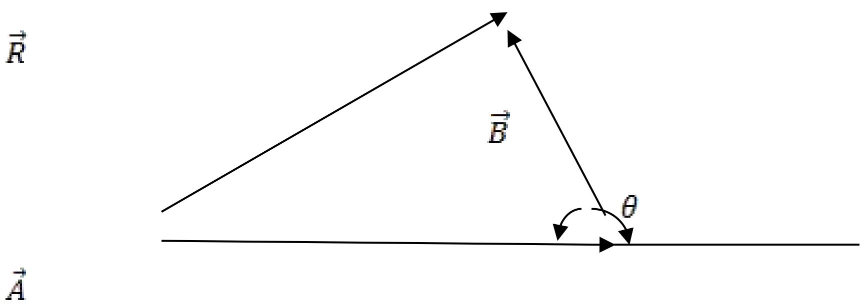
VECTORS CONTINUED

GEOMETRICAL METHOD

The two vectors being added $\vec{A}$ and $\vec{B}$ for example are represented by the two adjascent sides of a triangle inclined to each other at the angle between the two vectors. The third side of the triangle represents the resultant of the two vectors which is also a vector. If α is the angle between $\vec{A}$ and $\vec{B}$, the resultant vector $\vec{R}$ is shown below.



The addition of the two vectors$\vec{A}$ and $\vec{B}$ is given by the cosine rule

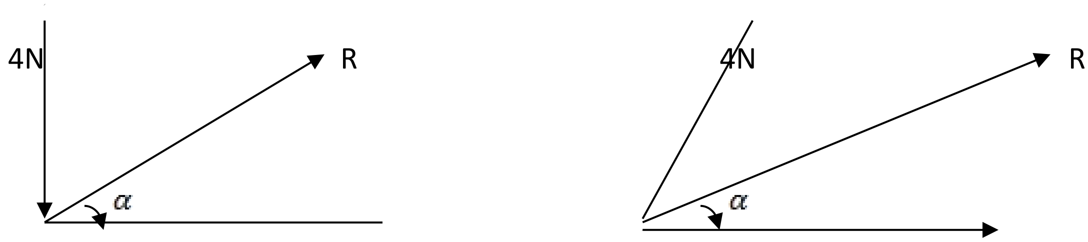
$$R^2 = A^2 + B^2 - 2AB\cos(180 - \theta)$$

But $cos(180 - \theta) = -cos\theta$

$$R^2 = A^2 + B^2 + 2AB\cos\theta$$

EXAMPLE

1. Two forces $3N$ and $4N$ act at right angle on a body. Find the resultant force acting on the body and the direction of this force. If the two forces are now inclined at an angle of 60° then find the resultant force.



3N Fig (1) 3N

(i) Right angle (ii) Acute angle

The two forces and the resultant R are shown in fig 1. From fig. 1(i) we have,

$R = (4^2 + 3^2)^{\frac{1}{2}}$

The direction is given by the angle α and tan $\alpha = \frac{4}{3}$

$\alpha = tan^{-1}(\frac{4}{3}) = 53.1°$

Similarly from fig.1(ii)$R^2 = 4^2 + 3^2 - 2(4)(3)\cos(180 - 60)$

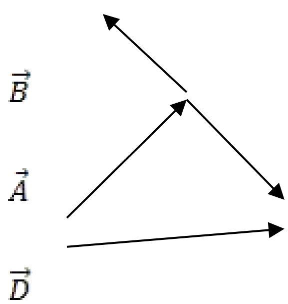
$R = \sqrt{37}$ N

$R = 6.08N$

<u>SUBTRACTION OF VECTORS</u>

Subtraction of vectors is similar to the addition of vectors.

Note that $\vec{D} = \vec{A} - \vec{B}$ is the same thing as $\vec{D} = \vec{A} + (-\vec{B})$

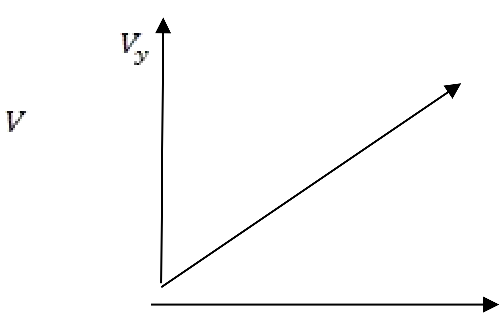


If convenient scales are used and the lengths and angles are carefully measured, the geometrical method can be carried out by scale diagram.

<u>ANALYTICAL METHOD</u>

The geometrical method is not convenient when more than two vectors can only be combined or added in pairs. The analytical method allows us to combine or add all the vectors at once by breaking them into their components.

Any vector V can be considered as the sum of two or more vectors. Each set of vectors, which when added gives a vector V is called the set of components of V. The most commonly used are the two rectangular components V_x and V_y



$V_x = V cos\theta$

$\theta V_x V_y = V sin\theta$

$V = iV_x + jV_y$

i and j show the directions of the vectors V_x and V_y. After breaking each vector into its

Components, we can add components in the same directions like ordinary numbers. In

Three directions, we have $V = iV_x + jV_y + kV_z$

$V = V_1 + V_2 + V_3 + \cdots + V_n$

$V = i(V_{1x} + V_{2x} + V_{3x} + \cdots + V_{nx}) + j\left(V_{1y} + V_{2y} + V_{3y} + \cdots + V_{ny}\right)$

$+ k(V_{1z} + V_{2z} + V_{3z} + \cdots + V_{nz})$

EXAMPLE

1. Three forces F_1, F_2 and F_3 act on an object. If

$\vec{F_1} = 4i - j\vec{F_2} = -3i + 2j$ and $\vec{F_3} = -3j$.

Find the resultant force.

SOLUTION

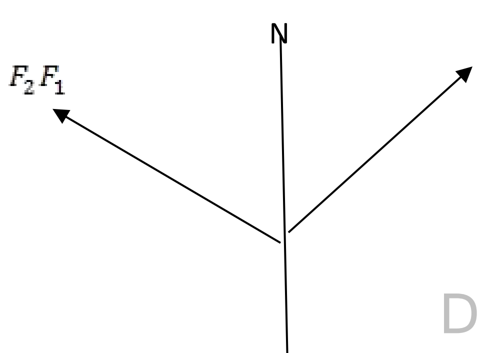
The resultant force is given by $\vec{F} = i(4 - 3) + j(-1 + 2 - 3)$

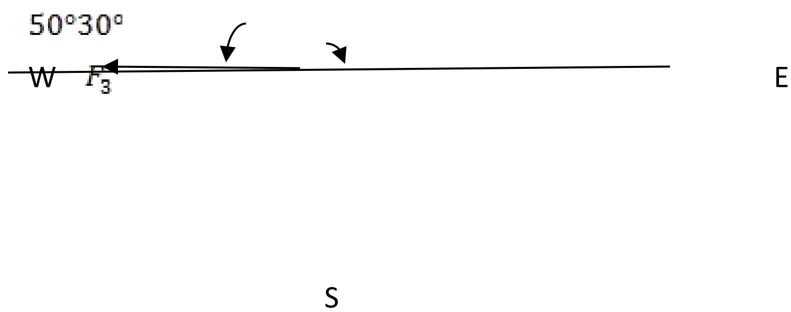
$= i - 2j$

$|F| = \sqrt{(1)^2 + (-2)^2}$

$= \sqrt{5}$ N

2. If three forces $F_1 = 20N$ at $30°NE, F_2 = 50N$ $along$ W and $F_3 = 40N$ $50°NW$ act on a body. Find the resultant force in magnitude and direction.





$\vec{F} = \vec{F_1} + \vec{F_2} + \vec{F_3}$

$\vec{F_1} = 20cos30°\, i + 20sin30° j \quad = 17.32i + 10j$

$\vec{F_2} = -50i$

$\vec{F_3} = (-40cos50°)i + (40sin50°)j = -25.71i + 30.64j$

$F = (17.32 - 50 - 25.71)i + (10 + 30.64)j$

$F = -58.39i + 40.64j$

$|F| = \vec{\ }\sqrt{(-58.39)^2 + (40.64)^2}$

$= 71.14N.$

Direction: $tan\theta = \frac{40.64}{-58.39} = -0.6960$

$\theta = 34.8°$ *to the* $x -$ *axis in the 2nd quadrant*

MULTIPLICATION OF VECTORS

There are three types of vectors multiplication operations.

