A PROGRAMMED APPROACH

CHAPTER ONE

ORDINARY DIFFERENTIAL EQUATIONS

INTRODUCTION

finition: An ordinary differential equation is an equation which presses a relationship between the derivatives of an unknown nction, the independent variable and the unknown function itself.

amples: (1) $y^{I} = \sin x$

(2) $y^{II} + 5y = e^{x}$

(3) $y^{III} - y = 0$

he differential equation is called ordinary because the dependent ariable depends only on a single variable (x). But if it depends on nore than one variable, it is called partial differential equation.

Definition: A partial differential equation is an equation involving ne or more partial derivatives of an (unknown) function of two or nore independent variables.

Examples: (1) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

(One – dimensional Wave equation)

(2) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

(One – dimensional Heat equation)

(3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(Two – dimensional Laplace equation)

(4) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$

(Two – dimensional Poisson equation)

(5) $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

(Two – dimensional Wave equation)

(6) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

(Three – dimensional Laplace equation)

1.1 Order of a differential equation

The order of a differential equation is given by the highest derivative involved in the equation.

Examples:

(1) $\frac{xdy}{dx} - y = 0$ is an equation of the Ist order

(2) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$ is an equation of the 2nd order.

1.2 Solution

A solution of an ordinary differential equation (ODE) is a function which satisfies the equation at all points in the domain of the function.

For example:

$y = Ae^{x} + Be^{-x}$ is the solution to the equation $y^{II} - y = 0$. Verify.

Note: (1) A differential equation together with an initial condition is called an initial value problem (IVP). With x as the independent variable, it is of the form:

$$y^{I} = f(x, y),\; y(x_o) = y_o$$

Where x_o and y_o are given values

(2)The initial condition defines the situation at some fixed instant and the solution of the initial value problem describes what happens later.

1.3 Formation of Differential Equations

Differential equation may be formed in practice from a consideration of the physical problems to which they refer. Mathematically, they can occur when arbitrary constants are eliminated from a given function.

Examples:

(1) Form a differential equation from the function, $y = x + \frac{A}{x}$

Solution

Given: $y = x + \frac{A}{x}$ (1)

Required: To form differential equation from (1)

Proof: From $y = x + \frac{A}{x}$

$y^I = 1 - Ax^{-2}$ {Differentiating equation (1)} (2)

From the given function:

$$\frac{A}{x} = y - x$$

or A = $x(y - x)$ (3)

Substituting A in equation (2), we have

$$y^I = 1 - x(y - x)x^{-2}$$

$$= 1 - \frac{(y-x)}{x} = \frac{x-y+x}{x}$$

$$y^I = \frac{2x-y}{x}$$

$xy^I = 2x - y$ (This is an equation of the first order).

(2) Form a differential equation from the function:

$y = A\sin x + B\cos x$ where A and B are arbitrary constants.

Solution

Given: $y = A\sin x + B\cos x$ (1)

Required: To form a differential equation from equation (1)

Proof: Differentiating Equation (1), we have

$$y^I = A\cos x - B\sin x$$ (2)

And $y^{II} = -A\sin x - B\cos x$ (3)

$$y^{II} = -(A\sin x + B\cos x)$$

But $y = A\sin x + B\cos x$

$\therefore y^{II} = -y$

Or $y^{II} + y = 0$ This is a second order equation.

Note: (1) A Ist order differential equation is derived from a function having one arbitrary constant.

(1) A 2nd order differential equation is derived from a function having two arbitrary constants.

(2) An nth order differential equation is derived from a function having n-arbitrary constants.

1.4 Forms of First Order Differential Equations

1. Separable differential equations
2. Exact differential equations
3. Homogeneous equations
4. Non Homogeneous equations
5. Linear equations
6. Bernoulli equation.

SEPARABLE DIFFERENTIAL EQUATIONS

Any differential equation in the form:

$$g(y)dy = f(x)dx \tag{1}$$

is called a separable equation, because the variables x and y are separated so that x appears only on the right and y appears only on the left.

To solve (1), we integrate on both sides wrt x, obtaining

$$\int g(y)\frac{dy}{dx}dx = \int f(x)dx + C \tag{2}$$

This reduces to

$$\int g(y)\,dy = \int f(x)dx + C \tag{3}$$

If we assume that f and g are continuous functions, the integrals in equation (3) will exist, and by evaluating these integrals, we obtain the general solution of (1).

Solved examples

(1) Solve the initial value problem (IVP):

$$y^{I} = \frac{-y}{x}, y(1) = 1.$$

Solution

$y^I = \frac{-y}{x}$

Separating the variables, we have

$\int \frac{dy}{y} = -\int \frac{dx}{x}$

$Iny = -ln\, x + ln\, A$

$lny = ln\, \frac{A}{x}$

or $y = \frac{A}{x}$

At $y(1) = 1,\ x = 1, y = 1$

$\Rightarrow\ 1 = \frac{A}{1}$

or A = 1

$\therefore y = \frac{1}{x}$

or $xy = 1$

(2) Solve the differential equation: $y^I = 1 + x + y^2 + xy^2$

Solution

$y^I = 1 + x + y^2 + xy^2$

$\frac{dy}{dx} = 1(1 + x) + y^2(1 + x)$

$=(1 + x)(1 + y^2)$

Separating variables, we obtain

$\int \frac{dy}{1+y^2} = \int (1 + x) dx$

Arctan y $= x + \frac{x^2}{2} + C$

(3) Solve the initial value problem (IVP):

$\begin{cases} [\sin(x + y) + \sin(x - y)]\, dx + \sec y\, dy = 0 \\ y(0) = \pi/4 \end{cases}$

Solution

$[\sin(x + y) + \sin(x - y)]dx + \sec y\ dy = 0$

$[\sin x \cos y + \cos x \sin y + \sin x \cos y - \cos x \sin y]dx +$

$\sec y\, dy = 0$

$2\sin x \cos y\, dx + \sec y\, dy = 0$

$\sec y\, dy = -2 \sin x \cos y\, dx$

Dividing through with $\cos y$, we have

$\frac{1}{\cos^2 y}\, dy = -2 \sin x\, dx$

$\sec^2 y\, dy = -2 \sin x\, dx$

Integrating both sides wrt x, we have

$\int \sec^2 y\, dy = -2 \int \sin x\, dx$

$\tan y = 2 \cos x + C$

Applying the initial condition, $i.e., x = 0,\ y = \frac{\pi}{4}$

$\Rightarrow \tan\left(\frac{\pi}{4}\right) = 2\cos(0) + C$

$\frac{1}{\sqrt{2}} = 2 + C$

or $C = \frac{1}{\sqrt{2}} - 2 = \frac{\sqrt{2}}{2} - 2 = \frac{(\sqrt{2}-4)}{2}$

$\therefore \tan y = 2\cos x + \left(\frac{\sqrt{2}-4}{2}\right)$

(4) Solve the differential equation $\begin{cases} y' = \cos(x-y) + \cos(x+y) \\ y(0) = 0 \end{cases}$

Solution

$y' = \cos(x - y) + \cos(x + y)$

$\frac{dy}{dx} = \cos x \cos y + \sin x \sin y + \cos x \cos y - \sin x \sin y$

$\frac{dy}{dx} = 2cosx\ cosy$

Separating variables, we have

$\frac{dy}{\cos y} = 2\cos x\, dx$

Integrating both sides wrt x $gives$

$\int \frac{dy}{\cos y} = 2\int \cos x\, dx + C$

$\ln|\sec y + \tan y| = 2sinx + C$

Applying the given condition, $i.e., y(0) = 0\ \ i.e., x = 0, y = 0$

$\Rightarrow \ln|\sec(0) + \tan y(0)| = 2\sin(0) + C$

$\ln|1| = 0 + c$

or $C = 0$

$\therefore \ln|\operatorname{Sec} y + \tan y| = 2\sin x$

(5) Solve the IVP; $y^I = -2xy,\ y(0) = 1$

Solution

Given $y^I = -2xy$ (1)

Required: To solve equation (1)

Procedure: $\frac{dy}{dx} = -2xy$

Separating variables,

$\frac{dy}{y} = -2x\, dx$

Integrating both sides wrt x, we have

$\int \frac{dy}{y} = -2\int x\, dx$

$\ln y = -\frac{2x^2}{2} + C$

$\ln y = -x^2 + C$

Applying the condition, $i.e., x = 0,\ y = 1$

$\Rightarrow \ln(1) = -(0)^2 + C$

$0 = C$

$\therefore \ln y = -x^2$

or $y = e^{-x^2}$

(6) Solve $y' + Cosec\, y = 0$

Solution

$y' + CSCy = 0$

$\frac{dy}{dx} + CSCy = 0$

$\frac{dy}{dx} = -CSCy$

Separating variables, we have

$\frac{dy}{dx} = \frac{-1}{\sin y}$

$Siny\, dy = -dx$

Integrating both sides wrt x gives

$\int \sin y dy = -\int dx$

$-cosy = -x + C$

$x - \cos y = C$

Practice problems 1

(1) Solve the IVP; $y' = y^{1/3}$, $y(0) = 0$

(2) Solve the IVP; $y' = y^2$, $y(0) = 1$

(3) Solve the IVP;

(a) $\frac{dy}{dx} = \frac{y\cos x}{1+2y^2}$, $y(0) = 1$

(b) $\frac{dy}{dx} = \frac{x(x^2+1)}{4y^3}$, $y(0) = \frac{-1}{\sqrt{2}}$

EXACT DIFFERENTIAL EQUATIONS

Any first order differential equation of the form:

$M(x, y)dx + N(x, y)dy = 0$ (1)

is called an exact differential equation if the differential form:

$M(x, y)dx + N(x, y)dy$ is exact. That is, the form is the differential

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \quad (2)$$

of some function $U(x, y)$. Then the differential equation (1) can be written; du = 0

The only necessary but also sufficient condition for equation (1) to be an exact differential equation is that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (3)$$

i.e., $M_y = N_x$

Solved examples

(1) Show that the following equations are exact

(a) $(3x^2y - 2y^3 + 3)dx + (x^3 - 6xy^2 + 2y)\,dy = 0$

(b) $2x\sin y - y\sin x + (x^2\cos y + \cos x)y' = 0$

(c) $(2y\,\sin x\cos x + y^2\sin x)dx + (\sin^2 x - 2y\cos x)dy = 0$

Solution

(a). Given: $(3x^2y - 2y^3 + 3)dx + (x^3 - 6xy^2 + 2y)dy = 0$ (1)

Required: To show that equation (1) is exact.

Procedure: Comparing equation (1) with the equation below

$M(x,y)dx + N(x,y)dy = 0$

$\Rightarrow M = 3x^2y - 2y^3 + 3; \quad N = x^3 - 6xy^2 + 2y$

Testing for exactness:

$\frac{\partial M}{\partial y} = 3x^2 - 6y^2; \quad \frac{\partial N}{\partial x} = 3x^2 - 6y^2$

Therefore, since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 3x^2 - 6y^2$

The given equation is exact.

(b) Given: $(2x\sin y - y\sin x) + (x^2\cos y + \cos x)\frac{dy}{dx} = 0$ (1)

Required: To show that equation (1) is exact.

Procedure: Comparing equation with the standard exact equation:

$M = 2x\sin y - y\sin x$

$N = x^2\cos y + \cos x$

Testing for exactness;

$\frac{\partial M}{\partial y} = 2x\cos y - \sin x$

$\frac{dN}{dx} = 2x\cos y - \sin x$

$\therefore$ Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x\cos y - \sin x$,

The given equation is exact.

(c) Given: $(2y\sin x\cos x + y^2\sin x)dx + (\sin^2 x - 2y\cos x)dy = 0$ (1)

Required: To show that equation (1) is exact.

Procedure: Comparing equation with the standard equation

$M = 2ysinx\ cosx + y^2 sinx$

$N = sin^2x - 2ycos\ x$

$\frac{\partial M}{\partial y} = 2\ sinx \cos x + 2y\ sin\ x$

$\frac{\partial N}{\partial x} = 2 \sin x \cos x + 2y\ sin\ x$

$\therefore$ Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2sinx \cos x + 2y\ sin\ x,$

The given equation is exact.

(2) Solve the initial value problem (IVP):

$\begin{cases} 2xcosy + 3x^2y)dx + (x^3 - x^2 \sin y - y)dy = 0 \\ y(0) = 2 \end{cases}$

Solution

$(2xcosy + 3x^2y)dx + (x^3 - x^2 \sin y - y)dy = 0$ (1)

$M = 2x \cos y + 3x^2y = \emptyset_{xy}(x, y)$

$N = x^3 - x^2 \sin y - y = \emptyset_{yx}(x, y)$

Testing for exactness;

$\frac{\partial M}{\partial y} = -2x \sin y + 3x^2;\ \frac{\partial N}{\partial x} = 3x^2 - 2x \sin y$

$\therefore$ Since $\frac{\partial M}{\partial y} = -2x \sin y + 3x^2 = \frac{\partial N}{\partial x},$

The equation is exact.

$\emptyset_x = \int M(x, y)dx + h(y)$

$= \int (2xcosy + 3x^2y)\ dx + h(y)$

$\emptyset_x = x^2 \cos y + x^3y + h(y)$ (2)

Differentiating (2) with y, we have

$$N = \frac{\partial}{\partial y}\left[x^2 \cos y + x^3 y + h'(y)\right]$$

$$N = -x^2 \sin y + x^3 + h(y)$$

But $N(x, y) = x^3 - x^2 \sin y - y$

$$\Rightarrow N = -x^2 \sin y + x^3 + h'(y) = x^3 - x^2 \sin y - y$$

Comparing terms, we have

$$h'(y) = -y$$

Integrating the above wrt y,

$$\int h'(y) = -\int y\, dy + k$$

$$h(y) = -\frac{y^2}{2} + k$$

Substituting $h(y)$ in equation (2), we have

$$\emptyset_x = x^2 \cos y + x^3 y - \frac{y^2}{2} + k$$

Let $\emptyset_x - k = A$

$$\Rightarrow A = x^2 \cos y + x^3 y - \frac{y^2}{2}$$

Applying the initial condition,

i.e., at $y(0) = 2,\ i.e.\ \ x = 0, y = 2$

$$\Rightarrow A = 0 + 0 - \frac{(2)^2}{2}$$

$$A = -2$$

$$\therefore\ x^2 \cos y + x^3 y - \frac{y^2}{2} = -2$$

OR $\quad 2x^2 \cos y + 2x^3 y - y^2 + 4 = 0$

(3) Find the value of b for which the differential equation given below is exact

$(ye^{2xy} + x)dx + (bxe^{2xy})dy = 0$

Hence or otherwise, find its solution.

Solution

Given: $(ye^{2xy} + x)dx + \ (bxe^{2xy})dx = 0$ (1)

Required: To find (i) value of b given that equation (1) is exact

(ii) the solution of equation (1)

Procedure: Comparing equation (1) with the standard equation, we have

$M(x,y)dx + N(x,y)dy = 0$

$i.e., M(x,y) = ye^{2xy} + x = \ \emptyset_{xy}(x,y)$

$N(x,y) = bxe^{2xy} = \emptyset_{yx}(x,y)$

$\frac{\partial M}{\partial y} = \ e^{2xy} + 2xye^{2xy} = \ (e^{2xy} + 2xye^{2xy})$

$\frac{\partial N}{\partial x} = be^{2xy} + 2bxye^{2xy} = b\ (e^{2xy} + 2xye^{2xy})$

But equation (1) is given as an exact equation. That is

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Longrightarrow\ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$= (e^{2xy} + 2xye^{2xy}) = .b\ (e^{2xy} + 2xye^{2xy})$

$\therefore\ b = \frac{e^{2xy}+2xye^{2xy}}{e^{2xy}+2xye^{2xy}} = 1$

$\therefore$ Equation(1) is given as

$(ye^{2xy} + x)\ dx + \ (xe^{2xy})dy = 0$

From $M(x,y) = ye^{2xy} + x = \emptyset_{xy}(x,y)$

$\frac{y^2}{2}$

$N(x,y) = xe^{2xy} = \emptyset_{yx}(x,y)$

$\emptyset_x = \int ye^{2xy} + x + h(y)$

$= y\int(e^{2xy} + x)dx + h(y)$

$= y\left[\frac{e^{2xy}}{2y}\right] + \frac{x^2}{2} + h(y)$

$\emptyset_x = \frac{e^{2xy}}{2} + \frac{x^2}{2} + h(y)$

$$\emptyset_x = \frac{e^{2xy}}{2} + \frac{x^2}{2} + h(y) \qquad (2)$$

Differentiating equation (2) wrt y, we have

$N(x,y) = \frac{\partial}{\partial y}\left[\frac{e^{2xy}}{2} + \frac{x^2}{2} + h(y)\right]$

$N(x,y) = \frac{2xe^{2xy}}{2} + h'(y) = xe^{2xy} + h'(y)$

But $N(x,y) = xe^{2xy}$

$\Longrightarrow xe^{2xy} + h'(y) = xe^{2xy}$

$h'(y) = 0$

Integrating both sides, we have

$\int h'(y) = \int 0dy$

$h(y) = k$

Substituting $h(y)$in Equation (2), we obtain

$\emptyset_x = \frac{e^{2xy}}{2} + \frac{x^2}{2} + k$

$\emptyset_x - k = \frac{e^{2xy}}{2} + \frac{x^2}{2}$

Let $A = \emptyset_x - k$

$\Rightarrow A = \frac{e^{2xy}}{2} + \frac{x^2}{2}$

$\therefore A = \frac{1}{2}(e^{2xy} + x^2)$

(4). Solve the equation: $(2y\, sinx \cos x + y^2 sinx)dx + (sin^2x - 2ycos\, x)dy = 0$

given that $y(0) = 3$

Solution

$(2y\, sinx \cos x + y^2 sinx)dx + (sin^2x - 2ycos\, x)dy = 0$ (1)

$M(x,y) = 2y\, sinx \cos x + y^2 sinx = \emptyset_{xy}(x,y)$ (2)

$N(x,y) = sin^2x - 2ycos\, x = \emptyset_{yx}(x,y)$ (3)

$\frac{\partial M}{\partial y} = 2sinx \cos x + 2y \sin x = M_y$

$\frac{\partial N}{\partial y} = 2sinx \cos x + 2y \sin x = N_x$

$\therefore$ *Since* $M_y = N_x = 2\sin x \cos x + 2\sin x \cos x$, it implies that the given equation is exact.

$\emptyset_x = \int(2y \sin x \cos x + y^2 \sin x)dx + h(y)$

$= 2y\int \sin x\, d(\cos x) + y^2 \int \sin x\, dx + h(y)$

$= 2y\frac{sin^2x}{2} + y^2(-\cos x) + h(y)$

$\emptyset_x = y\, sin^2x - y^2cosx + h(y)$ (4)

Differentiating equation (2)wrt. y, we have

$N(x,y) = \frac{\partial}{\partial y}[ysin^2x - y^2 \cos x + h(y)]$

$N(x,y) = sin^2x - 2y \cos x + h'(y)$

But $N(x,y) = sin^2x - 2y\cos x$

$\Rightarrow sin^2x - 2y\cos x + h'(y) = sin^2x - 2y\cos x$

$h'(y) = 0$

Integrating both side wrt. y, we have

$\int h'(y) = \int 0dy$

$h(y) = k$

Substituting $h(y)$in equation (4) , we have

$\emptyset_x = y\, sin^2x - y^2cosx + k$

Let $\emptyset_x - K = A$

$\Rightarrow A = y\, sin^2x - y^2cosx$

At $y(0) = 3,\ i.e.,\ x = 0, y = 3$

$\Rightarrow A = 3[\sin(0)]^2 - (3)^2\cos(0)$

$A = 3\ (0) - 9\ (1)$

$A = -9$

$\therefore\ -9 = ysin^2x - y^2\cos x$

OR $y\, sin^2x - y^2\cos x + 9 = 0$

(5). Determine the constants p and q if the equation:

$(3xy + qy^2 + e^x)dx + (px^2 + 6qxy)dy = 0$ is an exact equation

Solution

$(3xy + qy^2 + e^x)dx + (px^2 + 6qxy)dy = 0$ (1)

$M(x,y) = 3xy + qy^2 + e^x = \emptyset_{xy}(x,y)$

$N(x,y) = px^2 + 6qxy = \emptyset_{yx}(x,y)$

3x + 2

$M_y = 3x + 2qy$ (2)

$N_x = 2px + 6qy$ (3)

But $M_y = N_x$ (Given).

This implies that: equation (2) = equation (3)

i.e. $3x + 2qy = 2px + 6qy$

$(3 - 2p)x + (2q - 6q)y = 0$

$(3 - 2p)x - 4qy = 0$

Comparing terms, we have

$\Rightarrow [x];\ 3 - 2p = 0$

$P = {}^3/_2$

And $[y]\ - 4q = 0$

$\therefore p = {}^3/_2, q = 0$

(6) Solve the initial value problem (IVP):

$$\begin{cases} (\sin x \cosh y)\, dx - (\cos x \sinh y) dy = 0 \\ y(0) = 3 \end{cases}$$

Solution

$\{\sin x \cosh y\}dx - \{\cos x \sinh y\}dy = 0$ (1)

$M(x, y) = \sin x \cosh y$ (2)

$N(x, y) = -\cos x \sinh y$ (3)

$\frac{\partial M}{\partial y} = \sin x \sinh y$

$\frac{\partial N}{\partial x} = +\sin x \sinh y$

$\therefore$ *since* $M_y = \sin x \sinh y = N_x$, *the equation is exact.*

$\emptyset_x = \int\{\sin x\, coshy\}dx + h(y)$

$\emptyset_x = -\cos x\, coshy + h(y)$ (4)

Differentiating equation (4)with y, we have

$N(x,y) = \frac{\partial}{\partial y}[-\cos x\, coshy + h(y)]$

$N(x,y) = -\cos x\, sinhy + h'(y)$ (5)

Equating (3)and (5), we have

$-\cos x\, sinhy = -\cos x\, sinhy + h'(y)$

$\Rightarrow h'(y) = 0$

$\int h'(y) = \int 0dy$

$h(y) = k$

Substituting $h(y)$ in equation (4), we obtain

$\emptyset_x = -cosx \cosh + k$

Let $\emptyset_x - k = A$

$\Rightarrow A = -\cos x\, coshy$

At $y(0) = 3 \quad i.e., x = 0, y = 3, we\ obtain$

$A = -\cos(0)\cosh(3)$

$A = -(1)\cosh(3) = -\cosh(3)$

$\therefore -\cos x\, coshy = -\cosh 3 = 10.07$

$OR \quad \cosh 3 - \cos x\, coshy = 0$

HOW TO FIND INTEGRATING FACTORS

Suppose the equation: $M(x,y)dx + N(x,y)dy = 0$ (1)

is non-exact. To make it exact, we multiply it by a function say μ, which in general will be a function of both x and y. This results to the equation

$$\mu M dx + \mu N dy = 0, \tag{2}$$

This is exact. The exactness condition is:

$$\frac{d}{dy}(\mu M) = \frac{d}{dx}(\mu N)$$

By the product rule, with subscripts denoting partial derivatives. This gives

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x \tag{3}$$

case (1): Taking μ as a function of x.

$$i.e.\ \mu = f(x)$$

Then $\mu_y = 0$ and $\mu_x = \mu^1 = \frac{d\mu}{dx}$, so that equation (3) becomes

$$\mu M_y = \mu^1 N + \mu N_x$$

Dividing through by μN, we have

$$\frac{\mu M_y}{\mu N} = \frac{\mu^1 N}{\mu N} + \frac{\mu N_x}{\mu N}$$

$$\frac{M_y}{N} = \frac{\mu^1}{\mu} + \frac{N_x}{N}$$

Integrating both sides, we have

$$\int \frac{\mu^1}{\mu} = \int \frac{M_y - N_x}{N}$$

$$ln\mu = \int \frac{M_y - N_x}{N}.$$

OR $\mu_{(x)} = e^{\int \frac{M_y - N_x}{N}}$

$\therefore\ \mu_{(x)} = e^{\int \frac{M_y - N_x}{N}}$

Case (2): Taking μ as a function of y.

