FOURIER SERIES

A Fourier series expresses a Periodic function in terms of! Infinite number of Periodic terms that are harmonically related most Periodic function contains Infinite number of Periodic terms irrespective of shape.

Given a Periodic function f(t), its Fourier series will be expressed as;

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n w_0 t + b_n \sin n w_0 t$$

where $n = 1, 2, 3 \ldots$

a_0, a_n, b_n are called Linear Co-efficient

w_0: is called the Fundamental Frequency of the Periodic function f(t)

$$w_0 = \frac{2\pi}{T}$$

The Integral multiples of w_0 i.e. $2w_0, 3w_0, 4w_0 \ldots$ are known as the harmonic frequency f(t)

Note; A Periodic function must satisfy some condition to be Expressed as Fourier Series.

The condition is known as Dirchlet conditions.

They conditions are;

1 f(t) must be a single value

2 f(t) must have a finite number of discontinuities in Periodic interval

3 f(t) must have a finite number of maxima and minima in Periodic Interval

4 The Integral $\int_0^T |f(t)|\,dt$ must exist

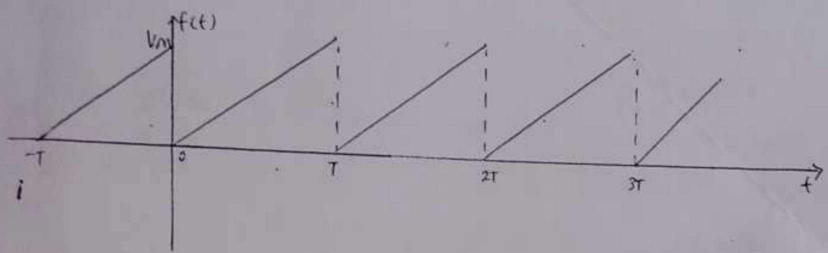
FORMULAR FOR SERIES FOURIER CO-EFFICIENTS

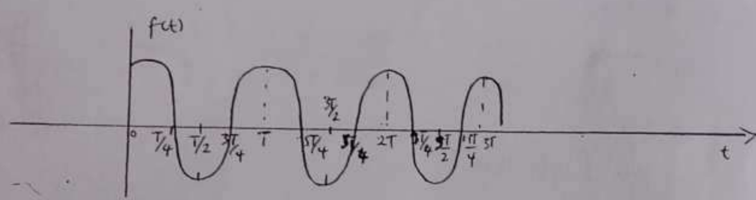
$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t \, dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t \, dt$$

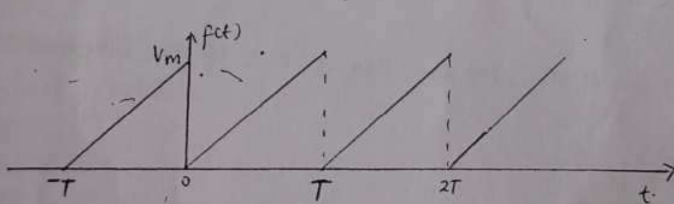
TYPES OF WAVES





Example 1;

Find the Fourier Series for the Periodic function shown below



Solution;

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos n\omega_0 t + b_n \sin n\omega_0 t\right]$$

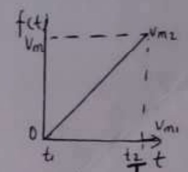
such that;

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t \, dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t \, dt$$

This is a triangular wave and follows a straight line graph pattern such that we will obtain our f(t) using the relationship : $Y = mx + c$



From the ~~grap~~ wave above we cut out the 1st wave pattern and since it obeys a straight line graph we will easily obtain our f(t)

$$Y = mx + c$$

were $Y = f(t)$

$$m = \text{slope} = \frac{\Delta V_m}{\Delta T} = \frac{V_{m_2} - V_{m_1}}{T_2 - T_1} = \frac{V_m - 0}{T - 0} = \frac{V_m}{T}$$

$$x = t$$

$$c = 0 \ \{\text{intercept}\}$$

$$f(t) = \frac{V_m}{T} t$$

$$a_0 = \frac{1}{T} \int_0^T \frac{V_m}{T} t \, dt$$

since $\frac{V_m}{T}$ is a constant, we can pull it out of the integration sign such that

$$a_0 = \frac{1}{T} \cdot \frac{V_m}{T} \int_0^T t \, dt$$

$$a_0 = \frac{V_m}{T^2} \int_0^T t \, dt$$

$$a_0 = \frac{V_m}{T^2} \left| \frac{t^2}{2} \right|_0^T$$

We then substitute t at various interval {higher interval - lower interval}

$$a_0 = \frac{V_m}{T^2} \left| \frac{T^2}{2} - \cancelto{0}{\frac{(0)^2}{2}} \right|$$

$$a_0 = \frac{V_m}{\cancel{T^2}} \times \frac{\cancel{T^2}}{2} = \frac{V_m}{2}$$

$$a_0 = \frac{V_m}{2}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t \, dt$$

we substitute for $f(t) = \frac{V_m}{T} t$

$$a_n = \frac{2}{T}\int_0^T \frac{V_m}{T} t \cos n\omega_0 t \, dt$$

Since $\frac{V_m}{T}$ is a Constant, we pull it out of the Integration sign

$$a_n = \frac{2}{T} \times \frac{V_m}{T}\int_0^T t\cos n\omega_0 t \, dt$$

$$a_n = \frac{2V_m}{T^2}\int_0^T t\cos n\omega_0 t \, dt$$

Since we have two functions, an algebraic function t and a Trignometric function $\cos n\omega_0 t$, We Integrate by parts

$$\frac{2V_m}{T^2}\int_0^T t\cos n\omega_0 t \, dt = uv - \int v \, du$$

From our knowlege of Types of function, the algebraic function Comes before the trignometric function, therefore Our $u = t$ and $dv = \cos n\omega_0 t$

