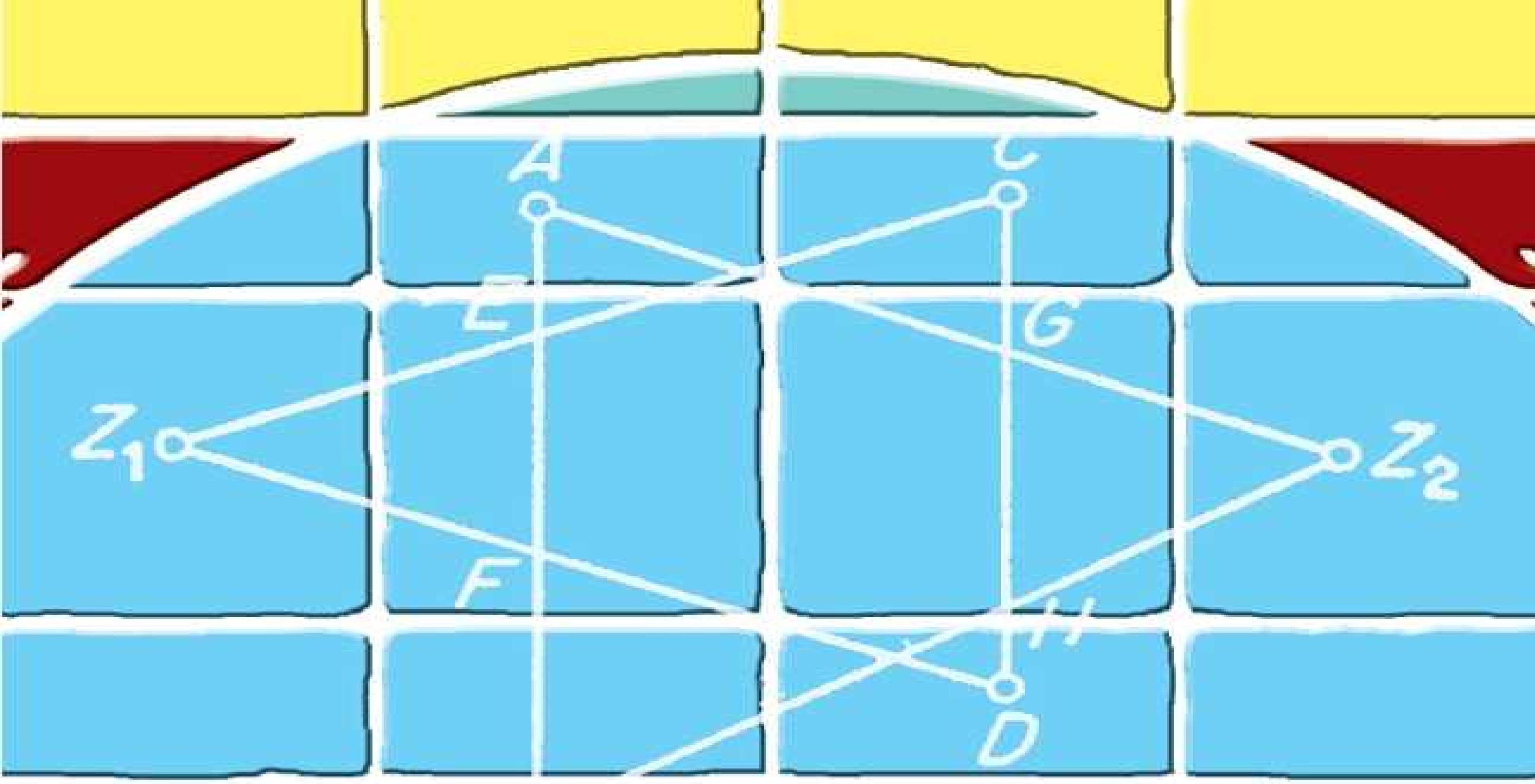

Introduction to the

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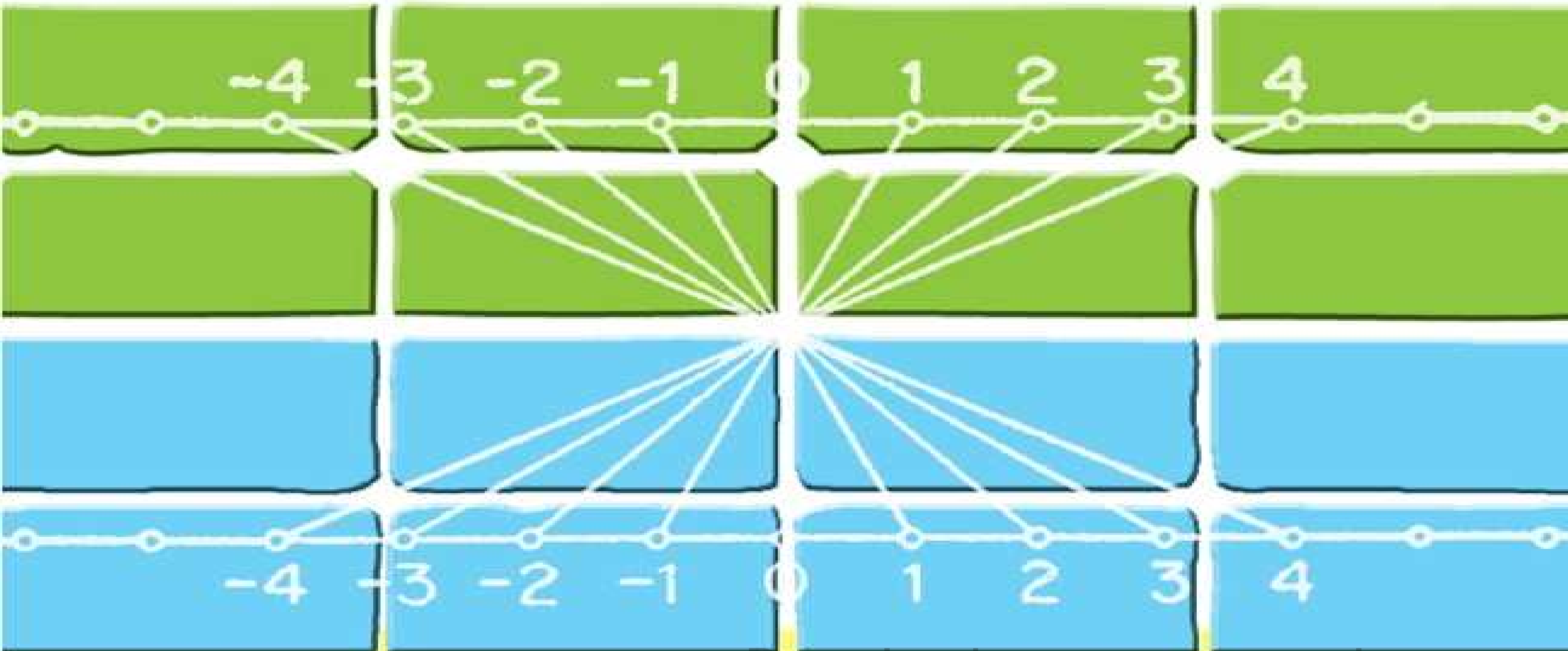
Joseph Breuer



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Joseph Breuer

Translated by
Howard F. Fehr

DOVER PUBLICATIONS, INC., MINEOLA, NEW YORK

Bibliographical Note

This Dover edition, first published in 2006, is an unabridged republication of the work originally published by Prentice-Hall, Inc., Englewood Cliffs, New Jersey, in 1958.

International Standard Book Number: eISBN 13: 978-0-486-15487-9

Manufactured in the United States of America
Dover Publications, Inc., 31 East 2nd Street, Mineola, N.Y 11501

PREFACE

Today questions concerning the foundations of mathematics are again receiving considerable attention. In particular, set theory has become an important area of investigation because of the way in which it seems to permeate so much of contemporary mathematical thought. The subject of set theory may be considered as originating with Georg Cantor, who attempted to organize concepts on collections of objects into a structure which could serve as a basis for a mathematical theory of the infinite. The infinite, in this sense, is not the "potential," unattainable type used in the theory of limits, but the "actual" or proper type conceived as a completely determined object lying beyond all finite magnitudes. How well Cantor succeeded is easily recognized in the extension of the theory from the original naive, plausible approach to the present-day abstract, axiomatic development which has become the basis of structure in algebra, geometry, and analysis.

In the English language there exists today only scattered fragmentary or advanced treatises on the subject. There is now a need for a treatment of set theory in English, from a less than abstract axiomatic approach, sufficiently elementary to serve as an introduction to the subject for college and high school instructors, college students, and interested laymen. This book meets that need. A naive approach, which depends upon observation of the concrete world for its development and meaning, is a natural way to introduce the subject, and this procedure is used in the following exposition. Little by little, certain properties and principles are developed, which in turn are used to prove further theorems concerning sets as collections of abstract entities. Thus one is led from concrete finite sets, to cardinal numbers, to infinite cardinals, and thence to ordinals via the use of ordinal-types.

Abstract set theory based on an axiomatic system is not treated here. For those who care to pursue the subject further, the appended bibliography provides directed study. The axiom of choice and its relation to the theorem of well-ordering have had tremendous effect on the whole development of set theory, but these are matters of concern to the mathematician rather than to the neophyte. This translation is offered with the hope that it will provide for the reader sufficient background and impelling interest for further study.

Howard F. Fekr

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INTRODUCTION

"Geometry and analysis, differential and integral calculus deal continually, even though perhaps in disguised expression, with infinite sets." Thus wrote F. Hausdorff (1914) in his *Fundamentals of the Theory of Sets*. To attain a genuine understanding and mastery of these various branches of mathematics requires a knowledge of their common foundation, namely, the theory of sets.

We may ask, what are the things with which mathematics concerns itself? They are, in every case, sets of numbers or sets of points—generally infinite sets, that is, sets which contain an infinite number of things.

The reader may question the idea of approaching the infinite by means of mathematical analysis, thus bringing it under the control of mathematical laws and formulas. But this approach is the essence of the theory of sets. For this purpose, our concept of the infinite must be separated from vague emotional ideas and from the infinite of nonmathematical realms (the infinite of metaphysics).

Before Cantor's time, the infinite in mathematics was an obscure and unpredictable area. Even Gauss, in 1831, was of the opinion that: "The infinite is only a 'manner of speaking' in that one actually talks of limits which certain ratios approach as closely as desired, while other ratios are permitted to grow larger without bounds." Gauss, himself, rejected the use of an "infinite number," as something which is never permitted in mathematics. He recognized the infinite only in the sense of a process of becoming infinite in the limit: $\lim_{x \to \infty}$ ….

One of Cantor's predecessors, Bolzano,* recognized that the infinite in mathematics was replete with paradoxes (contradictions) obstructing arithmetical treatment of the subject. It was Cantor, however, who taught us how to calculate with the "infinite" through his introduction of clearly determined and sharply differentiated infinite numbers, with well-defined operations upon them: "… It concerned an extension; that is, a continuation of the sequence of real integers beyond the infinite. As daring as this might seem, I not only express the hope, but also the firm conviction that, in time, this extension will come to be looked upon as thoroughly simple, acceptable, and natural."

After a ten-year delay, when he had come to recognize that his concepts were indispensable to the further development of mathematics, Cantor decided to publish his creation. In this work, he generalized the laws and rules applied to finite numbers so that they would extend beyond the domain of these numbers. He explained how one could compute with infinite sets, using the same methods that are applied to finite sets. With a few clearly defined concepts such as order (going back to Dedekind), power or cardinal number, denumerability, etc., he raised the theory of sets to a science which no longer contained fundamental barriers between the finite and the infinite—one that made the infinite understandable.

Today we know that Cantor, as Hilbert has said, thereby "created one of the most fertile and powerful branches of mathematics; a paradise from which no one can drive us out." The theory of sets stands as one of the boldest and most beautiful creations of the human mind; its construction of concepts and its methods of proof have reanimated and revitalized all branches of mathematical study. The theory of sets, indeed, is the most impressive example of the validity of Cantor's statement that, "The essence of mathematics lies in its freedom."

Mathematics exercises its freedom in asking questions. Who has not at some time posed questions of the following kind?

Are there more whole numbers than there are even numbers?

Does an unbounded straight line contain more points than a line segment?

Does a plane contain fewer points than space?

Are the rational points densely situated on the number scale?

In particular, what do $\infty + 1$, and $\infty \cdot 3$, and ∞^2 denote?

People refrained from discussing these questions publicly since such inquiries seemed naive or stupid and, above all, because they appeared to have no answer. However, the theory of sets gives clear answers possessing mathematical precision to all these questions, when the questions are properly phrased.

The foundation of the general theory of sets has now been established for over half a century. To understand it calls for scarcely any prerequisite technical knowledge. All that is necessary is an interest in establishing the "infinitely large" and a patience for grasping somewhat difficult concepts. Even though the theory of sets starts in the intuitive-concrete, it nevertheless climbs to a very high degree of abstraction.

This book is an *introduction* to the theory of sets. In the first few pages the fundamental concepts will be developed through the use of well-known finite sets. Although the theory of finite sets is nothing else than mere arithmetic and permutations and combinations, yet it helps to provide the terminology and symbolism of set theory. These concepts will provide the basis for the subsequent treatment of the infinite sets. The general theory of sets ends with a discussion of ordered sets. A few important theorems on point sets are appended in a supplement. Definitions that produce paradoxes are merely alluded to in the concluding paragraphs.

*In a work published in 1851 after his death.

2

FINITE SETS

I. Set, Element, Equality of Sets

1. What is a *set*? It is not that which we usually refer to in our everyday speech, when we speak of a large 3 of people, of ships, or of things. Rather:

> *A set is a collection of definite distinct objects of our perception or of our thought*, *which are called elements of the set**

2. The following are examples of sets:

(a) In Figure 1, the four persons sitting at the table form a set of four persons because they are four definite distinct objects of our perception. Father *A*, mother *A*, son Fred *A*, and son Peter *A* are to be considered as a whole, as a set called family *A*. The four chairs form a set of four elements; the four spoons, the four forks, the four knives, the four plates; each form a set of four elements. All the eating utensils can be considered as forming one set *B*, a set of 12 elements, provided we define the elements of *B* to consist of the eating utensils.

In the fruit bowl there is a set of seven pieces of fruit. We can also say: the bowl contains a set of four apples and set of three pears.

Notice that the elements belonging to a set are determined by the distinguishing characteristics of the set. For each thing considered, one must be able to say whether or not it is an element of the set. The set of all male members of family *A* hence contains the elements, father *A*, son Fred *A*, and son Peter *A*. The set of female members of family *A* contains only one element—mother *A*. In mathematics there can also be a set so small that it has only one element. In order to have greater generality, it is also convenient to have an empty set.

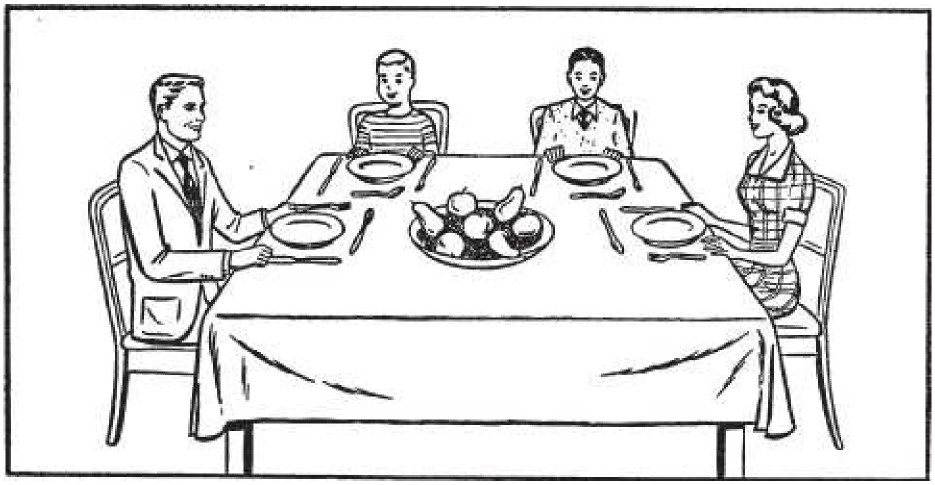


Figure 1.

An empty set contains no element.

The set of plums in the fruit bowl (Figure 1) is an example of an *empty set* or a *null-set*.

(b) If a senior class has 15 students, these 15 elements form the set of seniors. The defining property of this set is: each of its elements is a student of this senior class.

Suppose the classroom for these 15 students contains a set of 15 seats only. If the 15 senior students would occupy the 15 seats in every possible way, then, by the laws of permutations, * there would be 15! = 1,307,674,368,000 different arrangements. Thus the set of seating arrangements contains more than 1.3 trillion elements. (1.3 trillion seconds is more than 40,000 years!)

3. Combining physical objects into sets is much rarer in mathematics than is the construction of sets from abstract objects—objects of our thought. Examples of abstract objects are: numbers, points, triangles, and the like.

4. The following are examples of abstract sets:

(a) The set of all single-digit natural numbers contains the elements 1,2,3,4,5,6,7,8,9. These are nine definite and distinct objects all of which belong to a set M because they have the particular property of being single-digit natural numbers.

(b) Let the set N contain the numbers 9,8,7,6,5,4,3,2,1. To designate that certain things are elements of a set, we enclose them in braces. Thus we write:

$$M = \{1,2,3,4,5,6,7,8,9\};$$

$$N = \{9,8,7,6,5,4,3,2,1\}.$$

The statements "5 is an element of M" and "5 is an element of N" are respectively expressed in symbols by "$5 \in M$" and "$5 \in N$". The symbol "$\in$" is read "is an element of" or "belongs to"; correspondingly, the symbol "$\notin$" signifies "is not an element of."*

5. In the foregoing examples, the sets M and N contain the same elements. Except for the way in which they are arranged, there is no difference in the elements of each set. In this case we say that the sets M and N are equal. We write this $M = N$.

Two sets are equal if and only if they contain the same elements.

From this definition of equality of sets, we conclude that: the equality relation is reflexive, symmetric and transitive, that is; (a) $M = M$; (b) If $M = N$, then $N = M$; and (c) If $M = N$, and $N = P$, then $M = P$.

6. Cantor's definition of a set demands that the elements be definite and distinct. The same element cannot appear several times in the same set. Thus the letters *g,e,o,m,e,t,r,y,* form a set only after one of the two letters "*e*" is removed. Then

$$M = \{e,g,m,o,r,t,y\}.$$

7. Examples of sets from geometry:

(a) The set of lattice points† K, in the domain defined by the circular area $x^2 + y^2 \leqq 4$ contains 13 points (see Figure 2).

