
Thus,

Normal acceleration,

$$vw = \frac{v^2}{r} = rw^2 = r\dot{\psi}^2 \tag{8.7}$$

where

- ψ is the angle subtended by the position object from time t to $t+n,\ n > 0$
- w is the angular velocity.

8.8 Effective normal force

Effective force = mass × acceleration.
Thus, suppose an object of mass 'm' with acceleration 'a' maintain a centripetal movement about an origin. Then, the effective normal force without friction is,

$$F = ma = m\frac{v^2}{r} \tag{8.8}$$

where $a = \dfrac{v^2}{r}$ from (8.7).

8.9 Example

An object attached to the end of a string moves in angular path at a constant speed on a frictionless horizontal table.
The other end of the string is fixed at the centre of the circle. If the mass of the object is $2.5kg$, the radius of the circle is $7.5m$, and the speed of the object is $8m/s$. Find the tension in the string.

Solution

Given that:

$$m = 2.5kg$$
$$r = 7.5m$$
$$v = 8m/s$$

Using equation (8.8), we have

$$\text{Tension, } T = \frac{mv^2}{r} = 21N$$

Section B: The Projectile

8.10 Projectile

Figure 8.10

Suppose a particle H is projected from the origin at an angle ψ to the horizontal range OX with an initial velocity u. Assume YOX is the plane of the motion and OHP is the orbit of the particle. Then, at time $t = 0$, $x = 0$ and $y = 0$ with x and y horizontal and vertical distance covered respectively at time t. Furthermore, at time $t > 0$

$$\left.\begin{aligned} \frac{dx}{dt} &= \dot{x} = U\cos\psi \\ \frac{dy}{dt} &= U\sin\psi = \dot{y} \end{aligned}\right\} \tag{8.10.1}$$

Neglecting air resistance, the force acting on the projectile is its weight, mg. Resolving this vertically and horizontally gives:

$$\left.\begin{aligned} F &= m\ddot{x} = 0 \quad \text{(horizontal motion)} \\ F &= m\ddot{y} = -mg \quad \text{(vertical motion)} \end{aligned}\right\} \tag{8.10.2}$$

Integrating (8.8.2), we have

$$\left.\begin{aligned} \dot{x} &= c_1 \\ \dot{y} &= -gt + c_2 \end{aligned} \qquad \text{(verify)}\right\} \tag{8.10.3}$$

Comparing (8.8.1) and (8.8.3) at time $t = 0$ yields

$$\left.\begin{aligned} c_1 &= u\cos\psi \\ c_2 &= u\sin\psi \end{aligned}\right\} \tag{8.10.4}$$

and at time t, we have

$$\left.\begin{aligned} \dot{x} &= u\cos\psi \\ \dot{y} &= u\sin\psi - gt \end{aligned}\right\} \tag{8.10.5}$$

Using (8.8.3).
Integrating (8.8.5) yields

$$\left.\begin{aligned} x &= ut\cos\psi \\ y &= ut\sin\psi - \tfrac{1}{2}gt^2 \end{aligned}\right\} \tag{8.10.6}$$

Eliminating t from equation in (8.10.6) yields

$$y = x\tan\psi - x^2\left(\frac{g}{2u^2\cos^2\psi}\right) \tag{8.10.7}$$

Remarks

- Equation (8.10.6) defines the position of a projectile at time t seconds after firing.
- Equation (8.10.7) defines the orbit of the projectile at time t.

8.11 Maximum height of a projectile

When the projectile reached the maximum height say, $\bar{H}$, $\frac{dy}{dt} = 0$ and using equation (8.9.5), we have

$$t = \frac{u\sin\psi}{g} \tag{8.11.1}$$

which is the time required to attain the height. Then, substituting for t as given by (8.11.1) in (8.10.6) yields

$$y_{max} = \frac{u^2\sin^2\psi}{2g} \qquad \text{(Verify)} \tag{8.11.2}$$

8.12 Maximum horizontal distance travelled by the projectile

The projectile reached the maximum horizontal distance when $y(t) = 0$ for $t \neq 0$.
Then, with equation (8.10.6),

$$y = t\left(u\sin\psi - \frac{1}{2}gt\right) = 0$$

and

$$t = \frac{2u\sin\psi}{g}$$

Hence, the horizontal range is

$$\begin{aligned} R &= ut\cos\psi \\ &= \frac{u^2\sin 2\psi}{g} \qquad \text{(Verify)} \end{aligned} \tag{8.12.1}$$

8.13 Example

(1) A projectile is fired at an angle of elevation 30^0 with an initial speed $5m/s$.

  (a) When does the projectile reach its highest point?

  (b) How high does the projectile rise?

(2) A projectile is fired over horizontal ground at an initial speed of $60m/s$ at an angle of elevation of 45^0. Where will the projectile be in 5 seconds later? (Take $g = 10ms^{-2}$)

Solution

(1) $\psi = 30^0$
$u = 5m/s$

  (a) $t = \dfrac{u \sin 30^0}{g} = 0.25s$
  $\therefore$ it reached the maximum height in 0.25 seconds

  (b) $y_{max} = \dfrac{u^2 \sin^2 30^0}{2g}$
  $0.125m$
  $\therefore$ maximum height $= 0.125m$

(2) Position at $t = 5sec$ is (x, y) where

$$\begin{aligned} x &= ut\cos\psi \\ y &= ut\sin\psi - \frac{1}{2}gt^2 \\ \therefore \quad x &= 25\sqrt{3} \\ \therefore \quad y &= 25m \end{aligned}$$

8.14 Summary

In this lecture, we have equation for:

(i) Angular velocity, $w = \dfrac{d\psi}{dt}$

(ii) Angular acceleration,

$$\beta = \frac{dw}{dt} = \frac{d^2\psi}{dt^2} = \frac{d}{dt}\left(\frac{1}{2}\dot{\psi}^2\right)$$

(iii) Speed in a circular motion,

$$v = \frac{dl}{dt} = r\frac{d\psi}{dt}$$

(iv) Tangential acceleration in circular motion,

$$\frac{dv}{dt} = r\frac{d^2\psi}{dt^2}$$

(v) Normal acceleration in circular motion,

$$vw = \frac{v^2}{r} = rw^2 = r\left(\frac{d\psi}{dt}\right)^2$$

(vi) Effective normal force in circular motion,

$$F = \frac{mv^2}{r}$$

(vii) Maximum height of a projectile,

$$y_{max} = \frac{u^2 \sin^2 \psi}{2g}$$

(viii) Required time to get to the height,

$$t_y = \frac{u \sin \psi}{g}$$

(ix) Projectile horizontal range,

$$R = \frac{u^2 \sin 2\psi}{g}$$

(x) Required time to attain the range,

$$t_x = \frac{2u \sin \psi}{g}$$

8.15 Post-Test 8

(1) Find the equation for the motion of a projectile fired into the first quadrant from an arbitrary point (x_0, y_0).

(2) A ball is thrown from a stand $32m$ above the ground at an angle of 30^0. When and how far away will the ball strike the ground if its speed is

$32ms^{-1}$?

(3) A particle attached to the end of a string moves in angular path at a constant speed on a fixed frictionless horizontal table. The other end of the string is fixed at the centre of the circle. If the mass of the object is $1.25kg$, the radius of the circle is $0.75m$ and the speed of the object is $12m/s$. Find the tension in the string.

8.16 Solution to Pre-Test 8

Arc length,

$$\begin{aligned} s_1 s_2 &= 2r(360^0 - 260^0)\pi/360^0 \\ s_1 s_2 &= 20\pi m \end{aligned}$$

Contemporary Reading

1. O.O. Ugbebor and N.I. Akinwande: Analytical Geometry and Mechanics.

2. C.J. Tranter and C.G. Lambe: Advanced Level Mathematics, 4th Edition.

Lecture 9
Dynamics III

9.0 Introduction

Vibration or oscillatory motion occur in many practical situations. In this lecture, we shall study the motion of Simple Harmonic Motion (SHM), Elastic String, Simple Pendulum, and Conical Pendulum.