I nverse function eg. $\sin x^{-1}$
L ogarithmic function eg. $\log x$
A lgebraic function. eg t, t^2, x
T rignometric function eg. $\sin x, \cos t$
E xponential function eg. e^x

Let $u = t$

$$\frac{du}{dt} = 1$$

$$du = dt$$

$$dv = \cos n\omega_0 t$$

$$v = \frac{\sin n\omega_0 t}{n\omega_0}$$ {v is gotten from integrating dv}

Note; Prove that $\int \cos n\omega_0 t \, dt = \frac{\sin n\omega_0 t}{n\omega_0}$

Solution, Let $u_1 = n\omega_0 t$

$$\frac{du_1}{dt} = n\omega_0$$

$dt = \frac{du_1}{nw_0}$

Substituting for u_1 and dt, we have

$$\int \cos u_1 \frac{du_1}{nw_0}$$

Since nw_0 is a constant, we pull it out of Integration sign we then have that;

$$\frac{1}{nw_0}\int \cos u_1\, du_1 = \frac{1}{nw_0}\sin u_1$$

We then substitute for $u_1 = nw_0t$ from our Initial assumption

therefore $\frac{1}{nw_0} \times \sin nw_0t = \frac{\sin nw_0t}{nw_0}$

$$\frac{2V_m}{T^2}\int_0^T t\cos nw_0t\, dt = uv - \int v\, du$$

Let $u = t$, $du = dt$

$dv = \cos nw_0t\, dt$, $v = \frac{\sin nw_0t}{nw_0}$

$$\frac{2V_m}{T^2}\left[\frac{t\sin nw_0t}{nw_0} - \int \frac{\sin nw_0t}{nw_0}\, dt\right]_0^T$$

$$\frac{2V_m}{T^2}\left[\frac{t\sin nw_0t}{nw_0} - \frac{1}{nw_0}\int \sin nw_0t\, dt\right]_0^T$$

note; $\int \sin nw_0t\, dt = \frac{-\cos nw_0t}{nw_0}$

Prove; Let $u_2 = nw_0t$

$\frac{du_2}{dt} = nw_0$

$du_2 = nw_0\, dt$

$dt = \frac{du_2}{nw_0}$

substituting for u_2 and dt, we have;

$$\int \sin u_2 \frac{du_2}{nw_0} = \frac{1}{nw_0}\int \sin u_2\, du_2 = \frac{1}{nw_0}(-\cos u_2)$$

$u_2 = nw_0t$ from our Initial assumption

$= \frac{-\cos nw_0t}{nw_0}$

$$\frac{2V_m}{T^2}\left[\frac{t\sin n\omega_0 t}{n\omega_0} - \frac{1}{n\omega_0}\left[-\frac{1}{n\omega_0}\right]\cos n\omega_0 t\right]_0^T$$

$$\frac{2V_m}{T^2}\left[\frac{t\sin n\omega_0 t}{n\omega_0} + \frac{1}{n^2\omega_0^2}\cdot\cos n\omega_0 t\right]_0^T$$

but $a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt$

$$\Rightarrow \frac{2V_m}{T^2}\left[\frac{1}{n^2\omega_0^2}\cos n\omega_0 t + \frac{t}{n\omega_0}\sin n\omega_0 t\right]_0^T$$

we then Input our Limits

$\int^T$ (upper Limit)

$$\frac{1}{n^2\omega_0^2}\cos n\omega_0 T + \frac{T}{n\omega_0}\sin n\omega_0 T$$

$$\omega_0 = \frac{2\pi}{T}$$

$$\frac{1}{n^2\omega_0^2}\cos n\frac{2\pi}{\cancel{T}}\cdot\cancel{T} + \cancel{\frac{T}{n\omega_0}\sin n\frac{2\pi}{\cancel{T}}\cdot\cancel{T}} \to 0$$

Note $\cos 2\pi = 1$ $\quad \sin 2\pi = 0$

$\cos(0) = 1$ $\quad \sin(0) = 0$

for the upper Limit; we obtain $\frac{1}{n^2\omega_0^2}\cos n2\pi$

$\int_0 \to$ (Lower Limit)

$$\frac{1}{n^2\omega_0^2}\cos n\omega_0(0) + \cancel{\frac{t}{n\omega_0}\sin n\omega_0(0)} \to 0$$

but $\cos n\omega_0(0) = 1$

Therefore at Limit = 0 (lower Limit) we obtain $\frac{1}{n^2\omega_0^2}$

upper Limit − Lower Limit, we have that

$$\frac{2V_m}{T^2}\left[\frac{1}{n^2\omega_0^2}\cos n2\pi - \frac{1}{n^2\omega_0^2}\right]$$

$$a_n = \frac{2V_m}{T^2} \cdot \frac{1}{n^2 \omega_0^2} \left[\cos n2\pi - 1 \right]$$

but $\cos n2\pi = 1$

$$a_n = \frac{2V_m}{T^2} \cdot \frac{1}{n^2 \omega_0^2} \left[1 - 1 \right]$$

$$a_n = 0$$

Solve for b_n and check if you will obtain $\frac{-V_m}{\pi n}$ following similar procedure as in a_n.