(b) The set of lattice points, Q, on and within the square with vertices (0,2), (2,0), (0,–2), and (–2,0) contains the same elements. Hence the two sets of points are equal, and we write: $K = Q$.

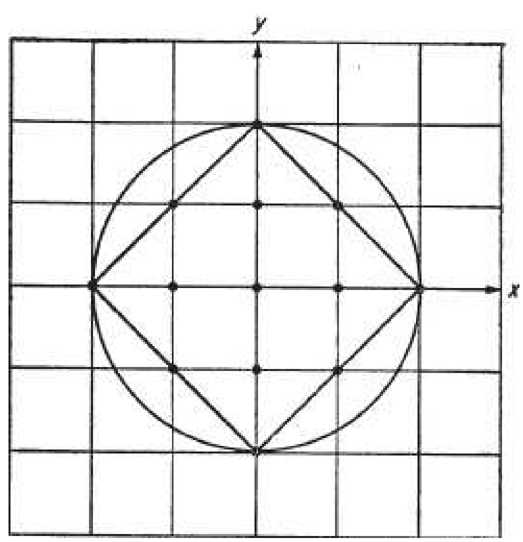


Figure 2. Sets of lattice points.

Exercises

1. Search your surroundings for "things" that may be combined as "elements of sets." (Hint: windows, seats, books, ….)
2. If in the preceding Example 7 we eliminate the lattice points from K and Q that lie on the circle and square, respectively, will the new sets be equal?
3. Is the set of lattice points lying in the region bounded by the circles $x^2 + y^2 = 12$ and $x^2 + y^2 = 14$, an empty set? (Draw a graph.)
4. Form the set of all proper fractions whose numerators and denominators are relatively prime* single-digit natural numbers.
5. Why does the set of improper fractions (values greater than one) whose numerators and denominators are relatively prime single-digit natural numbers contain exactly eight elements less than the set in Exercise 4?

II. Subset, Complementary Set, Union, Intersection

1. *A set N is called a subset of a set M*, (*and M a superset of N*) if every element of N is also an element of M. We write this: "$N \subseteq M$ if, given $a \in N$, then $a \in M$" The symbol "$\subseteq$" is read "is a subset of" or "is included in." If, furthermore, $N \neq M$, then N is called a *proper subset* of M. We write this: "$N \subset M$" and read it: "N is a proper subset of M." In this case, M contains at least one element that does not belong to N. In case $N = M$, N is called an *improper* subset of M. Note that every set is an improper subset of itself. We also agree that by definition:

The empty set is a subset of every set

2. *If N is a proper subset of M, then the set*, *R*, *of elements of M which do not belong to N, is called the complementary set to N over M*.

We write this "$R = M - N$."

3. *Examples*: A part of the class of seniors, which contained 15 elements [see Sec. I, 2b)], is used to form a soccer team. Then the set of players, F, is a subset of the set of seniors, P. All 11 elements of F are likewise elements of P. On the contrary, P contains elements that are not contained in F. Therefore, F is a proper subset of P. The complementary set R to F over P, ($R = P - F$), contains as elements the four nonplaying spectators from the senior class. Here R is also a subset of P. We can write these properties briefly thus:

$$F \subseteq P; \quad R = P - F; \quad R \subseteq P.$$

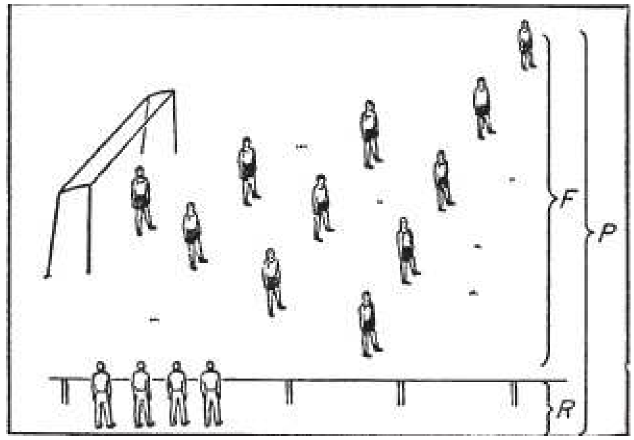


Figure 3. *P*, set; *F*, subset; *R*, complementary set.

The 11 students on the football team can occupy the 11 positions on the team in 11! different arrangements. Besides these 11! = 39,916,800, different arrangements, it is possible to select many other combinations of 11 players from the 15 seniors. If from the 15 elements belonging to set P, we construct all the possible different subsets of 11 elements, there will be

$$\binom{15}{11} = \frac{15!}{11!4!} = 1365^* \quad \text{of these subsets.}$$

Then the set of all possible team arrangements of 11 players from 15 seniors contains $11!\cdot\binom{15}{11}$ = 54,486,432,000 elements.

4. From the set $M = \{1,2,3\}$ we shall form all the possible subsets. First there is the null set, $M_0 = \{\ \}$. The subsets with only one element are: $M_{11} = \{l\}$ $M_{12} = \{2\}$, $M_{13} = \{3\}$. The subsets with two elements are: $M_{21} = \{1,2\}$; $M_{22} = \{1,3\}$; $M_{23} = \{2,3\}$. The improper subset is $M_3 = \{1,2,3\}$. Thus the set $M = \{1,2,3\}$ has exactly eight or 2^3 subsets.

Using the formula for the number of combinations of *n* things used 0,1,2,3,… ,*p*,…,*n* at a time,† it is easy to establish the following theorem.

For every set of n elements there are exactly 2^n subsets.

5. *The union of two sets is the set of all elements each of which belongs to at least one of the two sets.*

The *union* of two sets is symbolized by M ∪ N (read "the union of *M* and *N*") and the union set contains the elements of *M* and of *N*, except that elements contained in both *M* and *N* are used only once.

6. *The intersection of two sets is the set of elements each of which belongs simultaneously to both sets*.

The intersection of two sets is symbolized by $M \cap N$ (read "the intersection of *M* and *N*").

7. *Examples:*

(a) Consider the sets:

$$M = \{1,2,3,4,5,6,7,8,9\};$$
$$G = \{2,4,6,8\};$$
$$U = \{1,3,5,7,9\};$$
$$P = \{2,3,5,7\}.$$

Note that *M* contains all one-digit natural numbers; *G*, only the even, *U* only the odd, and *P* only the prime one-digit natural numbers. Now form subsets, complementary sets, unions, and intersections. The following are some of the possible relations:

(α) $G \subset M; \quad U \subset M; \quad P \subset M.$

(β) $G \cup M = M; \quad U \cup M = M; \quad P \cup M = M;$

$G \cup U = M; \quad (G \cup U) \cup P = M;$

$G \cup P = \{2,3,4,5,6,7,8\}; \quad P \cup U = \{1,2,3,5,7,9\}.$

(γ) $M - U = G; \quad M - G = U; \quad M - P = \{1,4,6,8,9\}.$

(δ) $M \cap G = G; \quad M \cap U = U; \quad M \cap P = P;$

$P \cap G = \{2\} \quad P \cap U = \{3,5,7\}; \quad G \cap U = \{\ \}.$

In the last intersection, note that *G* and *U* have no common elements—the elements of the one set are entirely different from the elements of the other set. We say these sets are *disjoint*. The intersection of two disjoint sets is the empty set.

(ε) The set *M* has $2^9 = 512$ subsets. They are: { }, {1}, {2}, … {9}, {1,2}, … {8,9}, {1,2,3}, … {1,2,3,4,5,6,7,8,9}.

(b) Computation with the sets *M* = {*m,o,r,g,e,n*}, *N* = {*n,a,c,h,t*}, and *P* =

$\{p,r,a,c,h,t\}$.

$$N \cup P = P \cup N = \{n,a,c,h,t,p,r\}; \quad N \cap P = P \cap N = \{a,c,h,t\};$$
$$(N \cap P) \cap M = M \cap (N \cap P) = \{\ \}.$$
$$M \cap N = N \cap M = \{n\};$$
$$M \cap (N \cup P) = (M \cap N) \cup (M \cap P) = \{n,r\};$$
$$(N \cap P) \subset N; \quad (N \cap P) \subset P; \quad N - (N \cap P) = \{n\}.$$

(The last equality says that the complementary set to the intersection $N \cap P$ over the set N is $\{n\}$.)

8. Some well-known laws of operation for ordinary arithmetic are also valid in the formation of unions and intersections of sets. This is illustrated in Example (b) above where we had:

(a) The commutative laws:

$$M \cup N = N \cup M; \quad M \cap N = N \cap M.$$

(b) The associative laws:

$$(M \cup N) \cup P = M \cup (N \cup P);$$
$$(M \cap N) \cap P = M \cap (N \cap P).$$

(c) The distributive law:

$$M \cap (N \cup P) = (M \cap N) \cup (M \cap P).$$

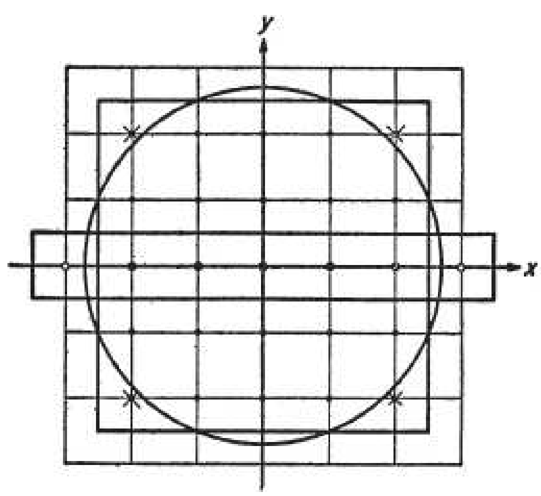


Figure 4. Sets of lattice points

9. *A geometrical example*: In Figure 4, there are three sets of lattice points—the set K of lattice points interior to the circle $x^2 + y^2 = 7$, which contains 21 points; the set Q of lattice points interior to the square with vertices (±2.5,±2.5), which contains 25 points; and the set R of lattice points interior to the rectangular area with vertices (±3.5,±0.5), which contains seven elements. Here, K is a proper subset of Q; we write $K \subset Q$. The complementary set M to K over Q, that is $M = Q - K$, contains four elements marked x in the figure. The intersection D_1 of R and K, that is $D_1 = R \cap K$, contains five elements designated by • . The intersection D_2 of R and Q ($R \cap Q = D_2$) contains the same five elements and $D_1 = D_2$. The complementary set N to D_1 over R or $N = R - D_1$ contains the two elements marked ∘ . The sets N and Q are disjoint and hence $N \cap Q = \{ \}$. The union $K \cup R$ contains 23 elements. Finally, the union $(K \cup Q) \cup R$ contains as elements all 27 marked lattice points.

Exercises

1. Construct all the subsets of the set $M = \{2,3,5,7\}$. How many are there?
2. Show that for two sets M and N, $N \subset M$, the following statements are equivalent: (a) $M \cap N = N$; (b) $M \cup N = M$.
3. From which of the following statements can we conclude that M and N are equal?

$$N \subseteq M \text{ and } M \subseteq N; \quad \text{(b) } M \cup N = M \text{ and } M \cap N = M$$

4. How many different basketball teams (each a subset of five elements) can be formed from a set of ten students? Disregard the actual position held by the players.
5. In a senior class, all students are preparing to take college entrance examinations. Is the set of students preparing for the examinations a proper or improper set of the senior class?

III. Equivalent Set, Cardinal Numbers

1. In our first consideration of sets we dealt with a set A containing the four members of the family called A [Sec. I]. We also recognized a set of four plates, a set of four chairs, a set of four knives, and a set of four apples. What is common to all these sets? Obviously, the property that each of the sets has the same number of elements, namely four. In general, what characteristic remains for a set when one disregards the physical nature of the elements of a set? It is the *number* of its elements; and it is in the number of elements that the previously mentioned sets agree.

2. In the second example (page 5), the set of senior students, P, contained 15 elements. The classroom contained a set of 15 seats, which we shall call *S*. The sets *P* and *S* have the same *power**; they are represented by the same cardinal number 15.

It is easy to establish the fact that the *different* sets *P* and *S* contain the same number of elements. For this purpose one needs only to count the set *P* of students: "1,2,3,…,15," and then the set *S* of seats: "1,2,3,…,15." But the fact can be established in a simpler way. A person who is unfamiliar with counting could merely request the young people to be seated. If every student finds a seat and no seats remain unoccupied, one recognizes by this very correspondence that the sets *P* and *S* have the same number of elements; they match each other; they are equivalent.

> *Two sets M and N are equivalent to each other if their elements can be related so that to every element of M there corresponds one and only one element of N, and conversely*.

Each element of *M* must correspond to a single element of *N,* and likewise each element of *N* must correspond to a single element of *M*. This kind of correspondence is called a one-to-one correspondence between the elements of *M* and *N*.

3. *An example*:

The set $B = \{a,b,c\}$ and the set $Z = \{1,2,3\}$ have the same number of elements (three); they are equivalent sets. The one-to-one correspondence (mapping of *B* upon *Z*) can be made in any one of the following ways, (3! or six of them).

(i)	(ii)	(iii)	(iv)	(v)	(vi)
$a \leftrightarrow 1$	$a \leftrightarrow 1$	$a \leftrightarrow 2$	$a \leftrightarrow 2$	$a \leftrightarrow 3$	$a \leftrightarrow 3$
$b \leftrightarrow 2$	$b \leftrightarrow 3$	$b \leftrightarrow 1$	$b \leftrightarrow 3$	$b \leftrightarrow 1$	$b \leftrightarrow 2$
$c \leftrightarrow 3$	$c \leftrightarrow 2$	$c \leftrightarrow 3$	$c \leftrightarrow 1$	$c \leftrightarrow 2$	$c \leftrightarrow 1$

In the first column, the correspondence assigns the element *a* of set *B* to the element 1 of set *Z*, and conversely the element 1 of *Z* to the element *a* of *B*, and so on.

4. *Two counter-examples*:

(a) The sets $A = \{a,b\}$ and $Z = \{1,2,3\}$ are not equivalent. Although every element of A can be ordered to some element of Z, for example $a \to 1$ and $b \to 3$, not every element of Z can be made to correspond uniquely to an element of A.

$$1 \to a$$

$$2 \to ?$$

$$3 \to b$$

(b) Figure 5 shows that frequently in a public conveyance the set of passengers is not equivalent to the set of seats.

Figure 5. Nonequivalent sets in a subway car.

5. The correspondence of two equivalent sets can be established by a function-equation, for example, $y = 2x + 3$, where x signifies an element of the set $X = \{1,2,3,\ldots,100\}$. The set Y thus becomes $\{5,7,9,\ldots,203\}$. A one-to-one correspondence of two equivalent sets is also called a *mapping*. The functional relation $y = 2x + 3$ maps the set X *onto* the set Y.

6. If the sets M and N are equivalent, we write in symbols: $M \sim N$. The property of equivalence is reflexive, symmetric, and transitive. That is:

(a) $M \sim M$; each set is equivalent to itself.
(b) If $M \sim N$, *then* $N \sim M$.

(c) If $M \sim N$ and $N \sim P$, then $M \sim P$.

7. The common property of all equivalent sets is their cardinal number, or their power, or their number of elements. We shall represent the cardinal number of a set by the corresponding small (lower case) letter of the alphabet or by placing the symbol representing the set between bars. Thus:

If $M = \{a,b,c,d,e\}$, then $|M| = |\{a,b,c,d,e\}| = m = 5$;
If $N = \{s,t,u,v,w\}$, then $|N| = |\{s,t,u,v,w\}| = n = 5$; $n = m$.

The cardinal number "5" denotes the power of all equivalent sets (that is, sets that contain five elements).

Now we are in a position to understand Cantor's definition of cardinal number.

> *The power or cardinal number of M is what we call the general concept, which by the aid of our active capacity for thought arises from the set M when we make abstractions from the nature of its various elements m and the order of their presentation.*

8. A set M has a greater cardinal number than a set N when N is equivalent to a subset of M, but M is not equivalent to a subset of N.

Examples.

Let $M = \{a,b,c,d\}$ and $N = \{1,2,3\}$. Then N is equivalent to $U = \{a,b,c\}$, which is a proper subset of M. Here $U \subset M$ and $N \sim U$. However, M is equivalent to no subset of N. Hence $|M| > |N|$, that is, $m > n$ or in this case $4 > 3$.

9. All the concepts and definitions of the theory of sets thus far discussed occur in the realm of finite sets—and only such sets have been recognized up to this point. The results appear to be self-evident and say nothing new. It is only in the application of these concepts and operations to infinite sets that it becomes evident how extensive and all-embracing they really are.

Exercises

1. Give several examples of equivalent sets, using instances in your immediate surroundings.
2. How many mappings of the set $M = \{a,b,c,d\}$ on itself are possible? Write out several of these mappings.
3. Given: (α) The set K of lattice points on and interior to the circle $x^2 + y^2 = 4$

(β) The set Q of lattice points on and interior to the square with vertices $(\pm 2, \pm 2)$

(a) Is either of the sets K and Q a proper subset of the other?

(b) Determine the intersection of K and Q.

(c) Determine the cardinal number $|Q - K|$.

(d) Excluding the points on the circle and on the square, are the sets of lattice points in their interiors equivalent sets?

4. Let $X = \{1,2,3,\ldots,10\}$. Which of the following rules can be used to produce sets equivalent to X?

(a) $y = 4x - 3$; (b) $y = 2^x$; (c) $y = x^3$; (d) $y = \sin \frac{\pi}{4}x$;

(e) $y = \ln x$

5. What property must a rule have in order to form equivalent sets?

6. Let K be the set of lattice points interior to the circle $x^2 + y^2 = 7$. Let P be the set of two-digit prime numbers. Which of the following three statements is correct?

(a) $|K| < |P|$; (b) $|K| = |P|$; (c) $|K| > |P|$

7. At a dance how can one determine whether the set of men, M, and the set of women, W, are equivalent sets?

8. In your classroom, is the set of blotters equivalent to the set of notebooks?

9. Give a rule (functional equation) that maps the set $X = \{1,2,3,\ldots,100\}$ onto the set $Y = \{2,4,6,\ldots,200\}$.

10. (a) Does the rule obtained as a solution to Exercise 9 map "all" natural numbers one-to-one with "all" the positive even numbers?

(b) Can we say that the set $N = \{1,2,3,\ldots,n,\ldots\}$ is equivalent to the set $G = \{2,4,6,\ldots,2n,\ldots\}$?

*The literal translation from the German is: A set is a bringing together into a whole sof definite well-distinguished objects of our perception or thought—which are to be called the elements of the set. (Georg Cantor)

*The number of permutations of n different things is $n!$. This is read "n-factorial" and is defined by the relation $n! = 1 \cdot 2 \cdot 3 \ldots \cdot n$.

*All the symbols are catalogued in a glossary on page 97.