9.1 Objectives

After this lecture, you should be able to:

(i) Describe the motion of object in a straight line towards or from a fixed point.

(ii) Describe the otion induced by Elastic string.

(iii) Deduce mathematical expression to describe the motion of a simple pendulum.

(iv) Describe the motion of a conical pendulum.

9.2 Pre-Test 9

Show that $y(t) = a \sin \psi t + b \cos \psi t$ is a solution of $y''(t) + \psi^2 y = 0$.

9.3 Simple Harmonic Motion (SHM)

Suppose a particle H moves in a straight line such that:

(i) its acceleration is always towards a fixed point on the line.

(ii) Acceleration is proportional to the distance of the particle from the point.

Then, the motion of the particle is described as Simple Harmonic

Figure 9.3

Let $OP = x$ then, the acceleration is

$$a = \frac{d^2x}{dt^2} = \ddot{x} = -\psi^2 x \tag{9.3.1}$$

where
x is the distance of the object from point O at time t
ψ is a positive constant.
From equation (9.2.1), we have

$$\frac{d^2x}{dt^2} + \psi^2 x = 0 \tag{9.3.2}$$

whose solution is

$$x(t) = a \sin \psi t + b \cos \psi t \tag{9.3.3}$$

where a and b are arbitrary constants.

9.4 Example

Suppose a particle moves with SHM from rest at a distance $k$cm from a fixed point 0. Find the position of the body at any time $t > 0$.

Solution

At time $t = 0$, $x = k$ and $\frac{dx}{dt} = 0$.
But at time t

$$x(t) = a \sin \psi t + b \cos \psi t$$

and

$$\frac{dx(t)}{dt} = a\cos\psi t - b\psi\sin\psi t$$

Thus, at $t = 0$

$$\begin{aligned} x(0) &= b = k \\ \frac{dx(t)}{dt} &= a\psi = 0 \quad \Rightarrow a = 0 \end{aligned}$$

Hence, $x(t) = k\cos\psi t$ is the position of the particle at any time t.

9.5 Elastic String

Suppose a light vertical perfectly elastic string of natural length $OB = b$ and modulus of elasticity N is attached to a fixed point O. A particle P of mass m is attached to the other free end. The forces acting on the particle are its weight, mg and the tension T on the string acting in upward direction. But

$$T = \frac{Nx}{b}$$

and the equation of motion is

$$F = ma = m\frac{d^2x}{dt^2} = mg - \frac{Nx}{b}$$

Thus,

$$\frac{d^2x}{dt^2} = -\frac{N}{mb}\left(x - \frac{mgb}{N}\right) \tag{9.5.1}$$

Let

$$\begin{aligned} y &= x - \frac{mgb}{N} \\ n^2 &= \frac{N}{mb} \end{aligned}$$

and equation (9.5.1) becomes

$$\frac{d^2y}{dt^2} = -n^2y$$

Hence, the motion is SHM with period

$$= \frac{2\pi}{\sqrt{\frac{mb}{N}}}$$

Figure 9.5

9.6 Example

A light elastic string of natural length $2a$ is fastened at one end to a fixed point. It hangs vertically and carries at its other end a particle of mass m. At equilibrium position, its length is $9a/4$.

(a) Find the period of small vertical oscillations of the particle.

(b) If the greatest acceleration during the oscillation is $\frac{1}{2}g$, find the amplitude.

Solution

(a) The extension at equilibrium is $\frac{1}{4}\alpha$.
Thus,

$$\frac{N\frac{1}{4}a}{2a} = mg$$

$$\Rightarrow \quad N = 8mg \quad \text{and} \quad n^2 = \frac{8mg}{m2a} \Rightarrow \quad n = 2\sqrt{\frac{g}{a}}.$$

$$\text{Period} = \frac{2\pi}{n} = \pi\sqrt{\frac{g}{a}}.$$

(b) Let $\bar{m}$ denote the amplitude. Then, the greatest acceleration occurs when $x = \bar{m}$ and $\frac{1}{2}g = \frac{8g\bar{m}}{2a}$

$$\therefore \quad \bar{m} = \frac{1}{8}a.$$

9.7 The Simple Pendulum

A simple pendulum consists of a light inelastic string of length l fixed at one end 0, with an object P of mass m attached to the free end. This is allowed to move under gravity in a vertical circle.

Figure 9.7

The tangential component of the forces is:

$$ml\frac{d^2\psi}{dt^2} + mg\sin\psi = 0 \tag{9.7.1}$$

and the normal component is

$$ml\frac{d\psi}{dt} = mg\cos\psi \tag{9.7.2}$$

From (9.6.1),

$$\frac{d^2\psi}{dt^2} = -\left(\frac{g}{l}\right)\sin\psi$$

If ψ is small enough, then,

$$\frac{d^2\psi}{dt^2} = -\left(\frac{g}{l}\right)\psi \tag{9.7.3}$$

Equation (9.7.3) describes a SHM with period,

$$T = 2\pi\sqrt{l/g}$$

Note:

(1) If the period $T = 2$ seconds, the pendulum is said to beat seconds.

(2) The frequency of SHM is the inverse of its period, $(f = \frac{1}{T})$. This is the number of complete oscillations in one seconds.

(3) For the simple pendulum,

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$$

9.8 Example

A pendulum clock gains 60 seconds per day. Find the adjustment necessary in the length of the pendulum which should beat seconds.

Solution

$$\frac{1}{f} = T = 2\pi\sqrt{\frac{l}{g}}$$

then

$$
\begin{aligned}
-logf &= \log T = \log\left(2\pi\sqrt{\frac{l}{g}}\right) \\
&= \log 2\pi + \frac{1}{2}\log l - \frac{1}{2}\log g.
\end{aligned}
$$

Differentiating,

$$
-\frac{df}{f} = \frac{dT}{T} = \frac{1}{2}\frac{dl}{l} - \frac{1}{2}\frac{dg}{g} \quad \text{(Why?)}
$$

If δ represents a small increment, then,

$$
\begin{aligned}
-\frac{\delta f}{f} &= \frac{\delta T}{T} = \frac{1}{2}\frac{\delta l}{l} - \frac{1}{2}\frac{\delta g}{g} \\
\Rightarrow \frac{\delta f}{f} &= -\frac{60}{24 \times 60 \times 60}
\end{aligned}
$$

$\delta_g = 0$, thus

$$
\delta l = -\frac{2l\delta f}{f} = \frac{2 \times 60l}{24 \times 60 \times 60} = \frac{l}{720}
$$

When $T = 2$.
From

$$
\begin{aligned}
T &= 2\pi\sqrt{\frac{l}{g}} \\
l &= g/\pi^2
\end{aligned}
$$

and $\delta l = *\frac{g}{720\pi^2}$

9.9 Conical Pendulum

An arrangement by which a particle attached by a string to a fixed point describe a horizontal circle is called conical pendulum

.

Figure 9.9

Let:
w denotes the angular velocity
l denotes the length of the string
F denotes the tensions on the string
ψ denotes the angle of inclination to the vertical.
$OR = h$
then, the acceleration of the particle is given by

$$a = w^2 l \sin\psi$$

and it is directed towards the centre 0.
The forces acting on the particles

$$ma = \bar{F}$$

$$mw^2(\sin\psi = F\sin\psi)$$

$$mg = F\cos\psi$$

Thus,

$$\begin{aligned} \frac{w^2 l}{g} &= \frac{1}{\cos\psi} \\ \Rightarrow \quad w^2 &= \frac{g}{l\cos\psi} = \frac{g}{h} \end{aligned}$$

$\therefore$ The time for one revolution (period) is

$$T = \frac{2\pi}{w} = 2\pi\sqrt{\frac{h}{g}}$$

9.10 Example

A conical pendulum consists of a particle of mass m attached to one end H of a light elastic string of natural length l. The modulus of elasticity is mg. The other end of the string is fired to a fixed point M. The particle describes a horizontal circle with constant angular velocity w.