(a) Multiplication of a vector by a vector to yield a vector.
For g example $F = m \times a$

(b) Multiplication of a vector by a vector to yield a scalar.
This is also called a dot product or a scalar product.
The dot product of two vectors $\vec{A}$ *and* $\vec{B}$ is defined as $\vec{A} \cdot \vec{B} = |A||B|cos\theta$ where θ is the angle between them.
Analytically, $\vec{A} \cdot \vec{B} = (iA_x + jA_y + kA_z) \cdot (iB_x + jB_y + kB_Z)$
$= i \cdot iA_xB_x + i \cdot jA_xB_y + i \cdot kA_xB_z + j \cdot i\, A_yB_x + j \cdot jA_yB_y + j \cdot kA_yB_z$

$$+ k \cdot iA_zB_x + k \cdot jA_zB_y + k \cdot kA_zB_z$$
$$since\ i \cdot i = j \cdot j = k \cdot k = 1$$
$$i \cdot j = i \cdot k = j \cdot k = 0$$
$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z$$

(c) Multiplication of a vector by a vector to yield a vector. This is also called a cross product. We can

also express the vector product of two vectors $A\ and\ B$ in terms of their components in three

dimensions such that $A \times B$ is written as

$$\vec{A} \times \vec{B} = (iA_X + jA_y + kA_z) \times (iB_x + jB_y + kB_z)$$

$$= iA_x \times iB_x + iA_x \times jB_y + iA_x \times kB_z + jA_y \times iB_x + jA_y \times jB_y + jA_y \times kB_z$$

$$+ kA_z \times iB_x + kA_z \times jB_y + kA_z \times kB_z$$

The product of two parallel vectors are zero, hence $i \times i = j \times j = k \times k = 0$

Similarly, $i \times j = k,\quad j \times k = i \quad k \times i = j \quad and \quad j \times i = -k$

The cross product of vectors $\vec{A}\ and\ \vec{B}$ is defined as $\vec{A} \times \vec{B} = |A||B|sin\theta$

$$\begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{A} \times \vec{B} = i(A_yB_z - A_zB_y) - j(A_xB_z - A_zB_x) + k(A_xB_y - A_yB_x)$$

<u>EXAMPLES</u>

1. Calculate the angle between the vectors $\vec{r} = i + 2j$ and $\vec{t} = j - k$

<u>SOLUTION</u>

$$\vec{r} \cdot \vec{t} = |\vec{r}||\vec{t}|\cos\theta$$
$$|\vec{r}| = \sqrt{(1)^2 + (2)^2} = \sqrt{5}$$
$$|\vec{t}| = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$$

$\vec{r} \cdot \vec{t} = (i + 2j) \cdot (j - k) = i \cdot j - i \cdot k + 2j \cdot j - 2j \cdot k = 0 - 0 + 2 - 0 = 2$

$cos\theta = \frac{\vec{r} \cdot \vec{t}}{|\vec{r}||\vec{t}|} = \frac{2}{\sqrt{5}\sqrt{2}} = \frac{2}{\sqrt{10}} = 0.632$

$\theta = cos^{-1}(0.632) = 50.8°$

2. Evaluate $(\vec{r} + 2\vec{t}) \cdot \vec{f}$ where

$\vec{r} = \vec{i} + 2\vec{j}$, $\vec{t} = \vec{j} - \vec{k}$ and $\vec{f} = \vec{i} - \vec{j}$

SOLUTION

$\vec{r} + 2\vec{t} = (\vec{i} + 2\vec{j}) + 2(\vec{j} - \vec{k}) = \vec{i} + 4j - 2k$

$(\vec{r} + 2\vec{t}) \cdot \vec{f} = (\vec{i} + 4\vec{j} - 2\vec{k}) \cdot (\vec{i} - \vec{j})$

$= \vec{i} \cdot \vec{i} + 4\vec{j} \cdot \vec{i} - 2\vec{k} \cdot \vec{i} - \vec{i} \cdot \vec{j} - 4\vec{j} \cdot \vec{j} + 2\vec{k} \cdot \vec{j}$

$= -3$

ASSIGNMENT

1. Prove that if two vectors have the same magnitudes v and make an angle θ, their sum has a magnitude $S = 2vcos\frac{1}{2}\theta$ and their difference is $D = 2vsin\frac{1}{2}\theta$.

2. Find the angle between $A = 2i + j$ and $B = -i + j$

3. If $\overrightarrow{d_1} + \overrightarrow{d_2} = 5\overrightarrow{d_3}$, $\overrightarrow{d_1} - \overrightarrow{d_2} = 3\overrightarrow{d_3}$ and $\overrightarrow{d_3} = 2i + 4j$. Find(i) $\overrightarrow{d_1}$ (ii) $\overrightarrow{d_2}$

4. Two vectors are given by

$\vec{a} = (4.0m)i - (3.0m)j + (1.0m)k$

$\vec{b} = (-1.0m)i + (1.0m)j + (4.0m)k$

In unit-vector notation, find (i) $\vec{a} + \vec{b}$ (ii) $\vec{a} - \vec{b}$ (iii) a third vector $\vec{c}$ such that $\vec{a} - \vec{b} + \vec{c} = 0$

KINEMATICS

This is the study of motion of objects. An object moves when it changes its position from point $P_1(x_{1,}y_{1,}z_1)$ in space to another point $P_2(x_{2,}y_{2,}z_2)$ in space in a time t, no matter how small the time interval is.

Displacement is defined as the change of an object's position that occurs during a period of time.

The displacement is given by $\Delta r = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$. Displacement is a vector quantity. Distance is the length of a path followed by a particle. If a displacement from point $P_1(2,2)$ to point $P_2(6,1)$ lie in a plane, the magnitude of displacement called the distance is

$$|\Delta r| = \sqrt{4^2 + (-1)^2} = \sqrt{17} = 4.12\Delta r = (6-2)i + (1-2)j = 4i - j$$

The average velocity is defined as the ratio of the displacement vector$\Delta\vec{x}$ for the motion of interest to the time interval Δt in which it occurs. When a particle has moved from position x_1 to position x_2 during a time interval $\Delta t = t_2 - t_1$,its average velocity during that interval is

$$\vec{V} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}$$

The instantaneous velocity is the rate at which a particle's position vector,$\vec{x}$, is changing with time at a given instant.

$$V = Lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

Example

The position of a particle moving along the x axis is given in centimeters by

$x = 9.75 + 1.50t^3$, where t is in seconds. Calculate

a. The average velocity during the time interval $t = 2.0s \ to \ t = 3.0s$

b. The instantaneous velocity at $t = 2.0s$

c. The instantaneous velocity at $t = 3.0s$

d. The instantaneous velocity at $t = 2.50s$

e. The instantaneous velocity when the particle is mid-way between its position at $t = 2.0s \ and \ t = 3.0s$

<u>SOLUTION</u>

a. Given that $x = 9.75 + 1.50t^3$. We obtain $x \ when \ t = 2.0s \ and \ t = 3.0s$

$x_{t=2} = 9.75 + 1.50(2.0^3) = 21.75cm$

$x_{t=3} = 9.75 + 1.50(3.0^3) = 50.25cm$

The average velocity during the time interval $2.0 \leq t \leq 3.0s \ is$

$$\bar{V} = \frac{\Delta x}{\Delta t} = \frac{50.25 - 21.75 \ cm}{3.0s - 2.0s} = 28.5cms^{-1}$$

$b.$ The instantaneous velocity is $v = \frac{dx}{dt} = 4.5t^2$, which, at time $t = 2.0s$ yields

$v = (4.5)(2.0^2) = 18.0cms^{-2}$

$c.$ At $t = 3.0s$, the instantaneous velocity $v = (4.5)(3.0^2) = 40.5cms^{-1}$

$d. At \ t = 2.50s$, the instantaneous velocity $v = (4.5)(2.50^2) = 28.1cms^{-1}$

$e.$ Let t_m stands for the moment when the particle is midway between x_2 and x_3 (that is, when the particle is at $x_m = \frac{x_2 + x_3}{2} = 36cm$)

Therefore,

$x_m = 9.75 + 1.5t_m{}^3 \Rightarrow t_m = 2.596s$

Thus, the instantaneous speed at this time is $v = (4.5)(2.596^2) = 30.3cms^{-1}$

The average speed involves the total distance covered which is independent of direction, so the average speed of a particle during a time interval Δt is defined as

Average speed $= \frac{Total\ distance}{\Delta t}$

Instantaneous speed is the magnitude of instantaneous velocity.

Whenever a particle's velocity changes, we define it as having an acceleration. The average acceleration $\bar{a}$, over an interval Δt is defined as

$$\bar{a} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$$

An object undergoes acceleration even if all that changes is only the direction of its velocity and not its speed.

The instantaneous velocity of a particle at any instant is the rate at which its velocity is changing at that instant.

$$a_{inst.} = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

A uniform motion is a motion in which the acceleration is constant that is, the average acceleration and instantaneous acceleration are equal. If we place our x-axis along the line of motion and expressing the acceleration along the line of motion of object we have

$$a_x = \frac{v_{2x} - v_{1x}}{t_2 - t_1} \ldots \ldots \ldots \ldots (1)$$

The subscript 1 and 2 refer to initial and final times, positions and velocities. From equation (1)

$$v_{2x} = v_{1x} + a_x(t_2 - t_1) \ldots \ldots \ldots \ldots \ldots \ldots (2)$$

$$v_{2x} = v_{1x} + a_x\Delta t$$

$$\Delta v_x = a\Delta t[a_x\ is\ constant]$$

In a similar manner, we can re-write the expression for the average velocity as

$$\bar{v}_x = \frac{x_2 - x_1}{t_2 - t_1} \dots\dots\dots\dots\dots (3)$$

$$x_2 = x_1 + \bar{v}_x(t_2 - t_1) \dots\dots\dots\dots\dots. (4)$$

But average velocity $\bar{v}_x = \frac{v_{1x}+v_{2x}}{2} \dots\dots\dots\dots\dots. (5)$

By substituting equation (2) into equation(5)

$$\bar{v}_x = \frac{1}{2}\left(v_{1x} + v_{1x} + a_x(t_2 - t_1)\right)$$

$$= v_{1x} + \frac{1}{2}a_x(t_2 - t_1) \dots\dots\dots.. (6)$$

By substituting equation (6) into equation(4) yields

$$x_2 - x_1 = v_{1x}(t_2 - t_1) + \frac{1}{2}a_x(t_2 - t_1)^2 \dots\dots\dots\dots\dots (7)$$

or

$$\Delta x = v_{1x}\Delta t + \frac{1}{2}a_x(\Delta t)^2$$

By solving for $(t_2 - t_1)$ in equation (2) and substituting in equation(7) gives

$$t_2 - t_1 = \frac{v_{2x} - v_{1x}}{a_x} \dots\dots\dots\dots\dots. (8)$$

$${v_{2x}}^2 = {v_{1x}}^2 + 2a_x(x_2 - x_1) \dots\dots\dots\dots. (9)$$

Examples

1. Spotting a police car, a boy applied a brake and his car slows down from a speed $100km/hr$ to a speed $80km/hr$ during a displacement of $88.0m$ at a constant acceleration

   (a) What is the magnitude of that constant acceleration?