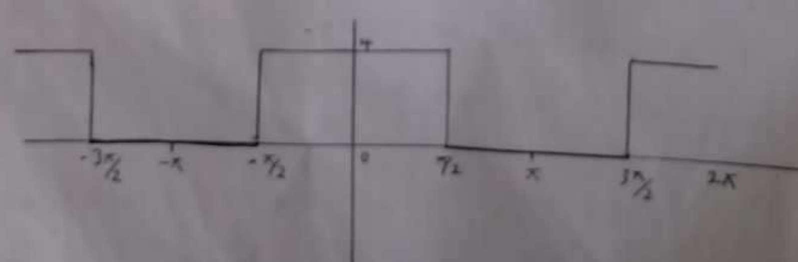
our Fourier series will then be;

$$\frac{V_m}{2} + \sum_{n=1}^{\infty} \left[(0)\cancel{\cos n\omega t}^{\,0} + \frac{-V_m}{\pi n} \sin n\omega t \right]$$

$$\frac{V_m}{2} + \sum_{n=1}^{\infty} \left[-\frac{V_m}{\pi n} \sin n\omega t \right]$$

$$\left[\frac{V_m}{2} - \frac{V_m}{\pi} \sin \omega t - \frac{V_m}{2\pi} \sin 2\omega t - \frac{V_m}{3\pi} \sin 3\omega t - \cdots - \frac{V_m}{\pi n} \sin n\omega t \right]$$

Example 2;



Imagine $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\, dx$

$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx$

$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx$

this is a triangular wave and the f(x) is the 1st value at the vertical axis such that;

$a_0 = \frac{1}{\pi} \int_0^{\pi/2} 4\, dx$.

$= \frac{1}{\pi} \left| 4x \right|_0^{\pi/2} = \frac{1}{\pi} \left| 2\pi - 0 \right| = \frac{2\pi}{\pi} = 2$

$a_n = \frac{2}{\pi} \int_0^{\pi/2} f(x) \cos nx\, dx$

$= \frac{2}{\pi} \int_0^{\pi/2} 4 \cos nx\, dx = \frac{8}{\pi} \int_0^{\pi/2} \cos nx\, dx$

note $\int \cos nx\, dx = \frac{\sin nx}{n}$

$\frac{8}{\pi} \left| \frac{\sin nx}{n} \right|_0^{\pi/2} = \frac{8}{\pi n} \left| \sin nx \right|_0^{\pi/2}$

$\Rightarrow \frac{8}{\pi n} \left| \sin n \tfrac{\pi}{2} - 0 \right| = \frac{8}{\pi n}$

Solve for $b_n = 0$ using similar method in a_n

$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$

$\Rightarrow f(x) = 2 + \frac{8}{\pi} \left\{ \cos x - \tfrac{1}{3} \cos 3x + \tfrac{1}{5} \cos 5x \ldots\ldots \right\}$

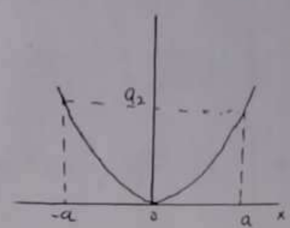
ODD & EVEN FUNCTIONS

Even,

A function f(x) is said to be even if :

$f(-x) = f(x)$

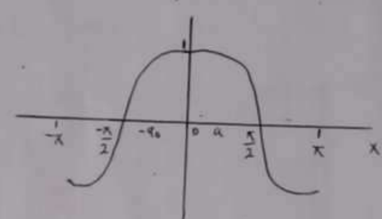
ie The function value for a particular negative value of x is the same as that of the corresponding positive value x, the graph of an even function is therefore symmetric about the y axis.



$y = f(x) = x^2$ is an even function

$f(-2) = 4 = F(2)$

$F(-3) = 9 = F(3)$ e.t.c.



$y = f(x) = \cos x$ is an even function

$\cos(-x) = \cos x$

$f(-a) = \cos a = f(a)$

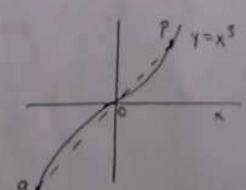
Note; for a function to be even, it must not pass through origin.

ODD FUNCTION

A function is said to be odd if;

$f(-x) = -f(x)$

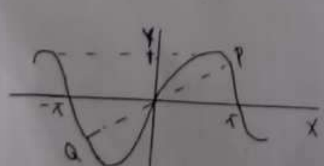
ie. The function value for a particular negative value of x is numerically equal to that of the corresponding value of x but opposite in sign. The graph of an odd function is thus symmetrical about the origin



$y = f(x) = x^3$ is an odd function because

$f(-2) = -8 = -f(2)$

$f(-5) = -125 = -f(5)$ e.t.c.