$i.e.\ \mu = f(y)$

Then $\mu_x = 0$ and $\mu_y = \mu^1 = \frac{dM}{dy}$, $so\ that\ equation$ (3)$becomes$

$\mu_y M + \mu M_y = \mu N_x$

Dividing through by μM, we have

$\frac{\mu_y M}{\mu M} + \frac{\mu M_y}{\mu M} = \frac{\mu N_y}{\mu M}$

m(x,y)dx+hy

$\frac{\mu_y}{\mu} + \frac{M_y}{M} = \frac{N_x}{M}$.

Rearranging, we have

$\frac{\mu_y}{\mu} = \frac{N_x - M_y}{M}$

Integrating both sides, we have

$\int \frac{\mu_y}{\mu} = \int \frac{N_x - M_y}{M}$

$ln\mu = \int \frac{N_x - M_y}{M}$

$\therefore\ \mu_{(y)} = e^{\int \frac{N_x - M_y}{M}}$

Example (7) Determine the integrating factor of the equation:

$(3x^2y + 2xy + y^3)dx + (x^2 + y^2)dx = 0$

Hence or otherwise, solve the equation.

Solution

Given: $(3x^2y + 2xy + y^3)dx + (x^2 + y^2)dx = 0$ (1)

Required: To find (i) the integrating factor of equation (1)

(ii) The solution of equation (1)

Procedure: comparing equation (1) with the standard exact equation, we have

$M(x,y)dx + N(x,y)dy = 0$

$M = 3x^2y + 2xy + y^3$

$N = x^2 + y^2$

Testing for exactness:

$$\frac{\partial M}{\partial y} = 3x^2 + 2x + 3y^2$$

$$\frac{\partial N}{\partial x} = 2x$$

$\therefore$ *since* $M_y \neq N_x$, *equation* (1) *is not exact.*

(1) We now determine its integrating factor, μ;

Taking the integrating factor as a factor of x, we have

$$\mu_x = e^{\int \frac{M_y - N_x}{N}} = e^{\int \frac{(3x^2+2x+3y^2)-2x}{x^2+y^2}}$$

$$= e^{\int \frac{3(x^2+y^2)}{(x^2+y^2)}} = e^{\int 3dx} = e^{3x}$$

$\therefore \mu_{(x)} = e^{3x}$. This is the integrating factor of equation (1).

(ii) Solving equation (1) to find its Solution.

Multiplying equation (1) by the integrating factor (IF), we have

$e^{3x}(3x^2 + 2xy + y^3)\,dx + e^{3x}(x^2 + y^2)\,dy = 0$

$(3x^2ye^{3x} + 2xye^{3x} + y^3e^{3x})dx + (x^2e^{3x} + y^2e^{3x})\,dy = 0$ (2)

$M = 3x^2ye^{3x} + 2xye^{3x} + y^3e^{3x} = \emptyset_{xy}(x,y)$

$N = x^3e^{3x} + y^2e^{3x} = \emptyset_{xy}(x,y)$

$M_y = 3x^2e^{3x} + 2xe^{3x} + 3y^2e^{3x}$ (3)

$N_x = 3x^2e^{3x} + 2xe^{3x} + 3y^2e^{3x}$ (4)

$\therefore$ *since* $M_y = N_x$, *the equation is now exact.*

$\emptyset_x = \int(3x^2ye^{3x} + 2xye^{3x} + y^3e^{3x})\,dx + h(y)$

$\emptyset_x = x^3y\,e^{3x} + \frac{y^3e^{3x}}{3} + h(y)$ [*use integration by parts*]

$\emptyset_x = x^2y\,e^{3x} + \frac{y^3e^{3x}}{3} + h(y)$ (5)

Differentiating the above equation wrt y, we have

$N(x,y) = \frac{\partial}{\partial y}\left[x^2ye^{3x} + \frac{y^3e^{3x}}{3} + h(y)\right]$

$= x^2e^{3x} + \frac{3y^2e^{3x}}{3} + h'(y)$

$N(x,y) = x^2e^{3x} + y^2e^{3x} + h'(y)$

But $N = x^2e^{3x} + y^2e^{3x}$

$\Rightarrow x^2e^{3x} + y^2e^{3x} + h'(y) = x^2e^{3x} + y^2e^{3x}\; h'(y) = 0$

$\int h'(y) = \int 0dy$

$h(y) = k$

Substituting $h(y)$ *in equation* (5), *we have*

$\emptyset_x = x^2ye^{3x} + \frac{y^3e^{3x}}{3} + k$

Let $\emptyset_x - k = A$

$\Rightarrow A = x^2ye^{3x} + \frac{y^3e^{3x}}{3}$

(8) Find an integrating factor and solve the IVP

$2sin\,(y^2)\,dx + xy\cos(y^2)\,dy = 0, \quad y(2) = \sqrt{\frac{\pi}{2}}$

Solution

Given: $2sin(y^2)\,dx + xy\cos(y^2)\,dx = 0$ (1)

Required: To find (a) the integrating factor of equation (1)

(b) the solution of equation (1)

Procedure: From equation (1);

$M = 2\sin(y^2)$

$N = xycos(y^2)$

$M_y = 4y\cos(y^2)$

$N_x = y\cos(y^2)$

$\therefore since\ M_y \neq N_x, the\ equation\ is\ not\ exact.$

(1) To find the integrating factor (IF)

Taking IF as a function of x, we have

$i.e.\ \mu_{(x)} = e^{\int\frac{M_y - N_x}{N}} = e^{\int\frac{4y\cos(y^2) - ycos(y^2)}{xycos(y^2)}}dx$

$= e^{\int\frac{3y\cos(y^2)dx}{xycos(y^2)}}$

$= e^{\int\frac{3}{x}dx} = e^{3lnx} = e^{lnx^3} = x^3$

$\therefore \mu_{(x)} = x^3 = The\ integrating\ factor.$

Multiplying equation (1) with the IF $(x^3), we\ have$

(b) $2x^3sin(y^2)dx + x^4ycos(y^2)dy = 0$ (2)

The new M and N now become;

$M = 2x^3 sin(y^2) = \emptyset_{xy}(x,y)$ (*)

$N = x^4 y\cos(y^2) = \emptyset_{yx}(x,y)$ (**)

$M_y = 4x^3 y\cos(y^2)$ (3)

$N_x = 4x^3 y\cos(y^2)$ (4)

$Therefore, since\ M_y = N_x = 4x^3 y\cos(y^2),$

$the\ equation\ is\ now\ exact.$

$\emptyset_x = \int 2x^3 \sin(y^2)\,dx + h(y)$

$\emptyset_x = \frac{2x^4}{4}\sin(y^2) + h(y)$

$\emptyset_x = \frac{x^4}{2}\sin(y^2) + h(y)$ (5)

Differentiating equation (5) wrt y, we have

$N(x,y) = \frac{\partial}{\partial y}\left[\frac{x^4}{2}\sin(y^2) + h(y)\right]$

$= \frac{2yx^4}{2}\cos(y^2) + h'(y)$

$N(x,y) = yx^4\cos(y^2) + h'(y)$ (6)

Equating equation (**) and (6), we have

$x^4 y\cos(y^2) = yx^4\cos(y^2) + h'(y)$

$h'(y) = 0$

$\int h'(y) = \int 0dy$

$h(y) = k$

Substituting $h(y)$ *in equation, we have*

$\emptyset_x = \frac{x^4}{2}\sin(y^2) + k$

Let $\emptyset_x - k = \Lambda$

$$\Rightarrow A = \frac{x^4}{2}\sin(y^2)$$

Substituting the boundary condition, we have

$i.e.\ y(2) = \sqrt{\pi/2}$ $i.e.\ x = 2;\ y = \pi/2$

$$\Rightarrow A = \frac{(2)^4}{2}\sin(\pi/2)^2$$

$$= 8\,sin\left(\pi^2/4\right)$$

$$\therefore \frac{x^4}{2}\,sin\,(y^2) = 8\sin\left(\pi^2/4\right)$$

OR $x^4\sin(y^2) = 16\sin\left(\pi^2/4\right)$

Note: We find the integrating factor (IF) using trial and error method. That is, you kept testing the integrating factor as a function of x and then as a function of y. Whichever one simplifies the computation gives the answer.

PRACTICE PROBLEMS 2

(1) Solve the equation:

$[y^3\,cos\,(xy) + 2xy]dx + [xy^2\cos(xy) - y\sin(xy) - 2x^2]dy = 0$

(2) Find an integrating factor by inspection or by using their formula and hence solve:

(a) $2x\tan y\,dx + sec^2 y\,dy = 0$

(b) $2\cosh x\cos y\,dx = \sinh x\,\sin y\,dy$

(c) $x^{-1} \cosh y \, dx + \sinh y \, dy = 0$

(3) Find the condition under which the equation;
$(ax^2 + bxy + cy^2)dx + (dx^2 + exy + fy^2)\,dy = 0$ is an exact equation.

(4) Solve $(6xy + 2y^2 - 5)dx + (3x^2 + 4xy - 6)dy = 0$

(5) Show that the given function is an integrating factor and hence, solve

(a) $2\cos y \, dx = \tan 2x \sin y \, dy, (\cos 2x)$

(b) $(2y + xy)\,dx + 2x\,dy = 0\,, \quad \left(\frac{1}{xy}\right)$

(c) $ydx + [y + \tan(x + y)]dy = 0, \quad \cos(x + y)$

(6) Solve the initial value problems:

(i) $2\sin(xy)dx + x\cos(xy)dy = 0\,,\ y(0) = \left(\frac{\pi}{2x}\right)$

(ii) $(2xy\,dx + dy)e^{x^2} = 0\,, y(o) = 2$

(iii) $ye^x\,dx + (2y + e^x)dy = 0\,, y(o) = -1$

CHAPTER THREE

HOMOGENEOUS EQUATIONS

This is determined by the fact that the total degree in x and y for each of the terms involved is the same. The key to solving every homogeneous equation is to substitute $y = vx$, where v is a function of x. This converts the equation into a form we can solve by separating variables.

i.e. $y = vx$

Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Solved Examples

(1) Solve the differential equation, $\frac{dy}{dx} = \frac{2xy+3y^2}{x^2+2xy}$

Solution

Given: $\frac{dy}{dx} = \frac{2xy+3y^2}{x^2+2xy}$ (1)

Required: To solve equation (1)

Procedure: From equation (1), it can be seen that the power of each of the terms is two. That is, they have equal degree of terms. Therefore the equation is homogenous.

Let $y = vx$

Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting y *and* y' *in equation* (1), *we have*

$$v + x\frac{dv}{dx} = \frac{2x\,(vx)+3(vx)^2}{x^2+2x\,(vx)}$$

$$v + x\frac{dv}{dx} = \frac{2x^2v+3x^2v^2}{x^2+2x^2v} = \frac{x^2\,(2v+3v^2)}{x^2(1+2v)}$$

$$v + x\frac{dv}{dx} = \frac{2v+3v^2}{1+2v}$$

$$\frac{xdv}{dx} = \frac{2v + 3v^2}{1 + 2v} - v$$

$$\frac{xdv}{dx} = \frac{2v+3v^2-v\,(1+2v)}{1+2v}$$

$$\frac{xdv}{dx} = \frac{2v+3v^2-v-2v^2}{1+2v}$$

$$x\frac{dv}{dx} = \frac{v^2+v}{1+2v}$$

Rearranging the above by separating variables, we have

$$\left(\frac{1+2v}{v+v^2}\right) dv = \frac{dx}{x}$$

Integrating the above equation, we obtain

$$\int \left(\frac{1+2v}{v+v^2}\right) dv = \int \frac{dx}{x}$$

$$\ln(v + v^2) = \ln x + \ln A$$

$$ln\,(v + v^2) = \ln A\,x$$

OR $v + v^2\,A\,x$

But $v = {}^{y}/_{x}$

This implies,

$$\frac{y}{x} + \left(\frac{y}{x}\right)^2 = A\,x$$

OR $xy + y^2 = Ax^3$

(2) Solve the IVP $\begin{cases}(x^3y - 2y^4)dx + (y^3x - 2x^4)dy = 0 \\ y(0) = 1\end{cases}$

Given: $\{(x^3y - 2y^4)dx + (y3x - 2x^4)\}dy = 0$ (1)

Required: To solve equation (1) given the initial condition $y(0) = 1$.

Procedure: From equation (1)

$$(x^3y - 2y^4)\,dx - (2x^4 - y^3x)dy = 0$$

OR $(x^3y - 2y^4)\,dx = (2x^4 - y^3x)dy$

OR $\frac{dy}{dx} = \frac{x^3y-2y^4}{2x^4 - y^3x}$ (2)

The above equation (2) is homogeneous since the powers of each of the individual term involved is same.

Let $y = vx$

Then, $\frac{dy}{dx} = v + x\,\frac{dv}{dx}$

Substituting y *and* y' *in equation, we have*

$$v + x\frac{dv}{dx} = \frac{x^3(vx)-2\,(vx)^4}{2x^4-(vx)^3x}$$

$$v + x\,\frac{dv}{dx} = \frac{x^4v-2x^4v^4}{2x^4 - x^4v^3}$$

$$v + x\frac{dv}{dx} = \frac{x^4(v-2v^4)}{x^4\,(2-v^3)}$$

$$V + x\frac{dv}{dx} = \frac{v-2v^4}{2-v^3}$$

$$x\frac{dv}{dx} = \frac{v-2v^4}{2-v^3} - v$$

$$= \frac{v-2v^4-v\,(2-v^3)}{2-v^3}$$

$$= \frac{v-2v^4-2v+v^4}{2-v^3}$$

$$= \frac{v-2v^4-2v+v^4}{2-v^3}$$

$$= -\frac{v-v^4}{2-v^3}$$

$$x\frac{dv}{dx} = -\frac{(v+v^4)}{2-v^3}$$ (

Rearranging equation (3) above, we have;

$$\left(\frac{2-v^3}{v+v^4}\right) dv = -\int\frac{dx}{x}$$

Integrating the above equation, we have;

$$\int\left(\frac{2-v^3}{v+v^4}\right) dv = -\int\frac{dx}{x}$$ (4

$$\frac{2-v^3}{v+v^4} = \frac{2-v^3}{v(1+v^3)} \equiv \frac{2-v^3}{v(1+v)(1-v+v^2)}$$

$$\therefore \frac{2-v^3}{v(1+v)(1-v+v^2)} \equiv \frac{A}{v} + \frac{B}{1+v} + \frac{Cv+D}{1-v+v^2}$$

Multiplying through by $v(1+v)(1-v+v^2)lcm, we\ have;$

$2 - v^3 = A(1+v)(1-v+v^2) + Bv\,(1-v+v^2) + (Cv+D)(v)(1+v$

Solving the above identity gives

$A = 2, B = -1, C = -2, and\ D = 1$

$$\therefore \frac{2-v^3}{v(1+v)(1-v+v^2)} \equiv \frac{2}{v} - \frac{1}{1+v} + \left(\frac{-2v+1}{1-v+v^2}\right)$$

$$= \frac{2}{v} - \frac{1}{1+v} - \frac{2v-1}{1-v+v^2}$$

But $\int\left(\frac{2-v^3}{v+v^4}\right) dv = -\int\frac{dx}{x}$

$$\Rightarrow \int \left[\frac{2}{v} - \frac{1}{1+v} - \frac{2v-1}{1-v+v^2}\right] dv = -\int \frac{dx}{x}$$

$$2\ln v = \ln(1+v) - \ln(1-v+v^2) = -\ln x + \ln A$$

$$(2)\ln v^2 - \ln(1+v) - \ln(1-v+v^2) = \ln\left(\frac{A}{x}\right)$$

$$\ln\left[\frac{v^2}{(1+v)(1-v+v^2)}\right] = \ln\left(\frac{A}{x}\right)$$

Cancelling ln, we have;

$$\frac{v^2}{1+v(1-v+v^2)} = \frac{A}{x} \qquad (5)$$

But $v = {}^{y}/_{x}$

Substituting v in equation (5) above gives

$$\frac{({}^{y}/_{x})^2}{(1+{}^{y}/_{x})\left(1-\frac{y}{x}+({}^{y}/_{x})^2\right)} = \frac{A}{x}$$

$$\frac{{}^{y^2}/_{x^2}}{(1+{}^{y}/_{x})\left(1-\frac{y}{x}+\frac{y^2}{x^2}\right)} = \frac{A}{x}$$

$$\frac{{}^{y^2}/_{x^2}}{\frac{(x+y\ (x^x-xy+y^2)}{x^3}} = \frac{A}{x}$$

OR $$\frac{y^2}{x^2} * \frac{x^3}{(x+y)(x^2-xy+y^2)} = \frac{A}{x}$$

$$\frac{xy^2}{(x+y)(x^2-xy+y^2)} = \frac{A}{x}$$

OR $$\frac{x^2y^2}{(x+y\ )(x^2-xy+\ y^2)} = A$$

Substituting the initial condition, $i.e.$ $y(0) = 1$

$i.e.$ $x = 0, y = 1$

$$\frac{(0)^2(1)^2}{(0+1)(0-0+1)} = A$$

$$\frac{0}{1} = A$$

OR $A = 0$

$$\therefore \quad \frac{x^2y^2}{(x+y)(x^2-xy+y^2)} = 0$$

(3) Solve $(x^2 + xy)\, dx = (xy - y^2)\, dx$

Solution

$$(x^2 + xy)dy = (xy - y^2)\, dx$$

Rearranging the above equation, we have

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2 + xy} \qquad (1)$$

The above equation is homogenous since the powers of the individual terms involved are same.

Let $y = vx$

Then, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting y and y' in equation (1), we have;

$$v + x\frac{dv}{dx} = \frac{x(vx) - (vx)^2}{x^2 + x(vx)}$$

$$= \frac{x^2v - x^2v^2}{x^2 + x^2v}$$

$$= \frac{x^2(v - v^2)}{x^2(1 + v)}$$

$$v + x\frac{dv}{dx} = \frac{v-v^2}{1+v}$$

$$x\frac{dv}{dx} = \frac{v-v^2}{1+v} - v$$

$$x\frac{dv}{dx} = \frac{v-v^2-v-v^2}{1+v}$$

$$x\frac{dv}{dx} = -\frac{2v^2}{1+v}$$

Separating the variables and integrating, we get

$$\int\left(\frac{1+v}{v^2}\right)dx = -2\int\frac{dy}{dx}$$

$$\int\left(v^{-2} + \frac{1}{v}\right)dv = -2\int\frac{dx}{x}$$

$$-\frac{1}{v} + \ln v = -2\ln x + C$$

But $v = {}^{y}/_{x}$

$$\Rightarrow -\frac{1}{({}^{y}/_{x})} + \ln({}^{y}/_{x}) = -2\ln x + C$$

$$-\frac{x}{y} + \ln({}^{y}/_{x}) = -lnx^2 + C$$

$$\text{OR} -\frac{x}{y} + \ln(yx) + lnx^2 = C$$

$$\frac{-x}{y} + ln(yx) = C$$

Homogenous equations of the form:

$$\frac{a_1+b_1y+c_1}{a_2+b_2y+c_2} = \frac{dy}{dx}$$

Forms

1. $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$

   $i.e. \quad a_1b_2 - a_2b_1 = 0$

2. $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$

   $i.e. \quad a_1b_2 - a_2b_1 \neq 0$

Example (1) Given that $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$,

where $a_1b_2 \neq_2 b_1$,

show that the equation can be reduced to homogeneous equation.

Solution

Given: $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ (1)

Where $a_1b_2 \neq a_2b_1$

Required: To show that equation (1) can be reduced to homogeneous equation.

Proof: let $x = X_0 + h$

and $y = Y_0 + k$

Then $\frac{dy}{dx} = \frac{dY_0}{dX_0}$

Substituting x, y *and* y' *in equation* (1), *we have*

$$\Rightarrow \frac{dY_0}{dX_0} = \frac{a_1(X_0+h)+b_1(Y_0+K)+C_1}{a_2(X_0+h)+b_2(Y_0+K)+C_2}$$

$$\frac{dY_0}{dX_0} = \frac{(a_1X_0+b_1Y_0)+(a_1h+b_1K+C_1)}{(a_2X_0+b_2Y_0)+(a_2h+b_2k+C_2)}$$

Let $a_1h + b_1k + c_1 = 0$

and $a_2h + b_2k + c_2 = 0$

This implies

$$\frac{dY_0}{dX_0} = \frac{a_1X_0+b_1Y_0}{a_2X_0+b_2Y_0}$$

The above equation is now homogeneous and of the form:

$$\frac{dY_0}{dX_0} = f\left(\frac{y}{x}\right)$$

Example (2) Given that $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$

Where $a_1b_2 = a_2b_1$, *show that the equation is separable.*

Solution

Given: $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ (1)

Where $a_1b_2 = a_2b_1$

Required: To show that equation (1) is separable

Proof: from $a_1b_2 = a_2b_1$ ……………………………(given)

We have (2)

$$\frac{a_2}{a_1} = \frac{b_2}{b_1}$$

Let equation (2) be equal to K.

$$\Rightarrow \frac{a_2}{a_1} = \frac{b_2}{b_1} = k$$

OR $a_2 = ka_1$ *and* $b_2 = kb_1$

Substituting a_2 *and* b_2 *in equation* (1), *we have*

$$\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{ka_1x+kb_1y+c_2}$$

$$\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{k(a_1x+b_1y)+c_2} \quad (3)$$

Let $u = a_1x + b_1y$

$$\frac{du}{dx} = a_1 + b_1\frac{dy}{dx}$$

$$\frac{du}{dx} - a_1 = b_1\frac{dy}{dx}$$

Dividing through by b_1, *we get*

$$\frac{1}{b_1}\frac{du}{dx} - \frac{a_1}{b_1} = \frac{dy}{dx}$$

OR $\frac{dy}{dx} = \frac{1}{b_1}\frac{du}{dx} - \frac{a_1}{b_1}$.