   (b) How much time is required for the given decrease in speed?

SOLUTION

(a) $a_x = \frac{{v_{2x}}^2 - {v_{1x}}^2}{2(x_2 - x_1)}$

$$v_{1x} = 100kmhr^{-1} = 27.78ms^{-1}$$

$$v_{2x} = 80kmhr^{-1} = 22.22ms^{-1}$$

$$\therefore\ a_x = \frac{(22.22ms^{-1})^2 - (27.78ms^{-1})^2}{2(88.0m)}$$
$$= -1.58ms^{-2}$$

$$(b) t_2 - t_1 = \frac{v_{2x} - v_{1x}}{a_x}$$
$$= \frac{22.22ms^{-1} - 27.78ms^{-1}}{-1.58ms^{-2}}$$
$$= 3.52s$$

2. A person walks first at constant speed of $5.0ms^{-1}$ along a straight line from point A to point B and then back along the line from B to A at a constant speed of $3.0ms^{-1}$. What is
   (a) Her average speed over the entire trip?
   (b) Her average velocity over the entire trip?

<u>SOLUTION</u>

(a) Let d represent the distance between A and B. Let t_1 be the time for which the walker has the higher speed in $5.0ms^{-1} = \frac{d}{t_1}$

Let t_2 represent the longer time for the return trip$= \frac{d}{t_2}$. Then the times are

$t_1 = \frac{d}{5.0ms^{-1}}$ and $t_2 = \frac{d}{3.0ms^{-1}}$. The average speed is

$$\bar{v} = \frac{Total\ distance}{Total\ time} = \frac{d + d}{\frac{d}{5.0ms^{-1}} + \frac{d}{3.0ms^{-1}}}$$

$$= \frac{2d}{\frac{(8.0ms^{-1})d}{(15.0m^2s^{-2})}} = \frac{2(15.0m^2s^{-2})}{(8.0ms^{-1})}$$
$$= 3.75ms^{-1}$$

(b) She starts and finishes at the same point A. With total displacement = 0, average velocity = 0

3. A $50.0g$ superball travelling at $25.0ms^{-1}$ bounces off a brick wall and rebounds at $22.0ms^{-1}$. A high speed camera records this event. If the ball is in contact with the wall for $3.50ms$, what is the magnitude of the average acceleration of the ball during this time interval?

   SOLUTION

   Choose the positive direction to be the outward direction, perpendicular to the ball.

   $v_f = v_i + at$

   $$\therefore a = \frac{\Delta v}{\Delta t} = \frac{22.0ms^{-1} - (-25.0ms^{-1})}{3.50 \times 10^{-3}s} = 1.34 \times 10^4 ms^{-2}$$

RELATIVE MOTION

When a motion is observed, it is always to a particular frame of reference. For example, the motion of a boat can be described relatively to

1. Another boat
2. The water
3. The stationary bank of the river.

The velocity assigned with each case is different. If a frame of reference is itself moving with respect to an observer with velocity v_1, while an object is moving relative to the frame of reference with v_2, the velocity v_0 of the object relative to the observer is

$v_0 = v_1 + v_2$

If you are in a car which a roadside observer sees as moving with $100kmhr^{-1}$ and another car overtakes your car by a velocity which you reckon as $20kmhr^{-1}$, the roadside observer will see this car as moving with

$v_0 = (100 + 20)kmhr^{-1}$

Generally, velocity of A relative to B is $v_{AB} = v_A - v_B$

Velocity of B relative to A is $v_{BA} = v_B - v_A = -v_{AB}$

<u>EXAMPLE</u>

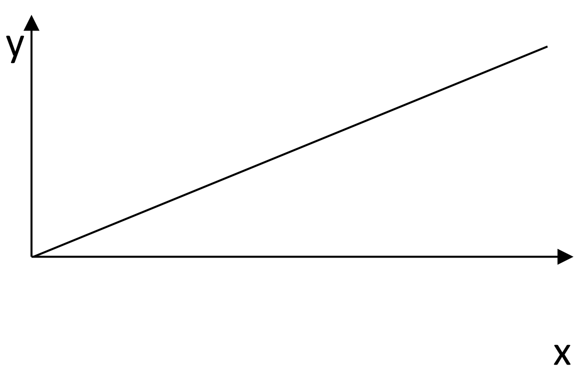
A boy runs inside a ship with a velocity $v_2 = 4i - 2j$ relative to the ship, while the ship itself is moving relative to the sea with velocity$v_1 = 50i + 30j$. The velocity of the boy relative to the sea is therefore

$$v_0 = v_1 + v_2$$
$$= (50i + 30j) + (4i - 2j)$$
$$= 54i + 28j$$

‘**PROJECTILE**

A projectile is an object that is launched into the air or atmosphere and is allowed to move freely under gravity. Examples of projectiles are: a stone shot out from a catapult; a bullet fired from a gun and a rocket launched from a rocket launcher.

When an object is projected upward at an angle to the vertical, its motion becomes two dimensional. Its displacement has both and components and the path is parabolic. Such a motion is called projectile motion. If the initial velocity of projection is at an angle to the horizontal, then the initial components of the velocity are



$u_x = u\cos\theta$

$u_y = u\sin\theta$

There is no acceleration in the x-direction because there is no force along this direction. The displacement in time is therefore

Along y-direction, there is a negative acceleration (g).The displacement is therefore

From equation (1)

By substituting equation (3) into equation (2) gives

Since the terms in the brackets are constants, we can write equation (4) as

Equation (5) is quadratic and gives a parabola which explains the parabolic nature of projectiles motions. The equation can also be shown to be a quadratic function of, which implies that there are two possible values of for the object to pass over the same height at a fixed horizontal distance.

The range R of the projectile is the horizontal distance covered before the object hits the ground and it is given as

where T = total time taken

at any instant is

At maximum height H,

Range,

Equation (11) shows that R is maximum when

The maximum height

EXAMPLES

1. Find the angle of projection at which the horizontal range and the maximum height of a projectile are equal.

SOLUTION

The horizontal range and maximum height are given in equations (11) and (13) respectively. To obtain the angle of projection, we equate these equations

;

2. A body projected upwards from the level ground at an angle of 50 with the horizontal has an initial speed of 40m/s. (a)How long will it take before it hits the ground?
   (b) How far from the starting point will the body hit the ground?
   SOLUTION
   (a) Choose upward as positive and place the origin at the launch point.

o find the time in air, we use and since at the end of flight,
 or . The first solution
Corresponds to the starting point The second solution is the required time

(b) The horizontal distance travelled,

A body is projected downward at an angle of 30 with the horizontal from the top of a building 170m high. Its initial speed is 40m/s.

(a) How long will it take before striking the ground?

(c) Find out how far from the foot of the building the body will strike and at what angle with the horizontal.

SOLUTION

Choose downward as positive and origin/ at the top edge of building

Using

Solving for t yields t = 4.2s

We need to the angle that the velocity makes with the x axis first before hitting the ground.

Using

ASSIGNMENT

(1)A hose lying on the ground shoots a stream of water upward at an angle of 40^0 to the horizontal. The speed of the water is $20ms^{-1}$ as it leaves the hose. How high up will it strike a wall which is 8m away?(5.33m)

(2) A ball is thrown upward at an angle of 30 to the horizontal and lands on the top of a building that is 20m away. The top edge is 5m above the throwing point. How fast was the ball thrown?(20m/s)

(3) A projectile is fired with initial velocity V_0 which is $95ms^{-1}$ at an angle 50^0. After 5 seconds, it strikes the top of a hill. What is the horizontal distance from the gun does the projectile land? (241m, 305m)

UNIFORM CIRCULAR MOTION

Some bodies are not free to move along a straight line but are constrained to move in a circular path. Examples of motion that are approximately circular include the revolution of the Earth around the sun, a race car zooming around a circular track, an electron moving near the centre of a large electromagnet and a stone tied to the end of a string that is whirled in a circle. Though the speed is constant in each case, the direction is constantly changing. Hence, the velocity is not constant. This implies acceleration.

A particle that travels around a circle or a circular arc at constant or uniform speed is said to be undergoing uniform circular motion. In circular motion, we talk about angular displacement which is analogous to linear displacement in linear motion. A relationship can be obtained between a, v and r as follows:

Average acceleration

Since delta t is a scalar, the acceleration must have the same direction as the difference between the two velocity vectors,. We define as zero at time and denote its location as. The object then moves at constant speed through an angle to a new location at time. In circular motion, velocity vectors are always perpendicular to their position vectors. This means that the angle between position vectors and is the same as the angle between velocity vectors and .