$y = f(x) = \cancel{\cos x}\ \sin x$ is an odd function

because; $\sin(-x) = -\sin x$

$\cancel{\cos(-x) = \cos x}$

$f(-a) = -f(a)$

Note: for a function to be an odd function, it must pass

For an Even Function:

$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt,$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\,dt$$

$$b_n = 0$$

For An Odd Function

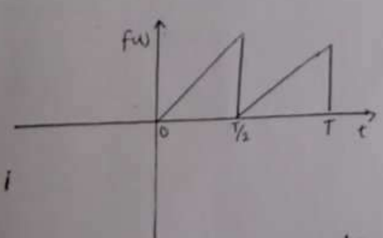
$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\,dt$$

HALF WAVE SYMMETRY

Half wave symmetry is a wave pattern that is not made up of a quarter wave.
It could be even Half wave symmetry and odd half wave symmetry.
example of Half wave symmetry



for an Even Half wave symmetry

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = 0$$

For an odd half wave symmetry

$a_0 = 0$

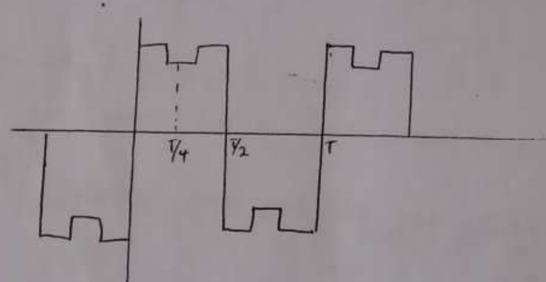
$a_n = \frac{4}{T}\int_0^{T/2} f(t) \cdot \cos n\omega_0 t \, dt$

$b_n = \frac{4}{T}\int_0^{T/2} f(t) \sin n\omega_0 t \, dt$

Quarter Wave SYMMETRY

A Quarter wave is a wave pattern that is made up of both half wave and a Quarter wave.
It can also be an even or an odd Quarter wave symmetry

example;



For a Quarter even symmetry

$a_n = 0$

$a_0 = 0$

$b_n = 0$

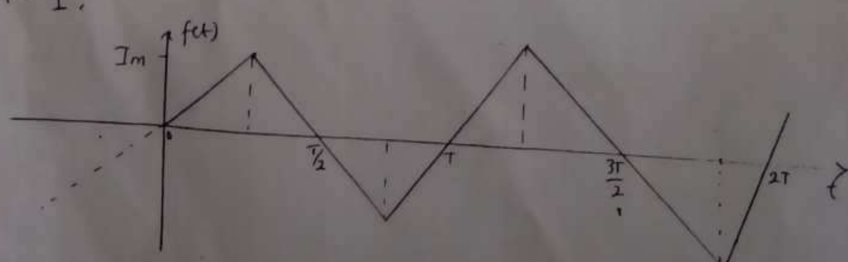
For a Quarter odd symmetry

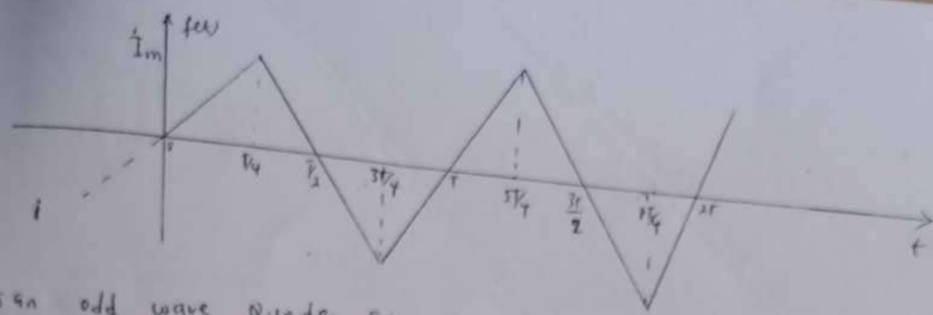
$a_n = 0$

$a_0 = 0$

$b_n = \frac{8}{T}\int_0^{T/4} f(t) \sin n\omega_0 t \, dt$

Example 1;





This is an odd wave Quarter Symmetry because, if we try to reproduce the wave pattern in negative part of the graph, it must pass through the origin.

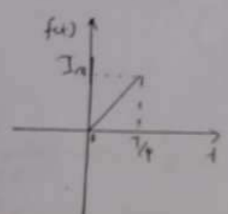
for an odd quarter wave symmetry;

$a_0 = 0$

$a_n = 0$

$$b_n = \frac{8}{T}\int_0^{T/4} f(t)\sin n\omega t \, dt$$

Since obeys a straight line graph; $Y = mx + c$, we can now determine our $f(t)$



$Y = mx + c$

$Y = f(t)$

$$m = \text{slope} = \frac{\Delta Y \text{axis}}{\Delta X \text{axis}} = \frac{I_m - 0}{T/4 - 0} = \frac{I_m}{T/4} = \frac{4I_m}{T}$$