†Lattice points are points with integral coordinates.

*Two natural numbers are relatively prime if they have no common divisor (other than 1). For example, 4 and 9 are relatively prime, but 4 and 6 are not.

*In the theory of permutations and combinations it is shown that the number of different combinations that can be formed from N different things, using P at a time, a $\binom{N}{P} = \frac{N!}{P!(N-P)!}$

† There are $\binom{n}{0}$ or one subset with 0 elelments; $\binom{n}{1}$ or n subsets with one element; $\binom{n}{2}$ or $\frac{n(n-1)}{1 \cdot 2}$ subsets with two elements; and so on. Then the total of subset is

$$1 + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n} = (1+1)^n = 2^n$$

*The word "power" is a translation for the German "Mächtigkeit." Perhaps a better translation would be "strength." In the rest of this work we shall use the phrase "cardinal number" as equivalent to "power" or "Mächtigkeit".

3

INFINITE SETS

VI. Equivalence and Transfinite Cardinal Numbers

1. In Exercise 9 above, the set of natural numbers $X = \{1,2,3,\ldots,100\}$ was placed in one-to-one correspondence with the set of even numbers $Y = \{2,4,6,\ldots,200\}$. If x is an element of set X, then the corresponding element y of set Y is given by the rule $y = 2x$. This rule (function-relation), however, furnishes much more than the correspondence of these two sets. It orders *every* natural number into a one-to-one correspondence with a positive even number. Thus the sets

$$N = \{1,2,3,\ldots,n,\ldots\} \quad \text{and} \quad G = \{2,4,6,\ldots,2n,\ldots\}$$

are equivalent sets. (See Exercise 10 above.) We say the sets N and G have the same "power"; they are characterized by the same cardinal number. If however, you attempt to establish the equality of the cardinal numbers by counting, your attempt would fail, for the sets N and G are infinite sets. The number of elements that each set contains is infinitely large.

The set of natural numbers thought of as ordered in their natural sequence, namely $N = \{1,2,3,\ldots\}$, forms an unbounded set (one without end or infinite) of definite and distinct elements, the number of which exceeds every finite cardinal number. Indeed, in this case we can no longer speak of the "number" of elements in the usual sense.

The elementary concept of number becomes meaningless for infinite sets, and gives us no help in answering the question, "Are there more natural numbers than there are positive even numbers?" However, the equivalence concept gives us the means to resolve this question. The sets N and G have the same cardinal number.

2. If we say that the set N is "infinite," this infinite is to be sharply differentiated from the "infinite" as we understand it in the sense of a limit process. For example, in mathematical analysis we say: $\lim_{x \to 0} \frac{1}{x} \to \infty$. As x becomes zero, $1/x$ becomes infinitely large, or more accurately, "$1/x$ can be made greater than any arbitrarily selected number, no matter how large, if x is chosen sufficiently small." This improper (potential) infinity of the limit process is something altogether different from the proper (actual) infinity of set N. The set $N = \{1,2,3,...\}$ displays a given existing infinity (not a becoming infinite).

3. The equivalence of sets N and G displays a surprising property of the infinite set G: G is a *proper subset* of N, and yet it is equivalent to N. In this case we have $G \subset N$ and $G \sim N$. For finite sets, as we have already shown, the statement "$A \subset B$" is inconsistent with the statement "$A \sim B$." For example, if $A = \{1,2,3\}$ and $B = \{1,2,3,4,5\}$, then $A \subset B$ and $|A| < |B|$ (since $3 < 5$). The cardinal number $|A| = 3$, and the cardinal number of the complementary set $|B - A| =2$ are each less than the cardinal number $|B| = 5$.

Dedekind* used precisely this property to define finite and infinite sets as follows:

> *If there exists no proper subset of M that is equivalent to M, then M is called a finite† set. If there is a proper subset of M that is equivalent to M, then M is called an infinite set.*

Note that Euclid's‡ axiom: "the whole is greater than any of its parts" is not valid for infinite sets.

4. *Examples*:

(a) The set of points on a straight line is an infinite set; the set of points on a ray or half-line is also an infinite set. These two sets have the same cardinal number. The equivalence of the set of points on segment AB and the set on ray s is established as shown in Figure 6 by using central projection from the point Z. The point P on AB corresponds to the point P' on s, and conversely.

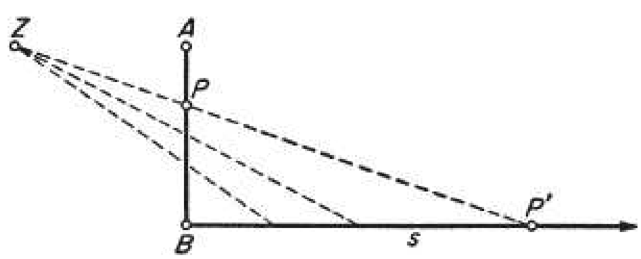


Figure 6. Point sets with the same cardinal number.

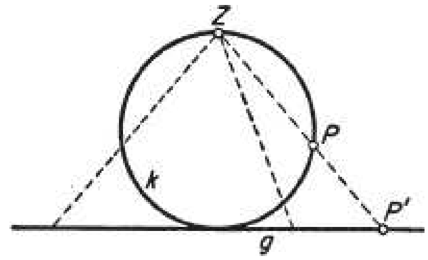


Figure 7. Point sets with the same cardinal number.

(b) Figure 7 shows that the set of points (other than z) on a circle k is equivalent to the set of points on a straight line g. A line through Z intersects the circle a second time at P and the line g in the corresponding point P'.

(c) The set of points on a line segment with length 1.5 inches is equivalent to the set of points on a line segment with length 3 inches. The correspondence is shown in Figure 8.

(d) The set of natural numbers $N = \{1,2,3,\ldots\}$ and the set of positive odd numbers $U = \{1,3,5,\ldots\}$ are equivalent sets. In this case $U \subset N$ and also $U \sim N$. The correspondence can be given by the rule $u = 2n - 1$.

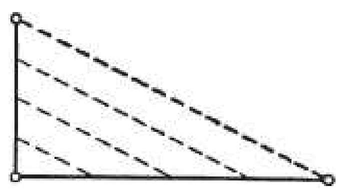


Figure 8. Equivalent sets of points.

5. Equivalent finite sets are represented by the same cardinal number. These finite cardinal numbers are the natural numbers arrived at by counting the number of elements in the set. In the case of infinite sets, equivalent sets (that is, sets of the same power) are likewise represented by the same *transfinite* cardinal number. These transfinite cardinal numbers represent an extension of the natural numbers. We shall study these transfinite cardinal numbers in the next few sections.

Exercises

1. Show the equivalence of the sets of points on two sides of a triangle.

2. Give a formula that maps $N = \{1,2,3,\ldots\}$ one-to-one onto the set $Z = \{1,10,100,\ldots\}$,

3. Using central projection, map the points of a semicircle onto the points of: (a) a straight line; (b) a ray; (c) a line segment.

V. Denumerable Sets

1. The simplest infinite set is the set of natural numbers, namely $N = \{1,2,3,\ldots\}$. We assign to this set the transfinite number* a, and write $|N| = a$.

All the sets having the cardinal number a are called *denumerable* sets. Denumerable sets are those sets which can be put into one-to-one correspondence with the set of natural numbers. In this respect, denumerable sets can have their elements ordered into a sequence, that is, they have a first element, a second, a third, and so on. Besides the set N, we have already recognized the denumerable sets

$$U = \{1,3,5,\ldots\} \quad \text{and} \quad G = \{2,4,6,\ldots\}.$$

2. The set of prime numbers form a denumerable set.

$$P = \{2,3,5,7,11,13,17,\ldots\}.$$

This set is infinite because no matter how large a prime number is given, there is always a greater prime number. We recall here Euclid's proof of this fact: let the first n prime numbers be given by $2,3,5,7,\ldots,P_n$. Then the number $z = (2 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot P_n) + 1$ is either a prime number that is greater than P_n, or z has a prime number factor that is greater than P_n.†

The prime numbers can be ordered into a sequence, for example, according to their numerical size. We can thus set up a correspondence in which $P \subset N$ and at the same time $P \sim N$:

$$N = \{1,2,3,4, 5, 6, 7, 8, 9,10,\ldots ,100,., 200,\ldots, 300,\ldots\};$$

$$P = \{2,3,5,7,11,13,17,19,23,29,\ldots ,541,\ldots ,1223,\ldots ,1987,\ldots\}.$$

The set of prime numbers is denumerable. $|P| = a$.

3. *The set of squares of the positive integers*, $Q = \{1,4,9,16,25,\ldots\}$, *is denumerable*. We have $Q \subset N$ and also $Q \sim N$. The correspondence of the elements of Q and N, is given by the rule $n = \sqrt{q}$ or $q = n^2$.

4. *The lattice points in a plane form a denumerable set*. Using the ordering scheme shown in Figure 9, the set of lattice points can be ordered into a sequence (that is, placed in one-to-one correspondence with the natural numbers). If the ordering starts at the origin (0,0), then the lattice point (0,1) is the second element, (1,1) the third, …, (3,2) is the thirty-second element, and so on.

5. *The set of positive and negative integers and zero is denumerable*. This set can be ordered into the following sequence:

$$Z = \{0,1,-1,2,-2,3,-3,4,-4,\ldots\}.$$

6. *The set of all rational numbers is denumerable*. By the set of all rational numbers we shall understand all fractions with relatively prime terms, that is, the numerator and denominator have no common factor other than 1. To form an ordered sequence of the rational numbers, we shall first order the positive rationals according to their increasing *height*, which is defined to be the *sum of the numerator and denominator*. Fractions having the same height will be ordered according to their increasing value.

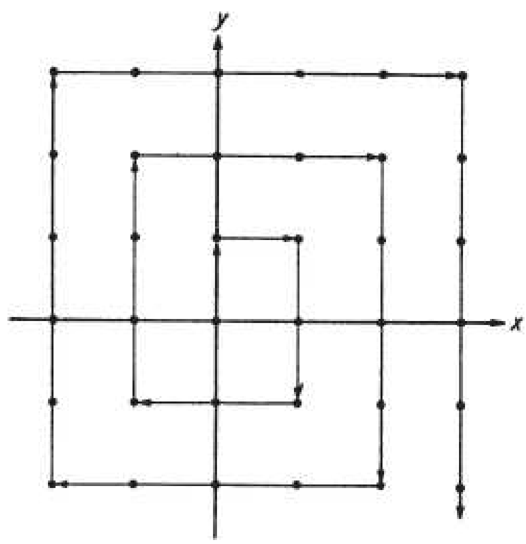


Figure 9. The set of lattice points in a plane is denumerable.

We place the number 0 as the first term of the sequence of rationals, and then exclude 0 as a term of all the remaining fractions. Then we have:

of height 2, the fraction $\frac{1}{1} = 1$;

of height 3, the fractions $\frac{1}{2}, \frac{2}{1}$ or 2;

of height 4, the fractions $\frac{1}{3}, \frac{2}{2}, \frac{3}{1}$ or 3 (the fraction $\frac{2}{2}$ is not in lowest terms and is eliminated);

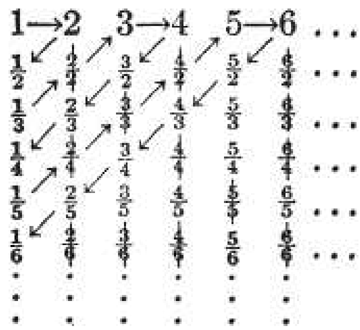
of height 5, the fractions $\frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}$ or 4, etc.

To include the negative rational numbers, we write the corresponding negative number after each positive rational number. The set of all rational numbers can thus be written in the following ordered sequence:

$$R = \left\{0,1,-1,\frac{1}{2},\frac{-1}{2},2,-2,\frac{1}{3},\frac{-1}{3},3,-3,\dots\right\}.$$

In this sequence every rational number has a definite position. The set R is a proper *superset* of all the various sets of numbers heretofore considered.

7. The denumerability of the set of rational numbers can also be established by the following method known as the *diagonal process*. In the accompanying diagram all fractions not in lowest terms



$(\frac{2}{2}, \frac{4}{2}, \frac{3}{3}, \frac{2}{4}, \dots)$ are cancelled. The diagram is then traversed in the diagonally arrowed path. In this manner, every positive rational number will be included. If the corresponding negative rational is placed after each positive rational, and the sequence begun with the rational 0, then all rationals will be included. We thus obtain

$$\bar{R} = \left\{0,1,-1,2,-2,\frac{1}{2},\frac{-1}{2},\frac{1}{3},\frac{-1}{3},3,-3,4,-4,\dots\right\}.$$

It is obvious that $\bar{R} = R$, and that the ordering of the two sets differs only within fractions having the same *height*. In the diagram, the fractions in any diagonal are all of the same height, only their Values alternately increase and decrease in successive diagonals.

8. One of the greatest and most beautiful achievements of Cantor tvas the proof that a set of numbers which is a proper superset of all heretofore considered sets of numbers (that is, in a naive sense contains still “more” elements) is also of the power a.

The set of all algebraic numbers is denumerable.

An algebraic number is any number, real or imaginary*, which is the root of an algebraic equation;

$$a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_2x^2 + a_1x + a_0 = 0.$$

In this equation it is understood that n is a natural number, and $a_n, a_{n-1},\dots,a_0$ are positive or negative integers or zero, and in particular $a_n \neq 0$.

The rational numbers are evidently only a "small" part of the algebraic numbers since each rational number is the root of the algebraic equation of the first degree:

$$mx - p = 0, \quad m \text{ and } p \text{ integers}; \qquad m \neq 0.$$

From this equation we obtain $x = \frac{p}{m}$. For $n = 2$, both irrational and imaginary numbers occur as roots, and are thus algebraic numbers. For instance, $x = \sqrt{2}$ is a solution to $x^2 - 2 = 0$, and $x = 1 + i$ is a solution to the equation $x^2 - 2x + 2 = 0$.

9. To develop the proof of the denumerability of the set of algebraic numbers, we shall order these numbers according to the *heights* of the algebraic equations. The height of an algebraic equation

$$a_nx^n + \dots + a_0 = 0$$

is defined to be that natural number

$$h = n + |a_n| + |a_{n-1}| + |a_{n-2}| + \dots + |a_2| + |a_1| + |a_0|.$$

Clearly, for a given height, there exists a definite finite number of algebraic equations; and to each algebraic equation of degree n there are at most n different algebraic numbers which are its roots.

The algebraic numbers are thus ordered into a sequence (denumerable set) according to the increasing heights of the algebraic equations for which they are the solutions. For different equations of the same height we shall order the numbers according to the increasing degrees of the equations. For the several solutions to the same equation we shall order the numbers according to their increasing value. Imaginary roots we shall order according to increasing values of their real part; for equal real parts of different complex numbers, according to the increasing values of the pure imaginary part.

Using these directions, we obtain the following ordering of the algebraic numbers:

(a) The height $h = 1$ is not possible, for then $n = 0$, $a_0 = \pm 1$, and we have the false statement ±1 = 0.

(b) For $h = 2$, $n = 1$ and $a_1 = \pm 1$, $a_0 = 0$. Thus we obtain the equations $\pm x = 0$, and the first algebraic number is 0.

(c) For $h = 3$, zhere are two possibilities:

(α) $n = 1$. If $a_1 = \pm 2$ and $a_0 = 0$ we obtain the equations $\pm 2x = 0$, and again we have the algebraic number 0. If $a_1 = \pm 1$ and $a_0 = \pm 1$, we obtain the equations $\pm x \pm 1 = 0$, and hence the next two algebraic numbers are –1 and +1.

(β) $n = 2$. If $a_2 = \pm 1$, then $a_1 = a_0 = 0$, and we obtain the equations $\pm x^2 = 0$, which merely furnish the algebraic number 0 once more.

(d) For $h = 4$ there are three possibilities:

(α) $n = 1$.

$\alpha_1 = \pm 1$; $a_0 = \pm 2$; $\pm x \pm 2 = 0$; solutions: –2 and +2.

$\alpha_1 = \pm 2$; $a_0 = \pm 1$; $\pm 2 \pm 1 = 0$; solutions: $\frac{-1}{2}$ and $\frac{+1}{2}$.

$\alpha_1 = \pm 3$; $a_0 = 0$; $\pm 3x = 0$; solution: 0.

(β) $n = 2$.

$a_2 = \pm 1$; $a_1 = 0$; $a_0 = \pm 1$; $\pm x^2 \pm 1 = 0$;

solutions: –1, 1, $-i$, i.

$a_2 = \pm 1$; $a_1 = \pm 1$; $a_1 = 0$; $\pm x^2 \pm x = 0$; solutions: –1, 0, +1.

$a_2 = \pm 2$; $a_1 = a_0 = 0$; $\pm 2x^2 = 0$; solution: 0. (γ) $n = 3$.

Then $a_3 = \pm 1$, $a_2 = a_1 = a_0 = 0$; $\pm x^3 = 0$; solution: 0.

The procedure can be extended indefinitely in this manner. The sequence of algebraic numbers thus starts with

$$A = \left\{0, -1, +1, -2, +2, \frac{-1}{2}, \frac{+1}{2}, -i, +i, \ldots\right\}.$$

In this sequence every algebraic number has a definite place. Thus the set of all algebraic numbers is denumerable.

10. The investigation of infinite sets according to their power has, in the case of all the sets considered, namely *G*, *U*, *Q*, *P*, *N*, *Z*, *R* and *A*, always led us to the same cardinal number *a*. Yet all of the sets discussed in this section have been quite different. Set *A* contained all the other sets as proper subsets. The sets *G*, *U*, *P* and *Q*, were each proper subsets of *N*. Further, *N* was a proper subset of *Z*, and *Z* a proper subset of *R*. Yet we demonstrated that all these sets were equivalent to the set of natural numbers, *N*. They all have the same cardinal number *a*.

From our development thus far, is it not plausible to expect that all infinite sets are equivalent, i.e., they are all denumerable? If this were the case, it would be uninteresting and the infinite would be without content; but in fact so far we have not learned of any non-denumerable set of numbers.

To be sure we have already considered equivalent sets of points, (e.g. the set of points on a line segment, a ray, a straight line, and a circle) whose cardinal numbers—as we shall next investigate—are greater than *a*.