But then

Also then

Thus the triangles shown in Fig1 are similar.

For small angles, . So we can write the ratio of magnitudes as

e can solve the similar triangle ratios for the change in velocity and substitute into the expression that defines the magnitude of acceleration in terms of the magnitude of velocity vector change over a change in time to get

owever, the speed v is given by the arc length along the circular path divided by the time interval, so that

(when the time interval becomes small)
o, we can replace in the expression for acceleration with the particle speed v to get

The angular velocity, (rad/s)

Fig 2
A relationship between and the instantaneous velocity v can be derived as follows:
Consider a motion in a circle as shown above

hen is very small,

V is called the tangential velocity. When the rotating body slows down or speeds up, the will be changing, there is an angular acceleration given byThe instantaneous velocity is constantly changing direction, that is, there is always an acceleration associated with circular motion. This is called radial or centripetal acceleration.

EXAMPLES

1. An Earth satellite moves in a circular orbit 640km above Earth's surface with a period of 98mins. What are the

(a) Speed of the satellite

(b) Magnitude of the centripetal acceleration of the satellite?

SOLUTION

(a) Since the radius of Earth is the radius of the satellite orbit is

Therefore, the speed of the satellite is

(b) The magnitude of the acceleration

2. A rotating fan completes 1200 revolutions every minute. Consider the tip of a blade, at a radius of 0.15m.

   (a) Through what distance does the tip move in one revolution?

   (b) What is the tip's speed?

   (c) What is the magnitude of its acceleration?

SOLUTION

(a) The circumference is

(b) With the speed is

The magnitude of the acceleration is

3. A boy whirls a stone in a horizontal circle of radius 1.5m and a height 2.0m above level ground. The string breaks and the stone flies off horizontally and strike the ground after travelling a horizontal distance of 10m. What is the magnitude of the centripetal acceleration of the stone during the circular motion?

SOLUTION

The stone moves in a circular path initially but undergoes projectile motion after the string breaks

Since , to calculate the centripetal acceleration of the stone, we need to know its speed during its circular motion(this is also the initial speed when it flies off). Taking the +y direction to be upward and placing the origin at the point where the stone leaves its circular orbit, then the co-ordinates of the stone during its motion as a projectile are given by and
since It hits the ground at and
Solving the y- component equation for the time, we obtain
which we substitute into the first equation

Therefore, the magnitude of the centripetal acceleration is

There are corresponding equations of motion for constant angular (rotational)motion as in the case of uniform linear motion.

L i n e a r m o t i o	Angular(rotational) motion

EXAMPLES

1. The angular position of a point on a rotating wheel is given by θ =2.0+4.0t^2+2.0t^3 where θ is in radians and is in seconds. At t = 0 what are

a. The point's angular position
b. Its angular velocity
c. What are its angular velocity at t = 4.0s?
d. Calculate its angular acceleration at t = 2.0s

SOLUTION

a. We evaluate the function at t =0 to obtain
b. The angular velocity as a function of time is given by
   which we evaluate at t = 0 to obtain
c. For t = 4.0s, we have
d. The angular acceleration as a function of time is

   When t = 0,

2. A disk, initially rotating at 120rad/s is slowed down with a constant angular acceleration of magnitude 4.0rad/s^2.
   a. How much time does the disk take to stop?
   b. Through what angle does the disk rotate during that time?

SOLUTION

(since it stops at time t)

3. A disk rotates about its central axis starting from rest and accelerates with constant angular acceleration. At one time it is rotating at 10rev/s , 60 revolutions later, its angular speed is 15rev/s. Calculate (a) the angular acceleration (b) the time required to complete the 60 revolutions (c) the time required to reach the 10rev/s angular speed, and (d) the number of revolutions from rest until the time the disk reaches the 10rev/s angular speed.

   SOLUTION

   The wheel starts from rest = 0 at time t=0. At time . At time its angular velocity . Between , it turns through where

   (a)Using

   = 1.04 rev/s

   (b) Using

   = 9.6s

ASSIGNMENT

A drum rotates around its central axis at an angular velocity of 12.60rad/s. If the drum then slows at a constant rate of 4.20rad/s^2

(a)How much time does it take and (b) through what angle does it rotate in coming to rest ? (3.0s, 18.9 rad)

NEWTON’S LAWS OF MOTION

To simplify things as we start, let us assume that any object considered

- **Has a constant mass**
- **Is not rotating**
- **Is in surrounding that is not accelerating**

NEWTON'S FIRST LAW

The laws of motion that relate external interactions between objects to their accelerations were first developed by Isaac Newton. Consider a body in which no force acts. If the body is at rest, it will remain at rest. If the body is moving, it will continue to move with a constant velocity.

Any object continues in a state of rest or uniform motion in a straight line unless it is compelled to change that state by external forces acting on it. Another way of putting this is that any object has a constant velocity unless it is acted upon by a resultant external force. This effectively defines force. A force is that which causes the velocity of an object to change. A force is necessary to accelerate an object or a force is that influence which acting alone can cause on object to be accelerated. A constant force acting on an object causes it to move along a straight line with a constant acceleration that is in the same direction as the force.

One Newton of force is defined to be the force necessary to impart an acceleration of 1 m/s^2 to the international standard kilogram.

NEWTON'S SECOND LAW

The first law defines force. If a fixed force is applied to two different masses, one large and one small, they will not have the same acceleration. The second law deals with the size of the effect that force has on objects.

The second law in formal language states that the rate of change of momentum of a body with time is directly proportional to the total force acting on it and occurs in the direction of the force.

Consider an object of mass m being pushed by a constant force F so that its velocity increases from u to v in time t.

Newton's second law gives

$$\frac{change\ in\ momentum}{time} \propto Force$$

$$\frac{\mathbf{mv} - \mathbf{mu}}{\mathbf{t}} \propto \mathbf{Force}$$

$$m\left(\frac{v-u}{t}\right) \propto Force$$

$$ma \propto Force$$

$$F = Kma$$

In S.I unit, 1N is the net force that produces an acceleration of $1ms^{-2}$ on a body of 1kg. By this definition, constant K above = 1

$$\therefore F = ma$$

***W*hen a large force acts on a body for a short time, the body receives what is normally described as a blow or impulsive force, measured by the product of force and time of action.**

$$I = ft$$

But $f = \frac{mv - mu}{t}$

$$I = mv - mu$$

Impulse = change in momentum = Force x time

EXAMPLE

A pitched baseball of mass 140g, in horizontal flight with a speed v_1 of mass 39.0m/s is struck by a bat. After leaving the bat, the ball travels in the opposite direction with speed v_2 also 39.0m/s

(a) What impulse J acts on the ball while it is in contact with the bat during the collision?

(b) The impact time for the baseball- bat collision is 1.20ms. What average net force acts on the baseball?

SOLUTION

(a)Let us choose the direction in which the ball is initially moving to be the negative direction.

$$\therefore J_x = P_{2x} - P_{1x} = mv_{2x} - mv_{1x}$$
$$= (0.140)(39.0) - (0.140)(-39.0)$$
$$= 10.9kgms^{-1}$$

The direction of the impulsive vector acting on the ball is in the direction in which the bat is swinging

(b) Average net force = $\frac{J_x}{t} = \frac{10.9}{1.20\times10^{-3}}$

$$= 9080N$$

ATWOOD MACHINE

When two objects of unequal mass are hung vertically over a frictionless pulley of negligible mass, the arrangement is called an Atwood machine. The device is sometimes used in the laboratory to measure the free-fall acceleration. As one object moves upward, the other object moves downward. Because the objects are connected by an inextensible string, their accelerations must be of equal magnitude. The objects in the Atwood machine are subjected to the gravitational force as well as to the forces exerted by the strings connected to them. The tension in the string on both sides of the pulley is the same.

When Newton's second law of motion is applied to object 1, we obtain

$$\sum F_y = T - m_1 g = m_1 a_y \dots\dots\dots\dots (1)$$

Similarly for object 2 we find

$$\sum F_y = m_2 g - T \dots\dots\dots\dots\dots.. (2)$$

When equation(2) is added to equation(1), T cancels and we have

$$-m_1 g + m_2 g = m_1 a_y + m_2 a_y$$

$$a_y = \frac{(m_2 - m_1)g}{(m_1 + m_2)} \dots\dots\dots\dots\dots (3)$$

When equation(3) is substituted into equation(1), we obtain

$$T = m_1 a_y + m_1 g = m_1(a_y + g) \dots\dots\dots.. (4)$$

When equation(3) is substituted into equation(4), we obtain

$$T = \left(\frac{2m_1m_2}{m_1+m_2}\right)g \dots\dots\dots\dots (5)$$

EXAMPLE

A light chord connecting objects of masses 10kg and 6kg passes over a light frictionless pulley

(a)What is the acceleration of the system?

(b)What is the tension in the chord?