(a) If the string makes an angle ψ with the downward vertical, show that $2lw^2 = g\sec^s \frac{1}{2}\psi$.

(b) If $\psi = 60^0$, find the time of revolution of the particle.

Solution

Figure 9.10

(a) Let $HM = x$, then

$$T = mg\left(\frac{x-l}{l}\right) \tag{9.10.1}$$

Radius of the circle,

$$r = x\sin\psi \tag{9.10.2}$$

The velocity,

$$v = rw \tag{9.10.3}$$

The effective normal force,

$$= mrw^2 \tag{9.10.4}$$

Balancing the forces and the reversed effective normal force, we have

$$T\cos\psi = mg \tag{9.10.5}$$

$$T\sin\psi = mrw^2 \tag{9.10.6}$$

From (9.10.1) and (9.10.5), we have

$$\cos\psi = \frac{l}{x-l} \tag{9.10.7}$$

and

$$x = l(1+\sec\psi) \tag{9.10.8}$$

But $rw^2 = g\tan\psi$ so that

$$\begin{aligned} w^2 &= \frac{s\tan\psi}{l(1+\sec\psi)\sin\psi} \\ &= \frac{g}{l(1+\cos\psi)} = \frac{g}{2l}\sec^2\frac{1}{2}\psi \end{aligned}$$

Using trig. identities

$$\therefore \quad 2lw^2 = g\sec^2\frac{1}{2}\psi$$

(b) When $\psi = 60^0$,
$\sec^2 \frac{1}{2}\psi = \frac{4}{3}$
$w^2 = \frac{4}{6}\frac{g}{l}$
Thus, the time of a revolution is

$$\frac{2\pi}{w} = \pi\sqrt{\frac{6l}{g}}$$

9.11 Summary

In this lecture, we have:
(i) $x(t) = a \sin \psi t + b \cos \psi t$ to describe the position of an object influenced by SHM
(ii) Period $= 2\pi\sqrt{\frac{mb}{N}}$ which is the period of motion under SHM.
(iii) $\ddot{\psi} = -\left(\frac{g}{l}\right)\psi$ to describe SHM of a simple pendulum with period, $T = 2\pi\sqrt{\frac{l}{g}}$.
(iv) $T = 2\pi\sqrt{\frac{h}{g}}$ describes time required for one revolution.

9.12 Post-Test 9

(1) A particle of mass $4kg$ is whirled round at the end of a string $50cm$ long so as to describe a horizontal circle, making 60 revolutions per minute. Calculate the tension in the string.

(2) A particle describing SHM is $8m$ from the central position when its speed is $12ms^{-1}$, and $6m$ from the centre when its speed is $8ms^{-1}$. Find the amplitude and period of the motion.

(3) A particle moving with SHM has a speed of $1ms^{-1}$ when passing through the centre 0 of its path with a period π seconds. Find its speed and acceleration when it is $0.75m$ from 0.

(4) A second pendulum is found to lose 20 beats per day when taken to the top of a mountain.
Calculate the change in the value of the acceleration due to gravity, g.

9.13 Solution to Pre-Test 9

$$y' = a\psi \cos \psi t - b\psi \sin \psi t$$

$$y'' = -a\psi^2 \sin \psi t - b\psi^2 \cos \psi t$$

$$y'' + \psi^2 y = -a\psi^2 \sin \psi t - b\psi^2 \cos \psi t + \psi^2(a \sin \psi t + b \cos \psi t) = 0$$

Hence,

$$y(t) = a \sin \psi t + b \cos \psi t$$

is a solution of

$$y'' + \psi^2 y = 0.$$

Contemporary Reading

1. O.O. Ugbebor and N.I. Akinwande: Analytical Geometry and Mechanics.

2. C.J. Tranter and C.G. Lambe: Advanced Level Mathematics, 4th Edition.

Lecture 10
Dynamic IV

10.0 Introduction

Force is that which changes or tends to change the state of rest or motion of a body in a certain direction.

A force acting on a body can be described completely by its:

(i) Magnitude

(ii) Line of action

(iii) Direction

Newton's First law states that everybody continues in its state of rest or uniform motion in a straight line unless acted upon by an external force.

Newton's Second law states that rate of change of momentum of a body is proportional to the external force applied and is in the direction of the force. Thus,

$$\frac{dp}{dt} = \frac{d(m\underline{v})}{dt} = m\frac{d\underline{v}}{dt} = m\underline{a} = \underline{F}$$

Newton's Third law states that to every action, there is an equal and opposite reaction.

10.1 Objectives

After this lecture, you should be able to write equation to describe:

(i) Motion on smooth and rough inclined plane.

(ii) Motion of connected particles.

(iii) Momentum and Impulse

(iv) Conservation of momentum.

10.2 Pre-Test 10

Given that

Find: (i) h_1, (ii) h_2 (iii) h_1/h_2

10.3 Motion on Smooth Inclined Plane

Consider figure (10.3)

Figure 10.3

Then, the forces acting on an object of mass m that is placed on an inclined frictionless (smooth) plane are:

(i) Weight of the object denoted by $W = mg$.

(ii) Reaction of the plane on the particle, $R = mg\cos\psi$

(iii) Parallel force acting along the plane, $f = mg\sin\psi$.

Suppose the object is moving up the plane with acceleration $\underline{a}$. Then, the equation of motion are:

$m\underline{a} = mg\sin\psi -$ (motion upward along the plane) (10.3.1)

$R = mg\cos\psi$ (10.3.2)

10.4 Motion on Rough Inclined Plane

Consider an object of mass m on rough plane inclined at an angle ψ to the horizontal.

Figure 10.4

There is frictional force which oppose motion along the plane. Thus, if the object moves up along the plane at angle ψ to the horizontal, the equations of motion:

$$m\underline{a} = mg\sin\psi - \mathcal{N}mg\cos\psi \tag{10.4.1}$$

$$R - mg\cos\psi = 0 \tag{10.4.2}$$

Hence, acceleration of motion is obtained by:

$$\underline{a} = g(\sin\psi - \mathcal{N}\cos\psi) \tag{10.4.3}$$

Remarks

(1) If there is no motion (i.e., $\underline{a} = 0$), the frictional force is sufficient to keep the object stationary.
Hence,

$$\sin\psi = \mathcal{N}\cos\psi \tag{10.4.4}$$

and the co-efficient of frictional,

$$\mathcal{N} = \tan\psi \tag{10.4.5}$$

(2) For the object to slide down the plane,

$$\sin\psi > \mathcal{N}\cos\psi \tag{10.4.6}$$

and

$$\mathcal{N} < \tan\psi \tag{10.4.7}$$

10.5 Example

An engine exerts a force of $37N$ on a train of mass $24kg$ and draws it up a slope of one in 120 against track resistance of $0.07N/kg$.

(i) Find the acceleration of the train.

(ii) Find the breaking force required on the return journey with steam shut off to prevent the acceleration exceeding $2m/s^2$.

Solution

Slope resistance $= 24 \times \frac{1}{120} \times 10 = 2N$
Track resistance $= 0.07 \times 24 \times 10 = 16.8N$
Total resistance $= 18.8N$
Engine pull $= 37N$
Net accelerating force $= 37N - 18.8N$
$\therefore$ acceleration $= \frac{18.2N}{24kg} \approx 0.76ms^{-2}$
$\therefore$ the acceleration is $0.76ms^{-2}$
(ii) Let BN be the required breaking force to keep the acceleration to $0.002m/s^2$ on the return journey.
Then, we have

$$\begin{aligned} -B + 2 - 16.8 &= 24 \times 0.002 \\ &= -14.848N \end{aligned}$$

$\therefore$ the required breaking force is $-14.85N$.

10.6 Motion of connected Objects

Let's consider the motion of two objects of mass M and m respectively over a light inextensible string. Suppose their displacement, velocity and acceleration are equal. The equation of motion are:

$$T - mg = ma \quad (10.6.1)$$

$$Mg - T = Ma \quad (10.6.2)$$

Thus, the tension, 'T' and acceleration, 'a', can be calculated using equations (10.6.1) and (10.6.2).