Exercises

1. How many mappings (one-to-one correspondences) are there for a denumerable set onto itself?
2. For any two rational numbers, no matter how close together, does there always exist another rational number with a value lying between them? Give an example.
3. Determine the algebraic numbers that are solutions of all the algebraic equations of height five.
4. Is sin 7°30' an algebraic number?
5. Show that the following expressions are valid:
   (a) $U \subset N$; $G \subset N$; $N \subset Z$; $Z \subset R$.
   (b) The sets U, G, N, Z and R have the same cardinal number, i.e., $U \sim G \sim N \sim Z \sim R$; all these sets are denumberable; all have the same transfinite cardinal number:

$$|U| = |G| = |N| = |Z| = |R| = \mathfrak{a}.$$

VI. Non-denumerable Sets

1. We now consider all the real numbers r in the interval $0 < r < 1$. All these numbers can be written in a unique fashion as infinite decimal fractions. For example:

$$\tfrac{1}{2} = 0.499999\ldots; \qquad \tfrac{2}{5} = 0.399999\ldots; \qquad \tfrac{1}{4} = 0.249999\ldots;$$
$$\tfrac{5}{12} = 0.416666\ldots; \quad \sqrt{\tfrac{1}{50}} = 0.141421\ldots; \quad \pi/6 = 0.523598\ldots;$$
$$\sin 4^0 = 0.069756\ldots; \qquad \log 1.09 = 0.037426\ldots.$$

If all these real numbers were denumerable, they could be written in some ordered sequence (perhaps according to value). Then there would be a first, a second, a third, and so on, as follows:

$$
\begin{array}{ll}
(1) & 0.\,Z_{11}\,Z_{12}\,Z_{13}\,Z_{14}\,\ldots. \\
(2) & 0.\,Z_{21}\,Z_{22}\,Z_{23}\,Z_{24}\,\ldots. \\
(3) & 0.\,Z_{31}\,Z_{32}\,Z_{33}\,Z_{34}\,\ldots. \\
(4) & 0.\,Z_{41}\,Z_{42}\,Z_{43}\,Z_{44}\,\ldots. \\
\cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdots. \\
\cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdots. \\
\cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdots.
\end{array}
$$

In these infinite decimals the Z_{ik} are digits from the set

$$\{0,1,2,3,4,5,6,7,8,9\}.$$

We now form a real number $r_1 = 0.a_1a_2a_3a_4 \ldots$, where $a_i \neq Z_{ii}$. That is, $a_1 \neq Z_{11}$; $a_2 \neq Z_{22}$; $a_3 \neq Z_{33}$; etc. In particular, let the digit a_i be any element other than Z_{ii} from the set: $\{0,1,2,\ldots,9\}$. Thus the formation of the real number r_j is possible in an infinite number of ways. However, no r_1 constructed in this manner is contained in the above written denumerable set of decimal fractions, because: r_1 differs from the first number at least in the first decimal place ($a_1 \neq Z_{11}$); it differs from the second number at least in the second decimal place ($a_2 \neq Z_{22}$); and so on. Thus there exists no denumerable set of real numbers in the interval $0 < r < 1$ that exhausts the set of *all* real numbers in this interval. We must conclude:

The set of all real numbers in the interval $0 < r < 1$ *is non-denumerable.*

2. The set of all real numbers in the interval $0 < r < 1$ can be placed in one-to-one correspondence with the points on a straight line segment *AB* of length one. Of course the end points of the segment are excluded as shown in Figure 10.

3. In a similar manner, the set of all real numbers in the domain $\infty < r < +\infty$ can be ordered in one-to-one correspondence with the points of a straight line. This is shown in Figure 11.

4. Figure 12 illustrates the equivalence of the set of all real numbers $-\infty < r < +\infty$ and the set of real numbers in the interval $0 < r < 1$. The segment *AB*, of length 1, is folded at its mid-point *M* and formed into a right angle at *M*. Then the points of the folded line segment *AB* are placed in one-to-one correspondence with the points of line *g* by central projection from point *Z*. Thus it follows that:

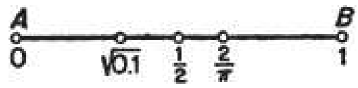


Figure 10. Mapping the number $0 < r < 1$ to the points of segment AB.

Figure 11. The number scale showing the equivalence of the real numbers to the points of a straight line.

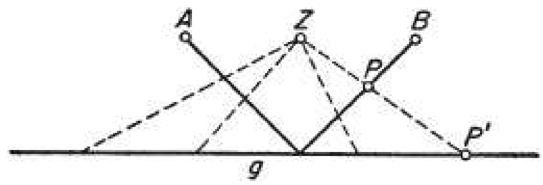


Figure 12. The set of points on the broken line AB and the straight line g are equivalent.

The set of all real numbers is non-denumerable.

In fact, the set of all real numbers has the same cardinal number as the set of real numbers in the interval $0 < r < 1$. The power, or cardinal number, of the set of all real numbers is called the *cardinal number of the continuum*, i.e., the cardinal number of the set of points of a continuous line segment or of a continuous unbounded line. We symbolize the power of this set by the transfinite cardinal number "c."*

5. The power of the set of all real numbers, or of the real numbers in the interval $0 < r < 1$, is the same as that of the set of points of the segment AB or of all points of the straight line g. These sets are equivalent to each other.

6. Thus we have discovered infinite sets that are of higher power, or have a greater cardinal number than any of the denumerable sets considered heretofore. It is this fact that gives meaning to the introduction of transfinite cardinal numbers. There is more than one such number. Not all infinite sets are equivalent or of the same power. There are "levels" in the infinite as well as in the finite. Later we shall see that besides the infinite cardinal numbers a and c there are still others, indeed an infinite number of them. First, however, we shall examine other sets with the cardinal number *c*.

7. Sets that are equivalent to a set with cardinal number *c* are;

(a) the set of all points interior to a square;
(b) the set of all points of a plane;
(c) the set of all points interior to a cube, and even;

(d) the set of all points in unbounded three-dimensional space.

The method of proof of these statements, which is quite simple, will be carried out here for the case of a cube having an edge one unit in length. In Figure 13, the cube is placed with three of its edges coinciding with rectangular space coordinate axes. In a Cartesian space-coordinate system, a point P interior to the cube has the real coordinates, (written as decimal fractions):

$$x = 0.\, a_1\, a_2\, a_3\, \ldots;$$

$$y = 0.\, b_1\, b_2\, b_3\, \ldots;$$

$$z = 0.\, c_1\, c_2\, c_3\, \ldots.$$

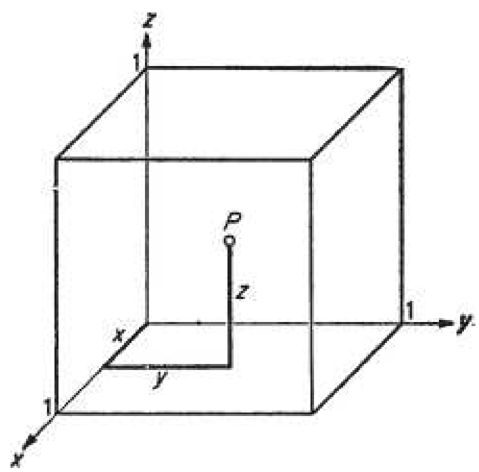


Figure 13. Point set of a cube.

Let us characterize this point P by the decimal d, formed from x, y and z, by writing

$$d = 0.\, a_1\, b_1\, c_1\, a_2\, b_2\, c_2\, a_3\, b_3\, c_3\, \ldots.$$

In this way, every point P interior to the cube is paired in a unique way with a definite decimal d, where $0 < d < 1$. We have already shown that the cardinal number of all decimals d for which $0 < d < 1$ is c. Hence, the cardinal number of all points within a cube with edge of unit length is also c.

Similarly, it can be shown that for every linear, planar, or spatial region the cardinal number of its set of points is c.

8. Previously (see Figure 9) we saw that the set of lattice points of a plane could be mapped onto the set of lattice points of a straight line (the set of integers Z). Both of these sets have the cardinal number a. Now we have obtained an even more remarkable result, namely: the sets of points of a straight line, of a plane and of space are equivalent—that is they can be placed in one-to-one correspondence. The *concept of dimension* is therefore of no significance in characterizing cardinal number. Linear, plane, and space point sets contain point sets of the same cardinal number c.

Nevertheless, there exists in the naive concept of dimension, something of significance for the theory of sets. By rather difficult investigation (done by Peano, Hilbert, and Brouwer) it has been proved that:

> *Between two continuums of different order*, *it is impossible to establish a one-to-one correspondence that maintains continuity*; *that is, a correspondence so that neighboring points of one continuum can be ordered to the neighboring points of the other continuum*.

9. The one-to-one correspondence of the point sets $0 < x < 1$ and $0 < y < \infty$, x and y real numbers, can also be established by the use of suitable function rules as in the following examples:

$$y = \frac{x}{1-x} \text{ (Fig. 14)}; \quad y = \tan\left(\frac{\pi}{2}x\right) \text{ (Fig. 15)}; \quad y = \frac{x^2}{1-x^2} \text{ (Fig. 16)}.$$

10. All the infinite sets thus far studied have had either the cardinal number a or the cardinal number c. We summarize them here.
(a) *Denumerable sets:*
The set of all natural numbers:

$$N = \{1,2,3,4,5,\ldots\}.$$

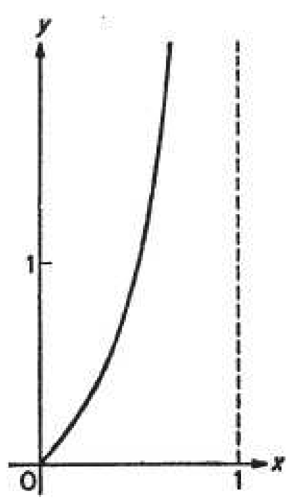


Figure 14.

$$y = \frac{x}{1 - x}$$

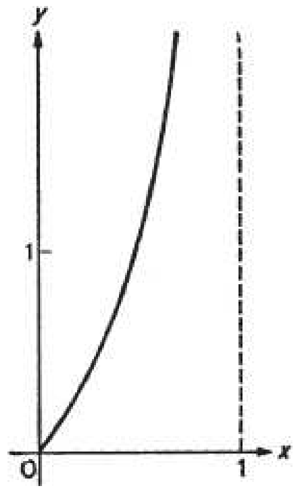


Figure 15.

$$y = \tan \frac{\pi}{2} x$$

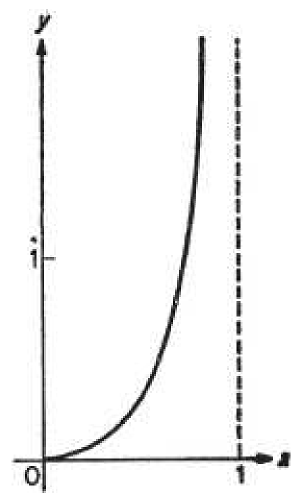


Figure 16.

$$y = \frac{x^2}{1 - x^2}$$

The set of all positive even integers:

$$G = \{2,4,6,8,10,\ldots\}.$$

The set of all positive odd numbers:

$$U = \{1,3,5,7,9,11,\ldots\}.$$

The set of all prime numbers:

$$P = \{2,3,5,7,11,\dots\}.$$

The set of squares of all integers:

$$Q = \{1,4,9,16,25,\dots\}.$$

The set of all integers:

$$Z = \{0,1,-1,2,-2,\dots\}.$$

The set of all rational numbers:

$$R = \left\{0,1,-1,\frac{1}{2},\frac{-1}{2},\dots\right\}.$$

The set of all algebraic numbers:

$$A = \left\{0,-1,1,-2,2,\frac{-1}{2},\frac{1}{2},\dots\right\}.$$

The set of all lattice points on a straight line: G_g.
The set of all lattice points on a plane: G_e.
For all of these sets we have the inclusion relations:

$$\begin{array}{ll} G \subset N & \\ U \subset N & N \subset Z;\quad Z \subset R;\quad R \subset A. \\ P \subset N & \\ Q \subset N & \end{array}$$

All of these sets are equivalent:

$$N \sim G \sim U \sim P \sim Q \sim Z \sim R \sim A \sim G_g \sim G_e$$

All of these sets have the same cardinal number; they are all denumerable:

$$|N| = |G| = |U| = |P| = |Q| = |Z| = |R| = |A| = |G_g| = |G_e| = \mathfrak{a}$$

(b) *Sets with the cardinal number of the continuum*:
The set of all real numbers in the interval $0 < r < 1$: R_{e_1}.
The set of all real numbers: R_e.

The set of all points on an open segment *AB*: P_{AB}.
The set of all points on a ray (Figure 10): P_s.
The set of all points on a straight line *g*: P_g.
The set of all points interior to a cube of edge 1: P_w.

For these sets we have the relations:

$$R_{e_1} \subset R_e; \quad R_{e_1} \sim R_e \sim P_{AB} \sim P_s \sim P_g \sim P_w;$$

$$|R_{e_1}| = |R_e| = |P_{AB}| = |P_s| = |P_g| = |P_w| = \mathfrak{c}.$$

11. Are there sets with the transfinite cardinal number m, where m is greater than the cardinal number *a* of the denumerable sets and m is also less than the cardinal number *c* of the continuum— that is, for which $a < m < c$? This question is to this day an unanswered one and is called the "problem of the continuum."

12. For our use in later study, we now examine a few important theorems.

(a) If a finite number of elements is added to or subtracted from a denumerable set, the new set is denumerable.

Proof: the union of a finite set $E = \{e_1,e_2,e_3\ldots,e_n\}$ and a denumerable set $B = \{b_1,b_2,b_3\ldots,\}$ can be written as the sequence

$$S = E \cup B = \{e_1,e_2,e_3,\ldots,e_n,b_1,b_2,b_3,\ldots\},$$

and *S* is therefore denumerable.

If a finite number of elements is subtracted from a denumerable set *S*, we obtain the denumerable complementary set exhibited by cancelling the elements of *S* that are subtracted.

Example:

Let $N = \{1,2,3,4,\ldots\}$ and let $P_1 = \{2,3,5,7\}$ (the set of one-digit prime numbers). Then,

$$\bar{P}_1 = N - P_1 = \{1,\not{2},\not{3},4,\not{5},6,\not{7},8,9,\ldots\} = \{1,4,6,8,9,\ldots\}.$$

(b) The union of two denumerable sets, or the union of a denumerable infinity of denumerable sets, is denumerable.

(c) If a denumerable set of elements is subtracted (removed or cancelled) from an infinite set, then if the resulting complementary set is still infinite, it has the same cardinal number as the original set.

Proof: let M be an infinite set. Let A be a denumerable subset of M. Let the complementary set to A over M be R, that is $M - A = R$. Assume that R is an infinite set. Then $M = A \cup R$. In the complementary set R let $\bar{A}$ be a denumerable subset and let the complementary set of $\bar{A}$ over R be $\bar{R}$, that is $R = \bar{A} \cup \bar{R}$. It is possible that $\bar{R}$ could be the empty set. We have now partitioned the set M into the subsets A, $\bar{A}$, and $\bar{R}$, no two of which have any elements in common; i.e., A, $\bar{A}$, and $\bar{R}$ are disjoint. Hence

$$M = A \cup \bar{A} \cup \bar{R}$$

Since A and $\bar{A}$ are denumerable, by theorem (b) above we have $A \cup \bar{A}$ and also $\bar{R} \sim \bar{R}$. Hence:

$$M = (A \cup \bar{A}) \cup \bar{R} \sim \bar{A} \cup \bar{R} = R \quad \text{or} \quad M \sim R.$$

In outline form, the proof appears as follows:

$$M = \begin{Bmatrix} A \\ R \end{Bmatrix} = \left.\begin{matrix} \begin{Bmatrix} A \\ \bar{A} \end{Bmatrix} \sim \bar{A} \\ \bar{R} \quad \sim \bar{R} \end{matrix}\right\} = R.$$

13. A *transcendental* number is a real or imaginary number, that is, not an algebraic number. We can now prove the important theorem:

> *The set of all real transcendental numbers is non-denumerable and has the cardinal number c.*

Proof: the set of all real numbers has the cardinal number c. By definition, the set of real algebraic numbers is the complement to the set of real transcendental numbers. If from the real numbers the denumerable set of real algebraic numbers is subtracted, then, by theorem (c) above, the complementary set (the set of all real transcendental numbers) has the same cardinal number c as the real numbers.

In the first place, this theorem demonstrates the existence of transcendental numbers. It further shows that there is an infinite number of transcendental numbers. Indeed, this set has a greater cardinal number than the set of algebraic numbers. As a rule, a real number is a transcendental number; the real algebraic numbers represent only the exceptional cases (a denumerable subset).

Nevertheless, to prove the_ transcendence of real numbers in special cases, (for example $2^{\sqrt{2}}$, e, π, sin 1, ln 2), that is, to prove that the number in question cannot be the solution to an algebraic equation is very difficult, and at present in some cases still impossible.

Exercises

1. Investigate the cardinal number of the set of points in a rectangle if the sides are segments of length two units and one unit.
2. Determine the cardinal number of the set of all (complex) numbers.
3. What is the cardinal number of the set of irrational algebraic numbers?
4. Does every infinite set have a denumerable subset?
5. Determine the cardinal number of the set of all powers m^n where m and n are natural numbers. (Use the diagonal process.)
6. Prove theorem (b) of the previous Section 12. Hint: for a denumerable number of denumerable sets use the diagonal process.

VII. Further Non-denumerable Sets

1. Up to this point we have learned of two levels of infinity; of two different kinds of cardinality of infinite sets, and of two transfinite cardinal numbers, a and c. We now seek others.

Consider the set, F, of all real functions, defined by $y = f(x)$ in the interval, $0 < x < 1$. By function, we shall mean the following kind of relationship: the independent variable assumes all real values in the interval $0 < x < 1$, but to each value of x there corresponds exactly one definite value of the dependent variable y. Under this condition, equal values of y can correspond to different values of x, but on the other hand to any value of x there is only one value of y. As x assumes every value in the interval $0 < x < 1$, $y = f(x)$ takes on another set of determinable values. (In the examples of Figures 17, 18, and 19, for instance, y took on the values $0 < y < \infty$). The definition of a particular function can be given by an equation, e.g., $y = x/(1 - x)$, or by a curve, or by some rule. However, special limitations (for example that the function must be continuous) shall not be made. In particular, two functions are considered to be different if they have different values at as much as a single value of x in the interval $0 < x < 1$. In the set under consideration, that of every possible conceivable function in the interval $0 < x < 1$, every element of the set is a function.

The set of all functions is surely an infinite set; for the simplest functions are given by $y = f(x) = c$, and the constant c can take on the non-denumerable set of real values, $0 < c < 1$.

We must now show that the cardinal number $\mathfrak{f}$ of the set of all real functions in the interval $0 < x < 1$ is greater than c. Let us assume that the set of all real functions of the interval $0 < x < 1$ has the cardinal number of the continuum.