Substituting $(a_1x + b_1y)$ *and* ${dy}/{dx}$ *in equation* (3), *we have*

$$\frac{1}{b_1}\frac{du}{dx} - \frac{a_1}{b_1} = \frac{u+c_1}{ku+c_2}$$

$$\frac{1}{b_1}\frac{du}{dx} = \frac{u+c_1}{ku+c_2} + \frac{a_1}{b_1}$$

$$\frac{1}{b_1}\frac{du}{dx} = \frac{b_1(u+c_1)+a_1(ku+c_2)}{b_1(ku+c_2)} \quad (4)$$

Multiplying equation (4) by b_1, *we have*

$$\frac{du}{dx} = \frac{b_1(u+c_1)+a_1(ku+c_2)}{ku+c_2}$$

Separating variables, we obtain

$$\int \frac{ku+c_2}{b_1(u+c_1)+a_1(ku+c_2)}\, du = \int dx$$

The above is the form required.

Example (3) Solve the differential equation; $\frac{dy}{dx} = \frac{x+y-3}{x-y-1}$

Solution

$$\frac{dy}{dx} = \frac{x+y-3}{x-y-1} \quad (1)$$

$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = -1 - 1 = -2 \neq 0$$

i.e., the equation is of the form $a_1b_2 \neq a_2b_1$

Let $x = X_0 + h\,, y = Y_0 + k$ *and* $\frac{dy}{dx} = \frac{dY_0}{dX_0}$

Substituting x, y *and* y^1 *in equation* (1), *we have*

$$\frac{dY_0}{dX_0} = \frac{X_0+h\ Y_0+k-3}{X_0+h-(Y_0+k)-1}$$

$$\frac{dY_0}{dX_0} = \frac{(X_0+Y_0)+(h+k-3)}{(X_0-Y_0)+(h-k-1)} \quad (2)$$

(3)

Let $h + k - 3 = 0$ *or* $h + k = 3$ (4)

and $h - k - 1 = 0$ *or* $h - k = 1$

Solving equations (2) and (3) simultaneously, we obtain

$h = 2,$ *and* $k = 1.$

Equation (2) now becomes

$$\frac{dY_0}{dX_0} = \frac{X_0+Y_0}{X_0-Y_0} \qquad (5)$$

Equation (5) above is homogeneous and can be used making the substituting;

$Y_0 = VX_0$

and $Y_0' = v + X_0\frac{dv}{dX_0}$

Substituting Y_0 *and* Y_0' *in equation* (5)*gives*

$$v + X_0\frac{dv}{dX_0} = \frac{X_0+VX_0}{X_0-VX_0}$$

$$= \frac{X_o + (1 + V)}{X_0(1 - v)}$$

$$v + X_0\frac{dv}{dX_0} = \frac{1+v}{1-v}$$

$$X_0\frac{dv}{dX_0} = \frac{1+v}{1-v} - v$$

$$X_0\frac{dV}{dX_0} = \frac{1+V-V\,(1-V)}{1-V}$$

$$= \frac{1 + V - V + V^2}{1 - V}$$

$$X_0\frac{dV}{dX_0} = \frac{1+V^2}{1-V}$$

Separating the variables and integrating wrt x we, have

$$\int\left(\frac{1-V}{1+V^2}\right) dv = \int\frac{dX_0}{X_0}$$

$$\int \left[\frac{1}{1+V^2} - \frac{V}{1+V^2}\right] dv = \int \frac{dX_0}{X_0}$$

$$\arctan V - \frac{1}{2}\ln(1+V^2) = \ln X_0 + C$$

But $V = {Y_0}/{X_0}$

$$\Rightarrow \arctan\left({Y_0}/{X_0}\right) - \frac{1}{2}\ln\left(1+\left(\frac{Y_0}{X_0}\right)^2\right) = \ln X_0 + C \quad (6)$$

But $x = X_0 + h$ and $y = Y_0 + k$

$\Rightarrow X_0 = x - h$ and $Y_0 = y - k$

or $X_0 = x - 2$ and $Y_0 = y - 1$

Substituting X_0 and Y_0 in equation (6), we have

$$\arctan\left[\frac{y-1}{x-2}\right] - \frac{1}{2}\ln\left[1+\left(\frac{y-1}{x-2}\right)^2\right] = \ln(x-2) + C$$

Example (4) Solve the differential equation

$$\left\{\frac{dy}{dx} = \frac{x+y-3}{3x+3y+1},\ y(-2) = \frac{1}{2}\right\}$$

Solution

$$\frac{dy}{dx} = \frac{(x+y)-3}{3(x+y)+1} \quad (1)$$

$$\begin{vmatrix}1 & 1\\3 & 3\end{vmatrix} = 3 - 3 = 0$$

i.e. the equation is of the form $a_1b_2 = a_2b_1$

Let $u = x + y$ where $u = f(x)$

$$\frac{du}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{du}{dx} - 1 = \frac{dy}{dx}$$

Substituting for y' and $(x + y)$ in equation (1), we get

$$\frac{du}{dx} - 1 = \frac{u-3}{3u+1}$$

$$\frac{du}{dx} = \frac{u-3}{3u+1} + 1$$

$$\frac{du}{dx} = \frac{u-3+3u+1}{3u+1}$$

$$\frac{du}{dx} = \frac{4u-2}{3u+1} \qquad (2)$$

The above equation is now separable

$$\left(\frac{3u+1}{4u-2}\right) du = dx$$

Integrating both sides, we have

$$\int \left(\frac{3u+1}{4u-2}\right) du = \int dx$$

$$\frac{3u}{4} + \frac{5}{8} \ln(4u - 2) = x + C$$

But $u = x + y$,

$$\Rightarrow \frac{3(x+y)}{4} + \frac{5}{8} \ln[4(x + y) - 2] = x + C$$

Applying the initial condition, ie $y(-2) = {}^{1}/_{2}$

$$\frac{3}{4}(-2 + {}^{1}/_{2}) + \frac{5}{8} \ln[4(-2 + {}^{1}/_{2}) - 2] = -2 + C$$

$$\frac{-9}{8} + \frac{5}{8} \ln[-8] = -2 + C$$

$$\frac{5}{8} \ln(-8) + 2 \frac{-9}{8} = C$$

$\frac{5}{8}\ln(-8) + \frac{7}{8} = C$

$\therefore \quad \frac{3(x-y)}{4} + \frac{5}{8}\, ln(4\,x + 4y - 2) = x + \frac{7}{8}$

PRACTICE PROBLEM 3

(1) Solve the following differential equations

(a) $\frac{dy}{dx} = \frac{2y-x+5}{2x-y-4}$

(b) $\frac{dy}{dx} = \frac{x+3y-5}{x-y-1}$

(c) $\frac{dy}{dx} = \frac{2x+3y+3}{x-2y-5}$

(2) Solve the differential equations

(a) $(2x - 2y + 1\,)dx - (2x + y - 2)dx = 0$

(b) $(2x - 5y + 3\,)dx - (2x + 4y - 6)dy = 0$

(c) $\frac{dy}{dx} = \frac{2x+2y-2}{x+y-5}$

(d) $\frac{dy}{dx} = \frac{-4x+3y+15}{2x+y-7}$

(3) Show that the equations in problems (a) to (h) are homogeneous and hence, find their solutions.

(a) $(x^2 + 3xy + y^2)\,dx - x^2\,dy = 0$

(b) $\frac{dy}{dx} = \frac{x+3y}{x-y}$

(c) $\frac{dy}{dx} = \frac{4y-3x}{2x-y}$

(d) $\frac{dy}{dx} = -\frac{4x+3y}{2x+y}$

(e) $\frac{dy}{dx} = \frac{x^x+xy+y^2}{x^2}$

(f) $\frac{dy}{dx} = \frac{x+y}{x}$.

(g) $2y\,dx - x\,dy = 0$

(h) $\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$

CHAPTER FOUR

LINEAR EQUATIONS

Any differential equation in the form

$$\frac{dy}{dx} + p(x)y = g(x) \qquad (1)$$

is said to be linear. Examples include:

(i) $y^I + 2y = e^{-x}$

(ii) $(1 + x^2)\frac{dy}{dx} + 4xy = (1 + x^2)^2$

(iii) $y^I - 2xy = x \ etc$

Linear equation are solved by multiplying the given equation with the integrating factor (μ).

Derivation of the formula for integrating factor.

The linear differential equation is of the form

$$y^I + p(x)\, y = g(x) \qquad (1)$$

Multiplying both sides of equation (1) by the integrating factor μ, we have

$$\mu[y^I + p(x)y] = \mu[g(x)] \qquad (2)$$

From the left hand side (LHS) of equation (2), we obtain

$$\mu[y^I + p(x)y] = [\mu y]^1$$

$$\mu y^I + \mu p(x)y = My^I + \mu^I y$$

Cancelling μy^I since its common, we have

$\mu p(x)y = \mu^I y$

Cancelling y, we have

$\mu p(x) = \mu^I$

$p(x) = \frac{\mu^I}{\mu^I}$

Integrating both sides, we have

$\int p(x) = \int \frac{\mu^I}{\mu}$

$\int p(x) = \ln \mu$

$or\ \mu = e^{\int p(x)dx}$

$\therefore \mu_{(x)} = e^{\int p(x)dx}$

SOLVED EXAMPLES

(1) Find the solution of the initial value problem (IVP):

$$\begin{Bmatrix} y^I - 2xy = x \\ y(0) = 0 \end{Bmatrix}$$

Solution

Given: $y^I - 2xy = x$ (1)

Required: To solve equation (1)

Procedure: (a) Equation (1) is a linear equation since it is of the form

$y^I + p(x)y = g(x)$

Where $p(x) = -2x, g(x) = x$

(b) We now determine the integrating factor of the equation which is given by

$\mu = e^{\int p dx} = e^{\int -2x dx} = e^{-\frac{2x^2}{2}} = e^{-x^2}$

(c) We now multiply equation (1) with the integrating factor to give

$e^{-x^2}[y^I - 2xy] = e^{-x^2}(x)$

$\left(e^{-x^2}y\right)^1 = xe^{-x^2}$

Integrating both sides, we have

$\int (e^{x2}y)^1 = \int xe^{-x^2}dx$

$e^{-x^2}y = -\frac{1}{2}e^{-x^2} + C$

Dividing through with e^{-x^2} gives

$y = -\frac{1}{2} + Ce^{x^2}$

Applying the initial condition gives

$i.e.\ \ y(0) = 0, we\ have$

$0 = -\frac{1}{2} + C\ (1)$

$\frac{1}{2} = C$

$or\ C = {}^1/_2$

$\therefore y = -\frac{1}{2} + \frac{1}{2}e^{x^2}$

$or\ \ y = \frac{1}{2}\left[e^{x^2} - 1\right]$

$\int xe^{-x^2}\ dx$

This is evaluated using substitution

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$\Rightarrow \quad \int ex^{-x^2}dx = \int xe^{-u}\frac{du}{2x}$

Cancelling x, we have

$\frac{1}{2}\int e^{-u}\,du = -\frac{1}{2}\,e^{-u} + C$

But $u = x^2$

$\therefore \int xe^{-x^2}\,dx = -\frac{1}{2}\,e^{-x^2} + C$

(2) Solve the initial value problem

$y^{l} - 2y = e^{2x}, \quad y(0) = 2$

Solution

$y^{l} - 2y = e^{2x}$ (1)

$p(x) = -2, \quad g(x) = e^{2x}$

$N = e^{\int pdx} = e^{\int -2dx} = e^{-2x}$

Multiplying equation (1) with the integrating factor (e^{-2x})

$e^{-2x}(y^{l} - 2y) = e^{-2x}(e^{2x})$

$(e^{-x}y)^1 = 1$

Integrating both sides with x, we have

$\int (e^{-2x}y)^1 = \int dx.$

$e^{-2x}y = x + C.$

Dividing through with e^{-2x}, we obtain (2)

$y = xe^{2x} + Ce^{2x}$

Applying the initial condition, $i.e.\, y(0) = 2, \; i.e. \; x = 0, y = 2,$

$\Rightarrow \quad 2 = 0 + C(1)$

$C = 2$

Substituting C in equation (2), we have

$y = xe^{2x} + 2e^{2x}$

$y = e^{2x}(x + 2)$

(3) Solve the initial value problem

$xy^1 + 2y = 4x^2, \qquad y(1) = 2.$

Solution

$xy^I + 2y = 4x^2$

Dividing through with x, we have

$y^I + \frac{2y}{x} = 4x$ (1)

The above equation is now linear since it is of the form;

$y^I + p(x)y = g(x)$

Where $p(x) = \frac{2}{x}, \; g(x) = 4x$

The integrating factor is given *by*

$\mu = e^{\int p dx} = e^{\int \frac{2}{x} dx} = e^{2lnx} = e^{lnx^2} = x^2$

Multiplying equation (1) with the integrating factor, we obtain

$x^2 \left[y^I + \frac{2}{x}y\right] = x^2\,(4x)$

$(x^2y)^I = 4x^3$

Integrating both sides, we have

$\int (x^2y)^I = \int 4x^3 \; dx$

$\frac{4x^4}{\;} + C$

$x^2y = x^4 + C$

Dividing through by x^2, we have

$y = x^2 + Cx^{-2}$ (2)

Applying the initial condition

$ie\ x = 1, y = 2,\ we\ have$

$2 = (1)^2 + c$

$2 - 1 = c$

$c = 1$

Substituting C in equation (2) we have

$y = x^2 + x^{-2}$

$or\ \ y = x^2 + \frac{1}{x^2}$

(4) Solve the initial value problem (IVP)

$y^I + 2y = g(x),\ y(0) = 0$

Where $g(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 0 \end{cases}$

<u>Solution</u>

Given: $y^I + 2y = g(x)$ (1)

For $g(x) = 1, 0 \le x \le 1$

$y^I + 2y = 1$ (2)

$p(x) = 2, g(x) = 1$

$\mu = e^{\int Pdx} = e^{\int 2dx} = e^{2x}$

Multiplying equation (2) with the integrating factor(e^{2x}), we have

$e^{2x}\ [y^I + 2y] = e^{2x}\ (1)$

$(e^{2x}y)^I = e^{2x}$

Integrating both sides wrt x, gives

$\int (e^{2x}y)' = \int e^{2x}\,dx$

$e^{2x}y = \frac{e^{2x}}{2} + C$

Dividing through by, e^{2x}, *we have*

$y_1 = \frac{1}{2} + Ce^{-2x}$ (3)

Applying the initial condition

i.e. $y(0) = 0$, *we obtain*

$0 = \frac{1}{2} + C(1)$

$C = -\frac{1}{2}$

Substituting C in equation (3), we get

$y = \frac{1}{2} - \frac{1}{2}e^{2x}$

$y = \frac{1}{2}[1 - e^{2x}]$ (4)

for $0 \leq x \leq 1$

For $g(x) = 0 \qquad x > 1$

The linear equation now becomes;

$y' + 2y = 0$ (5)

$\mu = e^{2x}$ [*The same as in previous case*].

Multiplying equation (5) by the IF (e^{2x}), we have

$e^{2x}[y' + 2y] = e^{2x}[0]$

$(e^{2x})' = 0$

Integrating both sides wrt x, gives

$\int (e^{2x}y)' = \int 0dy$

$e^{2x}y = C_2$

$\therefore y = C_2 e^{2x}$ (6)

Applying the initial condition

$i.e. \quad y = 0, x = 0, \qquad we\ obtain$

$0 = C_2(1)$

$let\ C_2 = 1$

$\Rightarrow y = e^{2x}$ (7)

Equating equations (4) and (7) at $x = 1$, we have

$\frac{1}{2}[1 - e^{-2}] = ke^{-2}$

Dividing through by e^{-2}, we get

$\frac{1}{2}e^2 - \frac{1}{2} = k$

$\therefore k = \frac{1}{2}[e^2 - 1]$

$y = k\,e^{-2x}$

$y = \frac{1}{2}[e^2 - 1]e^{-2x} \quad for\ x > 1$

(5) Solve the initial value problem

$y^I + p(x)y = 0, \quad y(0) = 1$

Where

$$p(x) = \begin{cases} 2, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

Solution

Given: $y^I + p(x)y = 0$ (1)

(i) For $p(x) = 2 \qquad 0 \le x \le 1$

Substituting $p(x) = 2\ in\ equation\ (1)\ , we\ have$

$y^1 + 2y = 0$ (2)

$M = e^{\int 2dx} = e^{2x}$

Multiplying equation (2) by IF (e^{2x}), *we get*

$e^{2x}(y' + 2y) = e^{2x}(0)$

$(e^{2x}.y)' = 0$

Integrating both sides, we have

$\int(e^{2x}y)' = \int 0dx$

$e^{2x}y = C_1$

$y = C_1 e^{-2x}$

Applying the initial condition, we have (3)

i.e. $x = 0, y = 1$

$\Rightarrow 1 = C_1(1)$

$C = 1$

$\therefore y = e^{-2x}$

for $0 \leq x \leq 1$ (4)

(ii) For $p(x) = 1 \qquad x > 1$

Equation (1) now becomes

$y' + y = 0$ (5)

$\mu = e^{\int pdx} = e^{\int 1dx} = e^x$

Multiplying equation (5) by the IF (e^x), *we have*

$e^x[y' + y] = e^x[0]$

$(e^x)' = 0$

Integrating both sides wrt x, we have

$\int(e^x y)' = \int 0dx$

$e^x y = C$

$\therefore y = C e^{-x}$ (6)

Applying the initial condition, *i.e.* $x = 0, y = 1$, *we get*

$1 = C\ (1)$

$C = 1$

$\therefore y = e^{-x}$ (7)

Equating (4) and (7), we have

$e^{-2x} = ke^{-x}$

For $x = 1$, *we have*

$e^{-2} = k\,e^{-1}$

$\therefore k = e^{-1}$

But $y_2 = k\,e^{-x}$

$y_2 = e^{-1}, e^{-x} = e^{-(1+x)}$ *for* $x > 1$

(6) Solve $y^I + \frac{2}{x}y = \frac{\cos x}{x^2}$,

Solution

$y^I + \frac{2}{x}y = \frac{\cos x}{x^2}$ (1)

$p(x) = \frac{2}{x}$, $g(x) = \frac{\cos x}{x^2}$

$\mu = e^{\int Pdx} = e^{\int\frac{2}{x}} = e^{2lnx} = e^{lnx^2} = x^2$

Multiplying equation (1) by the IF (x^2), *we have*

$x^2\left[y^I + \frac{2}{x}y\right] = x^2\left[\frac{\cos x}{x^2}\right]$

$(x^2.y)^I = \cos x$

Integrating both sides wrt gives

$\int(x^2.y^I)^I = \int\cos x\,dx$

$x^2.y = \sin x + C$

$$\therefore y = \frac{\sin x}{x^2} + \frac{c}{x^2}$$ (2)

PRACTICE PROBLEMS 4

(1) Solve $y^I + \left(\frac{2}{x}\right) y = \frac{1}{x^2}$

(2) Solve $y^I + \left(\frac{1}{x}\right) y = \frac{\cos x}{x}$

(3) Solve $y^I - 2y = x^2 e^{2x}$

(4) Solve $(1 + x^2)\, y^I + 4xy = (1 + x^2)^{-2}$

(5) Solve the following IVPs

(a) $y^I - y = 2x\, e^{2x}$, $y(0) = 1$

(b) $xy^I + 2y = \sin x \;\; y\left(\frac{\pi}{2}\right) = 1$

(c) $y^I + \frac{2}{x}\, y = \frac{\cos x}{x^2}$

CHAPTER FIVE

BERNOULLI'S EQUATIONS

Any differential equation in the form

$$\frac{dy}{dx} + p(x) = Q(x)\, y^n$$

is known as a Bernoulli's equation.

Steps involved in solving Bernoulli's equations

Given: $\frac{dy}{dx} + p(x)y = Q(x)y^n$ (1)

Divide equation (1) by y^n *to give*

$$y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = Q(x) \quad (2)$$

Let $V = y^{1-n}$ where v = f(x)

Then $\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$

Multiplying equation (2) by $(1-n)$, *we have*

$$(1-n)y^{-n}\frac{dy}{dx} + (1-n)p(x)y^{1-n} = (1-n)\,Q(x) \quad (3)$$

Putting

$y^{1-n} = V$ *and* $(1-n)y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$ *in equation* (3), *we have*

$$\frac{dv}{dx} + (1-n)p(x)V = (1-n)\,Q\,(x) \quad (4)$$

The above equation is now a linear equation and can be solved by using the integrating factor method.