SOLUTION

Let $m_1 = 10kg \; and \; m_2 = 6kg$

(1) $a = \frac{g(m_2-m_1)}{m_2+m_1}$

$$a = \frac{9.8(10 - 6)}{(10 + 6)}$$

$= 2.45ms^{-2}$

$$(2)\; T = \frac{g(2m_1m_2)}{(m_1+m_2)}$$

$$= \frac{9.8(2 \times 6 \times 10)}{(10+6)}$$

$= 73.5N$

A PASSENGER IN A LIFT

This is another instance where one can apply the second law of motion. There are three possibilities and each has different consequences. If R is the reaction on the passenger from the floor of the lift and W is his weight, then the following holds:

(1) When the lift is at rest,

$$R - W = 0 \text{ i.e } R = W$$

The man feels his real weight

(2) When the lift is ascending with an acceleration a,

$$R - W = ma$$

$$R = mg + ma$$

This is the apparent weight of the passenger and it shows why the passenger feels heavier than his normal weight.

(3) When the lift is descending with an acceleration a,

$$mg - R = ma$$

$$R = mg - ma$$

The passenger feels lighter than his real weight. When the lift moves down with an acceleration $a = g, R = 0$.

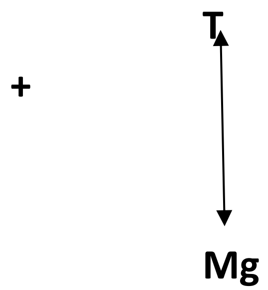
The man feels weightless.

EXAMPLE

An elevator cab and its load have a combined mass of 1600kg. Find the tension in the supporting cable when the cab, originally moving downward at 12m/s, is brought to rest with constant acceleration in a distance of 42m

SOLUTION

Consider the free-body diagram



Let T be the tension of the cable and mg be the force of gravity. If the upward direction is positive, then Newton's second law is

$$T - mg = ma$$

Where a is the acceleration.

Thus, T =m(g + a)

To determine the acceleration, a , we use the equation of motion

$$v^2 = {v_0}^2 + 2ay \qquad v = 0 \quad y = -42m \qquad v_0 = -12ms^{-1}$$

$$a = \frac{-{v_0}^2}{2y} = \frac{-(-12)^2}{2(-42)} = 1.71ms^{-2}$$

$$T = m(g + a)$$

$$= 1600(9.8 + 1.71)$$

$$= 1.8 \times 10^4 N$$

FRICTION

When one body slides or tends to slide over another body, the force that acts to oppose the relative motion between the two surfaces in contact is called the force of friction. This frictional force is always parallel to the surfaces that are in contact. The experimental fact about friction may be summarized by " laws of friction"

1. **A maximum frictional force on a body resting on another body is proportional to the normal force pushing the two surfaces together. For bodies at rest,**

   $F_s = \mu_s R$ where F_s = maximum force of static friction

   R = normal force

   Normal force is the resultant of all forces perpendicular to the surfaces.

   μ_s =coefficient of static friction.

   In other words, the coefficient of static friction =

   $$\frac{\text{Maximum force of friction}}{\text{Normal force}}$$

2. **If one body is sliding on another body, the force of friction is constant and independent of the relative velocity of the two surfaces. For bodies in motion,**

   $$F_k = \mu_k R$$

   Where F_k =sliding friction R = Normal force

   μ_k = coefficient of kinetic friction

   To a good approximation, static and kinetic frictional forces are independent on the surfaces that are in contact. Kinetic friction is almost independent of the relative

velocity of the object if the sliding velocity is not too great,
$\mu_k < \mu_s$

$R = mgcos\theta$

$$\mu = \frac{F}{mgcos\theta}$$

At the limiting point, when $mgsin\theta$ equals frictional force, the body is just set to move.

$$\mu = \frac{mgsin\theta}{mgcos\theta} = tan\theta$$

Where θ is the angle of inclination when the body is just set to move.

EXAMPLES

1. A $3.5kg$ block is pushed along a horizontal floor by a force $\vec{F}$ of magnitude 15N at an angle $\theta = 40°$ with the horizontal. The coefficient of kinetic friction between the block and the floor is 0.25. Calculate the magnitude of

   (a) Frictional force on the block from the floor

   (b)The block's acceleration

   SOLUTION

   We choose +x horizontally rightward and +y upward and observe that the 15N force has components $F_x = Fcos\theta$ and $F_y = -Fsin\theta$

   (a)We apply Newton's second law to the y axis

$F_N - Fsin\theta - mg = 0$

$F_N = 15sin40° + (3.5)(9.8) = 44N$

But $F_k = \mu_k F_N, \ \mu_K = 0.25$

$F_k = (0.25)(44) = 11N$

(b) We apply Newton's second law to the x-axis

$Fcos\theta - f_k = ma$

$$a = \frac{15cos40° - 11}{3.5}$$

$a = 0.14ms^{-2}$

2. The figure above shows an initially stationary block of mass m on a floor. A force of magnitude 0.500mg is then applied at upward angle $\theta = 20°$. What is the magnitude of the acceleration of the block across the floor if the friction coefficient are

(a) $\mu_s = 0.600 \ and \ \mu_k = 0.500$

(b) $\mu_s = 0.400 \ and \ \mu_k = 0.300$?

SOLUTION

The free- body diagram for the block is shown below, with $\vec{F}$ being the force applied to the block, $\overrightarrow{F_N}$ the normal force of the floor on the block, mg the force of gravity and $\vec{f}$ the force of friction. We take the + x direction to be horizontal to the right and the +y direction to be up.

$F_x = Fcos\theta - f = ma \ldots\ldots\ldots(1)$

$F_y = Fsin\theta + F_N - mg = 0 \ldots\ldots(2)$

Now, $f = \mu_k F_N$ and the second equation above gives

$F_N = mg - Fsin\theta$which yields

$f = \mu_k(mg - Fsin\theta)$. This expression is o substituted for in the first equation to obtain

$Fcos\theta - \mu_k(mg - Fsin\theta) = ma$

So the acceleration is

$$a = \frac{F}{m}(cos\theta + \mu_k sin\theta) - \mu_k g$$

(a) If we choose

$\mu_s = 0.600\ and\ \mu_k = 0.50$,then the magnitude of $\vec{f}$ has a maximum value of

$f_{max} = \mu_s F_N$

$= 0.600(mg - 0.50mgsin20°)$

$= 0.497mg$

on the other hand, $Fcos\theta = 0.50mgcos20°$

$= 0.470mg$

Therefore$Fcos\theta < f_{max}$ and the block remains stationary with a = 0

$(b)If\ \mu_s = 0.40$and $\mu_k = 0.30$, then the magnitude of $\vec{f}$ has a maximum value of

$f_{max} = \mu_s F_N$

$= 0.400(\mathrm{mg} - 0.50\mathrm{mgsin}20°) = 0.332\mathrm{mg}$

In this case, $Fcos\theta = 0.500mgcos20°$

$= 0.470mg > f_{max}$

Therefore, the acceleration of the block is

$$a = \frac{F}{m}(cos\theta + \mu_k sin\theta) - \mu_k g$$
$$= (0.500)(9.80)[cos20° + (0.30)sin20°] - (0.300)(9.80)$$
$$= 2.17ms^{-2}$$

Centripetal Force........

Conical Pendulum...

(2) A 15kg block rests on the surface of a smooth plane inclined at angle 30^0 to the horizontal. A light inextensible string passing over a small, smooth pulley at the top of the plane connects the block to another 13kg block hanging freely. Find the acceleration of the motion and the tension in the string.

<u>Solution</u>

For body 1, net force up the plane is $T - m_1 g sin30^0 = m_1 a$ ------------ (1)

$T - (15 X 9.8)sin30^0 = 15a$

$T = 15a + 73.5$

For body 2, net downward force is $m_2 g - T = m_2 a$ --------------- (2)

$(13)(18) - T = 13a$

$T = 127.4 - 13a$

From equations (1) & (2) $a = 1.93ms^{-2}$ $T = 102.4N$

(3). If the coefficient of kinetic friction between the plane and the 15kg mass is 0.25, find the acceleration of the resulting motion.

Solution

The upward motion of body 1 is now opposed by a frictional force = $\mu_s N$ = 0.25N

Where N is the normal reaction which is equal to $W\cos 30^0$ (by resolving normal to plane). Thus, net force up the plane in body 1 is

$T - m_1 g\sin 30^0 - 0.25(m_1 g\cos 30^0) = m_1 a$

$T = 15a + 105.3$

For body 2, $m_2 g - T = m_2 a$

$T = 127.4 + 13a$

$a = 0.79ms^{-2}$

A BICYCLE RIDER ON A ROUND TRACK

When a bicycle rider makes a bend, he must bend sideways in order that the ground may provide a frictional force F towards the center of curvature of the round track. This is called the centripetal force which keeps him in the round track.

The friction F is between the tyres and the ground and it is directed towards the centre. For equilibrium, the net moment about any point is zero.

Therefore, $Ra = Fh$

$$\frac{a}{h} = \frac{F}{R}$$

By substituting $\frac{a}{h} = tan\theta$ **and** $R = mg$ $\quad F = \frac{mv^2}{R}$

We obtain $tan\,\theta = \frac{v^2}{rg}$

Generally, θ is the angle to the vertical necessary for the rider to remain on track. Skidding occurs when $\frac{mv^2}{R} > \mu mg$

$\frac{v^2}{Rg} > \mu$ Or $\tan\theta > \mu$

BANKING OF CURVES

Diagram

Net force $F = ma_c = \frac{mv^2}{R}$ ------------- (1)

$$\tan\theta = \frac{Net\ force}{mg} \text{ --------------- } (2)$$

$$\therefore Net\ force = mg\tan\theta \text{ ----------- } (3)$$

From (1) and (2) $mg\tan\theta = \frac{mv^2}{R}$

$$\tan\theta = \frac{v^2}{Rg} \quad (\theta = angle\ of\ banking)$$

This formula shows that the angle of banking depends on the speed and on the radius of the curve. The absence of m from the formula shows that all cars and trucks require the same banking on the same curve at any given speed, regardless of their masses.