This set of functions would be equivalent to the set of real numbers $0 < r < 1$, and these sets could be placed in one-to-one correspondence. Call the function thus ordered to the point r; $f_r(x)$. We shall now construct a function $\phi(x)$, for which it will be shown that

$$\phi(r) \neq f_r(r)$$

for every value $x = r$ in the interval $0 < x < 1$. Such a function can be constructed in infinitely many ways; for example, by the condition that $\phi(r) = f_r(r) + 1$. This function $\phi(x)$ coincides with none of the functions $f_r(x)$, for we know that at least for the value $x = r$, $\phi(r) \neq f_r(r)$. Hence $\phi\{x)$ is not contained in the set of functions equivalent to the continuum, which we assumed contained all functions. The assumption that the set, F, of functions is equivalent to the continuum is false. We must therefore conclude:

The set F of all real functions in the interval $0 < x < 1$ *has a greater cardinal number than the continuum,* $\mathfrak{f} > c > a$.

2. The foregoing proof can also be looked upon as follows: no ordinary, constructed subset F_c of F, where F_c has the cardinal number c can exhaust the set F, because for any given subset F_c, elements of F can be produced which are not contained in F_c.

Example:

Consider the subset F_e of F that is put in correspondence to the real numbers $0 < r < 1$ by the relation $f_r(x) = \frac{x}{r}$. Thus: to the point $r = \frac{1}{2}$, there corresponds the function

$$f(x) = x \div \tfrac{1}{2} = 2x;$$

to the point $r = \frac{1}{4}$, there corresponds the function

$$f(x) = x \div \tfrac{1}{4} = 4x; \quad \text{etc.}$$

We now impose the condition on $\phi(x)$, that $\phi(r) \neq f_r(r)$ and specify in particular that $\phi(r) = f_r(r) + 1$. Since $f_r(r) = \frac{r}{r} = 1$, $\phi(r)$ always has the value 2. It is evident that the line $\phi(x) = 2$ is not contained in the set of lines $f_r(x) = \frac{x}{r}$. Thus $\phi(x)$ is an element of F that does not belong to

F_c.

3. Without proof, we state here that the cardinal number of the set of all continuous functions has "only" the cardinal number *c*. The continuous functions are the "exceptional cases" within the set of all real functions. It is therefore not surprising to find that the set of all differentiable functions also has the cardinal number of the continuum, but on the other hand, the set of all integrable functions has the cardinal number f of the set of all* functions, *F*. The differentiable functions are again the "exceptional cases" of the integrable functions. Here "integrability" is not to be confused with "representation of the integral function by an algebraic or elementary transcendental function." In this sense (sin *x*)/*x* is an integrable function.

4. Are there cardinal numbers beyond *a*, *c*, and *f*? The answer is that there are infinitely many transfinite cardinal numbers. We shall establish this fact by proving the following theorem.

> *For any arbitrary infinite set, there exists a set having a greater cardinal number than that of the selected set*.

This, of course, means that the sequence of transfinite cardinal numbers is not bounded above—there is no "greatest" cardinal number. By way of illustration, the set *U*(*M*) of all subsets of a. given set *M* has a greater cardinal number than the set *M* itself.

For finite sets, whose cardinal numbers are the natural numbers, this fact is easily established, since the finite set of *n* elements has 2^n subsets, and $2^n > n$.

5. To gain a better understanding of the general proof of this theorem for infinite sets, let us clarify its beautiful, but rather difficult line of thought by applying it first to a finite set as in the following example.

Example:

Consider a finite set of three elements, *M* = {1,2,3}. Then the set of subsets *U*(*M*) contains $2^n = 2^3$ = eight elements, namely:

$$U(M) = \{\{\ \}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

From *U*(*M*) we can remove a subset U_m that is equivalent to *M;* for example, the subset U_m = {{2}, {2,3}, {1,2,3}}. The equivalence of *M* and U_m may be established through a definite correspondence, perhaps like the following:

M		U_m
1	$\leftrightarrow$	{2,3}
2	$\leftrightarrow$	{1,2,3}
3	$\leftrightarrow$	{2}

In any such correspondence of the elements of M and U_m, two possibilities arise, namely:

An element of M appears in the corresponding subset element of U_m to which it is coordinated, (Class I), or The element of M does not appear in the corresponding subset element of U_m to which it is coordinated (Class II).

In our example, the elements of M are distributed as follows into the two classes:

Class I	*Class II*
(Elements of , which are like wise in the subsets to which they correspond.)	(Elements of ilf, which do not appear in the subsets to which they correspond.)
2 (contained in {1,2,3}).	1 (not contained in {2,3}). 3 (not contained in {2}).

Now form the set L which contains the elements of Class II: $L = \{1,3\}$. Then L is a subset of M, and hence $L \in U(M)$. However, L is not a member of U_m, i.e., $L \notin U_m$, which we shall now prove.

If L were contained in U_m, then the element m_1 paired with it in M, must be either in Class I or Class II. If m_1 fell in Class I, then m_1 would have to appear in L. But L contains only those elements of M which do not appear in the corresponding subset, namely the elements of Class II.

If m_1 fell in Class II, then according to the definition of Class II, it could not be in L.

Now L contains all the elements of Class II and therefore it must contain m_1. Hence L cannot be contained in U_m. Thus:

No subset U_m of $U(M)$ which is equivalent to M can possibly exhaust $U(M)$. Hence $U(M)$ has a greater cardinal number than M.

6. This typical method of reasoning in carrying out set-theoretical proofs can easily be generalized to the case where M is an infinite set.

Let M be an infinite set and $U(M)$ the set of all subsets of M. Then $|M| \leqq$

$|U(M)|$, that is, $U(M)$ has either the same cardinal number as M—since the set of all subsets with only one element is clearly equivalent to the set M—or $U(M)$ has a greater cardinal number than M. We shall prove $|U(M)| > |M|$.

Let U_m be a subset of $U(M)$ which is equivalent to M. Hence $U_m \subseteq U(M)$ and $U_m \sim M$. Let the elements of M correspond in a definite way to the elements of U_m. We now separate the elements of M into two classes as follows: Class I contains those elements of M which are also in the subset elements of U_m to which they correspond; Class II contains those elements of M which do not appear in the subset elements of U_m to which they correspond. The set of all elements of Class II is an element of $U(M)$, but not an element of U_m. (This is proved exactly as above for finite sets.) Thus, no subset of $U(M)$, no matter how it is formed to have the cardinal number $|M|$, can be equivalent to $U(M)$. We conclude that:

> *The set of all subsets of a set M has a greater cardinal number than the set M itself.*

For every transfinite cardinal number, a greater one can be determined.

7. The vague and naive concept of "infinitely large" within which there is no distinction as to size must therefore be revised. There is an infinite number of well-determined and distinct transfinite cardinal numbers that sharply determine the multiplicity of the infinite*, just as the finite cardinal numbers do for finite quantities.

Exercises

1. What is the cardinal number of the set of all lines $y = c_1x + c_2$ where c_1 and c_2 are real numbers?
2. To what set of numbers is the set of all circles $(x - a)^2 + (y - b)^2 = r^2$ equivalent? Here a, b, and r are real numbers.
3. Order according to size the various cardinal numbers, finite and infinite, thus far considered.

VIII. The Equivalence Theorem

1. In order to establish the equivalence of two finite sets M and N we try to map the elements of one set onto the other so that the elements are in one-to-one correspondence. If we succeed in making such a correspondence, we say the sets are equivalent, of the same power, and have the same cardinal number. We then write

$$M \sim N; \qquad |M| = |N|; \qquad \mathfrak{m} = \mathfrak{n}.$$

If, however, m comparing two infinite sets, it happens that M is equivalent to a subset N_1 of N, but N is not equivalent to any subset M_1 of M, we say that N is of greater power than M, that N has a greater cardinal number. We then write

$$|M| < |N|; \qquad \mathfrak{m} < \mathfrak{n}.$$

It was in this manner that we found $a < c < \mathfrak{f}$, and $|M| < |U(M)|$.

Sometimes, instead of comparing the sets M and N directly, it is simpler to attempt to set up a correspondence of the elements of one set with the elements of a subset of the other set. Then the following four cases can arise:

1.	M is equivalent to a subset of N, and	$M \sim N_1$
	N is equivalent to a subset M_1 of M	$N \sim M_1$
2.	M is equivalent to a subset N_1 of N, and	$M \sim N_1$
	N is equivalent to *no* subset M_1 of M	$N \nsim M_1$
3.	M is equivalent to *no* subset N_1 of N, and	$M \nsim N_1$
	N is equivalent to a subset M_1 of M	$N \nsim M_1$
4.	M is equivalent to *no* subset N_1 of N, and	$M \nsim N_1$
	N is equivalent to *no* subset M_1 of M	$N \nsim M_1$

Cases 2 and 3, as we have already seen, define the “greater than” and “smaller than” relationship among cardinal numbers:

$$\text{For 2,} \quad |M| < |N| \quad \text{or} \quad \mathfrak{m} < \mathfrak{n};$$

$$\text{For 3,} \quad |M| > |N| \quad \text{or} \quad \mathfrak{m} > \mathfrak{n}.$$

Case 4 evidently introduces a paradox, that neither of the two sets contains a subset equivalent to the other set. The cardinal numbers m and n would then be incomparable. That this case cannot arise, was asserted by G. Cantor, and for the present we must be satisfied with this allusion. The proof which was given later (after Cantor) with the help of the “well-ordering theorem” exceeds the level of difficulty of this book.

Of greatest interest to us now is Case 1 where $M \sim N_1$ and $N \sim M_1$; for then the so-called *equivalence theorem* states that the sets M and N are equivalent.

2. The equivalence theorem:

If both M is equivalent to a subset N_1 of N and N is equivalent to a subset M_1 of M, then the sets M and N are equivalent to each other.

They have the same transfinite cardinal number.

$$\begin{array}{lll} \text{Given:} & M \sim N_1 \quad \text{and} & N \sim M_1, \\ & M_1 \subset M \quad \text{and} & N_1 \subset N. \\ \text{To Prove:} & M \sim N. & \end{array}$$

Proof: It will be sufficient to prove that $M_1 \sim M$. Since the equivalence relation is reflexive and transitive, $M_1 \sim M$ and $N \sim M_1$ implies that $M \sim N$.

First, partition N so that N_r is the complementary set to N_1. Thus

$$N = N_1 \cup N_r, \tag{I}$$

where N_1 and N_r are disjoint and N_1 is the subset of N equivalent to M.

$$N_1 \sim M. \tag{II}$$

Partition M_1 so that it is the union of two disjoint subsets M_2 and M_r,

$$M_1 = M_2 \cup M_r, \tag{III}$$

and also M_2 is equivalent to N_1. Thus

$$M_2 \sim N_1. \tag{IV}$$

From II and IV we have $M_2 \sim M$.

Then

$$M_2 \subset M_1 \subset M.$$

The equivalence theorem is then proved if we can show that $M_2 \sim M$ and $M_2 \subset M_1 \subset M$ implies that $M_1 \sim M$.

We must therefore prove the very evident theorem: if a set M is equivalent to one of its subsets M_2, then it is also equivalent to *every* one of its subsets M_1 that lie between M and M_2. (Here, the statement "M_1 lies between M_2 and M" means: $M_2 \subset M_1 \subset M$.) The proof will be limited to proper subsets, for if $M = M_1 = M_2$, the theorem is obvious.

3. We now show that given the relations (a) $M \sim M_2$ and (b) $M_2 \subset M_1 \subset M$, it follows that $M_1 \sim M$. For the purpose of simplification we let $M_2 = A$, $M_1 - M_2 = B$, and $M - M_1 = C$. Thus we have

$$\begin{aligned} M &= A \cup B \cup C \\ M_1 &= A \cup B \qquad\qquad A, B, \text{ and } C \text{ are disjoint.} \\ M_2 &= A \end{aligned}$$

Then our theorem becomes

Given:

(a) $A \cup B \cup C \sim A$, and (b) $A \subset A \cup B \subset A \cup B \cup C$.

To prove:

$$\text{(c) } A \cup B \sim A \cup B \cup C.$$

By (a) there exists a mapping of *As* $\cup$ $B \cup C$ onto A, which is illustrated in Figure 17. Here A_1, B_1 and C_1 are subsets of A that are coordinated respectively to A, B, and C. The same rule also maps $A_1 \cup B_1 \cup C_1$ on A_1 where A_2, B_2, and C_2 are subsets of A_1 which are coordinated respectively with A_1, B_1 and C_1. This process can be repeated indefinitely. Thus we have:

$$\begin{aligned} A &= A_1 \cup B_1 \cup C_1; \quad A \sim A_1 \sim A_2 \sim A_3 \sim \ldots \\ A_1 &= A_2 \cup B_2 \cup C_2; \quad B \sim B_1 \sim B_2 \sim B_3 \sim \ldots \\ A_2 &= A_3 \cup B_3 \cup C_3; \quad C \sim C_1 \sim C_2 \sim C_3 \sim \ldots. \end{aligned}$$

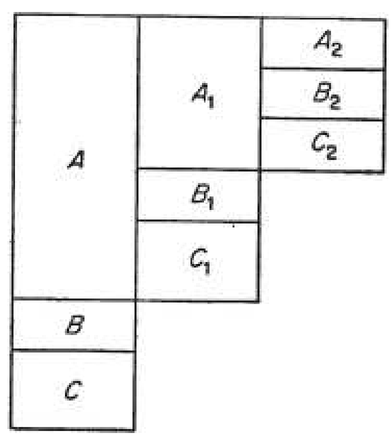


Figure 17. A mapping for the proof of the equivalence theorem.

In every case A_i, B_i and C_i are disjoint sets. The intersection of all the sets A_i is

$$D_A = A_1 \cap A_2 \cap A_3 \cap \ldots.$$

(Since the sequence A_1, A_2, A_3 … does not end, D_A can be an empty set.) Now, as the figure shows, we have

$$A \cup B \cup C = D_A \cup B \cup C \cup B_1 \cup C_1 \cup B_2 \cup C_2 \cup \ldots,$$

and

$$A \cup B = D_A \cup B \cup C_1 \cup B_1 \cup C_2 \cup B_2 \cup C_3 \cup \ldots,$$

On the right-hand side we have in each case the union of disjoint sets, and moreover, the two sets in each *column* on the right-hand side are equivalent. Hence the left-hand sides are also equivalent or $A \cup B \cup C \sim A \cup B$, that is, $M_1 \sim M$. (This proves the equivalence theorem, which in this form was first given by Bernstein.* The equivalence theorem will prove useful time after time.

4. Of the four different cases given in Section 1, Case 4, where $M_1 \subset M$, $N_1 \subset N$ and also $M \nsim N_1$ and $N \nsim M_1$ was eliminated as impossible.

Case 3:

$M_1 \subset M$, $N_1 \subset N$ and also $M \nsim N_1$ but $N \sim M_1$ implies $m > n$.

Case 2:

$M_1 \subset Ms$, $N_1 \subset N$ and also $M \sim N_1$ but $N \nsim M_1$ implies $m < n$.

Case 1:

$$M_1 \subset M, N_1 \subset N \text{ and also } M \sim N_1 \text{ and } N \sim M_1$$

is now decided by the equivalence theorem: $m = n$.

Of the three conditions $m > n$, $m = n$, and $m < n$, one and only one can be true in a given situation. Further, from Cases 1 and 2, it follows that if M is equivalent to a subset of N, that is $M \sim N_1$, then $m \leqq n$.

Exercises

1. Are the inconsistent relations of infinite sets: $M_1 \subset M$, $N_1 \subset N$, $M_1 \nsim N$, and $N \nsim M$ consistent for finite sets? (Under what conditions?)
2. Show, by the use of the equivalence theorem, that the sets $U = \{1,3,5,\ldots\}$ and $G = \{2,4,6,\ldots\}$ are equivalent.
3. Show, by the use of Figure 18, and with the help of the equivalence theorem, that the set of points on CD is equivalent to the set of points on AB.

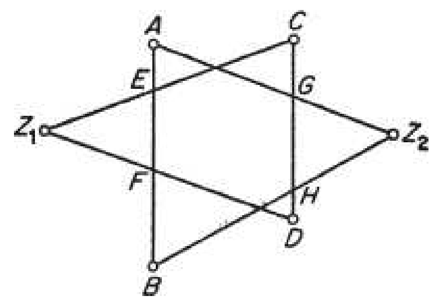


Figure 18. Application of the equivalence theorem to point sets.

IX. Sums and Products of Cardinal Numbers

1. Although definite transfinite cardinal numbers have now been created and their "greater-lesser" relationship developed in some detail, there still remains the task of discovering rules for computing with these numbers. For finite cardinal numbers, computations involving the rational operations have been commonplace since early childhood (even though we may have been able to count only to three). Further consideration gives us insight into the laws that are fundamental to elementary arithmetic. They are:

(a) the commutative laws:

$$a + b = b + a; \qquad a \cdot b = b \cdot a;$$

(b) the associative laws:

$$a + (b + c) = (a + b) + c; \qquad a(bc) = (ab)c;$$

(b) the distributive laws:

$$a(b + c) = ab + ac; \qquad (a + b)c = a \cdot c + b \cdot c.$$

Now we shall look for rules of operation for the transfinite cardinal numbers, at least for the "three first" cardinal numbers *a*, *c*, and *f*. Actually to call these cardinals the "first three" corresponds to a naive childhood conception about the infinite, for we do not know for certain whether or not these cardinals are really the "first three transfinite cardinals." (This is the problem of the continuum.)

In the following, by cardinal number we shall always understand a transfinite cardinal number. A set *M* representing the cardinal number m can be any particular set having the cardinal number *m* and in view of what we have learned about cardinal numbers, it can be replaced by any other set having this same cardinal number.