Solved examples

(1) Solve the initial value problem

($y(0) = {}^{1}/_{10}$)

Solution

$3y^{I} - 2y = -y^4 e^{3x}$

Dividing through by 3, we have

$y^{I} - \frac{2}{3}y = \frac{-y^4}{3}\, e^{3x}$

Dividing equation (2) with y^4, we have

$y^{-4}\frac{dy}{dx} - \frac{2}{3}y^{-3} = -\frac{1}{3}e^{3x}$

Let $V = y^{-3}$ $where\ v = f(x)$

The $\frac{dv}{dx} = -3y^{-4}\frac{dy}{dx}$

Multiplying equation (3) by $-3, we\ obtain$

$-3y^{-4}\frac{dy}{dx} + 2y^{-3} = e^{3x}$ (4)

Substituting

$y^{-3} = V\ and\ putting - 3y^{-4}\frac{dy}{dx} = \frac{dv}{dx}\ in\ equation\ (4), we\ have$

$\frac{dv}{dx} + 2v = e^{3x}$ (5)

The above equation is now linear in V. Its integrating factor is given by

$\mu = e^{\int 2dx} = e^{2x}$

Multiplying equation (5) with the IF $(e^{2x}), we\ have$

$e^{2x}[V^{I} + 2V] = e^{2x}(e^{3x})$

$(e^{2x}.V)^{I} = e^{5x}$

Integrating both sides with $x, we\ get$

$$\int (e^{2x}.V)^{I} = \int e^{5x}\,dx$$

$$(e^{2x}.V) = \frac{1}{5}\,e^{5x} + C$$

Dividing through by e^{2x}, *we have*

$$V = \frac{1}{5}\,e^{3x} + Ce^{-2x} \qquad (6)$$

But $V = y^{-3} = \frac{1}{y^{3}}$

Substituting V *in equation* (6) *gives*

$$\frac{1}{y^{3}} = \frac{1}{5}\,e^{3x} + Ce^{-2x}$$

$$\frac{1}{y^{3}} = \frac{e^{3x}}{5} + \frac{C}{e^{2x}}$$

$$\frac{1}{y^{3}} = \frac{e^{5x} + 5C}{5e^{2x}}$$

$$\therefore\ y^{3} = \frac{5e^{2x}}{e^{5x} + 5C}$$

Applying the initial condition, we have

i.e. $x = 0\,, y = {}^{1}/_{10}$

$$\Rightarrow \left(\frac{1}{10}\right)^{3} = \frac{5}{1 + 5c}$$

$$\frac{1}{1000} = \frac{5}{1 + 5c}$$

$$1 + 5C = 5000$$

$$5C = 49000$$

Let $5C = K$

$\Rightarrow k = 49000$

$$\therefore\ y^3 = \frac{5e^{2x}}{e^{5x} + 49000}$$

(2) Solve the equation; $x^2y - x^3\frac{dy}{dx} = y^4\cos x$

Solution

$x^2y - x^3\frac{dy}{dx} = y^4\cos x$

Rearranging equation (1), we have

$x^3\frac{dy}{dx} - x^2y = -y^4\cos x$

Dividing equation (2) by $x^3, we\ have$

$\frac{dy}{dx} - \frac{1}{x}y = \frac{-y^4}{x^3}\cos x$

Dividing equation (3) by $y^4, we\ get$

$y^{-4}\frac{dy}{dx} - \frac{1}{x}y^{-3} = -\frac{1}{x^3}\cos x$ (

Let $V = y^{-3}\ where\ V = f(x)$

Then $\frac{dv}{dx} = -3y^{-4}\frac{dy}{dx}$

Multiplying equation (4) by $-3, we\ have$ (5)

$-3y^{-4}\frac{dy}{dx} + \frac{3}{x}y^{-3} = \frac{3}{x^3}\cos x$

Substituting $y^{-3}\ and - 3y^{-4}\frac{dy}{dx}\ in\ equation\ (5), we\ have$ (6)

$\frac{dv}{dx} + \frac{3}{x}V = \frac{3}{x^3}\cos x$

Equation (6) above is now linear in V and can be solved using integrating factor (μ). The *IF* is given by

$e^{\int Pdx}\ e^{\int\frac{3}{x}dx} = e^{3lnx} = e^{lnx^3} = x^3$

Multiplying equation (6) by the IF (x^3), *we obtain*

$$x^3\left[\frac{dv}{dx}+\frac{3}{x}V\right] = x^3\left[\frac{3}{x^3}\cos x\right]$$

$$(x^3 V)^I = 3\cos x$$

Integrating equation (7) wrt x, *we have*

$$\int (x^3V)^I = 3\int \cos x\, dx$$

$$x^3v = 3\sin x + C$$

$$V = \frac{3}{x^3}\sin x + \frac{C}{x^3}$$

But V is given by

$$V = y^{-3} = \frac{1}{y^3}$$

Substituting V in equation (8) gives

$$\frac{1}{y^3} = \frac{3\sin x}{x^3} + \frac{C}{x^3}$$

$$y^3 = \frac{x^3}{3\sin x + C}$$

(3) Solve the equation; $x^2 y^I + 2xy - y^3 = 0,\ \ x > 0.$

(1)

Solution

(2)

$x^2y^I + 2xy - y^3 = 0$

Rewriting equation (1), we have

$x^2y^I + 2xy = y^3$

Dividing through by x^2, *we get*

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{y^3}{x^2} \qquad (3)$$

Dividing through by y^3, *we obtain*

$$y^{-3}\frac{dy}{dx} + \frac{2}{x}y^{-2} = \frac{1}{x^2} \qquad (4)$$

Let $V = y^{-2}$ *where* $V = f(x)$

Then $\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$

Multiplying through by -2, *we have*

$$-2y^{-3}\frac{dy}{dx} - \frac{4}{x}y^{-2} = -\frac{2}{x^2} \qquad (5)$$

Substituting y^{-2} *and* $-2y^{-3}\frac{dy}{dx}$ *in equation* (5), *we obtain*

$$\frac{dv}{dx} - \frac{d}{x}v = -\frac{2}{x^2} \qquad (6)$$

The above equation is now linear in V and its integrating factor is given by

$$\mu = e^{\int Pdx} = e^{\int\frac{-4}{x}} = e^{-4lnx} = e^{lnx^{-4}} = x^{-4} = \frac{1}{x^4}$$

Multiplying equation (6) by the integrating factor $\left({}^{1}/_{x^4}\right)$, *we have*

$$\frac{1}{x^4}\left[\frac{dv}{dx} - \frac{4}{x}V\right] = \frac{1}{x^4}\left[\frac{-2}{x^2}\right]$$

$$\left(\frac{1}{x^4}V\right)^{I} = \frac{-2}{x^6}$$

Integrating both sides wrt x, *we have*

$$\int\left(\frac{1}{x^4}V\right)^{I} = -2\int\frac{1}{x^6}dx = -2\int x^{-6}dx$$

$$\frac{1}{x^4}V = \frac{-2x^{-5}}{-5} + C$$

$$\frac{V}{x^4} = \frac{+2x^{-5}}{5} + C$$

$\therefore V = \frac{2}{5}x^{-1} + Cx^4$ (7)

But $V = y^{-2} = \frac{1}{y^2}$

$\Rightarrow$ *substituting V in (7) gives*

$$\frac{1}{y^2} = \frac{2}{5}x^{-1} + Cx^4$$

or $\frac{1}{y^2} = \frac{2}{5x} + Cx^4$

or $\frac{1}{y^2} = \frac{2 + 5Cx^5}{5x}$

or $y^2 = \frac{5x}{2 + 5Cx^5}$

or $y = \left[\frac{5x}{2 + 5Cx^5}\right]^{1/2}$

(4) Solve $y - 2x\frac{dy}{dx} = x(x+1)y^3$

Solution (1)

$$y - 2x\frac{dy}{dx} = x(x+1)y^3$$

Rearranging equation (1) gives us (2)

$$2x\frac{dy}{dx} - y = -x(x+1)y^3$$

Dividing through by $2x$, *we have;* (3)

$$\frac{dy}{dx} - \frac{y}{2x} = -\frac{(x+1)y^3}{2}$$

Dividing through by y^3 *gives;* (4)

$$y^{-3}\frac{dy}{dx} - \frac{1}{2x}y^{-2} = -\frac{(x+1)}{2}$$

Let $V = y^{-2}\ where\ V = f(x)$

Then $\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$

Multiplying equation (4) by $(-2)gives$

$-2y^{-3}\frac{dy}{dx} + \frac{1}{x}y^{-2} = (x+1)$ (5)

Substituting $y^{-2}\ and\ \left(-2y^{-3}\frac{dy}{dx}\right)\ in\ equaton\ (5)gives$

$\frac{dv}{dx} + \frac{1}{x}V = (x+1)$ (6)

The above is now linear and can be solved using integrating factor which is given by

$\mu = e^{\int Pdx} = e^{\int\frac{1dx}{x}} = e^{lnx} = x$

Multiplying equation (6) by the IF $(x), we\ have$

$$x\left[\frac{dv}{dx} + \frac{1}{x}V\right] = x\ (x+1)$$

$(xv)^I = x^2 + x$ (7)

Integrating equation (7) gives

$$\int (xv)^I = \int (x^2 + x)\ dx$$

$$xv = \frac{x^3}{3} + \frac{x^2}{2} + C$$

Dividing through by $x\ leads\ to$

$V = \frac{x^2}{3} + \frac{x}{2} + Cx^{-1}$

(8)

But $V = y^{-2} = \frac{1}{y^2}$

Substituting $V\ in\ equation\ (8)gives$

$$\frac{1}{y^2} = \frac{x^2}{3} + \frac{x}{2} + \frac{C}{x}$$

$$\frac{1}{y^2} = \frac{2x^3\ 3x^2 + 6c}{6x}$$

$$y^2 = \frac{6x}{2x^3 + 3x^2 + 6c}$$

$$\therefore \quad y = \left[\frac{6x}{2x^2 + 3x^2 + 6x}\right]^{1/2}$$

(5) Solve the differential equation; $xy^I - y - y^2e^{2x} = 0$

Solution

Given: $xy^I - y - y^2e^{2x} = 0$ (1)

Required: To solve equation (1)

Procedure: Rearranging equation (1) gives us;

$xy^I - y = y^2e^{2x}$ (2)

Dividing equation (2) by $x\ gives$

$\frac{dy}{dx} - \frac{1}{x}y = \frac{y^2e^{2x}}{x}$ (3)

Dividing equation (3) by $y^2\ gives$

$y^{-2}\frac{dy}{dx} - \frac{1}{x}y^{-1} = \frac{e^{2x}}{x}$ (4)

Let $V = y^{-1}\ where\ V\ is\ a\ function\ of\ x.$

Then $\frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$

Multiplying equation by $(-1)\,gives\ us$

$-y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = -\frac{e^{2x}}{x}$ (5)

Substituting y^{-1} *and* $\left(-y^{-2}\frac{dy}{dx}\right)$ *in* (5), *we obtain*

$$\frac{dv}{dx} + \frac{1}{x}V = -\frac{e^{2x}}{x} \qquad (6)$$

The above equation is linear and can be solved using integrating factor.

$$\mu = e^{\int Pdx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

Multiplying equation (6) by x *gives us*

$$x\left[\frac{dv}{dx} + \frac{1}{x}V\right] = x\left(\frac{-e^{2x}}{x}\right)$$

$$(xv)^1 = -e^{2x} \qquad (7)$$

Integrating both sides of equation (7) gives

$$\int (xv)^I = -\int e^{2x}\,dx$$

$$xv = -\frac{e^{2x}}{2} + C$$

$$V = \frac{-e^{2x}}{2x} + \frac{C}{x}$$

$$V = \frac{-e^{2x} + 2c}{2x}$$

But $V = y^{-1} = \frac{1}{y}$

$$\Rightarrow \qquad \frac{1}{y} = \frac{-e^{2x} + 2c}{2x}$$

$$or \qquad y = \frac{2x}{-e^{2x} + 2c}$$

(6) Solve the equation $\frac{dy}{dx} + y\tan x = y^3 \sec^4 x$

Solution

$\frac{dy}{dx} + y\tan x = y^3 sec^4 x$ (1)

Dividing both sides of equation (1) by y^3 *gives*

$y^{-3}\,\frac{dy}{dx} + \,y^{-2}\,tan\,x = sec^4 x$ (2)

Let $V = \,y^{-2}\,where\,V = f(x)$

The $\frac{dv}{dx} = -\,y^{-3}\frac{dy}{dx}\,(implicit\,differentiation)$

Multiplying both sides of equation (2) by (-1) *gives;*

$-y^{-3}\,\frac{dy}{dx} = -y^{-2}\tan x = \,-sec^4 x$ (3)

Substituting $y^{-2}\,and\,-(y^{-3}\frac{dy}{dx})\,in\,equation\,(3) gives;$

(4)

$\frac{dv}{dx} - V\tan x = sec^4 x$

The above equation is now linear and can be solved using integrating factor (IF).

$\mu = \,e^{\int Pdx} = \,e^{\int \tan x\,dx} = \,e^{\int\frac{\sin x}{\cos x}dx}$

$M = \,e^{-\int\frac{d(\cos x)}{\cos x}} = \,e^{-ln\cos x} = \,e^{[ln\cos x]^{-1}}$

$\therefore M = (\cos x)^{-1} = \frac{1}{\cos x} = \sec x$

Multiplying equation (4) by the integrating factor $(\sec x)$ *gives*

$\sec x\,\left[\frac{dv}{dx} - V\tan x\right] = \sec x\,(-sec^4 x)$

$(V\sec x)^I = \,-sec^5 x$

Integrating the above gives

$\int(V\sec x\,)^I = \,-\,\int sec^5 x\,dx$

$V\sec x = \,-\int sec^5 x\,dx$

PRACTICE PROBLEMS 5

(1) Solve the differential equation

$$\frac{dy}{dx} + y = y^2 (\cos x - \sin x)$$

(2) Solve $\frac{2dy}{dx} + y = y^3 (x - 1)$

(3) Solve $y + (x^2 - 4x)\frac{dy}{dx} = 0$

(4) Solve $2y - 3\frac{dy}{dx} = y^4 e^{3x}$

SECTION B

This section deals with the following:

- ✓ Introduction to Second Order Differential Equations
- ✓ Methods of solving Second Order Differential Equations which include
  - Method of undetermined coefficients
  - Method of Variation of parameters
  - Method of Reduction of Order and
  - Use of Laplace transforms.

CHAPTER SIX

SECOND ORDER DIFFERENTIAL EQUATIONS

The second order differential equation is of the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g(x) \quad (1)$$

Where $a, b, and\ c$ are constant coefficients and $g(x)$ is a given function of x.

It is important that when solving equation (1),

(a) The homogeneous part of the equation which is,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \quad (2)$$

is first solved. This is solved by making the substitutions;

$$\frac{d^2y}{dx^2} = m^2, \frac{dy}{dx} = m\ and\ y = 1$$

Implementing the substitutions give us

$$am^2 + bm + c = 0 \quad (3)$$

Equation (3) above is called the auxiliary equation and it is solved using factorization, completion of squares methods or any other method used in solving quadratic equations.

The nature of solution of equation (3) is

$$y = k_1e^{m_1x} + k_2e^{m_2x}$$

Where $k_1\ and\ k_2$ are arbitrary constants and m_1 and m_2 are the roots of the quadratic equation:

$$am^2 + bm + c = 0$$

(b) After solving the homogeneous part of equation (1) then, the second part which is the particular integral is now solved. This will be explained shortly.

Nature of solution (s)

(1) For real and distinct roots,

$m = m_1$ *and* $m = m_2$, *the solution is given by;*

$y = k_1 e^{m_1 x} + k_2 e^{m_2 x}$

(2) For real and equal roots,

i.e., $m_1 = m_2$, *the solution is given by;*

$y = e^{mx}(A + Bx)$

(3) For complex roots, *ie* $m = a + j\beta$

Then, $y = e^{ax}\{k_1 \cos \beta x + k_2 \sin \beta x\}$

(4) $\frac{d^2y}{dx^2} + n^2 y = 0$

The solution is $y = k_1 \cos nx + k_2 \sin nx$

(5) $\frac{d^2y}{dx^2} - n^2 y = 0$

Then, the solution is; $y = k_1 \cosh nx + k_2 \sinh nx$

Notes

(1) $y = k_1 e^{m_1 x} + k_2 e^{m_2 x}$ *is called the complementary function:*

(2) $y =$ x (*a function of x*) *is called the particular integral* (P.I)

(2) The complete general solution is given by
General Solution = {Complementary function} + {Particular Integral}

i.e. $y_T = y_{cf} + y_{PI}$

Solved examples

(1) Solve the diff...

$y^{(IV)} + 5y^{II} + 4y = 0$

$y(0) = 0,\ y^{I}(0) = 1,\ y^{II}(0) = 3,\ y^{III}(0) = 0$

Solution

Given: $y^{(IV)} + 5y^{II} + 4y = 0$ (1)

Required: To solve equation (1)

Procedure: Rewriting equation (1) to get its auxiliary equation, we have

$m^4 + 5m^2 + 4 = 0$ (2)

$m^4 + m^2 + 4m^2 + 4 = 0\ (factorizing)$

$m^2\,(m^2 + 1) + 4\,(m^2 + 1) = 0$

$(m^2 + 1)(m^2 + 4) = 0$

$m^2 + 1 = 0 \quad or \quad m^2 + 4 = 0$

$m^2 = -1 \quad or \quad m^2 = -4$

$\therefore m = \sqrt{-1} \quad or \quad m = \sqrt{-4}$

$m = \pm j1 \quad or \quad m = \pm j2$

The roots of the equation are complex.

For complex roots, we have

$y = e^{ax}\,\{k_1 \cos\beta x + k_2 \sin\beta x\}$

But in the given case, $a = 0,\ this\ implies$

$y = k_1 \cos\beta x + k_2 \sin\beta x$

Therefore, we have

$y = k_1 \cos x + k_2 \sin x + k_3 \cos 2x + k_4 \sin 2x$

Substituting the given conditions in equation (3), we have

$for\ y(0) = 0\ \ i.e.,\ x = 0, y = 0;$

$0 = k_1 \cos(0) + k_2 \sin(0) + k_3 \cos(0) + k_4 \sin(0)$

$0 = k_1 (1) + k_3 (1)$

$\therefore k_1 + k_3 = 0$ (4)

Differentiating equation (3), we have

$y^I = -k_1 \sin x + k_2 \cos x - 2k_3 \sin 2x + 2k_4 \cos 2x$ (3a)

Substituting $y^I(0) = 1, in\ equation\ (3a), we\ have$

$1 = -k_1 \sin(0) + k_2 \cos(0) - 2k_3 \sin(0) + 2k_4 \cos(0)$

$1 = k_2(1) + 2k_4 (1)$

$or\ \ k_2 + 2k_4 = 1$ (5)

Differentiating equation (3a), we have

$y^{II} = -k_1 \cos x - k_2 \sin x - 4k_3 \cos 2x - 4k_4 \sin 2x$ (3b)

Substituting

$y^{II}(0) = 3\ \ i, e., \ x = 0, \ y^{II} = 3$, in equation (3b), we have

$3 = -k_1 \cos(0) - k_2 \sin(0) - 4k_3 \cos(0) - 4k_4 \sin(0)$

$3 = -k_1 - 4k_3$

$or\ \ -k_1 - 4k_3 = 3$ (6)

Differentiating equation (3b), we have

$y^{III} = k_1 \sin x - k_2 \cos x + 8k_3 \sin 2x - 8k_4 cos$ (3c)

Substituting

$y^{III}(0) = 0, \ i.e., \ x = 0, \ y^{III} = 0, in\ equation\ (3c), we\ have$

$0 = k_1 \sin(0) - k_2 \cos(0) + 8k_3 \sin(0) - 8k_3 \cos(0)$

$0 = -k_2 - 8k_4$

$or\ \ -k_2 - 8k_4 = 0$ (7)

Bringing equations (4), (5), (6) and (7) together, we obtain

$k_1 + k_3 = 0$ (4)

$k_2 + 2k_4 = 1$ (5)

$-k_1 - 4k_3 = 3$ (6)

$-k_2 - 8k_4 = 0$ (7)

Solving equation (4) and (6) simultaneously, we have

$$\begin{array}{rl} k_1 + k_3 = 0 & (4) \\ -k_1 - 4k_3 = 3 & (6) \\ \hline -3k_3 = 3 & (4) + (6) \end{array}$$

$\therefore\ k_3 = {}^{-3}/_{3}$

$k_3 = -1$

Substituting k_3 *in equation* (4), *results*

$k_1 + (-1) = 0$

$k_1 - 1 = 0$

$k_1 = 1$

$\therefore\ k_1 = 1, \quad k_3 = -1$

Adding equations (5) and (7) gives

$$\begin{array}{rl} k_2 + 2k_4 = 1 & (5) \\ -k_2 - 8k_4 = 0 & (7) \\ \hline -6k_4 = 1 & \end{array}$$

$\therefore\ k_4 = {}^{-1}/_{6}$

Substituting k_4 *in equation* (5) *gives*

$k_2 + 2\left(-{}^{1}/_{6}\right) = 1$

$k_2 - {}^{1}/_{3} = 1$

$k_2 = 1 + {}^{1}/_{3}$

$k_2 = \frac{4}{3}$

$\therefore\ k_1 = 1, k_2 = \frac{4}{3}, k_3 = -1, k_4 = {}^{-1}/_{6}$

But $y = k_1 \cos x + k_2 \sin x + k_3 \cos 2x + k_4 \sin 2x$

$\Rightarrow y = \cos x + \frac{4}{3} \sin x - \cos 2x - \frac{1}{6} \sin 2x$

(2) For what values of the constant k *is* $y = e^{kx}$ a solution of the equation:

$y^{III} - 2y^{II} - y^{I} + 2y = 0$

Solution

$y^{III} - 2y^{II} - y^{I} + 2y = 0$ (1)

But $y = e^{kx}$

$y^{I} = ke^{kx}$

$y^{II} = k^2 e^{kx}$

$y^{III} = k^3 e^{kx}$

Substituting $y,\ y^{I},\ y^{II}$ *and* y^{III} *in equation* (1), *we have*

$k^3 e^{kx} - 2k^2 e^{kx} - ke^{kx} + 2e^{kx} = 0$

$(k^3 - 2k^2 - k + 2)\ e^{kx} = 0$

But $e^{kx} \neq 0$

Hence $k^3 - 2k^2 - k + 2 = 0$

$(K^3 - K)(2K + 2)$
$K(K^2 - 1)2(K^2 - 2)$

$(k^2 - 1)(k - 2) = 0$ *(factorizing)*

or $(k - 1)(k + 1)\ (k - 2) = 0$

$\therefore k = 1, -1$ *or* 2

$k_2 = \frac{4}{3}$

$\therefore\ k_1 = 1, k_2 = \frac{4}{3}, k_3 = -1, k_4 = {}^{-1}/_{6}$

But $y = k_1 \cos x + k_2 \sin x + k_3 \cos 2x + k_4 \sin 2x$

$\Rightarrow y = \cos x + \frac{4}{3} \sin x - \cos 2x - \frac{1}{6} \sin 2x$

(2) For what values of the constant k *is* $y = e^{kx}$ a solution of the equation:

$y^{III} - 2y^{II} - y^{I} + 2y = 0$

Solution

$y^{III} - 2y^{II} - y^{I} + 2y = 0$ (1)

But $y = e^{kx}$

$y^{I} = ke^{kx}$

$y^{II} = k^2 e^{kx}$

$y^{III} = k^3 e^{kx}$

Substituting $y,\ y^{I},\ y^{II}$ *and* y^{III} *in equation* (1), *we have*

$k^3 e^{kx} - 2k^2 e^{kx} - ke^{kx} + 2e^{kx} = 0$

$(k^3 - 2k^2 - k + 2)\ e^{kx} = 0$

But $e^{kx} \neq 0$

Hence $k^3 - 2k^2 - k + 2 = 0$

$(k^2 - 1)(k - 2) = 0$ (*factorizing*)

or $(k - 1)(k + 1)\ (k - 2) = 0$

$\therefore k = 1, -1$ *or* 2

$(K^3 - K)(2K + 2)$
$K(K^2 - 1)2(K^2 - 2)$

(3) Solve $\frac{d^2y}{dx^2} - \frac{12dy}{dx} + 36y = 0$

Solution

$\frac{d^2y}{dx^2} - \frac{12dy}{dx} + 36y = 0$

$\Rightarrow m^2 - 12m + 36 = 0$

$m^2 - 6m - 6m + 36 = 0$

$(m-6)(m-6) = 0$

$\therefore m = 6 \; twice$

The roots are equal and real.