$x = t$

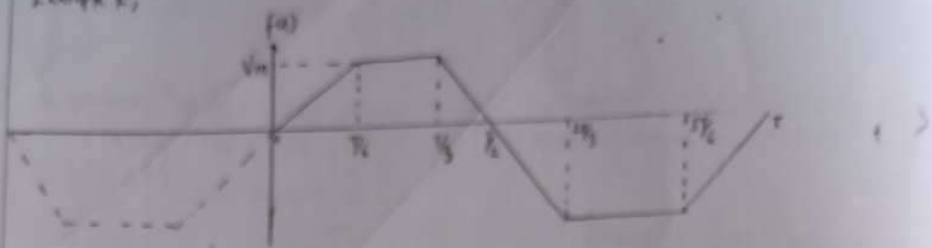
$c = 0 = \text{intercept}$

$$f(t) = \frac{4I_m}{T}t$$

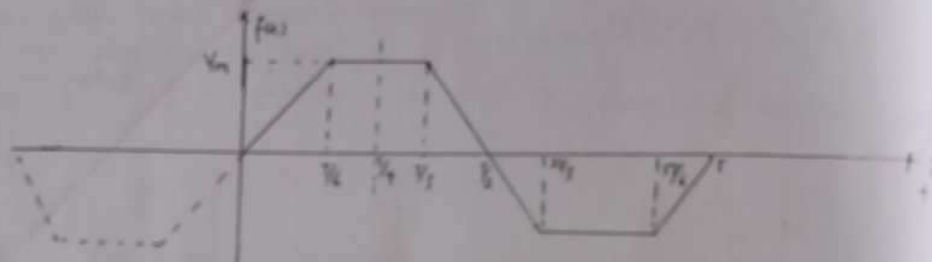
$$\therefore b_n = \frac{8}{T}\int_0^{T/4} \frac{4I_m}{T} t \sin n\omega t \, dt$$

$$= \frac{32}{T^2} I_m \int_0^{T/4} t \sin n\omega t \quad \{\text{Integrate by parts to get } b_n\}$$

Example 2;

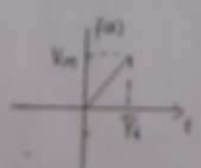


This is an odd quarter wave symmetry because if we divide half of the wave we are going to get equal quarter wave as shown



In these wave pattern, we have a triangular and rectangular wave this implies that we are going to have 2 f(t) that is one triangular and order for rectangular.

f(t) for triangular makes use of a straight line graph pattern; $y = mx + c$



$y = f(t)$

$m = \text{slope} = \dfrac{\Delta y\text{ axis}}{\Delta x\text{ axis}} = \dfrac{V_m - 0}{T/6 - 0} = \dfrac{V_m}{T/6} = \dfrac{6V_m}{T}$

$x = t$

$c = 0 = \text{intercept}$

$f(t) = \dfrac{6V_m}{T} t$

for rectangular wave pattern $f(t) = V_m$ {the value at the vertical axis

$$f(t) \begin{cases} \dfrac{6V_m}{T} t & ; 0 \le t \le T/6 \\ V_m, & T/6 \le t \le T/4 \end{cases}$$

$a_0 = 0$

$a_n = 0$

$$b_n = \frac{8}{T}\left[\int_0^{T/6} \frac{6V_m}{T} t \sin n\omega t \, dt + \int_{T/6}^{T/4} V_m \sin n\omega t \, dt\right].$$

SKETCHING OF WAVE FORM OVER A GIVEN INTERVAL

Example; 1

A Periodic signal f(t) is defined as f(t) over the Interval $\cancel{-\frac{T}{4} \leq t \leq \frac{3T}{4}}$ given ~~that~~ $f(t) = \frac{8}{T}t$ for $0 \leq t \leq T/8$ and $f(t) = -\frac{8}{T}t + 2$ for $T/8 \leq t \leq T/4$ where T is the period.

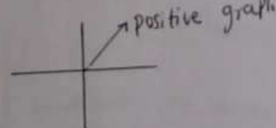
Sketch f(t) over the Interval $-T/4 \leq t \leq 3T/4$ given that f(t) is even quarter-wave Symmetry.

Sketch v(t) over the Interval $T/4 \leq t \leq 3T/4$ and find its Fourier co-efficients given that $v(t) = \frac{d[ft]}{dt}$

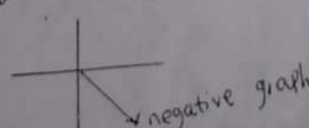
solution

1st step;

We will first satisfy the f(t)'s given above considering their limits. Notice that the first f(t): $\frac{8}{T}$ for $0 \leq t \leq T/8$ has a positive value thus it will follow the positive straight line graph pattern



The 2nd f(t) $-\frac{8}{T}t + 2$. for $T/8 \leq t \leq T/4$ has a negative sign and will follow a negative straight line graph



Since will were given the limit $0 \leq t \leq T/8$, our 1st value in the horizontal axis is $T/8$, 2nd value " then $T/8 + T/8 = T/4$ in that other till we reach the period T