2. The addition of two cardinal numbers is defined in terms of the union of two sets, which we have already defined. The *union* of two sets M and N having the cardinal numbers m and n is the set $S = M \cup N$, that is, the set of all elements that are contained in at least one of the sets M or N. We now think of M and N as disjoint (or what is the same thing, we replace M and N by arbitrary equivalent sets $\bar{M}$ and $\bar{N}$ that are disjoint) and define

$$\mathfrak{m} + \mathfrak{n} = |S| = |\bar{M} \cup \bar{N}|.$$

From this definition of addition of cardinal numbers, it is evident that the following laws hold:

$$\mathfrak{m} + \mathfrak{n} = \mathfrak{n} + \mathfrak{m}; \qquad (\mathfrak{m} + \mathfrak{n}) + \mathfrak{p} = \mathfrak{m} + (\mathfrak{n} + \mathfrak{p}).$$

Using the theorems which we have previously developed, we can obtain the following relations:

$$\mathfrak{a} + n = \mathfrak{a};$$

$$\mathfrak{a} + \mathfrak{a} = \mathfrak{a};$$

$$\mathfrak{a} + \mathfrak{a} + \mathfrak{a} = \mathfrak{a};$$

Example:

$$|\{1,3,5,\ldots\}| + |\{2,4,6,\ldots\}| = |\{1,2,3,4,5,6,\ldots\}|$$

$$\mathfrak{c} + n = \mathfrak{c};$$

$$\mathfrak{c} + \mathfrak{a} = \mathfrak{c};$$

$$\mathfrak{c} + \mathfrak{c} = \mathfrak{c};$$

$$\mathfrak{c} + n + \mathfrak{a} + \mathfrak{c} = \mathfrak{c} + \mathfrak{a} + \mathfrak{c} = \mathfrak{c} + \mathfrak{c} = \mathfrak{c}.$$

Example:

$$|\text{Point set: } 0 < x < 1| + |\text{Point set: } 1 \leqq x < 2| = |\text{Point set: } 0 < x < 2|.$$

Thus the rules for addition of transfinite cardinal numbers are quite different from those for finite cardinal numbers. For example, $a + a = a$, but for finite numbers $a + a \neq a$ if $a \neq 0$. Without proof, we shall agree that for any transfinite cardinal number m, the following rule will be valid

$$\mathfrak{m} + \mathfrak{a} + n = \mathfrak{m} \quad \text{and} \quad \mathfrak{m} + \mathfrak{m} = \mathfrak{m}.$$

3. It is not possible to define subtraction for transfinite cardinal numbers because—insofar as it is at all possible—it does not lead to a unique result. We show this in the following example. We assume that $\mathfrak{m} - \mathfrak{n} = |\bar{N}| = |M - N|$. Here M and N cannot be assumed as disjoint since $N \subseteq M$.

Let:	*Then:*	
$N = \{1,2,3,\ldots\}$	$N - N = \{\ \}$	or $\mathfrak{a} - \mathfrak{a} = 0$
$N_1 = \{2,3,4,\ldots\}$	$N - N_1 = \{1\}$	$\mathfrak{a} - \mathfrak{a} = 1$
$N_2 = \{3,4,5,\ldots\}$	$N - N_2 = \{1,2\}$	$\mathfrak{a} - \mathfrak{a} = 2$
$N_n = \{n+1,n+2,n+3,\ldots\}$	$N - N_n = \{1,2,3,\ldots n\}$	$\mathfrak{a} - \mathfrak{a} = n$
$G = \{2,4,6,\ldots\}$	$N - G = U$	$\mathfrak{a} - \mathfrak{a} = \mathfrak{a}$
$U = \{1,3,5,\cdots\}$		

4. Multiplication of two cardinal numbers is defined as follows: consider two cardinal numbers m and n as represented by the disjoint sets M and N. Now form the set P of ordered pairs of numbers (m,n) by taking each element of the set M and associating with it in turn each element of the set N. We call $P = M \times N$ the *product set* (or Cartesian product) of sets M and N.

The product of the cardinal numbers m and n is the cardinal number of the product set: $m \cdot n = |P| = |M \times N|$.

This definition is merely an extension of the common definition of multiplication for finite cardinal numbers, where 3·4 means the sum of three equal addends 4; or 3·4 = 4 + 4 + 4 = 12. This is exactly the meaning of our general definition $m \cdot n = |P|$. We illustrate our point by an example.

Example:

Let $M = \{a,b,c\}$ and $N = \{1,2,3,4\}$-Hence $|M| = 3$; $|N| = 4$.

$$P = M \times N = \{(a,1), (a,2), (a,3), (a,4), (b,1), (b,2), (b,3), (b,4), (c,1), (c,2), (c,3), (c,4)\}.$$

Hence

$$|P| = 12.$$

5. In the product of more than two factors, the product set is constructed in a similar manner. $P = A \times B \times C$ is the set of all possible ordered triads (a,b,c) where a is an element of A, b is an element of B, and c an element of C.

Example:

Let $A = \{1,2,3\}$, $B = \{g,h\}$, $C = \{\alpha,\beta\}$.

$$P = \{(1,g,\alpha), (1,g,\beta), (1,h,\alpha), (1,h,\beta), \\ (2,g,\alpha), (2,g,\beta), \ldots, (3,h,\beta)\}.$$

Then

$$|A| \cdot |B| \cdot |C| = |P| = 3 \cdot 2 \cdot 2 = 12.$$

For products, it is evident that the commutative, associative and distributive laws are valid:

$$\mathfrak{m} \cdot \mathfrak{n} = \mathfrak{n} \cdot \mathfrak{m} \quad (\text{because } M \times N = N \times M);$$

$$\mathfrak{m} \cdot (\mathfrak{n} \cdot \mathfrak{p}) = (\mathfrak{m} \cdot \mathfrak{n}) \cdot \mathfrak{p};$$

$$(\mathfrak{m} + \mathfrak{n})\mathfrak{p} = \mathfrak{m} \cdot \mathfrak{p} + \mathfrak{n} \cdot \mathfrak{p}.$$

In addition to these laws we also require that if one of the sets M or N is an empty set, then $M \times N = \{\ \}$ and $m \cdot n = 0$.

6. For our "first" transfinite cardinal numbers we have in particular the following valid relations:

$$n \cdot \mathfrak{a} = \mathfrak{a} \quad (\text{because } \mathfrak{a} + \mathfrak{a} + \mathfrak{a} + \ldots + \mathfrak{a} = \mathfrak{a}),$$

and

$$\mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a}.$$

To prove this relation, consider the set of lattice points on a straight line and the plane or the denumerability of ordered pairs of integers, by using the diagonal process.

$$
\begin{matrix}
(1,1) & (1,2) & (1,3) & \dots \\
(2,1) & (2,2) & (2,3) & \dots \\
(3,1) & (3,2) & (3,3) & \dots \\
\vdots & \vdots & \vdots &
\end{matrix}
$$

$$n \cdot \mathfrak{c} = \mathfrak{c} \quad (\text{because } \mathfrak{c} + \mathfrak{c} + \mathfrak{c} + \dots + \mathfrak{c} = \mathfrak{c}),$$

and

$$\mathfrak{c} \cdot \mathfrak{c} = \mathfrak{c}.$$

The point set: $0 < x < 1$ has the cardinal number c. The point set: $0 < y < 1$ has the cardinal number c. Then the product set with elements (x,y) also has the cardinal number c. To show this, look upon these elements as Cartesian coordinates. The product set can then be placed in one-to-one correspondence with the set of points interior to a square with unit edge. But this set has the cardinal number c.

$$\mathfrak{a} \cdot \mathfrak{c} = \mathfrak{c}.$$

Since $n < a < c$, we have $n \cdot c \leqq a \cdot c \leqq c \cdot c$. Therefore $c \leqq a \cdot c \leqq c$, that is $a \cdot c = c$.

The cardinal number of the product set P of two sets, which contains the elements is nothing more than the addition of m addends each n. For, to each selected m_1 of M there is a subset P_1 of P with the elements (m_1,n). This subset P_1 is equivalent to the set N, hence

$$\mathfrak{m} \cdot \mathfrak{n} = \mathfrak{n} + \mathfrak{n} + \mathfrak{n} + \dots \text{ (m times)}.$$

In the case of infinite sets we can also look upon $m \cdot n$ as an infinitely often repeated addition of equal addends.

7. Division of transfinite cardinal numbers, like subtraction, leads to ambiguity as is shown by the following attempts.

From $n \cdot c = c$, it would follow that

$$\mathfrak{c} \div \mathfrak{c} = n.$$

From $a \cdot c = c$, it would follow that

$$\mathfrak{c} \div \mathfrak{c} = \mathfrak{a}.$$

From $c \cdot c = c$, it would follow that

$$\mathfrak{c} \div \mathfrak{c} = \mathfrak{c}.$$

The extension of computational operations beyond the domain of finite numbers is possible only for the direct operations (addition and multiplication), not their inverses.

Exercises

1. Is it always true that if $m < n$ and $n \leqq p$), then $m < p$ and $m \cdot q \leqq n \cdot q \leqq p \cdot q$?
2. Compute the value of $a! = 1 \cdot 2 \cdot 3 \cdot 4$
3. For disjoint sets M and N, when does the relation $|M| \cdot |N| = |M \times N|$ hold?

X. Powers of Cardinal Numbers

1. As a final operation we define the *power* of cardinal numbers. The power m^n for finite cardinal numbers is defined by repeated multiplication, that is, there are n equal factors m. In exactly the same way we define powers with a transfinite cardinal number as base:

$$n \cdot \mathfrak{a}^3 \cdot \mathfrak{c}^2 = (n) \cdot (\mathfrak{a} \cdot \mathfrak{a} \cdot \mathfrak{a}) \cdot (\mathfrak{c} \cdot \mathfrak{c}) = (\mathfrak{a} \cdot \mathfrak{a})\mathfrak{c} = \mathfrak{a} \cdot \mathfrak{c} = \mathfrak{c}.$$

But what happens when the exponent is likewise a transfinite number? Here m^n will mean: the transfinite number m shall be combined by multiplication through tt factors. Corresponding to the definition of multiplication of cardinal numbers, we have to form the product $P_N(M) = M_1 \times M_2 \times M_3 \times \ldots$, where all the sets M_i have the same cardinal number m and the set $N = \{M_1, M_2, M_{3s} \ldots\}$ has the cardinal number n.

The elements of this product set consist of the n-tuple: $(m_1, m_2, m_3 \ldots)$ in which in turn equals all the elements of M_1 m_2 all the elements of M_2, etc. Then

$$\mathfrak{m}^{\mathfrak{n}} = |P_N(M)|.$$

Examples:

We shall illustrate the construction of the set $P_N(M)$ by two examples with finite sets.

(a) $2^3 = 8$. Let $N = \{1,2,3\}$; $M_1 = \{a,b\}$; $|M_i| = 2$;

$$M_2 = \{\alpha,\beta\}; \quad |N| = 3.$$

$$M_3 = \{A,B\};$$

$$P_N(M) = \{(a,\alpha,A), (a,\alpha,B), (a,\beta,A), (a,\beta,B), (b,\alpha,A), (b,\alpha,B), (b,\beta,A), (b,\beta,B)\};$$

$$|P_N(M)| = 8.$$

(b) $3^2 = 9$. Let $N = \{1,2\}$; $M_1 = \{a,b,c\}$; $|M_i| = 3$;

$$M_2 = \{A,B,C\}; \quad |N| = 2.$$

$$P_N(M) = \{(a,A), (a,B), (a,C), (b,A), (b,B), (b,C), (c,A), (c,B), (c,C)\};$$

$$|P_N(M)| = 9.$$

Thus the definition of power of finite cardinal numbers is contained in the foregoing general definition.

2. In defining power, Cantor introduced a very insightful and useful concept called a *covering set* (Belegungsmenge). Under this concept the power m^n is defined as follows: in the set N representing the cardinal number n, each element is "covered" by an arbitrary set M representing the cardinal m. In the foregoing example (a), the element 1 of N is covered by the set $M_1 = \{a,b\}$: the element **2** of N is covered by the set $M_2 = \{\alpha,\beta\}$; and the element 3 is covered by $M_3 = \{A,B\}$. The set of all possible sets arising from this coverage of the elements of N by sets M_i is called the covering set, symbolized by "N/M" The covering set has the cardinal number m^n, that is

$$m^n = |N/M|.$$

Example:

A covering set N/M of the set $N = \{1,2,3,4,5,6,7,8,9,10\}$ with the set $M = \{1,11,0\}$, represents the set of ten teams in a football lottery, listed as opponents to ten other teams in which every element of N (a definite game) is covered by the elements of M (win, lose, tie).

Evidently

$$|N/M| = 3^{10} = 59{,}049. *$$

3. The concept of *covering set* is closely related to the concept of function. As the independent variable x takes on every value in the set X (x is thus replaced by any element of X), the dependent variable y assumes all the values y of the set Y. Thus $y = f(x)$ is a function on X to Y. The function represents a definite covering of X by Y.

4. The elementary laws of exponents for the direct operations (addition and multiplication) are also valid for powers of transfinite cardinal numbers:

$$\mathfrak{m}^{\mathfrak{p}} \cdot \mathfrak{m}^{\mathfrak{n}} = \mathfrak{m}^{\mathfrak{p}+\mathfrak{n}}; \quad (\mathfrak{m}^{\mathfrak{n}})^{\mathfrak{p}} = \mathfrak{m}^{\mathfrak{n}\cdot\mathfrak{p}}; \quad \mathfrak{m}^{\mathfrak{n}} \cdot \mathfrak{p}^{\mathfrak{n}} = (\mathfrak{m} \cdot \mathfrak{p})^{\mathfrak{n}}.$$

Finally we define $m^{\circ} = 1$.

5. We defined the power set of a set N to be the set of all subsets of N, and designated this set by $U(N)$. We have already proved that $|U(N)| > |N|$. The power set of a set N can also be conceived as the covering set of N by the set $\{+,-\}$. *A subset of N is obtained if we select certain elements of N* (*those covered by* +) *and do not select the others* (*those covered by* –). Every subset of N can thus be considered as some definite coverage of the set N by the set $\{+,-\}$. The empty subset is that in which all the elements of N are covered by –. The improper subset is that in which all the elements are covered by +.

The set $\{+,-\}$ has the cardinal number 2. Thus the cardinal number of the power set $U(N)$ is $|U(N)| = 2^n$. We already have seen that $2^n > n$. From this it will follow that for every transfinite cardinal number there is a greater one. From $m = 2_n$ we have $m > n$. From $p = 2^{\mathrm{m}}$ we have $p > m > n$, etc. The question as to whether 2^n is the next greatest cardinal number after n or whether there is a cardinal number $\mathfrak{k}$ for which $n < \mathfrak{k} < 2^{\mathrm{n}}$ is still unanswered. For $n = a$ this question corresponds to the problem of the continuum, since as we shall soon see, $2^a = c$.

6. To determine the power set of a denumerable set N with the cardinal number $|N| = a$, we shall first consider the power 10^a (corresponding to our number system and its decimal notation). To represent a we choose the set $N = \{a_1,a_2,a_3,\ldots\}$ and interpret the elements a_n as digits in the decimal number $0\cdot a_1,a_2,a_3,\ldots$. Every element a_n of N is then covered by a set M_n whose cardinal number is $|M_n| = 10$. Select all these sets $M_n = \{1,2,3,4,5,6,7,8,9,0\}$. Then the covering set $P_N(M)$ is equivalent to the set of all decimal numbers between 0 and 1, for which we already know the cardinal number is c. Hence $10^a = c$.

For another system of notation consisting of n digits, a covering can be made in an analogous manner, so that for every cardinal number $n > 1$, the power n^a has the value c. In particular, $2^a = c$.

The cardinal number of the set of all subsets of a denumerable set is $\mathfrak{c}$.

7. Using all the rules of operation we now know, we can determine the following:

$$\mathfrak{c}^{n} = \mathfrak{c}, \quad \text{for} \quad \mathfrak{c}^{n} = (10^{\mathfrak{a}})^{n} = 10^{\mathfrak{a} \cdot n} = 10^{\mathfrak{a}} = \mathfrak{c}.$$

$$\mathfrak{c}^{\mathfrak{a}} = \mathfrak{c}, \quad \text{for} \quad \mathfrak{c}^{\mathfrak{a}} = (10^{\mathfrak{a}})^{\mathfrak{a}} = 10^{\mathfrak{a} \cdot \mathfrak{a}} = 10^{\mathfrak{a}} = \mathfrak{c}.$$

$$\mathfrak{a}^{\mathfrak{a}} = \mathfrak{c}, \quad \text{for} \quad 10^{\mathfrak{a}} \leq \mathfrak{a}^{\mathfrak{a}} \leq \mathfrak{c}^{\mathfrak{a}}; \quad \text{or} \quad \mathfrak{c} \leq \mathfrak{a}^{\mathfrak{a}} \leq \mathfrak{c}; \quad \mathfrak{a}^{\mathfrak{a}} = \mathfrak{c}.$$

The formula $c^n = c$ expresses the property previously determined that the set of all points of a straight line, of a plane, of three-dimensional space, and even of space of more dimensions (a denumerable number of dimensions) all have the cardinal number c. All these point sets can thus be placed in one-to-one correspondence with the set of points on an arbitrarily-selected small line segment.

$$n^{\mathfrak{c}} = \mathfrak{f}; \quad \text{for} \quad n^{\mathfrak{c}} = n^{\mathfrak{a} \cdot \mathfrak{c}} = \mathfrak{c}^{\mathfrak{c}} = \mathfrak{f}.$$

This follows from the fact that the set of all real functions in the interval $0 < x < 1$ is the covering set of the continuum, K, $(0 < x < 1)$ on the set of all real numbers R_{ei} $(0 < r < 1)$. That is $F = K/R_{e_1}$. Its cardinal number is $|K/R_{e_1}| = c^c = \mathfrak{f}$. Hence

$$\mathfrak{c}^{\mathfrak{c}} = \mathfrak{f},$$

$$\mathfrak{a}^{\mathfrak{c}} = \mathfrak{f}, \quad \text{for} \quad \mathfrak{a}^{\mathfrak{c}} = \mathfrak{a}^{\mathfrak{a} \cdot \mathfrak{c}} = (\mathfrak{a}^{\mathfrak{a}})^{\mathfrak{c}} = \mathfrak{c}^{\mathfrak{c}} = \mathfrak{f}.$$

$$\mathfrak{f}^{n} = \mathfrak{f}, \quad \text{for} \quad \mathfrak{f}^{n} = (\mathfrak{c}^{\mathfrak{c}})^{n} = \mathfrak{c}^{\mathfrak{c} \cdot n} = \mathfrak{c}^{\mathfrak{c}} = \mathfrak{f}.$$

$$\mathfrak{f}^{\mathfrak{a}} = \mathfrak{f}, \quad \text{for} \quad \mathfrak{f}^{\mathfrak{a}} = (\mathfrak{c}^{\mathfrak{c}})^{\mathfrak{a}} = \mathfrak{c}^{\mathfrak{c} \cdot \mathfrak{a}} = \mathfrak{c}^{\mathfrak{c}} = \mathfrak{f}.$$

$$\mathfrak{f}^{\mathfrak{c}} = \mathfrak{f}, \quad \text{for} \quad \mathfrak{f}^{\mathfrak{c}} = (\mathfrak{c}^{\mathfrak{c}})^{\mathfrak{c}} = \mathfrak{c}^{\mathfrak{c} \cdot \mathfrak{c}} = \mathfrak{c}^{\mathfrak{c}} = \mathfrak{f}.$$

We summarize the application of the laws of exponents to the cardinal numbers $n(>1)$, a, c, and $\mathfrak{f}$ as follows:

Exponent n:

$$m^{n} < \mathfrak{a}; \quad \mathfrak{a}^{n} = \mathfrak{a}; \quad \mathfrak{c}^{n} = \mathfrak{c}; \quad \mathfrak{f}^{n} = \mathfrak{f}.$$

Exponent a:

$$n^{\mathfrak{a}} = \mathfrak{a}^{\mathfrak{a}} = \mathfrak{c}^{\mathfrak{a}} = \mathfrak{c}; \quad \mathfrak{f}^{\mathfrak{a}} = \mathfrak{f}.$$

Exponent c:

$$n^{\mathfrak{c}} = \mathfrak{a}^{\mathfrak{c}} = \mathfrak{c}^{\mathfrak{c}} = \mathfrak{f}^{\mathfrak{c}} = \mathfrak{f}.$$

Exponent $\mathfrak{f}$:

$$n^{\mathfrak{f}} = \mathfrak{f}.$$

8. To conclude this section, we consider the set S of all real *continuous* functions, $y = f(x)$ in the interval $0 < x < 1$. We say a function is continuous if for every value x_0 in the domain of the definition, the value of the function is determined by the values of the function in the neighborhood of x_0. It may also help if we recall the definition of continuity from the study of differential calculus: Here

$$\lim_{x \to x_0} f(x) = f(x_0), \quad \text{or}$$

$$|f(x) - f(x_0)| < \epsilon \quad \text{for} \quad |x - x_0| < \delta(\epsilon),$$

or very briefly

$$\lim_{\Delta x \to 0} \Delta f = 0.$$

A continuous function is thus completely determined if its value is known for all rational values of x. The values of the function for irrational values of x are always as close as we care to make them to the values of the function for rational values of x. (For example, consider the approximation of irrational numbers by decimal fractions, as in $1.414 < \sqrt{2} < 1.415$.)