For real and equal roots;

$y = e^{mx}(A + Bx)$

$y = e^{6x}\{A + Bx\}$

(4) Solve the equation $\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 3y = 0$

Solution

$\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 3y = 0$

$m^2 + 4m + 3 = 0$

$m^2 + m + 3m + 3 = 0$

$m(m+1) + 3(m+1) = 0$

$(m+1)(m+3) = 0$

$\therefore m = 1 \; or \; -3$

For real and different roots,

$y = k_1e^{m_1x} + k_2e^{m_2x}$

$\Rightarrow y = k_1e^{-x} + k_2e^{-3x}$

Where k_1 *and* k_2 *are arbitrary constants.*

(5) Solve $\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 5y = 0$

Solution

$\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 5y = 0$

$m^2 + 4m + 5 = 0$ (*Auxilary equation*)

$m = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(5)}}{2(1)}$

$= \frac{-4 \pm \sqrt{16-20}}{2}$

$m = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm j2}{2.}$

$\therefore m = -2 \pm j1$

For complex roots, $y = e^{ax}\{k_1 \cos\beta x + k_2 \sin\beta x\}$

(6) Solve the initial value problem;

$9y^{II} + 6y^{I} + y = 0$

$y(0) = 4, \ y^{I}(0) = {-13}/{3}$

Solution

$9y^{II} + 6y^{I} + y = 0$ (1)

$9m^2 + 6m + 1 = 0$ (*Auxiliary equation*)

$9m^2 + 3m + 3m + 1 = 0$

$3m(3m + 1) + 1(3m + 1) = 0$

or $(3m + 1)(3m + 1) = 0$

$3m + 1 = 0 \quad or \quad 3m + 1 = 0$

$m = -1/_3$ twice

For real and equal roots;

$y = e^{mx}\{k_1 + k_2x\}$

$\Rightarrow y = e^{\frac{1}{3}x}\{k_1 + k_2x\}$

$or \; y = K_1e^{\frac{1}{3}x} + k_2xe^{-\frac{1}{3}x}$ (2)

Substituting

$y(0) = 4, \; i.e., \; x = 0, y = 4$ in equation (2), we have

$4 = k_1(e^0) + k_2(0)(e^0)$

$4 = k_1(1) + k_2(0)(1)$

$4 = k_1$

$\therefore k_1 = 4$

Differentiating equation (2) wrt x, we have

$y^I = -\frac{1}{3}k_1e^{-\frac{1}{3}x} + k_2\left\{e^{-\frac{1}{3}x} - \frac{1}{3}xe^{-\frac{1}{3}x}\right\}$ (3)

$y^I = -\frac{1}{3}k_1e^{-\frac{1}{3}x} + k_2e^{-\frac{1}{3}x} - \frac{1}{3}k_2xe^{-\frac{1}{3}x}$

Substituting $y^I(0) = {-13}/_3$ in equation (3), we obtain

$\frac{-13}{3} = -\frac{1}{3}k_1(e^0) + k_2(e^0) - \frac{1}{3}k_2(0)(e^0)$

$\frac{-13}{3} = -\frac{1}{3}k_1(1) + k_2(1) - \frac{1}{3}k_2(0)(1)$

$\frac{-13}{3} = -\frac{1}{3}k_1 + k_2$

But $k_1 = 4$

This implies

$\frac{-13}{3} = -\frac{1}{3}(4) + k_2$

$\frac{-13}{3} = -\frac{4}{3} + k_2$

or $k_2 = \frac{-13}{3} + \frac{4}{3} = \frac{-13+4}{3} = \frac{-9}{3} = -3$

$\therefore k_2 = -3$

But $y = k_1 e^{\frac{-1}{3}x} + k_2 x e^{\frac{-1}{3}x}$

$\Rightarrow y = 4e^{\frac{-1}{3}x} - 3xe^{\frac{-1}{3}x}$

or $y = e^{\frac{-1}{3}x}(4 - 3x)$

PRACTICE PROBLEMS 6

(1) Find a general solution of the following differential equations

(a) $4y^{II} + 4y^{I} - 3y = 0$ (b) $y^{II} + 9y^{II} + 20y = 0$ $y = e^{kx}$

(c) $9y^{II} - 30y^{I} + 25y = 0$ (d) $y^{II} + 2ky^{I} + k^2y = 0$

(2) Solve the initial problems given below.

(a) $y^{II} + 4y^{I} + 4y = 0$, $y(0) = 1$, $y^{I}(0) = 1$

(b) $y^{II} - y = 0$, $y(0) = 3$, $y^{I}(0) = -3$

(c) $4y^{II} - 4y^{I} - 3y = 0$, $y(-2) = e$, $y^{I}(-2) = -e/2$

(d) $y^{II} - k^2y = 0$, $(K \neq 0)$ $y(0) = 1$, $y^{I}(0) = 1$.

1 and or 2 and 2

$4m^2 + 4m + 3$

CHAPTER SEVEN

METHODS OF SOLVING SECOND ORDER DIFFERENTIAL EQUATIONS

- ✓ Method of undetermined coefficients
- ✓ Method of reduction of order
- ✓ Method of variation of parameters
- ✓ Use of Laplace Transforms.
- ✓ Method of D – operator

But in this book, we shall treat only the first four methods. The student should consult other books to see how to apply the D-operator method.

7.1 Method of undetermined coefficients

This method of solving second order differential equations applies to equations of the form

$$ay^{II} + by^{I} + cy = g(x) \qquad (1)$$

with constant coefficients and special right sides $g(x)$, namely, exponential functions, polynomial, cosines, sines or products of such functions. These $g(x)$ have derivatives of a form similar to $g(x)$ itself. This gives the key idea: choose for y_p (particular solution) a form similar to that of $g(x)$ and involving unknown coefficients to be determined by substituting that choice for y_p into (1)

7.2 Rules for the method of undetermined coefficients

(1) Basic Rule: If $r\ (x)$ in equation (1) is one of the functions in the first column in Table (A), choose the corresponding

function y_p in the second column and determine its undetermined coefficients by substituting y_p and its derivatives into equation (1).

(2) Modification rule: If a term in your choice of y_p happens to be solution of the homogeneous equation corresponding to equation (1), then multiply your choice of y_p by x ($or\ by\ x^2$ if this solution corresponds to a double root of the characteristic equation of the homogeneous equation).

(3) Sum Rule: If $g(x)$ is a sum of functions in several lines of Table (A), first column, then choose for y_p the sum of the functions in the corresponding lines of the second column.

Table A

Term in $g(x)$	Choice for y_p
ke^x	Ce^x
x^3	$k_1x^3 + k_2x^2 + k_3x + k_4$
$k\cos\beta x$	$k_1\cos\beta x + k_2\sin\beta x$
$k\sin\beta x$	$k_1\cos\beta x + k_2\sin\beta x$
$ke^{ax}\cos\beta x$	$e^{ax}[k_1\cos\beta x + k_2\sin\beta x]$
$ke^{ax}\sin\beta x$	$e^{ax}[k_1\cos\beta x + k_2\sin\beta x]$
$(x^2 + 3)$	$k_1x^2 + k_2x + k_3$

Solve Examples

(1) Solve the equation $\frac{d^2y}{dx^2} - \frac{5dy}{dx} + 6y = 2\sin 4x$

Solution

First, solve for the Homogeneous part of the equation.

i.e., $\frac{d^2y}{dx^2} - \frac{5dy}{dx} + 6y = 0$

$m^2 - 5m + 6 = 0$

$m^2 - 2m - 3m + 6 = 0$

$(m-2)(m-3) = 0$

$\therefore m = 2 \text{ or } m = 3$

For real and different roots,

$y_h = k_1e^{m_1x} + k_2e^{m_2x}$

$\Rightarrow y_h = k_1e^{2x} + k_2e^{3x}$

2sin4x

Second: solve for the particular integral, y_p.

Assume $y_p = k_3 \cos 4x + k_4 \sin 4x$

$y_p^I = -4k_3 \sin 4x + 4k_4 \cos 4x$

$y_p^{II} = -16k_3 \cos 4x - 16k_4 \sin 4x$

Substituting y_p, y^I and y_p^{II} in the given equation, we have

$\{(-16k_3 \cos 4x - 16k_4 \sin 4x) - 5(-4k_3 \sin 4x + 4k_4 \cos 4x)$
$+ 6(k_3 \cos 4x + k_4 \sin 4x)\} = 2 \sin 4x$

$\{-16k_3 \cos 4x - 16k_4 \sin 4x + 20k_3 \sin 4x - 20k_4 \cos 4x$
$+ 6k_3 \cos 4x + 6k_4 \sin 4x\} = 2 \sin 4x$

$\{(-16k_3 - 20k_4 + 6k_3) \cos 4x + (-16k_4 + 20k_3 + 6k_4) \sin 4x\} =$
$2 \sin 4x$

$\{(-10k_3 - 20k_4) \cos 4x + (-10k_4 + 20k_3) \sin 4x\} = 2 \sin 4x$

Comparing the coefficients of terms, we obtain

$[\cos 4x];\ -10k_3 - 20k_4 = 0$ (1) * 2

$[\sin 4x];\ \ 20k_3 - 10k_4 = 2$ (2)

$$-20k_3 \quad -40k_4 = 0 \qquad (3)$$

$$-50k_4 = 2 \qquad (2)+(3)$$

$$k_4 = \frac{-2}{50} = \frac{-1}{25}$$

Substituting k_4 *in equation* (1), *we have*

$$10k_3 + 20\left(\frac{1}{25}\right) = 0$$

$$10k_3 + \frac{4}{5} = 0$$

$$10k_3 = \frac{-4}{50}$$

$$or\ k_3 = \frac{-4}{50} = \frac{-2}{25}$$

$$\therefore\ k_3 = {}^{-2}/_{25},\ k_4 = {}^{-1}/_{25}$$

$Ae^{mx} + Bxe^{mx}$

But $y_p = k_3 \cos 4x + k_4 \sin 4x$

$$= \frac{-2}{25}\cos 4x - \frac{1}{25}\sin 4x$$

But $y_{Total} = y_{homogeneous} + y_{particular}$

$$= k_1 e^{2x} + k_2 e^{3x} - \frac{2}{25}\cos 4x - \frac{1}{25}\sin 4x$$

$$= k_1 e^{2x} + k_2 e^{3x} - \frac{1}{25}(2\cos 4x + \sin 4x)$$

(2) Solve the equation

$y''' + 3y'' = x^2 + \cos x$

Solution

$y''' + 3y'' = x^2 + \cos x$ (1)

First, solve for the homogeneous part of the equation.

$y''' + 3y'' = 0$

$m^3 + 3m^2 = 0$

$m^2\,(m + 3) = 0$

$m^2 = 0 \; or \; m + 3 = 0$

$\therefore m = 0 \; twice \; or \; m = -3$

$\Rightarrow \; y = 0, \; 0 \; or - 3$

$y_h = A + B\,x + Ce^{-3x}$

Where A, B and C are arbitrary constants.

Second, solve for the particular integral.

For x^2, $y_{p_1} = Dx^2 + E\,x + F$

For $\cos x$, $y_{p_2} = G\cos x + H\sin x$

$y_p = y_{p_1} + y_{p_2}$

$y_p = Dx^2 + E\,x + F + G\cos x + H\sin x$

$y_p^I = 2Dx + E - G\sin x + H\cos x$

$y_p^{II} = 2D - G\cos x - H\sin x$

$y_p^{III} = G\sin x - \cos x$

Substituting y_p^{III} and y_p^{II} in equation (1), we have

$G\sin x - H\cos x + 3\,(2D - G\cos x - H\sin x) = x^2 + \cos x$

$G\sin x - H\cos x + 6D - 3G\cos x - 3H\sin x = x^2 + \cos x$

$(G - 3H)\sin x + (-H - 3G)\cos x + 6D = x^2 + \cos x$ (2)

Comparing coefficients of terms, we obtain (3)

$\sin x$; $\quad G - 3H = 0$

$\cos x$; $\quad -H - 3G = 1$

CT; $\quad 6D = 0$

$\quad or \; D = {}^0/_6 = 0$

Solving equation (2) and (3) simultaneously, we have

$$G - 3H = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2) * 3$$

$$-3G - H = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$3G - 9H = 0 \qquad (4)$$

$$-10H = 1 \qquad (3)+(4)$$

$$\therefore H = {}^{-1}/_{10}$$

Substituting H in equation (3), gives

$$-({}^{-1}/_{10}) - 3G = 1$$

$$\frac{1}{10} + 3G = 1$$

$$3G = 1 - \frac{9}{10}$$

$$G = \frac{9}{30} = \frac{3}{10}$$

$$\therefore G = \frac{3}{10}, H = \frac{-1}{10}, D = 0, E = 0, F = 0$$

$$y_p = \frac{3}{10}\cos x - \frac{1}{10}\sin x = \frac{1}{10}(3\cos x - \sin x)$$

But $y_{Total}\ y_h + y_p$

$$= A + Bx + Ce^{-3x} + \frac{1}{10}(3\cos x - \sin x)$$

(3) Find the general solution of $y^{II} + 4y = 1 + x + \sin x$

Solution (1)

$$y^{II} + 4y = 1 + x + \sin x$$

First, solve for the homogeneous part of the equation.

This implies; (2)

$$y^{II} + 4y = 0$$

Turning this into auxiliary equation, we have

$m^2 + 4 = 0$

$m^2 = -4$

$m = \pm j2$

$\therefore m = \pm j2$

For complex roots, we obtain

$y = e^{ax}\{A\cos\beta x + B\sin\beta x\}$

$\Rightarrow y = A\cos 2x + B\sin 2x$

Where A and B are arbitrary constants.

Second, solve for the particular integral.

For $(1 + x)$; $\quad y_{p_1} = Cx + D$

For $\sin x$; $\quad y_{p_2} = E\cos x + F\sin x$

$y_p = y_{p_1} + y_{p_2} = Cx + D + E\cos x + F\sin x$

$y_p^I = C - E\sin x + F\cos x$

$y_p^{II} = -E\cos x - F\sin x$

Substituting y_p^{II} and y_p in equation (1), we have

$-E\cos x - F\sin x + 4(Cx + D + E\cos x + F\sin x) = 1x + \sin x$

$\{-E\cos x - F\sin x + 4Cx + 4D + 4E\cos x + 4F\sin x\} = \{1x + \sin x\}$

$4Cx + 4D + (-E + 4E)\cos x + (-F + 4F)\sin x = 1 + x + \sin x$

$4Cx + 4D + 3E\cos x + 3F\sin x = 1 + x + \sin x$

Comparing coefficients of terms, we obtain

$x;\quad 4C = 1$

$\therefore C = \frac{1}{4}$

Constant term (CT); $4D = 1$

$$\therefore D = \frac{1}{4}$$

$\cos x$; $3E = 0$

$$\therefore E = 0$$

$\sin x$; $3F = 1$

$$\therefore F = {}^{1}/_{3}$$

But $y_p = Cx + D + E\cos x + F\sin x$

$\Rightarrow y_p = \frac{1}{4}x + \frac{1}{4} + \frac{1}{3}\sin x$

But $y_{general} = y_h + y_p$

$y_{general} = A\cos 2x + B\sin 2x + \frac{1}{3}\sin x \; \frac{1}{4}(x+1)$

(4) Find the general solution of the differential equation

$y'' - 3y' - 4y = 3xe^{2x}$

Solution

$y'' - 3y' - 4y = 3xe^{2x}$ (1)

First, solve for the Homogeneous part of the equation.

$\Rightarrow y'' - 3y' - 4y = 0$

$m^2 - 3m - 4 = 0$ (*Auxiliary equation*)

$(m+1)(m-4) = 0$ (*Factorization*)

$\therefore m = -1$ or 4 (*The roots are real and distinct*).

$y_h = Ae^{-x} + Be^{4x}$ (2)

Second, solve for the particular integral.

$y_p = (Cx + D)e^{2x}$

$y_p' = Ce^{2x} + 2Cxe^{2x} + 2De^{2x}$

$\therefore\ y_p^I = (C + 2Cx + 2D)e^{2x}$

$y_p^{II} = 2Ce^{2x} + 2Ce^{2x} + 4Cxe^{2x} + 4De^{2x}$

$y_p^{II} = 4Ce^{2x} + 4Cxe^{2x} + 4De^{2x}$

$y_p^{II} = (4C + 4Cx + 4D)e^{2x}$

Substituting y_p, y_p^{II} *in equation* (1), *we have*

$(4C + 4Cx + 4D)e^{2x} - 3\,(C + 2Cx + 2D)e^{2x} - 4\,(Cx + D)e^{2x} = 3xe^{2x}$

$(C - 6D - 6Cx)e^{2x} = 3xe^{2x}$

Cancelling e^{2x}, *we get*

$C - 6D - 6Cx = 3x$

Comparing coefficients of terms, we have

$x\,;\qquad -6C = 3$

$\therefore C = {}^{-3}/_{6} = {}^{-1}/_{2}$

Constant term; $C - 6D = 0$

$-\frac{1}{2} - 6D = 0$

$\therefore D = -\,{}^{1}/_{12}$

But $y_p = (Cx + D)e^{2x}$

$= \left(-\frac{1}{2}x - \frac{1}{12}\right)e^{2x}$

$= -\frac{1}{2}\left(x + \frac{1}{6}\right)e^{2x}$

But $y_{general} = y_{CF} + y_p$

$= Ae^{-x} + Be^{4x} - \frac{1}{2}\left(x + \frac{1}{6}\right)e^{2x}$

(5) Solve the equation $y^{II} - 4y^{I} + 3y = 2e^{3x}$

Solution

$y^{II} - 4y^{I} + 3y = 2e^{3x}$ (1)

First, solve for the Homogeneous part of equation (1), we have

$y^{II} - 4y^{I} + 3y = 0$ (2)

$m^2 - 4m + 3 = 0$ (*Auxiliar equation*)

$(m-1)(m-3) = 0$

$\therefore m = 1 \text{ or } 3$ (*The roots are real and distinct*).

$y = Ae^{x} + Be^{3x}$ (3)

Second, solve for the particular integral.

But we see that e^{3x} is a solution of the homogeneous equation corresponding to a simple root. Hence, the modification rule applies in this case.

Assume $y_p = Cx\,e^{3x}$

$y_p^{II} = 3Ce^{3x} + 3Ce^{3x} + 9Cxe^{3x}$

$\therefore y_p^{II} = 6ce^{3x} + 9cxe^{3x}$

Substituting y_p, y_p^{I} *and* y_p^{II} *in equation* (1), *we have*

$9Cxe^{3x} + 6Ce^{3x} - 4\left(Ce^{3x} + 3Cxe^{3x}\right) + 3Cxe^{3x} = 2e^{3x}$

$(9C - 12C + 3C)xe^{3x} + (6C - 4C)e^{3x} = 2e^{3x}$

$2Ce^{3x} = 2e^{3x}$

Cancelling e^{3x}, *we get*

$2C = 2$

$\therefore C = 1$

$\therefore y_p = xe^{3x}$

But $y_{total} = y_h + y_p$

$= Ae^{x} + Be^{3x} + xe^{3x}$

(6) Find a general solution of the equation

$y^{II} + 10y^{I} + 25y = e^{-5x}$

Solution

$y^{II} + 10y^{I} + 25y = e^{-5x}$ (1)

First, solve for the homogeneous part of the equation.

$y^{II} + 10y^{I} + 25y = 0$

$\Rightarrow m^2 + 10m + 25 = 0$

$m^2 + 5m + 5m + 25 = 0$

$(m + 5)(m)5) = 0$

$\therefore m = -5 \; twice$

For real and equal roots;

$y_h = e^{mx}\,[k_1 + k_2x]$

$\Rightarrow y_h = e^{-5x}\,[k_1 + k_2x]$ (2)

$y_h = k_1e^{-5x} + k_2xe^{-5x}$

Second, solve for the particular integral.

Notice that e^{-5x} appears twice on the solution of the homogeneous part of the equation. Hence, the modification rule takes effect.

$\therefore y_p = k_3x^2e^{-5x}$

$y_p^I = -5k_3x^2e^{-5x} - 2k_3xe^{-5x}$

$y_p^{II} = 25k_3x^2e^{-5x} - 10k_3xe^{-5x} - 10k_3xe^{-5x} + 2k_3e^{-5x}$

$\therefore y_p^{II} = 25k_3x^2e^{-5x} - 20k_3xe^{-5x} + 2k_3e^{-5x}$

Substituting y_p, y_p^I *and* y_p^{II} *in equation* (1), *we have*

$\{25k_3x^2e^{-5x} - 20k_3xe^{-5x} + 2k_3e^{-5x} + 10\,(-5k_3x^2e^{-5x} + 2k_3xe^{-5x}) + 25k_3x^2e^{-5x}\} = e^{-5x}$

$\{(25k_3 - 50k_3 + 25k_3)x^2e^{-5x} + (-20k_3 + 20k_3)xe^{-5x} + 2k_3e^{-5x}\} = e^{-5x}$

$2k_3e^{-5x} = e^{-5x}$

Cancelling e^{-5x}, *we obtain*

$2k_3 = 1$

$\therefore\ k_3 = \frac{1}{2}$

$\therefore\ y_p = \frac{1}{2}x^2e^{-5x}$

But $y_{general} = y_h + y_p$

$= k_1e^{-5x} + k_2xe^{-5x} + \frac{1}{2}x^2e^{-5x}$

$= \left(k_1 + k_2x + \frac{1}{2}x^2\right)e^{-5x}$

Practice problems 7

(1) Find the general solution of the differential equation

$y^{II} - 3y^{I} + 2y = 3 + 2x^3$

(2) Find the general solution of the following equations

(a) $y^{II} + 6y^{I} + 9y = 2\sin 2x$

(b) $y^{II} + 8y^{I} + 16y = 64\cosh 4x$

(c) $y^{II} + 2y^{I} + 10y = 25x^2 + 3$

(d) $3y^{II} + 10y^{I} + 3y = 9x + 5\cos x$

(3) Solve the initial value problems

(a) $y^{II} + y^{I} = 2 + 2x + x^2,\ \ y(0) = 8,\ y^{I}(0) = -1$

(b) $y^{II} + 9y = 6\cos 3x \quad y(0) = 1,\ y^{I}(0) = 0$

CHAPTER EIGHT

METHOD OF VARIATION OF PARAMETERS

The general form of a second order differential equation is:

$$y^{II} + p(x)y^{I} + q(x)y = g(x) \qquad (1)$$

Where the functions p, q *and* g *are* continuous on the interval of interest. To use this method, known as the method of variation of parameters, it is necessary to know a fundamental set of solutions of the corresponding homogeneous equation.

$$y^{II} + p(x)y^{I} + q(x)y = 0 \qquad (2)$$

Suppose y_1 *and* y_2 are linearly independent solutions of the homogeneous equation (2). Then the general solution of equation (2) is

$$y_C(x) = C_1y_1(x) + C_2y_2(x)$$

The method of variation of parameters involves the replacement of the constants C_1 *and* C_2 by functions U_1 *and* U_2. We then seek to determine the two functions U_1 *and* U_2 so that

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \qquad (3)$$

satisfies the non-homogeneous differential equation (1).