To no the. 1st value in the vertical axis, we use the relationship: $y = mx + c$

$$y = f(t)$$

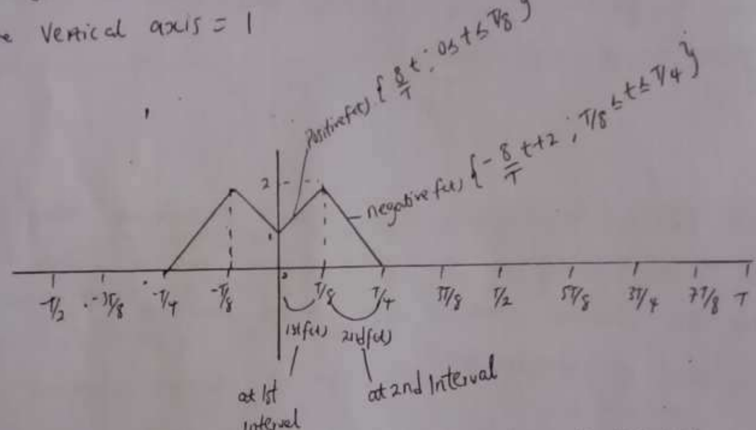
$$m = \text{slope} = \frac{\Delta \text{ Horizontal axis}}{\Delta \text{ Vertical axis}} = \frac{x}{T/8}$$

$$x = t$$

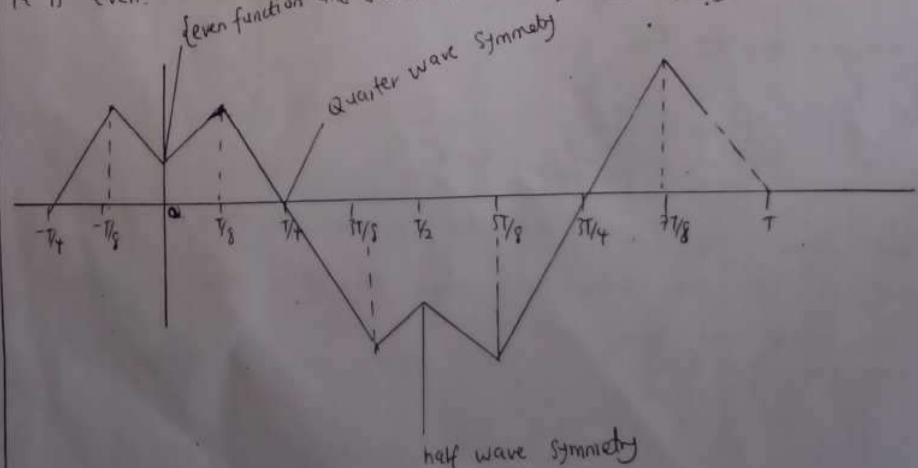
$$c = 0 = \text{Intercept}$$

We represent the unknown value in the vertical axis as x, but if we put $x = 1$ we will obtain $\frac{1}{T/8} = \frac{8}{T}$

Since putting $x = 1$ gives us the $f(t)$ $\frac{8}{T}$, so the 1st value that suites the Vertical axis = 1



2nd step: We are going to reproduce the wave to the period T, note: The wave cuts the vertical axis and didn't pass through the origin, thus it is even.



To sketch (vt) over the interval $-T/4 \le t \le 3T/4$ using the relationship

$$v(t) = \frac{d(f(t))}{dt}$$

v(t) at first interval $\frac{8}{T}t$; $0 \le t \le T/8$ is given as;

$$v(t) = \frac{d\left[\frac{8}{T}t\right]}{dt}$$

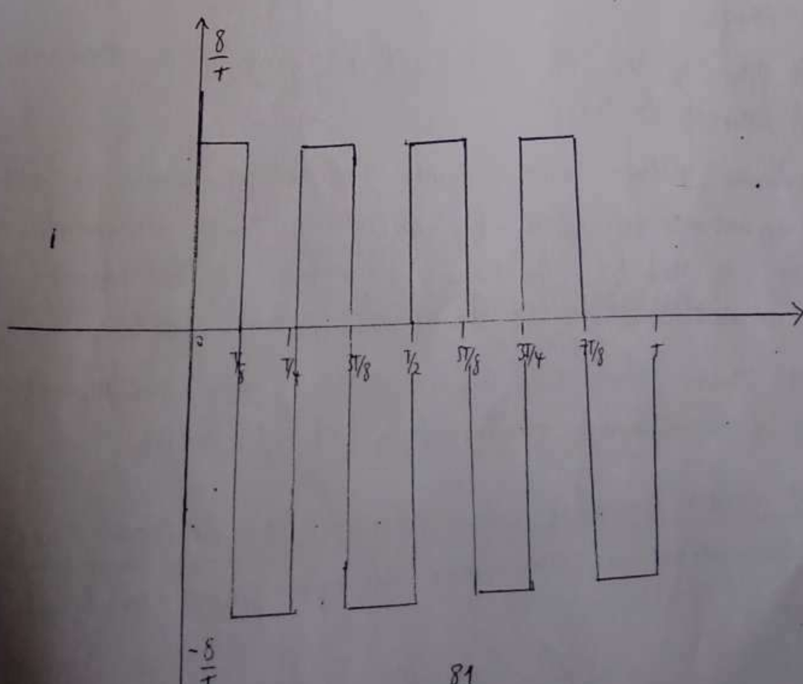
$$v(t) = \frac{8}{T}$$

$$v(t) = \frac{df(t)}{dt} \quad \text{at } T/8 \le t \le T/4$$

$$v(t) = \frac{d\left[-\frac{8}{T}t + 2\right]}{dt}$$ {we are differentiating with respect to t}

$$= -\frac{8}{T}$$

Note; v(t) is a rectangular wave so therefore v(t) over the interval $-T/4 \le t \le 3T/4$



CURVE FITTING AND EMPIRICAL LAW

EMPIRICAL LAW:

A Law which connects the two variables of a given data is k[nown] as empirical Law. several equations of different types can be obtained to express the given data approximately.