Figure 10.6

10.7 Example

A light inelastic string passes over a smooth peg. At each end of the string is attached masses of $3g$ and $5g$ respectively as shown below.

.

Figure 10.7

Calculate the:

(i) Acceleration of the system

(ii) Tension in the string.

Solution

From the system, the motion equations are:

$$
\begin{aligned}
5g - T &= 5a \\
\Rightarrow \quad T &= 5g - 5a \qquad (10.7.1) \\
\text{and} \quad T &= 3a + 3g \qquad (10.7.2)
\end{aligned}
$$

where g is the force of gravity.
Thus,

(i) $a = 2.5ms^{-2}$ (Verify)

(ii) Using equation (10.7.1),

$$T = 37.5N \quad \text{(Verify)}$$

10.8 Momentum and Impulse

Suppose a constant force F acts on an object of mass m for time t. Also, the velocity increases from u to v along the line of action of force. Thus,

$$F = ma \tag{10.8.1}$$

$$V = u + at \tag{10.8.2}$$

and

$$a = \frac{v - u}{t} \tag{10.8.3}$$

$$\Rightarrow \quad Ft = \frac{mv - mu}{t} \equiv \text{change in momentum}$$

10.9 Remarks

(1) If the force F is of variable magnitude,

$$F = m\frac{dv}{dt}$$

then

$$\begin{aligned} \int_0^t F dt &= m \int_0^t \frac{dv}{dt} dt \\ &= mv - mu \equiv \text{change of momentum} \end{aligned}$$

(2) A quantity $\int F dt$ or Ft is called the INPULSE acting on the body if F is constant.

10.10 Example

A body of mass m moving with velocity $a_i + b_j \ m/s$ is subjected to a force $x_i - y_j N$ for t seconds. Find the velocity with which the body moves after the force ceased.

Solution

$$\begin{aligned} Ft &= mv - mu \\ (x_i - y_j)t &= mv - m(a_i + b_j) \\ v &= (tx + ma)i + (mb - ty)j \end{aligned}$$

10.11 Conservation of Momentum-Impact

When two bodies moving in a straight line collide, there is a brief period of contact. During this period each body exerts a certain force on the other which vary in magnitude. At any instant, the forces exerted by each body on the other are equal and opposite. Hence, the total momentum gained by the two bodies is zero.

Figure 10.11

Let:
Velocity before impact $= u_1, u_2$
Velocity after impact $= v_1, v_2$
For M_1; the impulse is

$$Ft = M_1V_1 - M_iU_1$$

For M_2, the impulse is

$$Ft = M_2V_2 - M_2V_2$$

Hence,

$$M_1V_2 - M_1U_1 + M_2V_2 - M_2U_2 = 0$$

and

$$M_1V_1 + M_2V_2 = M_1U_1 + M_2U_2$$

$\therefore$ the total momentum before and after impact are equal. This is the principle of conservation of linear momentum

10.12 Example

A railway truck of mass $800kg$ running at $6km/h$ is over taken by a truck of mass $1200kg$ running at $10km/h$. After impact, the trucks begin to separate at $1km/h$. Find the

(i) Speed of the trucks after impact

(ii) Loss of kinetic energy in the impact.

Solution

(i) Let's the speed of the heavier truck after impact be $v\ km/h$. Thus, speed of the lighter truck is $(v+1)\ km/h$.
Hence,

$$\text{Total momentum after impact} = \text{Total momentum before impact.}$$

$$\begin{aligned} 1.2v + 0.8(v+1) &= 1.2 + 4.8 \\ v &= 0.8km/h \quad \text{(Verify)} \\ \text{and} \qquad v+1 &= 0.9km/h \end{aligned}$$

(ii) Loss of kinetic energy is

$$\begin{aligned} &= \left[\left(\frac{1}{2} \times 1.2(10^2 - (0.8)^2)\right) + \left[\frac{1}{2} \times 0.8(6^2 - 0.9^2)\right]\left(\frac{10^3}{3600}\right)^2\right] \\ &\approx 5.69 \text{ Joules} \qquad \text{(Verify)} \end{aligned}$$

10.13 Summary

In this lecture, we have:

(a) Equation for motion on smooth inclined plane for:

(i) reaction of the plane on the body,

$R = mg\cos\psi$

(ii) parallel force acting along the plane,

$f = mg\sin\psi$

(iii) upward motion along the plane,

$F = ma = mg\sin\psi$

(b) Equation for motion on rough inclined plane,

$F = m\underline{a} = mg\sin\psi - \mathcal{N}mg\cos\psi$

and acceleration of the motion,

$\underline{a} = g(\sin\psi - \mathcal{N}\cos\psi)$

where $\mathcal{N} = \tan\psi$.

(c) Equation for motion of connected objects as,

$T - mg = ma$

$Mg - T = ma$

where

T is Tension on the inelastic string

M, m are masses of the objects.

a is the acceleration in the string.

(d) Momentum of moving body,

$Ft = mv - mu$

where

u is initial velocity,

v is final velocity in the direction of force at time t.

(e) Conversation of linear momentum as

Total momentum before impact = Total momentum after impact.

Thus, $M_1U_1 + M_2U_2 = M_1V_1 + M_2V_2$.

10.14 Post-Test 10

(1) If a particle moving with SHM is at a distance α from a fixed point O, and is moving towards O with velocity P. Write an equation to find the distance of the particle at any time t.

(2) A ball of mass $200g$ moving at $15m/s$ hits a wall perpendicular and

rebounds with speed $6m/s$. Find the impulse given to the ball by the wall and the force between them if the contact lasts 0.01 seconds.

(3) Two bodies A and B of masses $3kg$ and $4kg$ collide when moving with velocities $4i+3j$ and $-5i+j$ respectively. After their impact, they move in directions parallel to the vectors j and $4i+j$ respectively. Find the speed of the bodies after impact and vector representing the impulse of A.

(4) An engine exerts a force of $35,000N$ on a train of mass 240 tonnes and draws it up a slope 1 in 120 against a resistance of $60N$/tonne. Find the:

  (a) Acceleration of the train

  (b) Breaking force that will be required on the return journey which will keep the acceleration ($g = 9.81m/s^2$).

(5) Find the acceleration when a force F acts on the body of mass m as given:

  (a) $F = 20N$, $m = 2kg$

  (b) $F = 5N$, $m = 750g$

  (c) $F = 4KN$, $m = 2.2t$

(6) What force will produce a velocity $6.4m/s$ in 4 seconds to a body of mass $5kg$ which was initially at rest?

10.15 Solution to Pre-Test 10

(i) $h_1 = 5\sin 30^0 = 2.5$

(ii) $h_2 = 5\cos 30^0 = 5\sqrt{3}/2$

(iii) $h_1/h_2 = 1/\sqrt{3}$

Contemporary Reading

1. O.O. Ugbebor and N.I. Akinwande: Analytical Geometry and Mechanics.

2. C.J. Tranter and C.G. Lambe: Advanced Level Mathematics, 4th Edition.

Lecture 11
Statistics I

11.0 Introduction (Moment of a force)

A single force on a body causes it to accelerate. If the body is at rest while several forces are acting on it, the forces are in Equilibrium. A force may have many different origins. This includes gravity, electrical and magnetic effects or wind, friction. This lecture consider effect of forces on rigid body.

11.1 Objectives

After this lecture, you should be able to understand:

(i) The Principle of Moment

(ii) Like Parallel Coplanar Forces

(iii) Unlike Parallel Coplanar Forces

(iv) Couple

(v) Two Non-Parallel Coplanar Forces.

11.2 Pre-Test 11

What is the weight of a body of mass $1.5kg$ at a point in space?
($g = 10ms^{-2}$).

11.3 The Moment of a Force

Suppose a beam of uniform length XY of mass m is placed on a pivot at its center of gravity G. Let's assume that weights W_1 and W_2 are placed at points P and Q respectively.