EXAMPLE

1. An airplane hovering around an airport in a circular track completes a circular turn in 2 minutes. If the speed of the airplane is $100ms^{-1}$. Determine
   (a) The radius of the circular track
   (b) The banking angle of the plane and
   (c) The centripetal force on the airplane, expressed as a fraction of its weight.

   Solution

(a) $T = 120s \qquad v = 100ms^{-1}$

$$T = \frac{2\pi r}{v} \qquad r = \frac{vT}{2\pi} = \frac{100 \times 120}{2\pi} = 1.91 \times 10^3 m$$

(b) The upward force F on the plane which keeps it in the air is provided by the air pressure which acts at normal to the wings. To make a turn, the wings have to be tilted (banked) at an angle θ to the horizontal. The situation then resembles that of a banked track.

$$\theta = tan^{-1}\left(\frac{v^2}{rg}\right) = tan^{-1}\left(\frac{100^2}{1.91 \times 10^3 \times 9.8}\right) = 28.1^\circ$$

(c) $\frac{F_c}{mg} = \frac{mv^2}{rmg} = \frac{v^2}{rg} = \frac{100^2}{1.91 \times 10^3 \times 9.8} = 0.53$

ASSIGNMENT

1. What is the maximum speed at which a car of mass m can go around an unbanked curve of radius 40m if the coefficient of static friction between the tyre and road is 0.7 ($16.6ms^{-1}$)
2. At what angle should a level curve of radius 200m be banked for cars and trucks travelling at $24.6ms^{-1}$
3. What is the proper speed for a car to go around a slippery curve of radius 50m if the road is banked at an angle of 25°

GRAVITATIONAL FIELD STRENGTH

The gravitational field strength at a point in a gravitational field is defined as the gravitational force of attraction per unit mass at that point.

Gravitational field strength, $g = \frac{gravitational\ force}{mass}$

$$g = \frac{\frac{GMm}{r^2}}{m}$$

$$g = \frac{GM}{r^2}$$

The gravitational field strength g is a vector having direction as well as magnitude. The addition of gravitational field must be done by vector addition. The unit of gravitational field strength is Nkg^{-1}. Since $F = ma$, it follows that acceleration is produced by gravitational field strength. The gravitational field strength on the earth's surface is $9.8Nkg^{-1}$ and it produces an acceleration of $9.81ms^{-2}$

Example

If the mass of the moon and its radius are $7.40 \times 10^{22}kg$ and $1.76 \times 10^{6}m$ respectively, find the free fall acceleration g of an object on or near the surface of the moon.

Solution

$$g_m = \frac{GM_m}{{r_m}^2}$$

$$G = 6.67 \times 10^{-11}Nm^2kg^{-2} \quad M_m = 7.40 \times 10^{22}kg \quad r_m = 1.76 \times 10^{6}m$$

$$g_m = \frac{(6.67 \times 10^{-11}Nm^2kg^{-2})(7.40 \times 10^{22}kg)}{(1.76 \times 10^{6}m)^2} = 1.71ms^{-2}$$

GRAVITATIONAL POTENTIAL

Besides gravitational field strength, there is another quantity associated with a gravitational field, that is, the gravitational potential. The gravitational potential,

V at a point in a gravitational field is the work done by the gravitational attraction to bring a unit mass from infinity to that point. The gravitational potential at infinity is assumed to be zero. Gravitational potential is a scalar quantity and its unit is Jkg^{-1}.

CALCULATION OF THE GRAVITATIONAL POTENTIAL DUE TO A SPHERICAL MASS

To find the gravitational potential at a point P distance r from the centre of the earth, consider a mass $m = 1\,kg$ at a distance x from the centre of the earth. The gravitational attraction on the $1\,kg$ mass is

$$F = \frac{GM \times 1}{x^2} \qquad M = mass\ of\ earth$$

When the 1 kg mass is moved a small distance dx towards the earth,

Work done, $dV = F\,dx$

$$= \frac{GM}{x^2}\,dx$$

To move the 1 kg mass from infinity $(x = \infty)$ to the point P $(x = r)$,

Work done $= \int_{\infty}^{r} F\,dx$

$$= \int_{\infty}^{r} \frac{GM}{x^2}\,dx$$

$$= -\frac{GM}{r}$$

Hence from the definition of gravitational potential,

$$V = -\frac{GM}{r}$$

The negative sign for V denotes that the gravitational potential at infinity is zero and decreases for points closer to the earth. On the surface of the earth $r = R$, radius of earth,

Gravitational potential $= -\frac{GM}{R}$

GRAVITATIONAL POTENTIAL ENERGY

The gravitational potential energy U of a body of mass m at a point in a gravitational field is defined as the work done to bring the body from infinity to that point.

Gravitational potential energy, $U = mV$

$$= -\frac{GMm}{r} \qquad \left(V = -\frac{GM}{r}\right)$$

For mass m at a distance r from earth which is of mass M.

The relationship between gravitational potential energy U and gravitational attraction F is

$$F = -\frac{dU}{dr} \quad \text{or } U = -\int F\,dr$$

Also, the gravitational field strength

$$g = -\frac{dV}{dr} \qquad (\text{ V = gravitational potential })$$

The gravitational field strength is numerically equal to the gravitational potential gradient.

SATELLITE

Any satellite is kept in its orbit by a gravitational attraction of the body about which it is rotating. Satellites are either artificial (man made) or natural (planets or moon)

Synchronous satellites are satellites that move round the earth at the same rate as the rotation of the earth about its axis. That is,

(a) Synchronous satellites have a period of 24hours and appear stationary when seen from the earth.

(b) They orbit round the earth in the same direction as the rotation of the earth, that is, west to east.

KEPLER'S LAWS OF PLANETARY MOTION

1. Each planet goes around the sun in an elliptical orbit with the sun at a focus of the ellipse
2. Imaginary lines from a planet to the sun sweeps out equal areas each second, whether the planet is close to or far from the sun.
3. The squares of the periods of revolution of the planets are proportional to the cubes of their mean distances from the sun. If T_1 and T_2 represent the periods of two planets, and if R_1 and R_2 represent their average distances from the sun then the third law may be expressed as follows:

$$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{R_1}{R_2}\right)^2$$

PROOF

$$\frac{GMm}{r^2} = m\omega^2 r$$

$$\frac{GM}{r^3} = \omega^2$$

But $\omega^2 = \left(\frac{2\pi}{T}\right)^2 = \frac{4\pi^2}{T^2}$

$$T^2 = \left(\frac{4\pi^2}{GM}\right)r^3$$

$T^2 \propto r^3$ since $\frac{4\pi^2}{GM}$ is constant

ENERGY OF SATELLITE

When a satellite of mass m is in an orbit of radius r from the centre of the earth, its gravitational potential energy is

$$U = -\frac{GMm}{r}, \quad M = mass\ of\ earth$$

The satellite also has kinetic energy of

$$K = \frac{1}{2}mv^2$$

But $\frac{mv^2}{r} = \frac{GMm}{r^2}$

$$mv^2 = \frac{GmM}{r}$$

$$\therefore K = \frac{1}{2}\frac{GMm}{r}$$

Hence, the total energy of the satellite

$$E = U + K$$

$$= -\frac{GMm}{r} + \frac{1}{2}\frac{GMm}{r}$$

$$= -\frac{GMm}{2r}$$

Example

As a result of air resistance, a satellite of mass 10.5kg loses energy slowly over a period of time and falls from an orbit of radius $7.5 \times 10^6 m$ to one radius $7.0 \times 10^6 m$. Find the change in the potential and kinetic energies of the satellite during this period and hence find the total loss of energy. What are the initial and final values of the speed of the satellite?