All real continuous functions are thus contained in a covering set B, which is formed by covering the denumerable set of all rational values of x by the set of all real numbers that has the cardinal number of the continuum c. The cardinal number of B is $|B| = c^a = c$.

The set S of all real continuous functions, however, is only a subset of B, and in fact a proper subset. This is because the conditions of continuity demand that neighboring rational values of x must correspond to neighboring values of y, and the correspondence is not entirely arbitrary. Hence $S \subset B$.

The set K of all constant functions $f(x) = c$ is again a proper subset of S, and $K \subset S$. The cardinal number of K is c. Hence $K \subset S \subset B$, and $K \sim B$. By the equivalence theorem, we conclude that $S \sim B$> that is $|S| = c$.

The set of all real continuous functions has the cardinal number c.

Exercises

1. Cover the set $N = \{1,2,3,\ldots,12\}$ by the set $M = \{I,II,0\}$, and determine the cardinal number of N/M.
2. By what set: ? < y < ? does the function $y = \sin x$ cover the set $0 < x < \pi/2$?
3. What is the set Y, for which the function $y = (-1)^x$ covers the set $X = \{1,2,3,4, \ldots\}$?
4. How many possible ways are there for assigning 16 students to six classes? (Hint: find the cardinal number of the set covering $N = \{1,2,3,\ldots,16\}$ with $M = \{I,II,\ldots,VI\}$.)
5. Prove, by the use of covering sets, that for all transfinite cardinals m, the relations $1^m = 1$ and m^1 — m hold.
6. What is the cardinal number of all real functions that have only rational values of y?
7. Prove the law $m^p \cdot n^p = (m \cdot n)^p$.

*Richard Dedekind (1831–1916).

†Finite means *ending*; infinite means transfinite, *not-ending*.

‡Euclid (c. 325 B.C.).

*a is the German letter *a* and is pronounced "ah."

†in 1896, Jacques Hadamard and Charles de la Vallée Poussin proved the prime number theorem: The number of prime numbers $\pi(n)$, less than or equal to n satisfies the relation:

$$\lim_{n\to\infty}\left[\pi(n) \div \frac{n}{\ln n}\right] = 1.$$

The following table illustrates this limit.

n	$\pi(n)$	$\frac{n}{\ln n}$	$\pi(n) \div \frac{n}{\ln n}$
100	25	21.7	1.151
1,000	168	144.7	1.160
10,000	1,229	1,085.7	1.132
100,000	9,592	8,685.8	1.104
1,000,000	78,498	72,382.4	1.084
10,000,000	664,579	620,420.6	1.071

This theorem not only proves that the prime numbers form an infinite set, since $\lim_{n\to\infty} \frac{n}{\ln n} = \infty$, but it also clarifies the nature of their distribution.

As of today, the questions are still unresolved as to whether the set of twin-primes (e.g., 17, 19 or 1949, 1951) or the set of prime number quartettes (e.g., 11,13,17, 19 or 299471, 299473, 299477, 299479) do or do not form an infinite set.

*Editor's note: Each number of elementary algebra is "complex," being of the form $a + ib$, where a and b

are real. If $b = 0$, the (complex) number is, in particular, real; if $a = 0$, then the number is "pure imaginary." Each (complex) number which is not real is "imaginary." For example, $1 + i$ is imaginary but not pure imaginary; 1 is real and is also complex. As long as one discusses real numbers only, the term "complex," while applicable, need not be mentioned.

*$\mathfrak{c}$ is the German small letter c, pronounced "tsay."

*The differential quotient and the integral, as is well-known, are related by the fundamental theorem of integral calculus. If one differentiates an integral of a continuous function with respect to its upper bound, one obtains the value of the integrand at the upper bound:

$$\frac{d}{dx}\int_a^x f(z)\, dz = f(x).$$

At first this fundamental theorem applied only to continuous functions, its underlying concept of "integral" being that of Bernhart Riemann (1826–66). This concept of integral was generalized by Henri Lebesgue (1875–1931) in connection with Cantor's theory of point sets, so that the fundamental theorem became valid without the limitation of continuity. Modern analysis even permits the differentiation of discontinuous functions—a very significant step for modern physics.

*G. Cantor called his area of study the "Theory of Multiplicities" (Mannigfaltigkeitslehre).

*F.Bernstein (a student of Cantor) born 1878 in Halle and, until 1933 Professor at Göttingen.

*For finite sets, the cardinal number of the covering set N/M is simply the number of permutations with repetitions of m things used n times. By the law of variations with repetitions this is $m \cdot m \cdot m \cdots$ (n factors) = m^n.

4

ORDERED SETS

XI. Ordered Sets and Order-Types

1. In grammar one is taught the difference between numbers that tell order and those that tell the size of groups (*ordinal* and *cardinal* numbers). As we learned to count, not only did we conceive of *three* as a plurality of three things, but we also became aware of what was meant by speaking of the *third* of these things (or the second, or the first).

In general, we were unable to think of the quantity three except as arranged in some order. Often, in counting, we pointed out a first, a second, and a third thing, probably with the use of our fingers. Our senses and our thinking almost always made us conceive sets of things in a definite spatial or temporal order.

The definition of the concept "set" as "a bringing together of definite distinct objects of our perception or our thought, to a whole," in the absence of some order of the elements, places considerable demands upon our powers of abstraction. Nevertheless, the absence of order and the absence of any reference to the nature of the elements, furnished us in the case of infinite sets with the concept of transfinite cardinal number. We came to understand such a number as represented by any one of the infinite number of equivalent sets having this number. Thus

$$|\{1,2,3,\ldots\}| = |\{2,4,6,\ldots\}| = |\{1,3,5,\ldots\}| = \mathfrak{a}.$$

We can now understand Cantor's remark when he said, "The cardinal number of M is the name of that general concept which arises out of the set M by means of our active powers of thought, and is apart from the nature of its various elements and the order in which they are given."

2. We shall now consider equivalent sets, again ignoring the special nature of the elements but paying attention, now, to the ordering of the elements. By doing this, we can determine fundamental differences in sets.

(a) The set N = {1,2,3,…} has a first, but no last element. Every element except the first has a definite predecessor. The element 5 precedes 6, and 7 follows after 6.

(b) The set $\bar{N}$ = {…3,2,1} has a last element, but no first element. Every element has a preceding element, and with the exception of the last element, also a successor.

(c) The set of all positive fractions ordered according to size has no first and no last element. No element has an immediate predecessor or successor. This follows from the fact that there is no smallest or greatest proper fraction, and between two fractions, no matter how close they are, there always exist other fractions (for example the arithmetic mean of the two fractions).

(d) The set Z = {0,1, –1,2, –2,…} has a first, but no last element. Every element except the first has an immediate predecessor and successor.

(e) The set $\bar{Z}$ = {… –3, –2, –1,0,1,2,3,…} has no first and no last element. Every element, however, has an immediate predecessor and immediate successor. Note that $Z = \bar{Z}$, that is, Z and $\bar{Z}$ have the same elements.

(f) In the set M = {1,3,5,…,2,4,6,…}, the element 2 and the element **1** each have no immediate predecessor.

Note that in the above equivalent sets, the elements are ordered in quite different manners. When we speak of *ordered sets* in what follows, we shall include in this concept of set a rule for ordering the set. By doing this we shall make accessible the mathematical understanding of the concepts of neighborhood, continuity, and dimension, all of which were lost in the generality of the equivalence relation.

3. A set M will be called an *ordered set*, if for any two of the elements a and b, one and only one of the two conditions is fullfilled:

either $a \prec b$, that is, a precedes b;
or $a \succ b$, that is, a comes after b.

This ordering of the elements has three characteristic properties:

(a) It is *non-reflexive;* it is never the case that $a \prec a$ or $a \succ a$.

(b) It is anti-symmetric; given $a \prec b$, it follows that $b \succ a$.

(c) It is transitive; from $a \prec b$ and $b \prec c$, it follows that $a \prec c$.

The signs “$\prec$” and “$\succ$” must not be confused with the greaters-smaller signs, “>“, and “<“ used with the cardinal numbers. In Example 2(a), N = {1,2,3,…}; “$\prec$” corresponds to “<“, and “$\succ$” to “>.” In Example 2(b), $\bar{N}$ = {… 3,2,1}, “$\prec$” corresponds to “>“, and “$\succ$” to “<”. In Example 2(c), “$\prec$” corresponds to “<“

and "$\succ$" to ">".

In Example 2(d), the directions for ordering can be given thus: of any two elements of different absolute value, the element of smaller absolute value comes before the element of greater absolute value; of two elements of the same absolute value, the positive element comes before the negative element.

In Example 2(e), $\bar{Z}$ = {…, – 3,–2,– 1,0,1,2,3,…}, "$\prec$" corresponds to "<" and "$\succ$" to ">".

In Example 2(f), the directions for ordering are: every odd number comes before every even number; of any two odd numbers, or even numbers, the smaller comes before the larger.

A set has a *first* element a, if for every other element $m \in M$, the relation $a \prec m$ is true. It has a *last* element b, if for every other element $m \in M$, the relation $m \prec b$ is true. An ordered set, at the most contains only one first element, and at the most only one last element.

If for three elements of a set, a, b, and c, it is known that $a \prec b \prec c$, then we say b *lies between* a and c. The sets N = {1,2,3,…} and $\bar{N}$ = {… ,3,2,1} as sets are *equal* since they each contain the same elements. We say $N = \bar{N}$. However, as *ordered sets* they are considered different sets, since their elements are arranged in different orders.

Every subset of an ordered set is also an ordered set, in which the principle of order for the whole set is maintained. By agreement, the empty set shall also be considered an ordered set.

4. In the case of ordered sets, the concept of equivalence is replaced by the sharper concept of *similarity*.

> *An ordered set M is similar to an ordered set N, when the elements of M and N can be put into one-to-one correspondence in such a manner that when for any two elements of M, m_1 and m_2 the relation $m_1 \prec m_2$ holds, then for the corresponding elements n_1 and n_2 of N, the relation $n_1 \prec n_2$ also holds.*

In this case we write: "$N \simeq M$" and say "N is *similar* to M"

Example:

Let N = {1,2,3,…};G = {2,4,6,…Then $N \simeq G$ since the ordering 1 ↔ 2, 2 ↔ 4, 3 ↔ 6, …, assigns every natural number (of N) to its double (in G). From this ordering, it follows that if $n_1 \prec n_2$ then $g_1 \prec g_2$; from $11 \prec 14$, it follows that $22 \prec 28$.

In contrast, the sets N = {1,2,3,…} and $\bar{N}$ = {…,3,2,1} are not similar although they are equivalent. Indeed, in N there is a first element that all the other elements follow, but in $\bar{N}$ there is no first element to which the 1 of $\bar{N}$ can be assigned. Also $\bar{N}$ has a last element but N does not.

Using the definition of similarity, it can be shown that ordered sets have the following properties:

(a) $M \simeq M$. Each ordered set is similar to itself. The similarity relation is reflexive.
(b) If $M \simeq N$, then $N \simeq M$. The similarity relation is symmetric.
(c) If $M \simeq N$ and $N \simeq P$, then $N \simeq P$ The similarity relation is transitive.

5. We already have seen that similar sets are always equivalent, but that equivalent sets are not always similar. In particular, using the example of the non-similar sets $N = \{1,2,3,\ldots\}$ and $\bar{N} = \{\ldots,3,2,1\}$, we reach the conclusion:

If, in two similar sets, the one set has a first (or last) element, then the other set also has a first (or last) element.

A still further consequence is the theorem:

If a set M is equivalent to an ordered set N, then M can be so ordered that M and N are similar to each other.

To prove this, we need only to make the order relation $n_1 \prec n_2$ which holds for the elements of the ordered set N apply also to the corresponding elements m_1, and m_2 of M to which n_1 and n_2 are coordinated, that is, to stipulate that $m_1 \prec m_2$.

Examples:

(a) The sets

$$N = \{1,2,3,\ldots\} \quad \text{and} \quad \bar{Z} = \{\ldots -3,-2,-1,0,1,2,3,\ldots\}$$

are equivalent; $N \sim \bar{Z}$; for both are denumerable. They are not similar, for N has a first element and $\bar{Z}$ does not. If we order $\bar{Z}$ in the manner

$$\bar{Z} = \{0,1,-1,2,-2,3,-3,\ldots\}; \quad \text{then} \quad N \backsimeq Z.$$

We could also obtain a mapping of similarity by reordering N into

$$\bar{\bar{N}} = \{\ldots 6,4,2,1,3,5,\ldots\}; \quad \text{then} \quad \bar{\bar{N}} \backsimeq \bar{Z}.$$

(b) The ordered set of points X in the interval $1 \leqq x \leqq 2$ and the ordered set of points $\bar{Y}$ in the interval $5 \leqq y \leqq 10$ are similar. The order relation is given by $y = 5x$. If $x_1 \prec x_2$ (which here means $x_1 < x_2$), then $y_1 \prec y_2$ (which here means $y_1 < y_2$).

(c) The ordered point sets X, $0 < x \leqq 1$ and Y, $0 \leqq y < 1$ are not similar to each other, if as a principle of ordering we take the respective x and y values according to their size. In this case X has a last element but no first, and Y has a first element but no last. The sets are equivalent. To make the sets similar, the correspondence of the points must be made by reordering one of the sets. A similarity correspondence can be given by the relation $y = 1 - x$. Then for $0 < x \leqq 1$, we have $1 > y \geqq 0$.

The point sets X, $(0 < x \leqq 1)$ and $\bar{Y}$, $(1 > \bar{y} \geqq 0)$ are similar. If $x_1 \prec x_2$ (or $x_1 < x_2$), then $\bar{y}_1 \prec \bar{y}_2$ (or $\bar{y}_1 > \bar{y}_2$).

(d) The following sets are equivalent, but obviously not similar:

$$N = \{1,2,3,\ldots\},$$

and

$$T = \{10,20,30,\ldots,11,21,31,\ldots 19,29,36,\ldots\}$$

6. Just as the concept of equivalence led to that of cardinal number, so the concept of similarity leads to that of *ordinal-type*. By the ordinal-type μ of a set M we shall understand any one* of the representatives of the class of all sets similar to M. Accordingly, to say "two sets have the same ordinal-type" means merely "the two sets are similar." Briefly, we shall write $\mu = [M]$ for the ordinal-type μ of set M.

The following examples will clarify and strengthen this symbolism.

$$M = \{1,4,7,10,13,\ldots\} \qquad M \neq N; \quad M \neq \bar{N}; \quad N = \bar{N};$$
$$M \subset N; \quad M \subset \bar{N}.$$
$$N = \{1,2,3,4,5,\ldots\} \qquad M \sim N \sim \bar{N}; \quad |M| = |N| = |\bar{N}| = \mathfrak{a}.$$
$$\bar{N} = \{\ldots,5,4,3,2,1\} \qquad [M] = \mu; \quad [N] = \nu; \quad [\bar{N}] = \bar{\nu};$$
$$M \backsimeq N; \quad M \not\backsimeq \bar{N}; \quad N \not\backsimeq \bar{N}.$$
$$\mu = \nu; \quad \mu \neq \bar{\nu}; \quad \nu \neq \bar{\nu}.$$

It can now be seen that equivalent sets may very well have different ordinal-types. These types are different, but not in relation to comparison of their sizes. They therefore lack the designating characteristic for numbers, namely: orderability according to size. Because of this fact we do not call them numbers, but ordinal-types.

7. Finite sets having the same cardinal number must always have the same ordinal-type. Another way of saying this is that equivalent finite sets are always similar to each other. No matter how four things are ordered, it is always possible to make the last element the fourth element. The sets

$$\{a,b,c,d\}; \quad \{1,2,3,4\}; \quad \{\alpha,\beta,\gamma,\delta\}$$

all have the same ordinal-type. For practical reasons we call this ordinal-type *four* and use the symbol "4." Thus the natural numbers obtain a double significance:

(a) They represent the numbers of finite sets (cardinal numbers).
(b) They represent ordinal-types (ordinal numbers).

8. We shall use small Greek letters to represent the ordinal-type of infinite sets. The ordinal-type of an infinite set, which by a reversal of arrangement of the elements will produce an ordered set of the ordinal-type μ we shall represent by $*\mu$ In particular, we shall write:

$$[N] = [\{1,2,3,\ldots\}] = \omega; \quad [\bar{N}] = [\{\ldots,3,2,1\}] = *\omega.$$

The ordinal-type of all the rational numbers according to (increasing) size is designated by η. The ordinal-type of all real numbers according to (increasing) size is designated by λ.

The adjacent diagram (Figure 19) shows the relations: $*\eta = \eta$; $*\lambda = \lambda$.

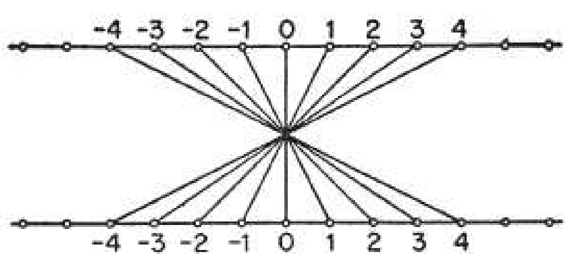


Figure 19. Equality of ordinal-types $*\eta = \eta$ and $*\lambda = \lambda$.