Then, the moment of the force, W_1 about the axis at G is

$$W_1 \times PG - \text{Anti clockwise}$$

and the moment of the force, W_2 about the axis at G is

$$W_2 \times QG - \text{Clockwise}$$

For the beam to be in equilibrium,

$$\begin{aligned} & W_1 \times PG = W_2 \times QG \\ \therefore \quad & W_1 \times PG - W_2 \times QG = 0 \end{aligned} \tag{11.3.1}$$

11.4 Remark

If a body is in equilibrium under the action of forces, then the sum of the moments of the forces about any arbitrary point is zero.

11.5 Example

A thin uniform rod AB of length $2m$ and mass $15kg$ is supported horizontally at its center of gravity G, and masses $6kg$ and $10kg$ are attached to the rod at point P, Q respectively, where $AP = 0.5m$, $AQ = 1.2m$. A third mass of 5kg is attached to the rod at a point R so that the rod is in equilibrium.

(a) Draw a sketch indicating all the forces

(b) Find the length AR

Solution

(a)

The force at:

$$
\begin{aligned}
P &= 6 \times 10 = 60N \\
R &= 5 \times 10 = 50N \\
Q &= 10 \times 10 = 100N
\end{aligned}
$$

(b) The moment of the force;
$60N$ about the axis at G is 60×0.5
$50N$ about the axis at G is $50 \times \bar{x}$ where $RG = \bar{x}$
$100N$ about the axis at G is 100×0.7
Thus,

$$
\begin{aligned}
60 \times 0.5 + 50 \times \bar{x} &= 100 \times 0.7 \\
\therefore \quad \bar{x} &= 0.8m
\end{aligned}
$$

But $AR = AG - RG$.
$\therefore \quad AR = 0.2m$ (Verify).

11.6 Like Parallel Coplanar Forces

Two parallel forces in the same direction are called "Like parallel forces". Let's assume the forces F_1 and F_2 are like parallel forces

.

Figure 11.6

Then, their resultant is obtained by $R = F_1 + F_2$.
Taking moment about B gives

$$F_1 \cdot A_1B = F_2 \cdot A_2B$$

and

$$\frac{F_1}{F_2} = \frac{A_2B}{A_1B}$$

This is the point at B at which the line of action of R divides the distance between the two like parallel forces.

11.7 Unlike Parallel Coplanar Forces

Two parallel forces acting in opposite directions are called “Unlike parallel forces”.

Figure 11.7

Taking moments about B, we have

$$F_1 \cdot A_1B = F_1 \cdot A_2B$$

Assume $F_2 + F_2 > F_1$, then $R = F_2 - F$ and if $F_1 > F_2$, $R = F_1 - F_2$.

11.8 Couple

A couple is a pair of equal and opposite parallel forces acting on a body. An example is the force exerted at the end of a spanner to turn a nut together with the equal and opposite force exerted by the nut on the spanner. If the unlike forces are equal in magnitude, then, their resultant, $R \neq 0$. Thus, the body will not be in equilibrium, but rather will experience a rotation.

11.9 Remarks

(a) The vector sum of the forces constituting a couple is zero.

(b) If line ABC is perpendicular to the lines of actions of the couple $F_1 - F_1$, the sum of the moment about the line ABC is

Figure 11.9

$$H = F \cdot AC - F \cdot AB = F \cdot BC \qquad (11.9.1)$$

$$H = F \cdot AB - F \cdot AC = -F \cdot BC \qquad (11.9.2)$$

Equations (11.9.1) and (11.9.2) depends on the sense of rotation taken about A.

11.10 Two Non-Parallel Coplanar Forces

Suppose two coplanar forces acting on a body are not parallel. Then, their lines of action intersect at a point.

Figure 11.10

The parallelogram law gives the resultant R. This acts at the common point of intersection, thus, the two forces, F_1 and F_2 is represented by side OA and OC respectively of the parallelogram. The diagonal OB represent their resultant.

11.11 Example

A uniform beam $AB, 20m$ long and of mass $20kg$ rest in a horizontal position on supports at A and B. An anti-clockwise couple of movement $250gN$ is applied at A, and a clockwise couple of $750gN$ at B. Find the reactions at the supports.

Figure 11.11

Let's assume that R and S are the reactions at the supports.
Then,

$$R + S = 200N$$

The couples are applied at the ends of the beam. But their total anti-clockwise moment about any point is $500N$
Hence,
$20 \times S - 2000 - 500 = 0$
$\therefore \quad S = 125N$ and $R = 75N$.

11.12 Summary

In this lecture, we have that:

(i) The principle of moment as

$F_1 x_1 - F_2 x_2 = 0$

where, F_i's are the forces, and x_i's are the distances form the pivot point, $i = 1, 2, \ldots, n, \quad n \in \mathbb{Z}^+$

(ii) Like parallel forces as

Resultant = Sum of forces

$R = F_1 + F_2 + \cdots + F_n$

pivoting point as

$\frac{F_1}{F_2} = \frac{x_2}{x_1}$

(iii) Unlike Parallel forces as

$F_1 \cdot x_1 = F_2 \cdot x_2$

and Resultant

$R = F_2 - F_1$ when $F_2 > F_1$

$R = F_1 - F_2$ whenever $F_1 > F_2$

(iv) Sum of moments about a straight line say ABC, $H = F \cdot b$

or $H = -F \cdot b$ and $F \cdot b - F \cdot b = 0$

where F is the force.

(v) The resultant of two non-parallel coplanar forces is

$R = (F_1^2 + F_2^2 - 2F_1F_2 \cos\theta)^{\frac{1}{2}}$

where θ is the angle between F_1 and F_2

or

$R = \frac{F_1 \sin\theta}{\sin\psi}$

11.13 Post-Test 11

(1) A uniform beam AB of mass $20kg$ has weight $30N$, $25N$ attached to the end-points A and B respectively. A weight $40N$ is attached to a point half-way between the centre of gravity and the end-point B. If the beam is suspended on a pivot at its centre of gravity what weight will be required at a point of distance $1/8$ of the length from point A to ensure equilibrium?

(2)

Find:

(a) R

(b) the angle between R and $50N$.

(3) A non-uniform beam XY of length $10m$ and mass M kg, rests horizontally on supports X and Y. The forces exerted on the supports at X and Y are $30g$ N and Rg N respectively. If the centre of gravity of the beam is $6m$ from X:

(a) Draw a sketch indicating all the forces.

(b) Find the numerical value of M.

(c) Find the value of R

(4) Consider

Is the system in equilibrium?

(5) Forces of $7N$ and $5N$ act on a body and the angle between them is 55^0. Find:

  (a) Their resultant

  (b) The angles its subtended by the two forces.

11.14 Solution to Pre-Test 11

$$\begin{aligned} \text{Weight} &= 1.5 \times 10\ N \\ &= 15\ N \end{aligned}$$

Contemporary Reading

1. O.O. Ugbebor and N.I. Akinwande: Analytical Geometry and Mechanics.

2. C.J. Tranter and C.G. Lambe: Advanced Level Mathematics, 4th Edition.

Lecture 12
Statistics II

12.0 Introduction

Suppose three or more forces act at a point, and are in equilibrium. The forces can be represented in magnitude and direction by the three sides of a triangle or the sides of a polygon taken in order.

12.1 Objective

After this lecture, you should be able to derive equation expressing the behaviour of three or more forces acting at a point.

12.2 Pre-Test 12

Write the equivalent expression for:

(i) $\tan 222^0 = \tan?$

(ii) $\cos(-120^0) = -\cos?$

(iii) $\cos 300^0 = \sin?$

(iv) $\sec 42^0 = cosec$

(v) $\tan 37^0 = \cot?$

12.3 The Triangle of Forces

Consider three strings fasten to a small ring so that they are all in tension. Thus, the strings experience three forces denoted by the tensions in the strings acting on the small ring and in equilibrium.