Differentiating equation (3) gives

$$y_p^I = u_1y_I^I + u_I^Iy_I + u_2y_2^I + u_2^Iy_2$$

$$y_p^I = (u_1^Iy_1 + u_2^Iy_2) + (u_1y_1^I + u_2y_2^I)$$

Simplifying this required that u_1 *and* u_2 *satisfy* (4)

$$u_I^Iy_1 + u_2^Iy_2 = 0$$

With this condition, we now have

$$y_p^I = u_1y_1^I + u_2y_2^I$$

Differentiating the above equation gives

$y_p^{II} = u_1^I y_1^I + u_1 y_1^{II} + u_2^I y_2^I + u_2$

Substituting y_p, y_p^I *and* y_p^{II} in equation (1) gives

$\{(u_1^I y_I^I + u_1 y_1^{II} + u_2^I y_2^I + u_2 y_2^{II}) + p(x)(u_1 y_1^I + u_2 y_2^I) +$

$q(x)\,(u_1 y_1 + u_2 y_2)\} = \; g(x)$ (5)

Simplifying the above equation (5) gives

$\{u_1\,(y_1^{II} + p(x)y_1^I + q(x)y_1) + u_2\,(y_2^{II} + p(x)y_2^I + q(x)y_2)$

$+ u_1^I y_1^I + u_2^I y_2^1\} = g(x)$

The terms in parentheses are zero since y_1 *and* y_2 are solutions of the homogeneous equation (2); hence the condition that y_p satisfy equation (1) leads to the requirement

$u_1^I y_1^I + u_2^I y_2^I = g(x)$ (6)

Rewriting equations (4) and (6), we have the following system of two equations.

$u_1^I y_1 + u_2^I y_2 = 0$ (4)

$u_1^I y_1^I + u_2^I = g(x)$ (6)

We can solve the above systems of equations in two unknown functions using crammer's rule.

Equations (4) and (6) can be rewritten as

$$\begin{pmatrix} y_1 & y_2 \\ y_1^I & y_2^I \end{pmatrix} \begin{pmatrix} u_1^I \\ u_2^1 \end{pmatrix} = \begin{pmatrix} 0 \\ g(x) \end{pmatrix}$$

$$\Delta_0 = \begin{vmatrix} y_1 & y_2 \\ y_1^I & y_2^I \end{vmatrix} = y_1 y_2^I - \; y_2 y_1^I = W\,(y_1, y_2)$$

Where $W(y_1, y_2)$ is called the wronskian.

$$\Delta_1 = \begin{vmatrix} 0 & y_2 \\ g(x) & y_2^I \end{vmatrix} = 0 - y_2\, g(x)$$

$$\therefore \Delta_1 = -y_2\, g(x)$$

$$\Delta_2 = \begin{vmatrix} y_1 & 0 \\ y_1^I & g(x) \end{vmatrix} = y_1 g(x) - 0 = y_1 g(x)$$

Hence, $u_1^I = \frac{\Delta_1}{\Delta_0} = \frac{-y_2\, g(x)}{y_1 y_2^I - y_2 y_1^I} = \frac{-y_2\, g(x)}{W(y_1, y_2)}$

$$u_2^I = \frac{\Delta_1}{\Delta_0} = \frac{y_1\, g(x)}{y_1 y_2^I - y_2 y_1^I} = \frac{y_1 g(x)}{W(y_1, y_2)}$$

From $u_1^I = \frac{-y_2 g(x)}{W(y_1, y_2)}$

Integrating both sides wrt x, we have

$$u_1 = \int \frac{-y_2 g(x)}{W(y_1, y_2)} = -\int \frac{y_2 g(x)}{W(y_1, y_2)}$$

Also, $u_2^I = \frac{y_1 g(x)}{W(y_1, y_2)}$

Integrating both sides wrt x, we obtain

$$u_2 = \int \frac{y_1 g(x)}{W(y_1, y_2)}$$

$$\therefore\ u_1 = -\int \frac{y_2 g(x)}{W(y_1, y_2)} \quad and \quad u_2 = \int \frac{y_1\, g(x)}{W(y_1, y_2)}$$

But $y_p = u_1 y_1 + u_2 y_2$ *i.e., equation* (3)

$$= y_1 \int \frac{-y_2 g(x)}{W(y_1, y_2)} + y_2 \int \frac{y_1 g(x)}{W(y_1, y_2)}$$

$$y_p = -y_1 \int \frac{y_2 g(x)}{W(y_1, y_2)} + y_2 \int \frac{y_1 g(x)}{W(y_1, y_2)}$$

$$y_p = -y_1\int^x \frac{y_2(t)g(t)}{W(y_1,y_2)} + y_2\int^x \frac{y_1(t)g(t)}{W(y_1,y_2)} \qquad (7)$$

When solving problems using the method of variation of parameters, it is usually better to start from the first principle instead of trying to memorize equation (7). Examples will help us to know how to apply equation (7) in solving problems and also on how to start from first principles. Don't bother yourself, you will soon be okay.

SOLVED EXAMPLES

(1) Determine the general solution of the equation;

$$y'' - 2y' + y = \frac{e^x}{1+x^2}$$

Solution

$$y'' - 2y' + y = \frac{e^x}{1+x^2} \qquad (1)$$

First, we solve for the Homogenous part of equation (1), *i.e.*;

$y'' - 2y' + y = 0$

$m^2 - 2m + 1 = 0 \quad (Auxiliary\ equation)$

$(m-1)(m-1) = 0$

$\therefore m = 1\ twice\ \ (verify).$

For real and equations;

$y = k_1 e^{mx}[k_1 + k_2 x]$

$\Rightarrow y_h = e^x(k_1 + k_2 x)$

Hence, the two independent solutions of the equation are e^x *and* xe^x

Second, solve for the non -homogeneous equation. (2)

Assume $y_p = u_1 e^x + u_2 x e^x$

Where u_1 *and* u_2 *are functions of* x.

The Wronskian, $W(y_i, y_2)$, *is given by*

$$W(y_1, y_2) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x(1+x) \end{vmatrix} = e^{2x} \quad (verify)$$

Using the formulae for finding u_1 *and* u_2, *we have*

$$u_1 = -\int \frac{y_2\, g(x)}{W(y_1, y_2)} = -\int \left(\frac{xe^x}{e^{2x}} * \frac{e^x}{1+x^2}\right) = -\int \frac{x}{1+x^2}$$

$$u_1 = -\frac{1}{2}\ln(1+x^2)$$

And $u_2 = \int \frac{y_1 g}{W(y_1, y_2)} = \int \left(\frac{e^x}{e^{2x}} * \frac{e^x}{1+x^2}\right) = \int \frac{1}{1+x^2} = arc\tan x$

$\therefore\ u_1 = -\frac{1}{2}\, ln(1+x^2)$ and $u_2 = \arctan x$

$\therefore\ y_p = -\frac{1}{2} e^x (1+x^2) + xe^x \arctan x$

But $y_{Total} = y_{CF} + y_p$

$$= k_1 e^x + k_2 x e^x - \frac{1}{2} e^x \ln(1+x^2) + xe^x \arctan x$$

(2) Determine the general solution of the equation

$y^{II} + y = \sec x, \qquad 0 < x < {}^{\pi}/_{2}$ (1)

Solution

$y^{II} + y = \sec x$

First, solve for the homogeneous par of equation (1), we get

$y^{II} + y = 0$

$\Rightarrow m^2 + 1 = 0$

$m^2 = -1$

$\therefore m = \pm\sqrt{-1}$

$= +i1$

For complex roots, $y = A\cos\beta x + B\sin\beta x$

$\Rightarrow y_h = A\cos x + B\sin x$

Where A *and* B *are arbitrary constants.*

Second, solve for the particular integral. We obtain

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

$$y_p = u_1\cos x + u_2\sin x$$

But $u_1 = -\int \frac{y_2 g(x)}{W(y_1,y_2)}$, $u_2 = \int \frac{y_1 g(x)}{W(y_1,y_2)}$

And $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1^I & y_2^I \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ \sin x & \cos x \end{vmatrix}$

$$W(y_1, y_2) = cos^2 x + sin^2 x = 1$$

$$g(x) = \sec x = \frac{1}{\cos x}$$

$$\therefore u_1 = \int \frac{-\sin x}{1}(\sec x)\,dx = -\int \sin x \frac{1}{\cos x}\,dx = -\int \frac{\sin x}{\cos x}\,dx$$

$$u_1 = \ln(\cos x)$$

$$u_2 = \int \frac{\cos x}{1} x \frac{1}{\cos x}\,dx = \int dx = x$$

$\therefore u_2 = x.$

But $y_p = u_1(x)\cos x + u_2(x)\sin x$

$= -ln\cos x(\cos x) + x\sin x$

$y_p = -\cos x\, ln(\cos x) + x\sin x$

But $y_{general} = y_h + y_p$

$= A\cos x + B\sin x + \cos x\ln(\cos x) + x\sin x$

(3) Find a general solution of the equation, $y^{II} + 9y = Csc3x$

Solution

$y^{II} + 9y = \text{Csc}\, 3x$

First, solve for the homogeneous part of equation (1), we have

$y^{II} + 9y = 0$

$m^2 + 9 = 0$

$m^2 = -9$

$\therefore m = \pm\sqrt{-9}$

$m = \pm j3$

For complex roots, we have

$y_h\ A\cos\beta x + B\ \sin\beta x$

$\Rightarrow y_h = A\cos\beta x + B\ \sin\beta x$

Second, solve for the particular integral, we obtain

$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

$y_p = u_1\cos 3x + u_2\sin 3x$

But $u_1 = -\int \frac{y_2 g(x)}{W(y_1,y_2)},\quad u_2 = \int \frac{y_1 g(x)}{W(y_1,y_2)}$

Where $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1^I & y_2^I \end{vmatrix} = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$

$W(y_1, y_2) = 3\cos^2 3x + 3\sin^2 3x = 3(\cos^2 3x + \sin^2 3x)$

$W(y_1, y_2) = 3(1) = 3$

$g(x) = \text{Csc}\, 3x = \frac{1}{\sin 3x}$

$\therefore u_1 = -\int\left(\frac{\sin 3x}{3} * \frac{1}{\sin 3x}\right)dx = -\int \frac{1}{3}dx = -\frac{1}{3}x$

$u_2 = \int\left(\frac{\cos 3x}{3} * \frac{1}{\sin 3x}\right)dx = \frac{1}{3}\int \frac{\cos 3x}{\sin 3x}dx = \text{In}(\sin 3x)$

But $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

$-\frac{1}{3}x\cos 3x + \sin 3x \ln(\sin 3x)$

$= -\frac{1}{3}x\cos 3x + \sin 3x \ln(\sin 3x)$

so, $y_{Total} = y_h + y_p$

$$= A\cos 3x + B\ sin3x - \frac{1}{3}x\cos 3x + \sin 3x \ln(\sin 3x)$$

4) Solve $y^{II} - 2y^{I} + y = \frac{e^x}{x^3}$

Solution

$$y^{II} - 2y^{I} + y = \frac{e^x}{x^3} \qquad (1)$$

First, solve for the homogeneous part of equation (1), we get

$y^{II} - 2y^{I} + y = 0$

$m^2 - 2m + 1 = 0 \qquad (Axiliary\ equation)$

$(m-1)(m-1)$

$m = 1\ twice$

$For\ real\ and\ distinct\ roots,$

$y = e^{mx}(A + Bx)$

$\therefore y_h = e^x(A + Bx) = Ae^x + Bxe^x$

Where $A\ and\ B\ are\ arbitrary\ constants.$

Second, solve for the particular integral, we obtain

$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

$y_p = u_1e^x + u_{2x}e^x$

Where $g(x) = \frac{e^x}{x^3}$ and $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1^I & y_2^I \end{vmatrix}$

$$W(y_1, y_2) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x(1+x) \end{vmatrix}$$

$$e^{2x}(1+x) - xe^{2x} = e^{2x} + xe^{2x} - xe^{2x}$$

$$W(y_1, y_2) = e^{2x}$$

But $u_1 = -\int \frac{y_2 g(x)}{W(y_1,y_2)}$ and $u_2 = \int \frac{y_1\, g(x)}{W(y_1,y_2)}$

$$u_i = -\int \left(\frac{xe^x}{e^{2x}} * \frac{e^x}{x^3}\right) dx = -\int x^{-2}\, dx$$

$$u_i = -\left[\frac{x^{-1}}{-1}\right] = \frac{1}{x}$$

$$u_2 = \int \left(\frac{e^x}{e^{2x}} * \frac{e^x}{x^3}\right) dx = \int x^{-3}\, dx = \left[\frac{x^{-2}}{-2}\right]$$

But $y_p = u_1 y_1 + u_2 y_2$

$$= \frac{1}{x} e^x - \frac{1}{2x^2} xe^x$$

$$= \frac{1}{x} e^x - \frac{1}{2x} e^x = \frac{1}{2x} e^x \left(1 - \frac{1}{2}\right) = \frac{1}{2x} e^x$$

But $y_{Total} = y_h + y_p$

$$= Ae^x + Bxe^x + \frac{1}{2x} e^x$$

$$= e^x \left(A + Bx + \frac{1}{2x}\right)$$

(5) Find a general solution, showing all details of your calculations of the differential equation: $y^{II} + 4y = 3\, Csc\, 2x$; $0 < x < \pi/2$

Solution

$y^{II} + 4y = 3\,\text{Csc}\,2x$ (1)

First, solve for the homogeneous part of the equation.

$y^{II} + 4y = 0$

$m^2 + 4 = 0$ $\quad$ (*Auxiliary equation*)

$\therefore m = \pm j2$

For complex roots, $y_h = A\cos\beta x + B\sin\beta x$

$\Rightarrow \; y_h = A\cos 2x + B\sin 2x$ (2)

Second, solve for the particular integral

Let $\quad y_p = u_1 y_1 + u_2 y_2$

$y_p = u_1\cos 2x + u_2\sin 2x$

$y_p^I = u_1^I\cos 2x - 2u_1\sin 2x + u_2^I\sin 2x + 2u_2\cos 2x$

Let $\quad u_1^I\cos 2x + u_2^I\sin 2x = 0$ (3)

$\therefore y_p^I = -2u_1\sin 2x + 2u_2\cos 2x$

$y_p^{II} = -2u_1^I\sin 2x - 4u_1\cos 2x + 2u_2^I\cos 2x - 4u_2\sin 2x$

Substituting y_p^{II} *and* y_p *in equation* (1), *we have*

$\{-2u_1^I\sin 2x - 4u_1\cos 2x + 2u_2^I\cos 2x - 4u_2\sin 2x$

$+ 4(u_1\cos 2x + u_2\sin 2x)\} = 3\,\text{Csc}\,2x$

$\{(-4u_1 + 4u_1)\cos 2x + (-4u_2 + 4u_2)\sin 2x - 2u_1^I\sin 2x$

$+ 2u_2^I\cos 2x\} = 3\,\text{Csc}\,2x$ (2)

$\Rightarrow -2u_1^I\,\sin 2x + 2u_2^I\cos 2x = 3\,\text{Csc}\,2x$

Solving equation (3) and (4) simultaneously, we have

$u_1^I \cos 2x + u_2^I \sin 2x = 0$ (3)

$-2u_1^I \sin 2x + 2u_2^I \cos 2x = 3 \operatorname{Csc} 2x$ (4)

$$\begin{pmatrix} cos2x & sin2x \\ -2sin2x & 2cos2x \end{pmatrix}\begin{pmatrix} u_1^I \\ u_2^I \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \operatorname{Csc} 2x \end{pmatrix}$$

Using crammer's rule, we obtain

$$\Delta o = \begin{vmatrix} cos2x & sin2x \\ -2sin2x & 2cos2x \end{vmatrix}$$

$\therefore\ \Delta o = 2\ cos^2\ 2x + 2\ sin^2 2x = 2(cos^2 2x + sin^2 2x) = 2(1) = 2$

$$\Delta_1 = \begin{vmatrix} 0 & sin2x \\ 3Csc2x & 2cos2x \end{vmatrix}$$

$\therefore\ \Delta_1 == 0 - 3Csc2x(sin2x) = -3$

$\Delta_2 = \begin{vmatrix} \cos 2x & 0 \\ -2 \sin 2x & 3 \operatorname{Csc} 2x \end{vmatrix} = 3 \cos 2x * \frac{1}{\sin 2x} - 0 = (3Csc2x * \frac{1}{sin2x})$

$\therefore\ \Delta_2 = 3 \cot 2x$

But $u_1^I = \frac{\Delta_1}{\Delta_0} = -\frac{3}{2}$

Also, $u_2^I = \frac{\Delta_2}{\Delta_0} = \frac{3 \cot 2x}{2} = \frac{3}{2} \cot 2x$

$\therefore\ u_1 = -\int \frac{3}{2}\, dx = -\frac{3}{2}x$

And $u_2 = \frac{3}{2}\int \cot 2x = \frac{3}{2}\int \frac{\cos 2x}{\sin 2x}\, dx = \frac{3}{4}\ ln(\sin 2x)$

But $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

$= -\frac{3}{2} x \cos 2x + \frac{3}{4} \sin 2x\ ln(\sin 2x)$

The general solution is given by

$$y_{general} = y_h + y_p$$

$$= A\cos 2x + B\sin 2x - \frac{3}{2}x\cos 2x + \frac{3}{4}\sin 2x\, ln(\sin 2x)$$

PRACTICE PROBLEMS 8

(1) Determine the general solution of the differential equation:

$y^{II} + y = \sec x, 0 < x < \frac{\pi}{2},$

using the method of variation of parameters.

(2) Determine a particular solution of the equation

$$y^{II} + 9y = 9\,sec^2 3x, \qquad 0 < x < \frac{\pi}{6}$$

In each of problems (3) through (6), determine the general solution using the method of variation of parameters.

(3) $y^{II} + y = \tan x$

(4) $y^{II} + 4y^{I} + 4y = x^{-2}e^{-2x}\,,\ x > 0$

(5) $y^{II} + 2y^{I} + y = 3e^{-x}$

(6) $y^{II} + 2y^{I} + y = \frac{e^{-x}}{cosx}$

$m^2 + m$

$m = 1$

CHAPTER NINE

METHOD OF REDUCTION OF ORDER

A very import and useful fact if the following: if one solution of a second order linear homogeneous differential equation is known, a second linearly in depend solution (and hence a fundamental set of solutions) can be determined. The procedure, which is due to D' Alembert, is usually referred to as the method of reduction of order.

Solved Examples

(1) Solve the differential equation $xy^{II} + 2y^{I} = 4x^3$

Solution

$xy^{II} + 2y^{I} = 4x^3$ (1)

Let $u = y^{I}$ and $u^{I} = y^{II}$

And substituting these in equation (1) gives

$xu^{I} + 2u = 4x^3$ (2)

Dividing equation (2) by x, we have

$u^{I} + \frac{2}{x}u = 4x^2$ (3)

Equation (3) is now linear in x and can be solved by the method used in solving first order linear equations. The integrating factor, is given by

$\mu = e^{\int P dx}$

$\mu = e^{\int \frac{2}{x} dx} = e^{2lnx} = e^{lnx^2} = x^2$

Multiplying both sides of equation (3) by the integrating factor (x^2), we have

$$x^2\left[u' + \frac{2}{x}u\right] = x^2(4x^2)$$

$$(ux^2)' = 4x^4$$

Integrating both sides wrt x, we obtain

$$\int (x^2u)' = \int (4x^4)\,dx$$

$$x^2u = \frac{4x^5}{5} + C$$

$$u = \frac{4x^3}{5} + \frac{C_1}{x^2}$$

But $u = \frac{dy}{dx}$ $\quad (i.e.,\ U = y')$

$$\Rightarrow \frac{dy}{dx} = \frac{4}{5}x^3 + \frac{C_1}{x^2}$$

$$dy = \left(\frac{4}{5}x^3 \frac{C_1}{x^2}\right)dx$$

Integrating both sides wrt x, we obtain

$$y = \int\left(\frac{4}{5}x^3 + \frac{C}{x^2}\right)dx$$

$$y = \frac{1}{5}x^4 - \frac{C_1}{x} + C_2$$

(2) Solve the equation $y'' - 2yy' = 0$

Solution

$y'' - 2yy' = 0$

Here, since x does not appear explicitly, replace y' with U and y'' with $U\left(\frac{dy}{dx}\right)$ *to get*

$$u\frac{du}{dy} - 2yu = 0$$

Dividing through by U, we have

$$\frac{du}{dy} = 2y$$

Separating variables gives

$$du = 2ydy$$

Integrating both sides wrt y, we have

$$u = y^2 + C_1$$

But $\quad u = \frac{dy}{dx}, this\ implies$

$$\frac{dy}{dx} = y^2 + C$$

$$or \quad \frac{dy}{y^2+C} = dx$$

Integrating both sides of this equation wrt x. The result of th integration depends on the sign of the arbitrary constant C.

$If \quad C > 0, we\ get$

$$\frac{1}{\sqrt{C}} tan^{-1}\left(\frac{y}{\sqrt{C}}\right) = x + k$$

$If\ C = 0, we\ get$

$$-\frac{1}{y} = x + k$$

And if $C < 0, we\ get$

$$\frac{1}{\sqrt{-C}} tanh^{-1}\left(\frac{y}{\sqrt{-C}}\right) = x + k$$

In each case, we can solve for y as a function of x, obtaining

$$y(x) = \begin{cases} \sqrt{C}\, tan\left[\sqrt{C}\,(x+k)\right] & if\ C > 0 \\ \dfrac{-1}{x+k} & if\ C = 0 \\ \sqrt{-C}\, \tan h\left[\sqrt{-C}(x+k)\right] & if\ C < 0 \end{cases}$$

(3) Solve the equation $xy^{II} - 2y^{I} = 1$

Solution

$xy^{II} - 2y^{I} = 1.$ (1)

Let $u = y^{I}$ *and* $u^{I} = y^{II}$ and substituting these in equation (1) gives

$xu^{I} - 2u = 1$ (2)

Equation is now a first order linear equation.

Dividing through by x gives

$$u^{I} - \frac{2}{x}u = \frac{1}{x} \quad (3)$$

The integrating factor is given by

$$\mu = e^{\int Pdx} = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = x^{-2} \quad (verify).$$

Multiplying equation (3) by the integrating factor, we obtain

$$x^{-2}\left(u^{I} - \frac{2}{x}u\right) = x^{-2}\left(\frac{1}{x}\right)$$

$$(ux^{-2})^{I} = \left(\frac{1}{x^3}\right)$$

Integrating both sides gives

$$ux^{-2} = -\frac{1}{2x^2} + C_1$$

$$i.e., \quad \frac{u}{x^2} = \frac{-1}{2x^2} + C_1$$

$$u = -\frac{1}{2} + C_1$$

But $\quad u = \frac{dy}{dx}$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{2} + C_1$$

$$dy = \left(-\frac{1}{2} + C_1\right) dx$$

Integrating both sides wrt x gives

$$y = -\frac{1}{2}x + C_1 x + C_2$$

(4) Given that: $y_1(x)$ is one of the solution of : $ay^{II} + by^{I} + Cy = 0$,
Find the second solution of the equation using reduction of order method.