CURVE FITTING

To find empirical law, a curve of 'best fit' which can pa[ss] through most of the points of given data or nearest to them is drawn. The process of finding such equation of best fit is known as Curve fitting..

The equation of the curve (best fit) is used to predict the unknown v[alues]. To obtain an equation representing the data, we apply graphical met[hod] or method of least square.

GRAPHICAL METHOD

If we are required to fit (a straight line) a Linear Law, $y = a$[...] to the given data with the help of the graph, we proceed as follow[s]

1. plot the given points
11. Draw the straight line of best fit such that, it may pass through m[ost] of the points or nearest to them
111. substitute the cordinate of two suitable points in the equation $y = a+bx$

Two simultaneous equations are obtained. on solving these equations we get the values of a and b. The Law $y = a+bx$ is determined. If a straight line is not suitable to the points of the data, a cu[rve] is drawn through them, from the shape of the curve, the equation of the curve can be transformed to the form of $y = a+bx$

DETERMINATION OF OTHER EMPIRICAL LAWS REDUCIBLE TO LINEAR FORM

By simple transformation, we can reduce the non-Linear to Linear form.

S/N	EQUATION OF CURVE	SUBSTITUTION	REQUIRED LINEAR EQUATION
1	$y = a + bx^2$	$y = Y$, $x^2 = X$	$Y = a + bX$
2	$xy = ax + by$ or $y = a + b\frac{y}{x}$ {Dividing through by x}	$y = Y$, $\frac{y}{x} = X$	$Y = a + bX$
3	$y = a + x + bx^2$ $y - x = a + bx^2$	$y - x = Y$, $x^2 = X$	$Y = a + bX$
4	$y = Ax^n$ $\log y = \log A + n\log x$	$\log y = Y$, $\log x = X$ $A = a$	$Y = a + nX$
5	$yx^n = m$ $\log y + n\log x = \log m$	$Y = \log y$, $X = \log x$ $a = \log m$	$Y = a + nX$
6	$y = Ax^b$ $\log y = \log A + b\log x$	$Y = \log y$, $a = \log A$ $X = \log x$	$Y = a + bX$
7	$W = ae^{nT}$ $\log_e W = \log_e a + \log_e e^{nT}$ $\log_e W = \log_e a + nT$	$\log_e W = Y$, $\log_e a = a$ $T = X$	$Y = a + nX$

Note; $\log_e = \ln$

$\log a = c$

$a = 10^c$

$\log_e a = c$

$a = e^c$

Example 1

Graphs of $y = ax^n$ where a and n are Constants ; Convert into Linear form.

Values of x and y are related by equation $y = ax^n$

x	2	5	12	25	32	40
y	5.62	13.8	52.5	11.2	160	200

Determine the values of Constant a and n

$y = ax^n$

$\log y = \log a + n \log x$

Let $\log y = Y$

$\log a = a$

$\log x = x$

$y = mx + c \equiv Y = a + nx$

or $Y = nx + a$

$\log x$	0.301	0.699	1.079	1.398	1.505	1.602
$\log y$	0.749	1.139	1.720	1.049	2.204	2.301

$P\{\overset{x}{0.5}, \overset{y}{0.94}\}$

$Q\{\overset{x}{1.7}, 2.45\}$

Recall ; $y = mx + c$

$0.94 = n(0.5) + a$ ---- ①

$2.45 = n(1.7) + a$ ---- ②

Solving the two equations simultaneously

[illegible]

Let $\log a = 0.31$

$a = 10^{0.31}$

$a = 2.041$

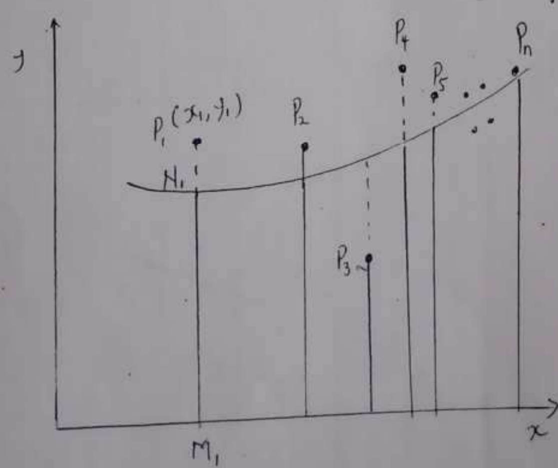
$y \doteq 2.041x^{1.26}$

PRINCIPLE OF LEAST SQUARES

The graphical method has the obvious draw back in that the straight drawn may not be ~~use~~ unique. The method of least square is probably the most systematic procedure to fit a unique curve through the given data

EMPIRICAL LAWS AND CURVE FITTING

Let $y = f(x)$ be the equation of curve to be fitted to the given data points $(x_1, y_1), (x_2, y_2) \ldots\ldots (x_n, y_n)$. At $x = x_1$, the observed or experimental value of the ordinate PM is y_i and the corresponding value on the fitting curve is NM, i.e. $[f(x_1)]$ the difference of the observed and expected (theoretical) value is $PN = P_1M_1 - N_1M = e$. This difference is called the error.



similarly

$$e_1 = y_1 - f(x_1)$$

$$e_2 = y_2 - f(x_2)$$

$$e_3 = y_3 - f(x_3)$$

$$e_n = y_n - f(x_n)$$

Some of the errors $e_1, e_2, e_3 \ldots e_n$ will be positive and others negative. To make all errors positive, we square each of the errors

$$S = e_1^2 + e_2^2 + e_3^2 \ldots e_n^2$$

The curve of the best fit is that for which the sum of the sq of errors (S) is minimum.