Figure 12.3.1

Figure 12.3.2
(Space Diagram)

Figure 12.3.3
(Force Diagram)

The lengths of the sides of the triangle ABC are proportional to the magnitudes of the forces which they represent. Therefore,

$$\frac{F_1}{AB} = \frac{F_2}{BC} = \frac{F_3}{AC}$$

12.4 Theorem (Lami's Theorem)

If three coplanar forces, acting at a point are in equilibrium, and straight lines be drawn parallel to the directions of the forces. The lengths of the sides of the triangle so formed are proportional to the magnitudes of the forces which they represent.

12.5 Remarks

Note that the directions in which the forces act indicated by the arrow-heads on the triangle are either in clockwise or anti-clockwise direction.

If the direction of one of the forces is reversed, the force is regarded as the resultant.

Figure 12.5 (F_3 as resultant)

2.6 Lami's Theorem

If three forces acting at a point are in equilibrium, each is proportional to the sine of the angle included between the other two.
That is,

$$\frac{F_1}{\sin C} = \frac{F_2}{\sin B} = \frac{F_3}{\sin A}$$

(Using figure 12.5).

12.7 Example

A mass of $20kg$ is supported by two strings of length $x_1\ cm$, $x_2\ cm$ knotted at O and attached at A and B to a horizontal beam. The angle between the strings is 120^0. The angle between the length x_1 cm and the vertical string holding the weight is 135^0, and that between x_2 and the vertical string is 105^0

(a) Sketch the space diagram

(b) Sketch the force diagram

(c) Find the tensions on the strings x_1 and x_2.

Solution

Let T_1, T_2 denote the tensions in x_1 and x_2.

(a)

Space diagram

(b)

Force diagram

(c) Using Lami's Theorem,

$$\frac{T_1}{\sin 105^0} = \frac{T_2}{\sin 135^0} = \frac{20}{\sin 120^0}$$

Hence,

$$\begin{aligned} T_2 &= \frac{20 \sin 135^0}{\sin 120^0} \\ &= 16.33 kgf \qquad \text{(Verify)} \\ T_1 &= 22.32\ kgf \qquad \text{(Verify)} \end{aligned}$$

12.8 The Polygon of Forces

You saw in this lecture that when three forces acting at a point are in equilibrium, they can be represented by the sides of a triangle. This principle can be extended to any number of forces.

Let F_1, F_2, F_3, F_4, F_5 be forces whose magnitude and directions are known, and which act at a point say O. Then, the space and force diagram are sketch below

Figure 12.8.1
(Space diagram)

Figure 12.8.2
(Force diagram)

12.8 Theorem (Polygon of Force)

If a number of forces acting at a point are in equilibrium, they can be represented in magnitude and direction by the sides of a polygon taken in order.

12.9 Example

Find by means of a polygon of force the resultant of four forces of $4N, 3N, 3N$ and $4N$ acting a shown below.

Figure 12.9.1

Solution

The force diagram is

Figure 12.9.2

where R is the resultant force. Resolving the forces along the Y axes, we have

$$\begin{aligned} F_1 &= 0\ N \\ F_2 &= 2.12\ N \\ F_3 &= 2.12\ N \\ F_4 &= -4\ N \end{aligned} \qquad \text{(Verify)}$$

and along the X-axis, we have

$$\begin{aligned} F_1 &= 4\ N \\ F_2 &= 2.12\ N \\ F_3 &= -2.12\ N \\ F_4 &= -40\ N \end{aligned} \qquad \text{(Verify)}$$

$\therefore$ total net forces along the axes are:

$$\begin{aligned} Y\text{-axes}, \quad R_1 &= 0.24\ N \\ X\text{-axes}, \quad R_2 &= 4.01\ N \end{aligned}$$

Thus,

.

Figure 12.9.3

$$R = 4.01\ N \qquad \text{(Verify)}$$

The angle which the resultant force makes with the X-axes is

$$\tan\psi = 0.05985 \qquad \text{(Verify)}$$
$$\therefore\ \psi = 3.43^0$$

12.10 System of Forces

Given a system of coplanar forces which are not in equilibrium, their resultant can be obtained using either of the following principles.

(i) The components of the resultant in two perpendicular directions will equal the sum of the components of the forces in the same direction or

(ii) The moment of the resultant about any point will equal the sum of the moments of the forces about the same point.

12.12 Post-Test 12

(1) Forces of 2 N and 6 N act along two strings respectively. The angle between them is 120^0. A third force acts along the third string all knotted at a point, and makes angle 120^0 with both string. If the direction of the resultant is perpendicular to one of the two strings, what is the magnitude of the force along the third string?

(2) Find the resultant in magnitude and direction of forces 10 $N, 20\ N, 30\ N$ and 40 N acting respectively in the directions $060^0, 120^0, 180^0; 270^0$

(3) Five strings are attached to a point mass. The tensions and directions of four of the five rings are 50 N, 060^0, 40 N, 090^0; 100 N, 270^0; 20 N, 330^0

  (a) Sketch the space diagram

  (b) Sketch the force diagram

  (c) Find the tension and direction of the fifth string assuming that all the five strings are coplanar.

(4) Three forces acting at the origin can be represented by the vectors, OA, OB, OC with the coordinate, of A, B, C given respectively by $(5,2)^0; (-3,8)^0, (-2,-10)^0$. Show that the forces are in equilibrium.

(5) Two parallel forces 50 N and 30 N act in lines which are 40 units apart. Find their resultant and where it acts if the forces are:

  (a) Like forces

  (b) Unlike forces

12.11 Summary

In this lecture, we have:

(a) Equations, expressing three forces acting at a point in equilibrium by:

(i) $\dfrac{F_1}{AB} = \dfrac{F_2}{BC} = \dfrac{F_3}{AC}$

where F_1, F_2, F_3 are the forces and AB, BC, AC are the sides of the triangle.

(ii) $\dfrac{F_1}{\sin C} = \dfrac{F_2}{\sin B} = \dfrac{F_3}{\sin A}$

where A, B, C are the angles subtended by the triangle.

(iii) Resultant of two forces in the clockwise direction is $R = F_1 + F_2$.

(b) that forces more than three can be represented by sides of a polygon taken in order.

12.13 Solution to Pre-Test 12

(i) $\tan 40^0$

(ii) $-\cos 60^0$

(iii) $\sin 30^0$

(iv) $cosec 48^0$

(v) $\cot 53^0$

Contemporary Reading

1. O.O. Ugbebor and N.I. Akinwande: Analytical Geometry and Mechanics.

2. C.J. Tranter and C.G. Lambe: Advanced Level Mathematics, 4th Edition.

Lecture 13
Centre of Mass and Moment Inertia I

13.0 Introduction

When we balance a flat plate on the tip of a finger, the finger tip is at the plate's centre of mass. A point at which the entire mass concentrated is called centre of mass. It is important to locate a centre of mass.

13.1 Objective

After this lecture, you should be able to write equation to find the center of mass of a body.

13.2 Pre-Test 13

Solve $\int_0^4 \left(2x + \frac{x^2}{4}\right) dx$.

13.3 Definition (Center of Mass)

Consider at any time t the particles of a system of masses $m_1, m_2, \ldots$ at the points whose position vectors are $I_1, I_2, I_3, \ldots$ respectively with reference to an origin O. Then, the position of the centre of mass is

$$\begin{aligned} I &= \frac{m_1 r_1 + m_2 r_2 + m_2 r_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} \\ &= \frac{\sum (mr)}{\sum m} \qquad (13.3.1) \end{aligned}$$

The velocity of the centre of mass is

$$\underline{\bar{v}} = \frac{\sum(m\underline{v})}{\sum m} \tag{13.3.2}$$

The acceleration of the centre of mass is

$$\underline{\hat{a}} = \frac{\sum(m\underline{a})}{\sum m} \tag{13.3.3}$$

13.4 Cartesian Formulae

Suppose the points $r_1, r_2, r_3, \ldots$ are given by Cartesian components (x_1, y_1, z_1), (x_2, y_2, z_2), $(x_3, y_3, z_3), \ldots$. Then, the components for the centre of mass are:

$$\left.\begin{aligned} \bar{x} &= \frac{\sum(mx)}{\sum m} \\ \bar{y} &= \frac{\sum(my)}{\sum m} \\ \bar{z} &= \frac{\sum(mz)}{\sum m} \end{aligned}\right\} \tag{13.4.1}$$

In particular, on the plane, we have the following:

$$\left.\begin{aligned} \bar{x} &= \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \\ \bar{y} &= m_1y_1 + m_2y_2 \\ \bar{z} &= \frac{m_1z_1 + m_2z_2}{m_1 + m_2} \end{aligned}\right\} \tag{13.4.2}$$

Thus, the centre of mass of any two points on the plane divides the line joining them in the inverse of the ratio of their masses.