Solution

Given: $ay^{II} + by^{I} + cy = 0$ (1)

Required: To find the second solution of equation (1) given that $y_1(x)$ *is a first solution of equation* (1).

Procedure: Let $y_2 = u_1(x)y_1(x)$

$$y_2^I = u_1^I y_1 + u_1 y_1^I$$

$$y_2^{II}(x) = u_1^{II} y_1 + u_1^{II} y_1^I + u_1^I y_1^I + u_1 y_1^{II}$$

$$y_2^{II}(2) = u_1^{II} y_1 + 2u_1^I y_1^I + u_1 y_1^{II}$$

Substituting y_2^{II}, y_2^I *and* y_2 *in equation* (2) *gives us;*

$$a[u_1^{II} y_1 + 2u_1^I y_1^I + u_1 y_1^{II}] + b[u_1^I y_1 + u_1 y_1^I] + c\,[u_1 y_1] = 0$$

$$au_1 y_1^{II} + 2au_1^I y_1^I + bu_1^I y_1 + au_1^{II} y_1 + bu_1 y_1^I + cu_1 y_1 = 0$$

$$u_1(ay_1^{II} + by_1^I + cy_1) + u_1^I(2ay_1^I + by_1) + u_1^{II}(ay_1) = 0$$

The expression in U_1 is equal to zero since it contains a solution of the equation.

$$i.e.,\ ay_1^{II} + by_1^I + cy_1 = 0$$

Hence, we have

$$U_1^I\,(2ay_1^I + by_1) + u_1^{II}(ay_1) = 0$$

$$u_1^I(2ay_1^I + by_1) = -au_1^{II}y_1$$

$$\frac{u_1^{II}}{u_1^I} = \frac{-(2ay_1^I + by_1)}{ay_1}$$

Integrating both sides wrt x, we have

$$\int \frac{u_1^{II}}{u_1^I} = -\int \frac{(2ay_1^I + by_1)}{ay_1}$$

$$ln\, u_1^I = -\int \frac{2ay_1^I + by_1}{ay_1} + C_1$$

$$u_1^I = e^{-\int \frac{2ay_1^I + by_1}{ay_1}} + C_1$$

Let $e^{C_1} = k_1$

$$\Rightarrow u_1^I = k_1 e^{-\int \frac{2ay_1^I + by_1}{ay_1}}$$

Integrating again wrt x gives

$$u_1 = \int k_1 e^{-\int \frac{2ay_1^I + by_1}{ay_1}} + k_2$$

But $y_2 = u_1 y_1$

$$\therefore\; y_2 = y_1 \int k_1 e^{-\int \frac{2ay_1^I + by_1}{ay_1}} + k_2$$

(5) Show that $y = x$ is a solution of the Legendre equation of order one

$$(1 - x^2)y^{II} - 2xy^I + 2y = 0, \qquad -1 < x < 1$$

And hence, find a second linearly independent solution.

Solution

Given: $(1-x^2)y^{II} - 2xy^I + 2y = 0$ (1)

Required: To show that $y = x$ *is a solution of equation* (1).

Proof:

(i) To show that $y = x$ *is a solution of equation* (1)

$y = x$

$y^I = 1$

And $y^{II} = 0$

Substituting values in equation (1) gives

$$(1-x^2)(0) - 2x(1) + 2(x)$$

$$0 - 2x + 2x = 0$$

$\therefore y = x$ *is a solution of equation* (1)

(ii) To find the second linearly independent solution using reduction of order method.

Let $y_2 = ux$

$$y_2^I = u^I x + u$$

$$y_2^{II} = u^{II}x + u^I + u^{II} = u^{II}x + 2u^I$$

Substituting y_2, y_2^I *and* y_2^{II} *in equation* (1) *gives us*

$$(1-x^2)[u^{II}x + 2u^I] - 2x\,[u^I x + u] + 2[ux] = 0$$

$$u^{II}x + 2u^I - u^{II}x^3 - 2u^I x^2 - 2u^I x^2 - 2ux + 2ux = 0$$

$$u^{II}(x - x^3) + u^I(2 - 2x^2 - 2x^2) = 0$$

$$u^{II}(x - x^3) + u^I(2 - 4x^2) = 0$$

$$u^{II}(x - x^3) = u^I(4x^2 - 2)$$

Let $u^I = V$ *and* $u^{II} = V^I$

Substituting, we have

$$v^I\,(x - x^3) = v\,(4x^2 - 2)$$

$$\frac{dv}{dx}\,(x - x^3) = v\,(4x^2 - 2)$$

Rearranging, we have

$$\frac{dv}{v} = \frac{4x^2-2}{x - x^3}$$

$$\frac{dv}{v} = \frac{4x^2-2}{x(1-x^2)}$$

$$\frac{dv}{v} = \frac{4x^2-2}{x(1+x)(1-x)}$$

Integrating both sides gives

$$\int \frac{dv}{v} = \int \frac{4x^2-2}{x(1+x)(1-x)}\,dx$$

$$\ln V = -2\ln x - \ln(1 + x) - \ln(1 - x) + \ln c$$

(Verify using partial fraction)

$$\ln V = \ln\frac{C}{x^2(1 + x)(1 - x)}$$

Let $\quad C = 1$

$$\Rightarrow \ln V = \ln[\frac{1}{x^2(1+x)(1-x)}]$$

Cancelling ln gives us

$$V = \frac{1}{x^2(1+x)(1-x)}$$

But $V = \dfrac{dv}{dx}$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{x^2(1+x)(1-x)}$$

$$\int du = \int \frac{1}{x^2(1-x)(1+x)}dx$$

$$\frac{1}{x^2(1-x)(1+x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(1-x)} + \frac{D}{1+x}$$

Multiplying through by the $lcm\ [x^2\ (1-x^2)], we\ have$

$1 = A\,x\,(1-x)(1+x) + B\,(1-x)(1+x) + Cx^2(1+x) +$

$Dx^2\,(1-x)$

Putting $x = 0$;

$1 = A\,(0) + B\,(1) + C\,(0) + D\,(0)$

$\therefore B = 1$

Put $x = 1$

$1 = A(0) + B(0) + C(2) + D(0)$

$1 = 2C$

$\therefore C = \frac{1}{2}$

Put $x = -1$

$1 = A(0) + B(0) + C\,(0) + D(2)$

$\therefore D = \frac{1}{2}$

But $1 = Ax\,(1-x^2) + B(1-x^2) + Cx^2(1+x) + Dx^2(1-x)$

$1 = A(x-x^3) + B\,(1-x^2) + C(x^2+x^3) + D(x^2-x^3)$

Comparing coefficients of terms, we have

$x^3\ ;\ 0 = -A + C - D$

Substituting values of $C\ and\ D\ in\ the\ above\ gives\ us;$

$0 = -A + \frac{1}{2} - \frac{1}{2}$

$\therefore 0 = A$

$\therefore A = 0, B = 1, C = \frac{1}{2}$ and $D = \frac{1}{2}$

We have

$$\frac{1}{x^2(1-x)(1+x)} = \frac{1}{2} + \frac{1}{2(1-x)} + \frac{1}{2(1+x)}$$

From $\int du = \int \frac{1}{x^2(1-x)(1+x)} dx$

$$u = \int \left[\frac{1}{x^2} + \frac{1}{2(1-x)} + \frac{1}{2(1+x)}\right] dx$$

$$u = -\frac{1}{x} - \frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x) + C$$

$$u = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) - \frac{1}{x} \quad (Assuming\ C = 0)$$

But $y_2 = ux$

$$\Rightarrow y_2(x) = x\left[\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) - \frac{1}{x}\right]$$

$$y_2(x) = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1$$

(6) Given that $y_1(x) = x^{-1}$ *is a solution of the equation* $2x^2y^{II} + 3xy^{I} - y = 0, \ x > 0,$

Find a second linearly independent solution.

Solution

Given: $2x^2y^{II} + 3xy^{I} - y = 0$ (1)

Required: To find the second linearly independent solution

Proof: Assume $y_2(x) = x^{-1}v$

$$y_2^{I} = -x^{-2}v + x^{-I}v^{I}$$

$$y_2^{II} = 2x^{-3}v - x^{-2}v^{I} - x^{-2}v^{I} + x^{-1}v^{II}$$

$$y_2^{II} = 2x^{-3}v - 2x^{-2}v^{I} + x^{-1}v^{II}$$

Substituting y_2, y_2^{I} *and* y_2^{II} *in equation* (1) *gives*

$$2x^2\,[2x^{-3}v - 2x^{-2}v^{I} + x^{-1}v^{II}] + 3x[-x^{-2}v + x^{-1}v^{I}] - x^{-1}v = 0$$

$$4x^{-1}v - 4v^{I} + 2xv^{II} - 3x^{-1}v + 3v^{I} - x^{-1}v = 0$$

$$2xv^{II} - v = 0$$

$$2xv^{II} = v^{I} \qquad (2)$$

Let $\quad u = v^{I}$ *and* $u^{I} = v^{II}$, *this implies*

$$2xu^{I} = u$$

$$2x\,\frac{du}{dx} = u$$

$$2\frac{du}{u} = \frac{dx}{x} \qquad (separating\ variables);$$

Integrating both sides with x gives us;

$$2\ln u = \ln x + \ln A$$

$$\ln u^2 = \ln(Ax)$$

$$u^2 = Ax \qquad [cancelling\ ln]$$

$$u = Ax^{\frac{1}{2}}$$

$$u = x^{\frac{1}{2}} \qquad (Assume\ A = 1)$$

But $u = \frac{dv}{dx}$

$$\Rightarrow \quad \frac{dv}{dx} = x^{1/2}$$

$$\int dv = \int x^{1/2}\,dx$$

$$v = \frac{x^{3/2}}{3/2} + C$$

$$v = \frac{2}{3} = x^{3/2} + C$$

But $y_2(x) = x^{-1}v$

$$\Rightarrow \quad y_2(x) = x^{-1}\left[\frac{2}{3}x^{3/2} + C\right]$$

$$= \frac{2}{3}x^{1/2} + Cx^{-1}$$

Practice problems 9

(1) Solve the differential equation: $x^2y^{II} = y^{I}(3x - 2y^{I})$

2 (i) Show that $y = x$ *is a solution of the equation*

$(x^2 + 1)y^{II} - 2xy^{I} + 2y = 0$

(ii) Obtain the second solution by reduction of order and

(iii) Show that these solutions form a fundamental set.

(3) Solve the following equations:

(a) $y^{II} + (y^{I})^2 = 0$

(b) $y^{II} = 1 + (y^{I})^2$

(c) $2yy^{II} = (y^{I})^2$

(d) $y^{II} - k^2y = 0$

(e) $yy^{II} - (y^{I})^2 = 0$

General practice problems 10

In each of problems I through 8, show that the given function $\emptyset$ is a solution of the differential equation for any choices of the constants C_1 and C_2.

(1) $y^{II} - 5y^{I} + 4y = 0$; $\quad \emptyset(x) = c_1e^{x} + c_2e^{4x}$

(2) $y^{II} - y^{I} - 2y = 0$; $\quad \emptyset(x) = C_1e^{-x} + C_2e^{2x}$

(3) $y^{II} + 8y^{I} + 16y = 0$; $\quad \emptyset(x) = e^{-4x}(c_1 + c_2x)$

(4) $y^{II} + 16y = 0$ $\quad \emptyset(x) = c_1\cos(4x) + c_2\sin(4x)$

(5) $y^{II} - 16y = 0 \quad \emptyset(x) = c_1e^{4x} + c_2e^{-4x}$

(6) $y^{II} + 4y = 2x \quad \emptyset(x) = c_2\cos(2x) + c_1\sin(2x) + \frac{1}{2}x$

(7) $y^{II} - 9y = 3 \quad \emptyset(x) = c_1e^{3x} + c_2e^{-3x} - {}^1/_3$

(8) $y^{II} + y^{I} - 12y = x^2 - 1;$

$$\emptyset(x) = \left[c_1e^{-4x} + c_2e^{3x} - \frac{1}{12}x^2 - \frac{1}{72}x + \frac{59}{864}\right]$$

Solve the following initial value problems

(9) $y^{II} + 2y^{I} + 3y = 0 \qquad y(0) = 2,\ y^{I}(0) = -3$

(10) $y^{II} - 6y^{I} + 9y = 0 \qquad y(0) = 3,\ y^{I}(3) = 2$

(11) $y^{II} - y^{I} + y = 0, \qquad y(0) = 2,\ y^{□}(0) = 0$

(12) $y^{II} - y^{I} - 6y = 0 \qquad y(1) = 4,\ y^{I}(1) = 7$

(13) $y^{II} - 4y^{I} + 4y = 0; \qquad y(0) = 3,\ y^{I}(0) = 2.$

(14) *Determine the three linearly independent solutions of the differential equation;*
$y^{III} + 5y^{II} + 7y^{I} + 3y = 0.$
Hence, compute their wronskian

Hint: $$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1^I & y_2^I & y_3^I \\ y_1^{II} & y_2^{II} & y_3^{II} \end{vmatrix}$$

If $W(y_1, y_2, \ldots) > 0$, *the solution set are linearly independent*

If $W(y_1, y_2, \ldots) < 0$, *the solution set are linearly dependent.*

(15) Determine the general solution using the method of variation of parameters of the differential equation

$y^{II} + y = \sec x, 0 < x < \frac{\pi}{2}.$

Find an integral factor and hence solve the following differential equations:

(16) $2\cosh x\, cosy\, dx = \sinh x\, siny\, dy$

(17) $(2\, cosy + 4x^2)\, dx = x\, siny\, dy$

(18) $x^{-1}\, coshy\, dx + \sinh y\, dy = 0$

(19) $2x\, tany\, dx + sec^2 y\, dy = 0$

Find the general solutions of the following differential equations.

(20) $y' + ky = e^{-kx}$

(21) $x^3 y' + 3x^2 y = \frac{1}{x}$

(22) $y' + ysin\, x = e^{\cos x}$

(23) $xy' = 2y + x^3 e^x$

CHAPTER TEN

INTRODUCTION TO LAPLACE TRANSFORMS

If $f(x)$ represents some expressions in x defined for $x \geq 0$, the Laplace transform of $f(x)$, denoted by $L\{f(x)\}$, is defined to be:

$$L\{f(x)\} = \int_{x=0}^{\infty} e^{-sx} f(x)dx$$

Where s is a variable whose values are chosen so as to ensure that the semi – infinite integral converges.

Example (1) Find the Laplace transform of $f(x) = 3 \; for \; x \geq 0$.

Solution

From $L\{f(x)\} = \int_{x=0}^{\infty} e^{-sx} f(x)dx$

$so \; L\{3\} = \int_{x=0}^{\infty} e^{-sx} \; 3dx$

$= 3\left[\frac{e^{-sx}}{-s}\right]_0^{\infty} = 3\left[0 - \left(-\frac{1}{s}\right)\right] = 3\left(\frac{1}{s}\right) = \frac{3}{s}$

Notice that $S > 0$ *is demanded because if* $S < 0$, *then* $e^{-sx} \to \infty$ *as* $x \to \infty$ *and if* $S = 0$,

Then $L\{3\}$ *is not defined.*

$\therefore L\{3\} = \frac{3}{s}$ *provided* $S > 0$

By the same reasoning, if k is some constant, then

$L\{k\} = \frac{k}{s}$ provided $s > 0$

Example (2)

Find the Laplace transform of

$f(x) = e^{-kx},\ x \geq 0$ *where k is a constant.*

Solution

From $L\{f(x)\} = \int_{x=0}^{\infty} e^{-sx}\ f(x)dx$

This implies

$L\{e^{-kx}\} = \int_0^{\infty} e^{-sx}\ (e^{-kx})dx$

$$= \int_0^{\infty} e^{-(s+k)x}\ dx$$

$$= \left[\frac{e^{-(s+k)x}}{-(s+k)}\right]_0^{\infty} = \left[0 - \left(\frac{-1}{s+k}\right)\right] = \frac{1}{s+k}$$

$\therefore L\{e^{-kx}\} = \frac{1}{s+k}$ *provided* $s > -k$

10.1 THE INVERSE LAPLACE TRANSFORM

The Laplace transform is an expression in the variable s which is denoted by $F(s)$. It is said that $f(x)$ *and* $f(s) = L\{f(x)\}$ form a transform pair. This means that if $F(s)$ is the Laplace transform of $f(x)$, *then* $f(x)$ is the inverse Laplace transform of $F(s)$. *We write:*

$f(x) = L^{-1}\{f(s)\}$

There is no simple integral definition of the inverse transform, so you have to find it by working backwards.

For example:

If $f(x) = 4$, *then the Laplace transform*

$L\{f(x)\} = F(s) = \frac{4}{s}$

$f(x) = \mathcal{L}^{-1}\{f(s)\}$	$F(s) = \mathcal{L}\{f(x)\}$
(1) k	$\frac{k}{2}$ $\quad S > 0$
(2) e^{-kx}	$\frac{1}{S+k}$ $\quad S > -k$

$$L\{e\} = \frac{e}{S} \text{ provided } S > 0$$

(c) $f(x) = e^{2x}$

Because $L\{e^{-kx}\} = \frac{1}{S+k}$ provided $S > -k$,

$$L\{e^{2x}\} = \frac{1}{S-2} \text{ provided } S > 2$$

(d) $f(x) = -5e^{-3x}$

$$L\{-5e^{-3x}\} = \int_{x=0}^{\infty} e^{-sx}\,(-5e^{-3x})\,dx$$

$$= -5\int_{x=0}^{\infty} e^{-sx}\,.\,e^{-3x}dx = -5L\{e^{-3x}\}$$

$$\therefore L\{-5e^{-3x}\} = \frac{-5}{S+3} \text{ provided } S > -3$$

(e) $f(x) = 2e^{7x-2}$

$$L\{2e^{7x-2}\} = \int_{x=0}^{\infty} e^{-sx}\,(2e^{7x-2})dx$$

$$= 2e^{-2}\int_{x=0}^{\infty} e^{-sx}\,.\,e^{7x}dx = 2e^{-2}\,L\{e^{7x}\}$$

$$L\{2e^{7x-2}\} = \frac{2e^{-2}}{S-7} \text{ provided } S > 7.$$

Example (4)

Find the inverse Laplace transform of each of the following:

(a) $F(s) = \frac{-3}{4S}$ (b) $F(s) = \frac{1}{2S-3}$

Solution

(a) $f(s) = \frac{1}{2S-3}$

$f(s)$ 1 $\frac{1}{}$

So that $f(x) = L^{-1}\left\{\frac{1}{2S-3}\right\} = L^{-1}\left\{\frac{\frac{1}{2}}{S-{}^{3}/_{2}}\right\} = \frac{1}{2}\, e^{\frac{3x}{2}}$

(b) $F(s) = \frac{-3}{4S} = \left(\frac{-3/4}{S}\right)$

So the $L^{-1}\left\{\frac{-3}{4S}\right\} = L^{-1}\left\{\frac{-3/4}{S}\right\} = \frac{-3}{4}$

10.2 LAPLACE TRANSFORM OF A DERIVATIVE

Before you can use the Laplace transform to solve a differential equation, you need to know the Laplace transform of a derivative.

$L\{f^{I}(x)\} = sF(s) - f(0)$

So the Laplace transform of the derivative of $f(x)$ is given in terms of the Laplace transform of $f(x)$ itself and the value of $f(x)$ when$x =$ 0.

10.3 TWO PROPERTIES OF LAPLACE TRANSFORM

Both the Laplace transform and its inverse are linear transforms, by which is meant that:

(1) The transform of a sum (or difference) of expressions is the sum of the individual transforms. That is:

$L\{f(x) \pm g(x)\} = L\{f(x)\} \pm L\{g(x)\}$

And $L^{-1}\{f(s) \pm G(s)\} = L^{-1}\{F(s)\} \pm L^{-1}\{G(x)\}$

(2) The transform of an expression that is multiplied by a constant is the constant multiplied by the transform of the expression. That is:

$L\{K f(x)\} = kL\{f(x)\}$ *and* $L^{-1}\{k f(s)\} = kL^{-1}\{F(s)\}$

Where K is a constant.

10.4 REVISION SUMMARY

(1) If $F(s)$ is the Laplace transform of $f(x)$ then the Laplace transform of $f^I(x)$ is

$L\ \{f^I(x)\} = sF(s) - f(0)$

(2)(a) The Laplace transform of a sum (or difference) of expressions is the sum (or difference) of the individual transforms. That is:

$$L\{f(x) \pm g(x)\} = L\{f(x)\} \pm L\{g(x)\}$$

And $L^{-1}\{F(s) \pm G(s)\} = L^{-1}\{F(s)\} \pm L^{-1}\{G(s)\}$

(b) The transform of an expression multiplied by a constant is the constant multiplied by the transform of the expression. That is:

$L\{K f(x)\} = KL\{f(x)\}$ *and* $L^{-1}\{K F(s)\} = kL^{-1}\{F(s)\}$

Where K is a constant.

(3) To solve a differential equation of the form:

$af^I(x) + bf(x) = g(x)$ *given that* $f(0) = K$

Where a, b and k are known constants and $g(x)$ is a known expression in x:

(a) Take the Laplace transform of both sides of the differential equation

(b) Find the expression $F(s) = L\{f(x)\}$ in the form of an algebraic fraction

(c) Separate $F(s)$ into its partial fractions

(d) Find the inverse Laplace transform $L^{-1}\{F(s)\}$ to find the solution $f(x)$ of the differential equation.

SOLVED EXAMPLES/APPLICATIONS

(1) Solve the differential equation: $3f'(x) - 2f(x) = 4e^{-x} + 2$

Where $f(0) = 0$.