Solution

Potential energy $= -\frac{GMm}{r}$

$$P.E_1 = -\frac{GMm}{r_1} = -\frac{(6.67 \times 10^{-11})(6.0 \times 10^{24})(10.5)}{7.5 \times 10^6}$$

$$= -5.6 \times 10^9 J$$

$$P.E_2 = -\frac{GMm}{r_2} = -\frac{(6.67 \times 10^{-11})(6 \times 10^{24})(10.5)}{7.0 \times 10^6}$$

$$= -6.0 \times 10^9 J$$

Change in potential energy $\Delta P.E = P.E_2 - P.E_1$

$$= -6.0 \times 10^9 J - (5.6 \times 10^9 J)$$

$$= -4.0 \times 10^8 J$$

Kinetic energy K.E $= \frac{GMm}{2r}$

$$K.E_1 = \frac{GMm}{2r_1} = \frac{(6.67 \times 10^{-11})(6.0 \times 10^{24})(10.5)}{2 \times 7.5 \times 10^6} = 2.8 \times 10^9 J$$

$$K.E_2 = \frac{GMm}{2r_2} = \frac{(6.67 \times 10^{-11})(6.0 \times 10^{24})(10.5)}{2 \times 7.0 \times 10^6} = 3.0 \times 10^9 J$$

Change in kinetic energy $\Delta K.E = K.E_2 - K.E_2$

$$= (3.0 - 2.8) \times 10^9 J$$

$$2.0 \times 10^8 J$$

Total energy, T.E = K.E + P.E

$$T.E_1 = K.E_1 + P.E_1 = (2.8 - 5.6) \times 10^9 J = -2.8 \times 10^9 J$$

$$T.E_2 = K.E_2 + P.E_2 = (3.0 - 6.0) \times 10^9 J = -3.0 \times 10^9 J$$

Total loss of energy $= T.E_1 - T.E_2$

$$= (-2.8 \times 10^9 J) - (-3.0 \times 10^9 J) = 2 \times 10^8 J$$

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

$$v^2 = \frac{GM}{r}$$

$$v_1^{\,2} = \frac{GM}{r_1} = \frac{(6.67\times10^{-11})(6.0\times10^{24})}{7.5\times10^{6}} = 5.3 \times 10^{7}$$

$$v_1 = 7.3 \times 10^{3} ms^{-1}$$

$$v_2^{\,2} = \frac{GM}{r_2} = \frac{(6.67 \times 10^{-11})(6.0 \times 10^{24})}{7.0 \times 10^{6}} = 5.7 \times 10^{7}$$

$$v_2 = 7.5 \times 10^{3} ms^{-1}$$

ESCAPE VELOCITY

This term is often called escape velocity but since only magnitude of the velocity is needed and not its direction, it is more accurately called the escape speed. The escape speed can be obtained by realizing that the kinetic energy per unit mass = gain in potential energy in leaving the gravitational well.

$$\frac{1}{2}mv^2 = \frac{GMm}{r}$$

$$\frac{1}{2} \times 1\,kg \times v^2 = \frac{GM}{r} \times 1kg$$

$$v^2 = \frac{2GM}{r}$$

Since $\frac{GMm}{r^2} = mg$

$$\frac{GM}{r} = rg$$

$$v^2 = rg$$

$$v = \sqrt{rg}$$

If the average radius of the earth is $6.4 \times 10^{6}m$**, then** $v = 11.2\ kms^{-1}$**.The escape speed** v **is** $\sqrt{2rg}$ **and is therefore** $\sqrt{2} \times$ **the speed needed for a circular**

orbit. The kinetic energy in orbit is therefore half of the kinetic energy needed to escape.

CONSERVATIVE AND DISSIPATIVE FORCES

A force is said to be conservative if the work done by it on a particle to move the particle between two points depends only on these points and not on the path taken. Gravitational force is conservative as shown below.

For the body of mass m either through path 1 or 2 to the ground level, the work done (potential energy) is the same. The work done in moving round a closed path (mABm) in a conservative field is zero.

$$W = \oint \vec{F}.\vec{dr} = 0$$

Force F is said to be dissipative or non-conservative if the work done by it between two points depends on the paths taken between the two points. An example is frictional force in which the work done in a closed path is not zero.

$$W = \oint \vec{F}.\vec{dr} \neq 0$$

EXAMPLES

1. A satellite is sent to a distance where it revolves about the earth with an angular velocity equal to that at which the earth rotates so that it always remains above the same point on the earth. Determine the radius of such an orbit if the mass of the earth is $5.98 \times 10^{24} kg$.

   Solution

   The orbital velocity of the satellite $v = \sqrt{\frac{GM_e}{r}}$

   Period of the satellite $T = 24hrs = 24 \times 3600s = 8.64 \times 10^4 s$

But $T = \frac{2\pi r}{v} = \frac{2\pi r}{\sqrt{\frac{GM_e}{r}}} = 8.64 \times 10^4 s$

By substituting $G = 6.67 \times 10^{-11} Nm^2 kg^{-2}$ $M_e = 5.98 \times 10^{24} kg$

$r = 4.2 \times 10^7 m$

2. Certain neutrons stars (extremely dense stars) are believed to be rotating at about one revolution per second. If such a star has a radius of 20km, what must its mass be so that objects on its surface will be attracted to the star and not be thrown off by the radial rotation?

Solution

If m_s = mass of star m_0 = mass of objects r_s = radius of star

v = Velocity of object (and star) = ωr_s where ω is the angular velocity.

$$F_g = \frac{Gm_s m_0}{{r_s}^2} = \frac{m_0 v^2}{r_s}$$

$$v = \omega r_s = \frac{(2\pi) r_s}{T} = 2\pi r_s = 1.26 \times 10^5 ms^{-1}$$

$$m_s = \frac{r_s v^2}{G} = \frac{(2 \times 10^4)(1.26 \times 10^5)^2}{6.67 \times 10^{-11}} = 4.74 \times 10^{24}\ kg$$

3. At what altitude above the earth's surface would the acceleration due to gravity be $4.9 ms^{-2}$? Assume that the mean radius of the earth is $6.4 \times 10^6 m$ and the acceleration due to gravity on the earth surface is $9.8 ms^{-2}$

Solution

$$F_g = \frac{Gm_e m}{r^2} = mg \quad \therefore r^2 = \frac{Gm_e}{g} = \frac{Gm_e}{4.9} ----------1$$

On the earth's surface, $g = \frac{Gm_e}{r^2}$ $\therefore gr^2 = Gm_e$

$$Gm_e = (9.8)(6.4 \times 10^6)^2 = 4.01 \times 10^{14}$$

Equation (1) then gives $r^2 = \frac{(4.01 \times 10^{14})}{4.9}$

$r = 9.05 \times 10^6 m$ from the earth's centre

Or $(9.05 - 6.40) \times 10^6 m$ from the earth's surface

Assignment

The mass of the earth is specified as $5.98 \times 10^{24}\ kg$ and its radius is $6.4 \times 10^6 m$. The mass of the moon is $7.4 \times 10^{22}\ kg$ while its radius is $1.6 \times 10^6 m$. Given that the distance from the earth to the moon is $4 \times 10^8 m$ and $G = 6.67 \times 10^{-11} Nm^2kg^{-2}$ Determine:

(a) the acceleration due to gravity on the moon.

(b) the gravitational potential energy of the moon with respect to the earth.

WORK AND ENERGY

Work done by an external force on a body is the product of the force (F) and displacement (s), provided the force acts in the same direction as the displacement.

Diagram 1

Then work done = F X S

If the force acts an angle θ to displacement (diagram 2), the work done is the resolved component of the force along the direction of displacement multiplied by the displacement , that is, $W = FS \cos\theta$

Diagram 2.

Hence, the work done is defined as the scalar product of force F that moves its point of application through a displacement S

$W = \vec{F}.\vec{S} = |\vec{F}||\vec{S}| \cos\theta$ where θ is the angle between $\vec{F}$ and $\vec{S}$. The unit of work done is Joule (J)

Examples

1. A body moves through a distance $\vec{S} = 5i + 2j - 3k$ from the origin $(0, 0, 0)$ under the influence of a constant force $\vec{F} = 4i + 3j - 2k$.
   Determine: (a) the work done

   (b) the angle between $\vec{F}$ and $\vec{S}$

   <u>Solution</u>

(a) Work done, W $= \vec{F}.\vec{S} = (4i + 3j - 2k).(5i + 2j - 3k)$

$$W = (4 \times 5) + (3 \times 2) + (-2 \times -3) = 32J$$

(b) $|\vec{F}| = \sqrt{4^2 + 3^2 + (-2)^2} = \sqrt{29}N = 5.4N$

$$|\vec{S}| = \sqrt{5^2 + 2^2 + (-3)^2} = \sqrt{38}m = 6.16m$$

$$W = |\vec{F}||\vec{S}|cos\,\theta \qquad cos\,\theta = \frac{W}{|\vec{F}||\vec{S}|} = \frac{32}{5.4 \times 6.16} = 0.9621$$

$$\theta = cos^{-1}(0.9621) = 15.8°$$

2. A force $F = (6i - 2j)N$ acts on a particle that undergoes a displacement $\Delta r = (3i + j)m$. Find (a) the work done by the force on the particle
   (b) the angle between F and Δr
   <u>Solution</u>
   (a) $W = F.\Delta r = F_x x + F_y y = (6.0)(3.0) + (-2)(1) = 16.0\,J$

   (b) $\theta = cos^{-1}\left(\frac{F.\Delta r}{|F||\Delta r|}\right) = cos^{-1}\left(\frac{16}{\sqrt{(6.0^2 + (-2.0)^2)(3.0^2 + 1.0^2)}}\right) = 36.9°$

3. A Force $F = (4xi + 3yj)N$ acts on an object as the object moves in the x- direction from the origin to $x = 5.0m$. Find the work done on the object by the force.
   <u>Solution</u>
   $W = \int_{x_1}^{x_2} F.dr = \int_0^5 (4x\,i + 3y\,j)N.dx\,i = \int_0^5 4x\,dx + 0 = 50J$

POWER

Power is the rate of doing work. If an amount of work Δw is done in a small interval of time Δt, then the power P is

$$P = \frac{\Delta w}{\Delta t}$$

The unit of Power is Js^{-1} which is also called Watt

Since $W = FS$ $\quad \frac{dw}{dt} = \frac{d}{dt}(FS)$

For a constant force,

$$\frac{dw}{dt} = F\frac{ds}{dt} = F.V$$

$V = \frac{ds}{dt}$ = instantaneous velocity of displaced body.