13.5 Continuously Distributed Mass System

Let v be the volume of a region containing a system of particles. Let δM be the mass of the particle in a region of volume of element δV. Assume $\lim_{\delta V \to 0} \dfrac{\delta M}{\delta V}$ exist. Then, we say that V is occupied by a continuously distributed mass system. The density of the system

$$\rho = \frac{dM}{dV} \tag{13.5.1}$$

This is the mass per unit volume. Similarly, for plane lamina, the density σ is the mass per unit area obtained by

$$\sigma = \frac{dM}{dS} \tag{13.5.2}$$

where S in the surface area.
For rods or strings the density λ is the mass per unit length obtained by

$$\lambda = \frac{dM}{dS} \tag{13.5.3}$$

where S is the arc-length.

13.5.1 Example

A thin strip of uniform density stretches along the x-axis from $x = a$ to $x = b$. Show that the centre of mass is located halfway between the two ends.

Centre of mass,

$$\hat{r} = \frac{\int_a^b x\rho dx}{\int_a^b \rho dx}$$

where ρ is the density of the rod.

Since ρ is constant, we have

$$\begin{aligned} \hat{r} &= \frac{\displaystyle\int_a^b xdx}{\displaystyle\int dx} \\ &= \frac{a+b}{2} \end{aligned}$$

$\therefore$ The centre of mass is located halfway between the rod.

13.6 Three-dimensional continuous body, centre of mass

For a 3-dimensional continuous body, the coordinate of the centre of mass is obtained by

$$\begin{aligned} \bar{x} \lim \frac{\sum(x\delta M)}{\sum \delta M} &= \lim \frac{\sum(\rho x \delta x)}{\sum \rho \delta x} \\ &= \frac{\displaystyle\int (\rho x dx)}{\displaystyle\int \rho dx} \end{aligned} \tag{13.6.1}$$

$$\bar{y} = \frac{\displaystyle\int \rho y y x)}{\displaystyle\int \rho dy} \tag{13.6.2}$$

$$\bar{z} = \frac{\displaystyle\int \rho z y z)}{\displaystyle\int \rho dz} \tag{13.6.3}$$

13.7 Example

A mental rod with one end at the origin and the other end at $x = 10$ thickness from left to right so that its density, instead of being a constant mass per unit length is

$$\rho(x) = 1 + \frac{x}{10}\ kg/m.$$

Find the rod's centre of mass.

Solution

$$\bar{x} = \frac{\int_0^{10} x\rho dx}{\int \rho dx} = \frac{\int_0^{10} x\left(1+\frac{x}{10}\right)dx}{\int \left(1+\frac{x}{10}\right)dx}$$

$$= \frac{\left(\frac{x^2}{2}+\frac{x^3}{30}\right)\Big|_0^{10}}{\left(x+\frac{x^2}{20}\right)\Big|_0^{10}}$$

$$= 5.56m \ \ (2d.p.)$$

$\therefore$ the mass is located at the point $5.56m$.

13.8 Definition

The centre of mass of a region of volume V is called its CENTROID.

13.9 Remarks

(i) The center of mass of uniform symmetrical body is at the centre of symmetry. For instance, uniformly circular or elliptical ring or disc, uniform rectangular plate or prism, uniform sphere or ellipsoid.

(ii) For a uniform triangular lamina, the centre of mass is along the median. It is the point where the median meet. This is $\frac{2}{3}$ along the median from any vertex.

13.10 Summary

In this lecture, we have that:

(i) the centre of mass of a body as $\dfrac{\sum(mI)}{\sum m}$, m is the body mass and I is the position vector.

(ii) the velocity of the centre of mass as $\bar{v} = \dfrac{\sum(m\underline{v})}{\sum m}$, m is the velocity.

(iii) $\bar{a} = \dfrac{\sum(ma)}{\sum m}$, a is the acceleration.

(iv) the Cartesian equation for centre of mass in term of: the coordinate

$$\bar{x} = \frac{\sum(mx)}{\sum x}, \quad \bar{y} = \frac{\sum(mx)}{\sum m}, \quad \bar{z} = \frac{\sum(mz)}{\sum m}$$

(v) the density of a system as

$\rho = \dfrac{dM}{dV}$, mass per unit volume

$\sigma = \dfrac{dM}{dS}$, mass per unit area, S is the surface area

$\lambda = \dfrac{dM}{d\bar{S}}$, mass per unit length, $\bar{S}$ = arc-length.

(vi) Centre of mass for 3-dimensional body as

$$\bar{x} = \frac{\int(\rho x dx)}{\int \rho dx}$$

$$\bar{y} = \frac{\int(\rho y dy)}{\int \rho dy}$$

$$\bar{z} = \frac{\int(\rho z dz)}{\int \rho dz}$$

13.11 Post-Test 13

(1) The distance of the centre of mass of a rod AB from the end B is $\frac{1}{8}$ of the length of the rod. If the line density is proportional to a certain power of the distance from A. Find the index of the power. (Hint: Let $\lambda = ax^n$ where n is the required index, and 'a' is a constant).

(2) The ends of two thin uniform rods of equal length are welded together to make right-angled frame. The density of the rod, $\lambda(x) = 1 + \frac{x}{L}$, $0 \leq x \leq L$. Find the centre of mass of the rod.

13.12 Solution to Pre-Test 13

$$\int_0^4 \left(2x + \frac{x^2}{4}\right) dx = 21.33 \quad \text{(Verify)}$$

Contemporary Reading

O.O. Ugbebor and N.I. Akinwande: Analytical Geometry and Mechanics.

C.J. Tranter and C.G. Lambe. Advanced Level Mathematics, 4th edition.

Lecture 14
Centre of Mass and Moment of Inertia II

14.0 Introduction

In this lecture, we will discuss the moment of inertial of a particle relative to the line of action.

14.1 Objective

After this lecture, you should be able to define and write an expression to calculate moment of inertia of a rectangular and circular body.

14.2 Pre-Test 14

(1) If a cone has height 'h' and base radius $\underline{b}$, find:

  (a) the volume of the cone

  (b) the mass of the cone if its density is $3/\pi\ hr$.

(2) Find the area of the region bounded by the curve $y = x^2 + 1$, the ordinates, $x = 1$, $x = 2$ and the x-axis.

14.3 Definitions

14.3.1 Moment of Inertia

Let r be the perpendicular distance of a particle of mass m from a given line. The moment of inertia, I of the particle relative to the line is obtained by

$$\begin{aligned} I &= m_1r^2 + m_2r^2 + \cdots + m_kr^2 \\ &= \sum(mr^2) \end{aligned} \qquad (14.3.1a)$$

Let $M = \sum m$, the radius of gyration, K of the system is defined by

$$K^2 = \frac{T}{M} \qquad (14.2.1b)$$

But for continuously distributed system, equation (14.3.1) and (14.3.2) becomes

$$I = \int r^2 dM \qquad (14.3.1c)$$

$$K^2 = \frac{\int r^2 dM}{\int dM} \qquad (14.3.1d)$$

respectively.
Let:

(i) $dM = \rho dV$ for volume

(ii) $dM = \sigma dS$ for surface area

(iii) $dM = \lambda d\bar{S}$ for lines in equation (14.3.1d).