METHOD OF LEAST SQUARES

Let $y = a+bx$, be the straight line to be fitted to the given data points $(x_1, y_1), (x_2, y_2) \ldots (x_n, y_n)$. Let y_{t_1} the theoretical value for x_1

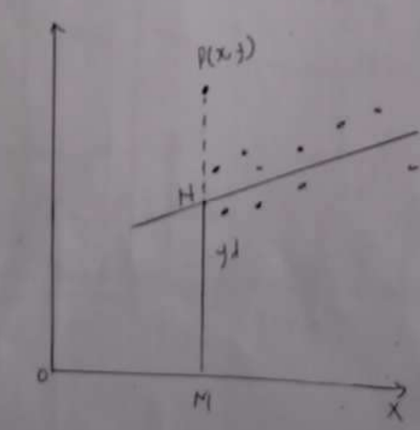
then $e_1 = y_1 - y_{t_1}$

$$e_1 = y_1 - (a + bx_1)$$

$$e_1^2 = (y_1 - a - bx_1)^2$$

$$S = e_1^2 + e_2^2 + e_3^2 \ldots e_n^2 = \Sigma e_i^2$$

$$S = \sum_{i=1}^{n} (y_i - a - bx_i)^2$$



for S to be minimum

$$\frac{\partial s}{\partial a} = \sum_{i=1}^{n} 2(y_i - a - bx_i)(-1) = 0$$

OR;

$$\Sigma (y - a - bx) = 0 \quad \ldots (2)$$

{To generalise y_i, y_i is written as y}

$$\therefore \frac{\partial s}{\partial b} = \sum_{i=1}^{n} 2(y_i - a - bx_i)(-x_i) \text{ or}$$

$$\Sigma (xy - ax - bx^2) = 0$$

on simplification equation (2) and (3) becomes;

$$\left.\begin{aligned} \Sigma y &= na + b\Sigma x \\ \Sigma xy &= a\Sigma x + b\Sigma x^2 \end{aligned}\right\} \text{Normal equation.}$$

This equations is then use to obtain the constants a and b;

Example 1;

find the best values of a and b so that $y = a + bx$ fits the data given in the data

x	0	1	2	3	4
y	1.0	2.9	4.8	6.7	8.6

Solution, $y = a + bx$ - - - * {The equation is already in a straight line pattern}

x	y	xy	x^2
0	1.0	0	0
1	2.9	2.9	1
2	4.8	9.6	4
3	6.7	20.1	9
4	8.6	13.4	16
$\Sigma x = 10$	$\Sigma y = 24.0$	$\Sigma xy = 67.0$	$\Sigma x^2 = 30$

Normal equation; $\Sigma y = na + b\Sigma x$

$$\Sigma xy = a\Sigma x + b\Sigma x^2$$

on Putting the values of Σx, Σy, Σxy, Σx^2 into the given equation

$$24 = 5a + 10 \quad \cdots (1)$$

$$67 = 10a + 30b \quad \cdots (2)$$

on solving (1) and (2) simultaneously we have;

$a = 1$

$b = 1.9$

on substituting the values of a & B in equation A, we get

$$y = 1 + 1.9x \quad = y = a + bx$$

Example 2;

Find least squares Polynomial approximation of degree two to the

x	0	1	2	3	4
y	-4	-1	4	11	20

Solution; Let the equation of the Polynomial be;

$$y = an + bx + cx^2$$

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

X	Y	XY	X^2	X^2Y	X^3	X^4
0	-4	0	0	0	0	0
1	-1	-1	1	-1	1	1
2	4	8	4	16	8	16
3	11	33	9	99	27	81
4	20	80	16	320	64	256
$\Sigma x = 10$	$\Sigma y = 30$	$\Sigma xy = 120$	$\Sigma x^2 = 30$	$\Sigma x^2y = 434$	$\Sigma x^3 = 100$	$\Sigma x^4 = 354$

on putting the values of Σx, Σy, Σxy, Σx^2, Σx^2y, Σx^3, Σx^4

$$30 = 5a + 10b + 30c \quad \text{--- (1)}$$
$$120 = 10a + 30b + 100c \quad \text{--- (2)}$$
$$434 = 30a + 100b + 354c \quad \text{--- (3)}$$

On solving equation (1) & (2) & (3) simultaneously we have;

$a = -4$, $b = 2$, $c = 1$

The required Polynomial ; $y = -4 + 2x + x^2$

Try and solve many examples yourself;

Marwi {okammuta} wishes you "A's" in ENG 307 Exams