Solution

$3f^{I}(x) - 2f(x) = 4e^{-x} + 2$ (1)

Taking the Laplace transforms of both sides of equation (1), we have

$3\,[s\,F(s) - f(0)] - 2\,F(s) = \frac{4}{S+1} + \frac{2}{S}$ (2)

Substituting values into equation (2), we have

$$3[s\,F(s) - 0] - 2\,F(s) = \frac{4}{S+1} + \frac{2}{S}$$

$$3s\,F(s) - 2\,F(s) = \frac{4}{S+1} + \frac{2}{S}$$

$$(3S-2)\,F(s) = \frac{4S + 2\,(S+1)}{S(S+1)}$$

$$(3S-2)\,F(s) = \frac{6S+2}{S(S+1)}$$

$$F(s) = \frac{6S+2}{S(S+1)(3S-2)}$$

Transforming the above into partial fractions gives

$$\frac{6S+2}{S(S+1)(3S-2)} \equiv \frac{A}{S3} + \frac{B}{S+1} + \frac{C}{3S-2}$$

Multiplying through by $S(S+1)(3S-2)(Lcm)gives$

$$6S + 2 \equiv A\,(S+1)(3S-2) + BS(3S-2) + CS(S+1)$$

Putting $S = 1\ gives$

$$6(-1) + 2 = A(0) + B(-1)(3S-2) + CS(S+1)$$

$$-6 + 2 = -B(-5)$$

$$-4 = 5B$$

$$\therefore B = \frac{-4}{5}$$

Putting $S = 0$ gives

$$6(0) + 2 = A(0+1)(0-2) + B(0) + C(0)$$

$$0 + 2 = A(1)(-2)$$

$$2 = -2A$$

$$\therefore A = \frac{-2}{2} = -1$$

$$\therefore A = -1$$

Putting $S = {}^2/_3$ *gives*

$$6({}^2/_3) + 2 = A(0) + B(0) + C\,({}^2/_3)\left(\tfrac{2}{3} + 1\right)$$

$$4 + 2 = C\left(\tfrac{2}{3}\right)\left(\tfrac{5}{3}\right)$$

$$6 = \frac{10\,C}{9}$$

$$\therefore C = \frac{6 * 9}{10} = \frac{27}{5}$$

$$\therefore A = -1,\ B = \frac{-4}{5}\ and\ C = \frac{27}{5}$$

But $F(s) = \dfrac{6S + 2}{S(S+1)(3S-2)} \equiv \dfrac{A}{S} + \dfrac{B}{S+1} + \dfrac{C}{3S-2}$

Substituting the values of $A, B\ and\ C$ into the above expression gives

$$F(s) = \frac{-1}{S} - \frac{4}{5}\left(\frac{1}{S+1}\right) + \frac{27}{5}\left(\frac{1}{3S-2}\right)$$

Rearranging the above expression, we obtain

$$F(s) = \frac{27}{5}\left(\frac{1}{3S-2}\right) - \frac{4}{5}\left(\frac{1}{S+1}\right) - \frac{1}{S}$$

$$F(s) = \frac{27}{15}\left(\frac{1}{S-{}^2/_3}\right) - \frac{4}{5}\left(\frac{1}{S+1}\right) - \frac{1}{S}$$

Taking the inverse Laplace transform gives

$$f(x) = \frac{9}{5}e^{\frac{2x}{3}} - \frac{4}{5}e^{-x} - 1$$

(2) Solve the differential equation $f^1(x) + f(x) = e^{-x}$

Where $f(0) = 0$.

Solution

$f^I(x) + f(x) = e^{-x}$ (

Taking the Laplace transform of equation (1) gives

$\{sF(s) - f(0)\} + F(s) = \frac{1}{S+1}$ (

Substituting the value $f(0) gives$

$[sF(s) - 0] + F(s) = \frac{1}{S+1}$

$sF(s) + F(s) = \frac{1}{S+1}$

$(s + 1)\ F(s) = \frac{1}{S+1}$

$F(s) = \frac{1}{(S+1)(S+1)} = \frac{1}{(S+1)^2}$

Taking the inverse Laplace transform of the above gives:

$f(x) = xe^{-1}$ $\{Refer\ to\ table\ 1\}$

10.5 GENERATING NEW TRANSFORMS

Deriving the Laplace transform of $f(x)$ often requires one to integrat by parts, sometimes repeatedly. However, because $\mathcal{L}\{f^I(x)\} =$ $s\mathcal{L}\{f(x)\} - f(0)$, you can sometimes avoid this involved proces when you know the transform of the derivative $f^I(x)$. Take as an example, the problem of finding the Laplace transform of the expression:

$f(x) = x$.

$\ldots = 1\ and\ f(0) = 0$

So that substituting in the equation:

$\mathcal{L}\{f^{I}(x)\} = s\mathcal{L}\{f(x)\} - f(0),$

gives

$\mathcal{L}\{1\} = s\mathcal{L}\{x\} - 0$

That is:

$\frac{1}{s} = s\,\mathcal{L}\{x\}$

$\therefore\ \mathcal{L}\{x\} = \frac{1}{s^2}$

Example (3) Find the Laplace transform of $f(x) = x^2$

Solution

$f(x) = x^2$

$f^{I}(x) = 2x\ \ and\ f(0) = 0$

Substituting in

$\mathcal{L}\{f^{I}(x)\} = s\mathcal{L}\{f(x)\} - f(0)$ gives:

$\mathcal{L}\{2x\} = s\,\mathcal{L}\{x^2\} - 0$

$2\mathcal{L}\{x\} = s\mathcal{L}\{x^2\}$

$\frac{2}{s^2} = s\,\mathcal{L}\{x^2\}$

$\therefore\ \mathcal{L}\{x^2\} = \frac{2}{s^3}$

Example (4) Show that $\mathcal{L}\{xe^{-x}\} = \frac{1}{(s+1)^2}$

Solution

Required: To show that $\mathcal{L}\{xe^{-1}\} = \frac{1}{(s+1)^2}$

$f(x) = xe^{-x}, \; f^{I}(x) = e^{-x} - xe^{-x} \; and \; f(0) = 0$

Substituting in:

$\mathcal{L}\{f^{I}(x)\} = s\,\mathcal{L}\{f(x)\} - f(0)$

gives:

$\mathcal{L}\{e^{-x} - xe^{-x}\} = s\mathcal{L}\{xe^{-x}\} - 0$

That is:

$\mathcal{L}\{e^{-x}\} - \mathcal{L}\{xe^{-x}\} = s\,\mathcal{L}\{xe^{-x}\}$

$\therefore \mathcal{L}\{e^{-x}\} = s\,\mathcal{L}\{xe^{-x}\} + \mathcal{L}\{xe^{-x}\}$

$\mathcal{L}\{e^{-x}\} = (s+1)\,\mathcal{L}\{xe^{-x}\}$

giving

$\frac{1}{s+1} = (s+1)\,\mathcal{L}\{xe^{-x}\}$

and so:

$\mathcal{L}\{xe^{-x}\} = \frac{1}{(s+1)(s+1)} = \frac{1}{(S+1)^2}$

10.6 LAPLACE TRANSFORM OF HIGHER DERIVATIVES

The Laplace transforms of derivatives higher than the first are readily derived.

Let $F(s) and\, G(s)$ be the respective Laplace transforms of $f(x) and\, g(x)$. That is:

$\mathcal{L}\{f(x)\} = F(s)$ so that $\mathcal{L}\{f^{I}(x)\} = sF(s) - f(0)$

and $\mathcal{L}\{g(x)\} = G(s)$ and $\mathcal{L}\{g^{1}(x)\} = sG(s) - g(0)$

Now let $g(x) = f^{1}(x)$ so that $\mathcal{L}\{g(x)\} = \mathcal{L}\{f^{1}(x)\}$

where:

$g(0) = f^{I}(0) \; and \; G(s) = sF(s) - f(0)$

Now because $g(x) = f^{I}(x)$

$$g^{I}(x) = f^{II}(x)$$

This means that

$\mathcal{L}\{g^{I}(x)\} = \mathcal{L}\{f^{II}(x)\} = sG(s) - g(0)$

$= s\,[sF(s) - f(0)] - f^{I}(0)$

So;

$\mathcal{L}\{f^{II}(x)\} = s^2 F(s) - sf(0) - f^{I}(0)$

By a similar argument, it can be shown that

$\mathcal{L}\{f^{III}(x)\} = s^3F(s) - s^2f(0) - sf^{I}(0) - f^{II}(0)$

$\mathcal{L}\{f^{IV}(x)\} = sF(s) - s^3f(0) - s^2f^{I}(0) - sf^{II}(0) - f^{III}(0)$

Table of Laplace transforms (2)

$f(x) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(x)\}$	
(1) k	$\frac{k}{s}$	$s > 0$
(2) e^{-kx}	$\frac{1}{S+1}$	$s > -k$
(3) xe^{-kx}	$\frac{1}{(S+k)^2}$	$s > -k$
(4) x	$\frac{1}{s^2}$	$s > 0$
(5) x^2	$\frac{1}{s^3}$	$s > 0$
(6) $\sin kx$	$\frac{k}{s^2+k^2}$	$s^2 + k^2 > 0$
$\cos kx$	$\frac{s}{s^2+k^2}$	$s^2 + k^2 > 0$

10.7 LINEAR, CONSTANT- COEFFICIENT, INHOMOGENEOUS DIFFERENTIAL EQUATIONS

The Laplace transform can be used to solve equations of the form:

$$a_n f^n(x) + a_{n-1} f^{(n-1)}(x) + \ldots + a_2 f^{II}(x) + a_1 f^I(x) + a_0 f(x) = g(x)$$

Where $a_n, a_{n-1}, \ldots \ldots \ldots \ldots a_2, a_1, a_0$ are known constants, $g(x)$ is a known expression in x and the values of $f(x)$ and its derivatives are known at $x = 0$. This type of equation is called a linear, constant coefficient, inhomogeneous differential equation and the values of $f(x)$ and its derivatives $x = 0$ are called boundary conditions.

The procedure for solving the equations of second and higher order is the same as that for solving the equations of the first order. Namely:

(a) Take the Laplace transform of both sides of the differential equation

(b) Find the expression $F(s) = \mathcal{L}\{f(x)\}$ in the form of an algebraic fraction

(c) Separate $F(s)$ into its partial fractions

(d) Find the inverse Laplace transform $\mathcal{L}\{F(s)\}$ to find the solution $f(x)$ to the differential equation.

Solved examples

(1) Use the Laplace transform to solve each of the following differential equations

(a) $f^{II}(x) - 4f(x) = \sin 2x$ *where* $f(0) = 1$ *and* $f^I(0) = -2$.

(b) $f^{II}(x) + 5f^I(x) + 6f(x) = 2e^{-x}$ *where* $f(0) = 0$ *and* $f^I(0) = 0$.

Solution

(a) $f^{II}(x) - 4f(x) = \sin 2x$ *where* $f(0) = 1$ *and* $f^I(0) = -2$

Transforming both sides of the equation into Laplace transforms:

$$[s^2F(s) - sf(0) - f^I(0)] - 4F(s) = \frac{2}{S^2+4} \quad (1)$$

Substituting values into equation (1), we obtain

$$[s^2F(s) - s(1) - (-2)] - 4\,F(s) = \frac{2}{S^2+4}$$

$$s^2F(s) - S + 2 - 4\,F(s) = \frac{2}{S^2+4}$$

$$s^2\,F(s) - 4F(s) = \frac{2}{S^2+4} + S - 2$$

$$(S^2 - 4)\,F(s) = \frac{2+S\,(S^2+4) - 2(S^2+4)}{S^2+4}$$

$$(S^2 - 4)\,F(s) = \frac{S^3 - 2S^2 + 4S - 6}{S^2+4}$$

$$\therefore F(s) = \frac{S^3 - 2S^2 + 4S - 6}{(S^2-4)(S^2+4)}$$

$$F(s) = \frac{S^3 - 2S^2 + 4S - 6}{(S-2)(S+2)(S^2+4)}$$

Resolving $F(s)$ into partial fractions, we have

$$\frac{S^3 - 2S^2 + 4S - 6}{(S-2)(S+2)(S^2+4)} \equiv \frac{A}{S-2} + \frac{B}{S+2} + \frac{CS+D}{S^2+4}$$

Multiplying both sides with $(S - 2)\,(S + 2)(S^2 + 4)Lcm,$ *we have*

$$S^3 - 2S^2 + 4S - 6 = A\,(S + 2)(S^2 + 4) + B\,(S - 2)(S^2 + 4) + (CS + D)(S - 2)(S + 2)$$

Putting $S = 2$

$$(2)^3 - 2\,(2)^2 + 4\,(2) - 6 = A\,(2 + 2)(2^2 + 4) + B(0) + 0$$

$$8 - 8 + 8 - 6 = A(4)(8)$$

$$2 = 32A$$

$$\therefore A = \frac{2}{32} = \frac{1}{16}$$

Putting $S = -2$;

$$(-2)^3 - 2\,(-2)^2 + 4\,(-2) - 6 = A\,(0) + B\,(-2 - 2)(4 + 4) + 0$$

$-8-8-8-6 = B(-4)(8)$

$+30 = +32B$

$\therefore B = \frac{30}{32} = \frac{15}{16}$

From the expression:

$S^3 - 2S^2 + 4S - 6 \equiv A(S+2)(S^2+4) + B(S-2)(S^2+4) + (CS+D)(S-2)(S+2)$

$S^3 - 2S^2 + 4S - 6$

$\equiv A(S^3 + 2S^2 + 4S + 8) + B(S^3 - 2S^2 + 4S - 8) + CS^3 + DS^2 - 4CS - 4D$

$S^3 - 2S^2 + 4S - 6 \equiv \{(A+B+C)S^3 + (2A - 2B + D)S^2 + (4A + 4B - 4C)S + (8A - 8B - 4D)\}$

Comparing the coefficients of terms we have

$[S^3]: 1 = A + B + C$

i.e.,

$A + B + C = 1$ (3)

But $A = \frac{1}{16}$ and $B = \frac{15}{16}$

Substituting values in equation (3) gives

$\frac{1+15}{16} + C = 1$

$\frac{16}{16} + C = 1$

$1 + C = 1$

$\therefore C = 1 - 1$

$C = 0$

$[S^2]: -2 = 2A - 2B + D$

$or\ 2A - 2B + D = -2$

$$2\left(\frac{1}{16}\right) - 2\left(\frac{15}{16}\right) + D = -2$$

$$\frac{2}{16} - \frac{30}{16} + D = -2$$

$$D = -2 - \frac{2}{16} + \frac{30}{16}$$

$$D = \frac{-32 - 2 + 30}{16}$$

$$D = \frac{-4}{16} = \frac{-1}{4}$$

$\therefore\ A = \frac{1}{16},\qquad B = \frac{15}{16},\qquad C = 0$ and $D = \frac{-1}{4}$

But $F(s) = \frac{A}{S-2} + \frac{B}{S+2} + \frac{Cs+D}{S^2+D}$

Substituting the values of A, B, C *and* D into the expression above gives us

$$F(s) = \frac{1}{16}\left(\frac{1}{S-2}\right) + \frac{15}{16}\left(\frac{1}{S+2}\right) - \frac{1}{4}\left(\frac{1}{S^2+4}\right)$$

Making some transformations, we have

$$F(s) = \frac{1}{16}\left(\frac{1}{S-2}\right) + \frac{15}{16}\left(\frac{1}{S+2}\right) - \frac{1}{4} * \frac{1}{2}\left(\frac{2}{S^2+2^2}\right)$$

$$F(s) = \frac{1}{16}\left(\frac{1}{S-2}\right) + \frac{15}{16}\left(\frac{1}{S+2}\right) - \frac{1}{8}\left(\frac{2}{S^2+2^2}\right)$$

Taking the inverse Laplace transform of the above expression gives the solution of the equation as

$$f(x) = \frac{1}{16}e^{2x} + \frac{15}{16}e^{-2x} - \frac{1}{8}\sin 2x$$

i.e.,

$$f(x) = \frac{e^{2x}}{16} + \frac{15e^{-2x}}{16} - \frac{\sin 2x}{8}$$

(b) $f^{II}(x) + 5f^{I}(x) + 6f(x) = 2e^{-x}$

Where $f(0) = 0$ *and* $f^{I}(0) = 0$ (1)

Taking the Laplace transform of equation (1) gives

$$[S^2 F(s) - sf(0) - f^I(0)] + 5\,[s\,F(s) - f(0)] + 6F(s) = \frac{2}{S+1}$$

Substituting values, we obtain

$$[S^2 F(s) - s(0) - 0] + 5\,[s\,F(s) - 0] + 6F(s) = \frac{2}{S+1}$$

$$S^2 F(s) + 5\,F(s) + 6\,F(s) = \frac{2}{S+1}$$

$$(S^2 + 5 + 6)\;F(s) = \frac{2}{S+1}$$

$$(S+2)(S+3)\,F(s) = \frac{2}{S+1}$$

$$\therefore\; F(s) = \frac{2}{(S+1)(S+2)(S+3)}$$

Resolving $F(s)$ into partial fractions, we have

$$F(s) = \frac{2}{(S+1)(S+2)(S+3)} \equiv \frac{A}{S+1} + \frac{B}{S+2} + \frac{C}{S+3}$$

Solving the above gives $A = 1,\; B = -2$ *and* $C = 1$.

$$\therefore F(s) = \frac{1}{S+1} - \frac{2}{S+2} + \frac{1}{S+3}$$

Taking the inverse Laplace transform of both sides gives

$$f(x) = e^{-x} - 2e^{-2x} + e^{-3x}$$

(2) Solve the differential equation

$$2f^{II}(x) - f^{I}(x) - f(x) = \sin x - \cos x$$

Where $f(x) = 0$ *and* $f^{I}(0) = 0$

Solution

$2f^{II}(x) - f^{I}(x) - f(x) = \sin x - \cos x$ (1)

Where $f(0) = 0$ *and* $f^{I}(0) = 0$

Transforming equation (1) into Laplace transform gives us

$$2[S^2F(s) - sf(0) - f^{I}(0)] - [sF(s) - f(0)] - F(s)$$
$$= \frac{1}{S^2+1} - \frac{1}{S^2+1}$$

Substituting values, we have

$$2[S^2F(s) - s(0) - 0] - [sF(s) - 0] - F(s) = \frac{1-S}{S^2+1}$$

$$2S^2F(s) - sF(s) - F(s) = \frac{1-S}{S^2+1}$$

$$(2s^2 - s - 1)\ F(s) = \frac{1-s}{s^2+1}$$

$$(2s+1)(s-1)\ F(s) = \frac{1-s}{s^2+1}$$

$$F(s) = \frac{1-s}{(2s+1)(s-1)(s^2+1)} = \frac{A}{2s+1} + \frac{B}{s-1} + \frac{Cs+D}{s^2+1}$$

Multiplying through with $(2s+1)(s-1)(s^2+1)lcm, we\ have$

$$1 - s = A(s-1)(s^2+1) + B(2s+1)(s^2+1)$$
$$+ (Cs+D)(2s+1)(s-1)$$

Putting $s = 1$;

$$1 - 1 = A(0) + B(2+1)(1+1) + 0$$

$$0 = B(3)(2)$$

$$0 = 5B$$

$\therefore B = \frac{0}{5} = 0$

Putting $s = -1/2$

$1 - (-1/2) = A\left(-\frac{1}{2} - 1\right)\left(\frac{1}{4} + 1\right) + 0 + 0$

$1 + \frac{1}{2} = A\left(-\frac{1-2}{2}\right)\left(\frac{1+4}{4}\right)$

$\frac{3}{2} = A\left(\frac{-3}{2}\right)\left(\frac{5}{4}\right)$

$\frac{3}{2} = -\frac{15A}{8}$

$\therefore A = -\frac{3x\,8}{2 * 15} = \frac{-24}{30} = \frac{-4}{5}$

$\therefore A = -\frac{4}{5}, \qquad B = 0$

From:

$1 - s = A(s-1)(s+1) + B(2s+1)(s^2+1) +$
$(Cs + D)(2s+1)(s-1)$

$1 - s = \{A\,(s^3 + s - s^2 - 1) + B(2s^3 + 2s + s^2 + 1) +$
$(Cs + D)(2s^2 - s - 1)\}$

$1 - s = \{A(s^3 - s^2 + s - 1) + B(2s^3 + s^2 + 2s + 1) + 2Cs^3 - Cs^2 - Cs +$
$2Ds^2 - Ds - D\}$

$1 - s = \{(A + 2B + 2C)s^3 + (-A + B - C + 2D)s^2 + (A + 2B - C - D)s +$
$(-A + B - D)\}$

Comparing coefficients of terms, we have

$[S^3]\ :\ 0 = A + 2B + 2C$

$0 = -\frac{4}{5} + 0 + 2C$

$$\frac{4}{5} = \frac{2C}{1}$$

$$\therefore C = \frac{4 \times 1}{2 \times 5} = \frac{2}{5}$$

(CT) : $1 = -A + B - D$

$$1 = -\left(\frac{-4}{5}\right) + 0 - D$$

$$1 = \frac{4}{5} - D$$

$$D = \frac{4}{5} - 1$$

$$D = \frac{4-5}{s} = -\frac{1}{5}$$

$$\therefore A = \frac{-4}{5}, \quad B = 0, \quad C = \frac{2}{5}, \quad D = \frac{-1}{5}$$

$$\therefore F(s) = \frac{-4}{5}\left(\frac{1}{2s+1}\right) + \frac{\frac{2s}{5} - \frac{1}{5}}{s^2+1}$$

$$F(s) = \frac{-4}{5}\left(\frac{1/2}{s + 1/2}\right) + \frac{\frac{2s-1}{5}}{s^2+1}$$

$$= -\frac{4}{5} * \frac{1}{2}\left(\frac{1}{s + 1/2}\right) + \frac{1}{5}\left(\frac{2s-1}{s^2+1}\right)$$

$$= \frac{-2}{5}\left(\frac{1}{s + 1/2}\right) + \frac{1}{5}\left(\frac{2s-1}{s^2+1}\right)$$

$$= \frac{-2}{5}\left(\frac{1}{s + 1/2}\right) + \frac{1}{5}\left[\frac{2s}{s^2+1} - \frac{1}{s^2+1}\right]$$

$$= \frac{-2}{5} e^{-1/2x} + \frac{1}{5}\left[2\left(\frac{s}{s^2+1}\right) - \frac{1}{s^2+1}\right]$$

$$= \frac{-2}{5} e^{-1/2x} + \frac{1}{5} [2\cos x - \sin x]$$

$$= \frac{-2}{5} e^{-1/2x} + \frac{2}{5}\cos x - \frac{1}{5}\sin x$$

EXERCISES

Solve each of the following differential equations:

(i) $f^{II}(x) - 5f^{I}(x) + 6f(x) = 1$ *where* $f(0) = 0$ *and* $f^{I}(0) = 0$

(ii) $2f^{II}(x) - f^{I}(x) - f(x) = e^{-3x}$
Where $f(0) = 2$ *and* $f^{I}(0) = 1$

(iii) $f(x) + f^{I}(x) - 2f^{II}(x) = xe^{-x}$ *where* $f(0) = 0$
and $f^{I}(0) = 1$

(iv) $f^{II}(x) + 16f(x) = 0$ *where* $f(0) = 1$ *and* $f^{I}(0) = 4$

(v) $2f^{II}(x) - f^{I}(x) - f(x) = \sin x - \cos x$
Where $f(0) = 0$ *and* $f^{I}(0) = 0$.