Then, we have

(i) $K^2 = \dfrac{\int r^2 \rho dV}{\int \rho dV}$

(ii) $K^2 = \dfrac{\int r^2 \sigma dS}{\int \sigma dS}$

(iii) $K^2 = \dfrac{\displaystyle\int r^2 \lambda dS}{\displaystyle\int \lambda dS}$

14.4.2 Parallel Axes Theorem

Let

- I denotes the moment of inertia of a system of particles about any axis.
- I_m denotes the moment of inertia about the parallel axis through the centre of mass of the system.
- M denotes the total mass.
- d denote the distance between the axes.

Then,

$$\left.\begin{aligned} I &= I_m + Md^2 \\ K^2 &= K_m^2 + d^2 \end{aligned}\right\} \tag{14.4.2}$$

14.4.3 Perpendicular Axes Theorem

The moment of inertia about the z-axis for a system of particles in the OXY plane is

$$I_z = I_x + I_y \tag{14.4.3}$$

where I_x, I_y, I_z are the moment of inertia about the rectangular OX, OY and OZ

14.4.4 Uniform Rod of Length $2a$

Figure 14.4.4

14.4.4a

Moment of inertial about the axis perpendicular to the rod through the center $OA = OB = d$ is

$$I = \int_{-d}^{d} \lambda x^2 dx$$

where λ is mass per unit length. It is a constant. Thus,

$$I = \frac{2}{3}\lambda d^3$$

Let the total mass, $M = 2d\lambda$, then

$$I = \frac{1}{3}Md^2$$

and

$$K^2 = \frac{1}{3}d^2$$

14.4.4b

Moment of inertial about the axis through one of the end-point perpendicular to the rod is

$$I = \frac{1}{3}Md^2 + Md^2 = \frac{4}{3}Md^2$$

and

$$K^2 = \frac{4}{3}d^2$$

Using equation (14.4.3).

14.4.5 Uniform Rectangular Lamina sides $2a$ by $2b$

Figure 14.4.5

14.4.5a
Moment of inertia about an axis through the centre parallel to a side:

$$\begin{aligned} I_x &= \frac{1}{3}Ma^2 \\ I_y &= \frac{1}{3}Mb^2 \end{aligned}$$

14.4.5b
Moment of inertia about an axis through the centre perpendicular to the Lamina,

$$I_z = I_x + I_y = \frac{1}{3}M(a^2 + b^2)$$

14.4.6 Thin Uniform Ring with radius, r

Figure 14.4.6

14.4.6a

Moment of inertia about an axis through the centre perpendicular to the plane of the ring,

$$I = \int r^2 dM = Mr^2.$$

14.4.6b

Moment of inertia about a diameter: $(X, Y$-axes)

$$\begin{aligned} I_z &= I_x + I_y = Mr^2 \\ I_x &= I_y = \frac{1}{2}Mr^2 \end{aligned}$$

14.4.7 Uniform Disc radius 'a'

Figure 14.4.7

14.4.7a

Moment of inertia about an axis through the centre perpendicular to the plane of a disc is

$$I = \int_0^a 2\pi r dr, dr^2 = \frac{1}{2}\pi a^4 \sigma$$

But $M = \pi a^2 \sigma$.
$\therefore \quad I = \frac{1}{2} M a^2$.

14.4.7b

Moment of inertia about a diameter is $I = \frac{1}{4} M a^2$.

14.4.8 Other spherical shapes with radius 'a'

For uniform spherical shell with radius 'a' moment of inertia about any diameter is

$$I = \frac{2}{3} M a^2$$

14.4.8b

For uniform solid spherical of radius 'a', the moment of inertia about any diameter is:

$$I = \frac{2}{5} M a^2$$

14.4.8c

The moment of inertia about the cylinder's axis for uniform circular cylinder ring of radius 'a', length $2l$ is

$$I = \frac{1}{2} M a^2$$

14.4.8d

The moment of inertia about the line through the centre of cylinder perpendicular to its axis for uniform circular cylindrical ring radius, 'a', length $2l$ is

$$I = M\left(\frac{1}{4} a^2\right)$$

14.5 Example

(1) Find the moment of inertia of a uniform solid hemisphere of mass M and radius 'a' about a diameter of the circular boundary

Figure 14.5

(2)(a) Find the moment of inertia of a uniform square lamina of mass M and side $2a$ about a diagonal.

(b) Find the radius of gyration.

Solution

(1) The whole body of mass say M is halved. Thus, the inertial I is also halved.

$$\begin{aligned} \therefore \quad \frac{1}{2}I &= \frac{2}{5}\frac{M}{2}a^2 \\ &= \frac{2}{5}Ma^2 \end{aligned}$$

Since the moment of inertia about any diameter, $I = \frac{2}{5}Ma^2$

Moment of inertia about x

$$I_x = \frac{1}{3}Ma^2$$

Moment of inertia about y

$$I_y = \frac{1}{3}Mb^2$$

Therefore, moment of inertia about z,

$$\begin{aligned} I_z &= I_x + I_y \\ &= \frac{1}{3}M(a^2 + b^2) \end{aligned}$$

Since the plane is a square, we have

$$b = a$$

and

$$I_z = \frac{1}{3}M(2a^2)$$

Let AC and BD be x and y axis respectively. By the diagram above,

$$I_z = I_x + I_y$$

and $I_x = I_y$

$$\therefore \quad I_x = I_y = \frac{1}{2}I_z = \frac{1}{2}\left(\frac{2}{3}Ma^2\right)$$

$$\therefore \quad I = \frac{1}{3}Ma^2$$

(2) $K^2 = \frac{T}{M}$
$\dfrac{1}{3}a^2$ (Verify); $\therefore$ the radius of gyration is $\frac{1}{3}a^2$.

14.6 Summary

In this lecture, we have that the:

(i) Moment of inertia of a particle

$$I = \sum(mr^2) \qquad \text{(a)}$$

(ii) Radius of gyration

$$K = \left(\frac{I}{M}\right)^{1/2} \text{ where } M \text{ is the particle mass} \qquad \text{(b)}$$

(iii) Moment of inertia for continuously distributed particle,

$$I = \int r^2 dM \qquad \text{(c)}$$

(iv) Radius of gyration for continuously distributed particle,

$$K = \left(\frac{\int r^2 dM}{\int dM}\right)^{1/2} \qquad \text{(d)}$$

(v) Radius of gyration for particle in terms of volume, surface area and arc length as:

$$K = \left(\frac{\int r^2 \rho dV}{\int \rho dS}\right)^{1/2} \text{ where } S \text{ is density of the particle}$$

$$K = \left(\frac{\int r^2 \sigma dV}{\int \sigma dS}\right)^{1/2} \text{ where } S \text{ is surface area of the particle}$$

$$K = \left(\frac{\int r^2 \lambda dS}{\int \lambda d\bar{S}}\right)^{1/2} \text{ where } \lambda \text{ is arc length respectively.}$$

(vi) Parallelogram Axes law as:

$$I = I_m + Md^2 \text{ and } K = (K_m^2 + d^2)^{1/2}$$

(vii) Perpendicular Axes theorem as $I_z = I_x + I_y$

14.8 Post-Test 14

A rod of uniform thickness has half of its length composed of one metal and the other half of another metal. The centre of mass is distance $\frac{1}{7}$ of the whole length from the centre of the rod. Find the ratio of the densities of the metals.

14.9 Solution to Pre-Test 14

(1) Volume $= \dfrac{1}{3}\pi r^2 h$

Mass $= \dfrac{1}{3}\pi r^2 h \cdot \dfrac{3}{\pi\, hr} = r$ units.

(2) $\dfrac{dy}{dx} = x^2 + 1$

$y = \displaystyle\int_1^2 (x^2+1)dx = 3\frac{1}{3}$ units.

Contemporary Reading

1. O.O. Ugbebor and N.I. Akinwande: Analytical Geometry and Mechanics.

2. Bostock, L. and Clandler, S.: Pure Mathematics 2, 1979, U.K. Stanley Thomas (Publishers) Ltd.