If $\vec{F}$ acts at an angle θ to v, then

$P = (Fcos\theta).(v)$ or the scalar product of F and v

Thus Power $= \vec{F}.\vec{v}$

Example

The electric motor of a model train accelerates the train from rest to $0.620ms^{-1}$ in $21.0ms$. The total mass of the train is $875g$. Find the average power delivered to the train during the acceleration.

Solution

Average power $= \frac{W}{t} = \frac{K_f}{t} = \frac{mv^2}{2t} = \frac{(0.875)(0.620)^2}{2 \times 21 \times 10^{-3}} = 8.01W$

ENERGY

Energy is the capacity of a body to do work. The energy acquired by a body by virtue of its position is called potential energy while that due to motion is its kinetic energy.

The work-energy theorem states that the change in energy of a body is equal to the work done on the body. Thus, if a body of mass m which is lifted to height h, the force needed for lifting it is equal to the weight mg of the body and the work done on the body is

Work done = Force x distance

Or W =mgh

This according to the work- energy theorem is also the gravitational potential energy, E_p acquired by the body by its virtue of its position (height, h) relative to the ground.

Examples

1. A 0.60kg particle has a speed of $2.0ms^{-1}$ at point A and kinetic energy of $7.50J$ at point B. What is (a) its kinetic energy at A? (b) Its speed at B? (c) The total work done on the particle as it moves from A to B?

   Solution

   (a) $K_A = \frac{1}{2}(0.60)(2.0)^2 = 1.20J$

   (b) $\frac{1}{2}mv_B{}^2 = K_B \therefore v_B = \sqrt{\frac{2K_B}{m}} = \sqrt{\frac{2 \times 7.50}{0.60}} = 5.0ms^{-1}$

   (c) $\sum W = K_B - K_A = \frac{1}{2}m(v_B{}^2 - v_A{}^2) = \frac{1}{2}(0.60)(5^2 - 2^2)$

   $$\sum W = 6.30 \text{ J}$$

2. A 3.0kg object has a velocity $(6.0i - 2.0j)ms^{-1}$
   (a) What is its kinetic energy at this time?
   (b) Find the total work done on the object if its velocity changes to $(8.0i + 4.0j)ms^{-1}$.

   Solution

   $(a)\ v_i = (6.0i - 2.0j)ms^{-1} |v_i| = \sqrt{(6.0)^2 + (-2.0)^2} = \sqrt{40}ms^{-1}$

   $$K_i = \frac{1}{2}mv_i{}^2 = \frac{1}{2}(3.0)(40) = 60.0J$$

(b)

$v_f = (8.0i + 4.0j)ms^{-1}$ $|v_f| = \sqrt{(8.0)^2 + (4.0)^2} = \sqrt{80}ms^{-1}$

$K_f = \frac{1}{2}mv_f^2 = \frac{1}{2}(3.0)(80) = 120\,J$

$\sum W = K_f - K_i = (120 - 60)J = 60\,J$

ASSIGNMENT

1. **A 12.0N force with a fixed orientation does work on a particle as the particle moves through the three- dimensional displacement $\vec{d} = (2.0i - 4.0j + 3.0k)m$. What is the angle between the force and the displacement if the change in the particle's kinetic energy is $30.0\,J$? ($62.3°$)**

2. **A single force acts on a 3.0 kg particle-like object whose position is given by $x = 3.0t - 4.0t^2 + 1.0t^3$, with x in metres and t in seconds. Find the work done on the object by the force from $t = 0$ to $t = 4.0s$**

PRINCIPLE OF LINEAR MOMENTUM CONSERVATION

The linear momentum of a particle or an object of mass m moving with velocity v is defined as the product of the mass and velocity, $p = mv$

Linear momentum is a vector quantity. If a particle is moving in an arbitrary direction, p must have three components, p_x, p_y and p_z given by

$p_x = mv_x$, $p_y = mv_y$, $p_z = mv_z$

The law of conservation of momentum states that whenever two or more particles in an isolated system interact, the total momentum of the system remains constant.

Total momentum before collision = Total momentum after collision

ELASTIC AND INELASTIC COLLISION

A collision is said to be elastic if kinetic energy before collision is equal to kinetic energy after collision. We have an inelastic collision if kinetic energy before collision is not equal to kinetic energy after collision

Diagram 1

Momentum before collision = Momentum after collision

$m_1u_1 + m_2u_2 = m_1v_1cos\theta + m_2v_2cos\emptyset$ (Along x- axis)

Before collision, momentum along y = 0

After collision, $m_1v_1sin\theta - m_2v_2sin\emptyset$

$m_1u_1 + m_2u_2 = m_1v_1cos\theta + m_2v_2cos\emptyset$ ------------- (1)

$0 = m_1v_1sin\theta - m_2v_2sin\emptyset$ --------------- (2)

From equation (2), $m_2v_2sin\emptyset = m_1v_1sin\theta$

$$sin\,\emptyset = \frac{m_1v_1sin\theta}{m_2v_2}$$

COEFFICIENT OF RESTITUTION

Diagram 2

CONSERVATION PRINCIPLE

1. Conservation of momentum: $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$ ------(3)
2. Conservation of energy: $\frac{1}{2}m_1{u_1}^2 + \frac{1}{2}m_2{u_2}^2 = \frac{1}{2}m_1{v_1}^2 + \frac{1}{2}m_2{v_2}^2$ (4)

   From equation (3)

   $m_1(u_1 - v_1) = m_2(v_2 - u_2)$ ------------- (5)

From equation (4)

$$m_1({u_1}^2 - {v_1}^2) = m_2({v_2}^2 - {u_2}^2)$$ ------------- (6)

By re-writing equation (6)

$$m_1(u_1 - v_1)(u_1 + v_1) = m_2(v_2 - u_2)(v_2 + u_2)$$ ------------ (7)

By dividing equation (7) by equation (5)

$$\frac{m_1(u_1 - v_1)(u_1 + v_1)}{m_1(u_1 - v_1)} = \frac{m_2(v_2 - u_2)(v_2 + u_2)}{m_2(v_2 - u_2)}$$ -------------- (8)

$$u_1 + v_1 = u_2 + v_2$$ ------------------ (9)

$$u_1 - u_2 = v_2 - v_1$$

$$\frac{v_2 - v_1}{u_1 - u_2} = e$$ ------------- (10)

"e" is called the coefficient of restitution

The coefficient of restitution is the ratio of velocity of separation to velocity of approach.

In elastic scattering , e = 1

In elastic scattering , $0 < e < 1$

In coalition, e = 0

It can be shown that when two masses m_1 and m_2 collide and their initial velocities are u_1 and u_2 respectively, the loss of kinetic energy due to the impact is

$$\Delta(K.E) = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(u_1 - u_2)^2(1 - e^2)$$ ------------- (11)

If $\Delta(K.E) = 0$ then e = 1 (Elastic scattering)

<u>Examples</u>

1. An object of mass $3.0kg$, moving with an initial velocity of $5.0i\ ms^{-1}$, collides with and sticks to an object of mass $2.0kg$ with an initial velocity of $-3.0j\ ms^{-1}$. Find the final velocity of the composite object.

Solution

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2)v_f$$

$$(3.0)(5.0i) - (2.0)(3.0j) = 5.0v_f$$

$$v_f = (3.0i - 1.20j)ms^{-1}$$

2. Two particles with masses m and $3m$ are moving towards each other along the x- axis with the same initial speeds v_i. Particle m is travelling to the left, while particle $3m$ is travelling to the right. They undergo an elastic glancing collision such that m is moving downwards after the collision at right angles from its initial direction.
   (a) Find the final speeds of the two particles.
   (b) What is the angle θ at which the particle $3m$ is scattered ?

Solution

x- component of momentum for the system of the two objects is

$$p_{1ix} + p_{2ix} = p_{1fx} + p_{2fx}$$

$$-mv_1 + 3mv_1 = 0 + 3mv_{2x} \quad \text{----------- (1)}$$

y- component of momentum of the system is

$$0 = -mv_{1y} + 3mv_{2y} \quad \text{------------ (2)}$$

By conservation of energy of the system we have

$$\frac{1}{2}mv_1^2 + \frac{1}{2}(3m)(v_1)^2 = \frac{1}{2}mv_{1y}^2 + \frac{1}{2}(3m)\left(v_{2x}^2 + v_{2y}^2\right) \text{----(3)}$$

From equation (1) we have $v_{2x} = \frac{2}{3}v_1$

Also from equation (2) we have $v_{1y} = 3v_{2y}$

From equation (3), we obtain $v_{2y} = \frac{\sqrt{2}v_1}{3}$

∴ The object of mass m has final speed $\sqrt{2}v_1$

The object of mass $3m$ moves at

$$\sqrt{{v_{2x}}^2 + {v_{2y}}^2} = \sqrt{\frac{4}{9}{v_1}^2 + \frac{2}{9}{v_1}^2} = \sqrt{\frac{2}{3}}v_1$$

(c) $\theta = tan^{-1}\left(\frac{v_{2y}}{v_{2x}}\right) = tan^{-1}\left(\frac{\sqrt{2}}{3}v_1.\frac{3}{2v_1}\right) = 35.3^{\circ}$

ASSIGNMENT

A 3.0kg particle has a velocity of $(3.0i - 4.0j)ms^{-1}$.

(a) Find its x and y components of momentum

(b) Find the magnitude and direction of its momentum.