9. We close this section with a study of computation with ordinal-types. First we consider addition. Given two disjoint sets with the ordinal-types $\mu = [M]$ and $\nu = [N]$, we define

$$\mu + \nu = [M] + [N] = [M \cup N].$$

In forming the union of *M* and *N*, we shall treat it as an *ordered union* of the disjoint sets *M* and *N*. In this union all elements of *M* shall occur before those of *N*. In general, an ordered union is not symmetric (commutative), that is,

$$[M \cup N] \neq [N \cup M].$$

Examples:

$$\text{(a)}: \; [\{1,2,3,\ldots\}] + [\{a\}] = [\{1,2,3,\ldots,a\}]; \quad \omega + 1 = (\omega + 1).$$
$$[\{a\}] + [\{1,2,3,\ldots\}] = [\{a,1,2,3,\ldots\}]; \quad 1 + \omega = \omega.$$

Thus $\omega + 1$ is different from $1 + \omega$; the addition of ordinal-types is not commutative.

$$\text{(b)}: \; [\{\bar{1},\bar{2},\bar{3},\ldots\}] + [\{1,2,3,\ldots,n\}] = [\{\bar{1},\bar{2},\bar{3},\ldots,1,2,3,\ldots,n\}];$$
$$\omega + n = \omega + n.$$
$$[\{1,2,3,\ldots,n\}] + [\{\bar{1},\bar{2},\bar{3},\ldots\}] = [\{1,2,3,\ldots,n,\bar{1},\bar{2},\bar{3},\ldots\}];$$
$$n + \omega = \omega.$$

The ordinal-type $\omega + \omega$ is obtaines, e.g., by the addition of

$$[\{1,3,5,\ldots\}] + [\{2,4,6,\ldots\}] = [\{1,3,5,\ldots,2,4,6,\ldots\}] = \omega + \omega.$$

Further examples:

$$\omega + {}^*\omega = [\{1,3,5,\ldots\}] + [\{\ldots,6,4,2\}] = [\{1,3,5,\ldots,6,4,2\}].$$
$${}^*\omega + \omega = [\{\ldots,6,4,2\}] + [\{1,3,5,\ldots\}] = [\{\ldots,6,4,2,1,3,5,\ldots\}].$$

The order-type ${}^*\omega + \omega$ thus represents the previously given ordered set

$$\bar{Z} = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\}.$$

According to the definition of addition of ordinal-types, the associative law will hold, that is

$$(\alpha + \phi) + \pi = \alpha + (\phi + \pi).$$

Examples of this law are:

$$(\omega + 1) + {}^*\omega = \omega + (1 + {}^*\omega) = [\{1,2,3,\ldots,a,\ldots,\bar{3},\bar{2},\bar{1}\}].$$

$$({}^*\omega + 1) + \omega = {}^*\omega + (1 + \omega) = {}^*\omega + \omega = [\{\ldots,\bar{3},\bar{2},\bar{1},a,1,2,3,\ldots\}].$$

In the case of finite ordinal-types, addition is identical with that of the cardinal numbers. The laws of addition for cardinal numbers are retained for addition of ordinal-types.

$$[\{a,b,c,d\}] + [\{1,2,3\}] = [\{a,b,c,d,1,2,3\}]; \quad 4 + 3 = 7.$$

$$[\{1,2,3\}] + [\{a,b,c,d\}] = [\{1,2,3,a,b,c,d\}]; \quad 3 + 4 = 7.$$

It is also possible to define multiplication of infinite ordinal-types as a repeated ordered addition of equal addends. Since this multiplication is non-commutative, it is particularly necessary to differentiate between the *multiplicand* (the ordinal-type of the set that is repeatedly added) and the *multiplier* (the ordinal type that tells how often, and in what sequence the addition is to be carried out). For example*

$$\omega \cdot 2 = \omega + \omega = [\{1,3,5,\ldots,2,4,6,\ldots\}].$$

$$2 \cdot \omega = [\{a_1,b_1\}] + [\{a_2,b_2\}] + \ldots = [\{a_1,b_1,a_2,b_2,a_3,b_3,\ldots\}] = \omega.$$

Similarly, $n \cdot \omega = \omega$. However, the ordinal-types $\omega \cdot 2$, $\omega \cdot 3$, $\omega \cdot 4,\ldots, \omega \cdot n$ are all different. Further,

$$[\{a_1,a_2,a_3,\ldots,b_1,b_2,b_3,\ldots,c_1,c_2,c_3,\ldots\}]$$

$$= \omega + \omega + \omega + \ldots + \omega = \omega \cdot \omega = \omega^2.$$

The point sets P_g of a straight line, where P_1 is the interval $0 < x < 1$, P_2 the interval $1 < x < 2$, …, P_n the interval $n - 1 < x < n$ are all similar sets. That is $P_1 \simeq P_2 \simeq P_3 \simeq \ldots \simeq P_n$. The ordinal types of each of these sets are equal, that is,

$$[P_g] = [P_1] = [P_2] = [P_3] \ldots = [P_n] = \lambda.$$

If we form the union of P_1, $x = 1$, and P_2, we obtain

$$P_1 \cup \{x = 1\} \cup P_2 = P,$$

where P is the point set of the interval $0 < x < 2$. Adding the ordinal-type, we obtain

$$\lambda + 1 + \lambda = \lambda.$$

By a corresponding union of the intervals ($0 \leqq x < 1$); ($1 \leqq x < 2$); ($2 \leqq x < 3$), …, ($n - 1 \leqq x < n$), we obtain the rule:

$$(1 + \lambda) + (1 + \lambda) + \ldots + (1 + \lambda) = (1 + \lambda) \cdot n = (1 + \lambda),$$

and finally

$$(1 + \lambda) \cdot \omega = 1 + \lambda.$$

Exercises

1. Compute $1 + \omega + {}^*\omega + 1 + \omega$. Give an example for this addition.
2. What is the ordinal-type of all fractions in the interval $0 \leqq x \leqq 1$, ordered according to (increasing) size?
3. Establish the following laws:

(a) $\omega + 1 \neq \omega + 2 \neq \omega + 3 \neq \omega + n.$
(b) $1 + \omega = 2 + \omega = 3 + \omega = n + \omega = \omega.$
(c) ${}^*\omega + 1 = {}^*\omega + 2 = {}^*\omega + 3 = {}^*\omega + n = {}^*\omega.$
(d) $1 + {}^*\omega \neq 2 + {}^*\omega \neq 3 + {}^*\omega \neq n + {}^*\omega.$
(e) $\omega + n + \omega = \omega + (n + \omega) = \omega + \omega.$
(f) $n + \omega + \omega = (n + \omega) + \omega = \omega + \omega.$

XII. Well-ordered Sets and Ordinal Numbers

1. *An ordered set is said to be "well-ordered" if it, and every one of its non-empty subsets, contains a first element.* This requirement separates from the collection of all ordered sets those that have the property characteristic for counting, namely: there is in the set (and also in every subset) a first element; and every element except for the last has an immediate successor.

We have already dealt with well-ordered sets in such examples as

$$N = \{1,2,3,\ldots\};$$
$$Z = \{0,1,-1,2,-2,3,-3,\ldots\};$$
$$M = \{1,3,5,7,\ldots,2,4,6,8,\ldots\}.$$

According to the definition, the following sets are *not* well-ordered:

$$\bar{N} = \{\ldots,3,2,1\};$$
$$\bar{Z} = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\};$$

The set of all rational numbers in the interval $0 < x < 1$;
The set of all real numbers in the interval $0 < x < 1$; etc.

These sets have no first element.

2. The following theorems are an immediate consequence of the definition of well-ordered sets:

(a) *Every subset of a well-ordered set is well-ordered.*
(b) *Every ordered set which is similar to a well-ordered set, is itself well-ordered.*
(c) *Every finite set is well-ordered.*

The empty set is assumed to be well-ordered.

3. If, from a well-ordered set M, a subset A is selected to contain all the elements of M that appear before a definite element m of M, then the subset A is called the *segment* of M determined by m. The segment determined by the first element of M is the null-set. In the well-ordered set

$$\{1,3,5,\ldots,2,4,6,\ldots\}$$

the element 2 determines the segment $A = \{1,3,5,\ldots$

4. A rigorous study of well-ordered sets which, carried through in detail, would lead beyond the scope of this text has produced very significant results, the most important of which are summarized in the following theorems, given without proof.

(a) *Two well-ordered sets are either similar to each other or one of them is similar to a segment of the other. Therefore, two well-ordered sets M and N are always comparable as to their cardinal numbers. This means that one and only one of the conditions* m = n, m < n, *and* m > n *holds.*
(b) *A well-ordered set M can be mapped into a set similar to itself only through an identity mapping.*
(c) *If M is a well-ordered set and M is similar to* $N (M \simeq N)$, *then there is only one similarity mapping of M onto N.*
(d) *A well-ordered set is similar to none of its segments.*

5. The crown of all theorems about well-ordered sets is the so-called *well-ordering theorem*, which Cantor had accepted as true, but that was first proved rigorously by Zermelo in 1904:

Every set can be well-ordered.

Thus any arbitrary set can be reordered into a well-ordered set. Unfortunately, however, even though the existence of well-ordering has been proved, still no way has been found for the actual construction of this well-ordering. Even for relatively simple sets, such as those with the cardinal number c, to this day the construction of the well-ordering has not been solved.*

6. Since all well-ordered sets are comparable we now have the means of distinguishing among the corresponding ordinal-types. The ordinal-types of well-ordered sets are called *ordinal numbers*. If the well-ordered set M is similar to a segment of a well-ordered set N, we say the ordinal number $\mu = [M]$ is smaller than the ordinal number $v = [N]$: $\mu < v$.

Every well-ordered set M, and every set similar to M has a unique definite ordinal number μ that can be compared with regard to magnitude to all other ordinal numbers (ordinal-types of well-ordered sets). For the ordinal-types which we have already studied, there are the ordinal numbers

$$\omega,\ \ \omega + 1,\ \ \omega + 2,\ \ \omega \cdot 2,\ \ \omega^2.$$

The ordinal-types $*\omega$, $*\omega + \omega$, η, and λ are not ordinal numbers.

7. In the case of finite sets; cardinal number, ordinal-type, and ordinal number coincide. For example, the set of 15 seniors (Figure 6) has the cardinal number "15." Its 15! = 1,307,674,368,000 different possible arrangements, are similar sets of the ordinal-type "15." If we consider each of these more than 1.3 trillion representations of the senior class, we will always reach the same conclusion—that the last element is the fifteenth.

8. In the case of infinite sets, a reordering of the set leads to different ordinal numbers. The set $N = \{1,2,3,\ldots\}$ can be reordered in many ways, for example:

$$N_1 = \{2,3,4,\ldots,1\}; \qquad N_2 = \{3,4,5,\ldots,1,2\};$$
$$N_3 = \{1,3,5,\ldots,2,4,6\}; \quad N_4 = \{\ldots,3,2,1\};$$
$$N_5 = \{1,3,5,\ldots,6,4,2\}; \quad N_6 = \{1,11,21,\ldots,2,12,22,\ldots\}.$$

All of these sets are equal since they contain the same elements. They are equivalent and have the same cardinal number a. All of these sets are ordered. Their ordinal-types are:

$$[N] = \omega; \qquad [N_1] = \omega + 1; \quad [N_2] = \omega + 2$$
$$[N_3] = \omega + \omega = \omega \cdot 2; \quad [N_4] = *\omega; \qquad [N_5] = \omega + *\omega;$$
$$[N_6] = \omega \cdot 10.$$

All the ordinal-types are different. Among these sets there are no similar sets.

The sets N, N_1, N_2, N_3 and N_6 are well-ordered. Their ordinal numbers are ω, $\omega + 1$, $\omega + 2$, $\omega \cdot 2$ and $\omega \cdot 10$, respectively. Thus we have $\omega < \omega + 1 < \omega + 2 < \omega \cdot 2 < \omega \cdot 10$. Here $\omega + 1 > \omega$, for N_1 is similar to the segment of N_1 determined by the element "1."

The sets N_4 and N_5 are not well-ordered. In N_4 there is no first element. In N_5, although there is a first element, the subset defined as the "set of even numbers" has no first element.

9. The rules of operation for calculating with the ordinal numbers are the same as for calculating with ordinal-types. The ordinal number that is greater than and immediately following the ordinal number μ is evidently $\mu + 1$. This fact enabled Cantor to create an uninterrupted number sequence extending beyond the infinite.

$$\begin{array}{cccccc}
0 & 1 & 2 & 3 & \ldots & \omega \\
 & \omega + 1 & \omega + 2 & \omega + 3 & \ldots & \omega \cdot 2 \\
 & \omega \cdot 2 + 1 & \omega \cdot 2 + 2 & \omega \cdot 2 + 3 & \ldots & \omega \cdot 3 \\
 & \vdots & \vdots & \vdots & & \vdots \\
 & \ldots & \ldots & \ldots & \ldots & \omega \cdot \omega = \omega^2 \\
 & \omega^2 + 1 & \omega^2 + 2 & \omega^2 + 3 & \ldots & \omega^2 \cdot 2
\end{array}$$

ana so forth.

In general, then, $\omega^n + \omega^{n-1} \cdot n_1 + \omega^{n-2} \cdot n_2 + \ldots$ is any transfinite ordinal number. The same goes for ω^ω, ω^{ω^ω}, etc. We can now count even beyond the infinite.

With every ordinal number there is also associated a definite cardinal number. Thus the well-ordered sets with the ordinal numbers ω, $\omega + 1$, $\omega + 2$, $\omega + n$, $\omega \cdot 2$, …, ω^2, all have the cardinal number a. To every cardinal number, there belongs a whole class of ordinal numbers, called the "associated" number class.

10. We shall have to be satisfied with this brief look into the theory of well-ordered sets. It is sufficient for us to realize that as Hilbert said—"It concerns the wonderful flowering of the mathematical spirit and one of the highest performances of pure, intellectual, human creation."

The usual sobriety of mathematics does not in the least prevent us from appreciating Cantor's statement with reference to the equations

$$1 + \omega = \omega; \qquad \omega + 1 = \omega + 1:$$

"When the finite is placed in relation to the infinite, as one can plainly see, everything happens. If it (the finite) comes first, it goes into the infinite and disappears there. If, however, it knows this and takes its place after the infinite, then it remains preserved and joins itself to a new and modified infinity."

Exercises

1. Which of the following sets are well-ordered?

$$Z = \{0,1,-1,2,-2,3,-3,\ldots\};$$

$$\bar{Z} = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\};$$

$$Z_1 = \{0,1,2,3,\ldots,-1,-2,-3,\ldots\};$$

$$Z_2 = \{1,2,3,\ldots,0,-1,-2,-3,\ldots\};$$

$$Z_3 = \{1,2,3,\ldots,-1,-2,-3,\ldots,0\}.$$

2. Give the ordinal-types of the sets Z, $\bar{Z}$, Z_1, Z_2. Z_3. Which are ordinal numbers?
3. Can a well-ordered set contain a subset of the ordinal-type $^*\omega$?
4. For every ordinal number μ is it true that $\mu < \mu + 1$?
5. Is there more than one way in which the not well-ordered set

$$\bar{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}.$$

can be mapped on itself so as to produce a similar set?

6. Are $\eta + 1$ and $1 + \eta$ ordinal numbers?

*Editor's note: Cantor and some other writers consider the order-type (in each case) to be an unambiguous abstraction, rather than being an arbitrary, or even uniquely selected, representative of the class.

*"·" is to be read "multiplied by." Here ω is the multiplicand, 2 the multiplier.

*The particular significance of the well-ordering theorem lies in the possibility that we can apply to any arbitrary well-ordered sets, the method of complete induction (inference from n to $n + 1$) which is well known to us for denumerable sets. We wish to prove that a certain property E belongs to all elements of a well-ordered set. To do this we prove that the property E belongs to an element as soon as it applies to all preceding elements, and in particular to the first element.

Then the property E must belong to all the elements of the set. For suppose there were elements which did not have the property E, then there would have to be a first element e which does not have this property. All previous elements, however, have the property E, hence e also has it. From this contradiction it follows that all the elements have the property E. According to the well-ordering theorem, every set can be well-ordered, and hence complete induction can be applied to it.

5

POINT SETS

XIII. Accumulation and Condensation Points

1. As a simple application of set theory, let us finally consider sets of points. We have already treated several examples of point sets: the lattice points in a straight line, in a square, in a cube and in a plane; the points of a line segment, of a straight line, of a plane, of a cube, etc. In what follows we shall limit our discussion to *linear point sets*; that is, sets of points on a straight line. We can think of this straight line (Figure 15) as a number scale; then it represents a similarity mapping of the ordered set of real numbers on the points of this scale.

If the point P_1 is the image of the real number z_1 and P_2 the image of z_2 then from the relation $z_1 \prec z_2$ (or $z_1 < z_2$) we also have the relation $P_1 \prec P_2$ (or, P_1 lies to the left of P_2). Thus all assertions about point sets are at the same time assertions about real numbers ordered according to their numerical size. We shall therefore use the following manner of speaking that will lead to no misunderstanding.

Rational points are points that are the images of rational numbers. *Irrational points* are points that are the images of irrational numbers. To fix the exact concept of *interval* we shall write:

a,b to represent the closed interval, including the bounds a and b;
a,b to represent the open interval, excluding the bounds a and b;
a,b to represent the half-open interval, including a but excluding b:
a,b to represent the half-open interval, excluding a but including b.

The point set X in the interval (0,1) is determined by $0 < x < 1$; correspondingly, 0,1 means the same thing as $0 \leqq x \leqq \mathbf{1}$.

Examples: The following are some more examples of point sets:

The set of all lattice points of a straight line,

$$\{\ldots,-2,-1,0,1,2,\ldots\}.$$

The set of all points: $\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\}$.
The set of all points of the interval 0,1.
The set of all rational points of the interval (0,1.
The set of all irrational points of the interval (0,2).

2. We call P an *accumulation* point of a point set M, if in every neighborhood of P there is an infinite number of points of M. For a linear point set M, the *neighborhood* of P shall mean the set of all points whose distance, right or left from P, is less than any pre-assigned positive number ε, no matter how small. We can also say: P is an accumulation point of M, if in any neighborhood of P, no matter how small, there exists at least one point of M different from P.

Examples:
(a) The set of lattice points of a straight line has no accumulation point.

(b) The set of all rational points consists only of accumulation points, for between any two rational numbers, no matter how close in value, there always exists an infinite number of other rational numbers. (To grasp this, think of a continued formation of the arithmetic means.)

(c) The set of all points of a straight line, and the set of all irrational points of a straight line consist entirely of accumulation points.

(d) The set $P_1 = \{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\ldots\}$ has one accumulation point 0, which itself is not an element of the set P_1.

(e) The set $P_2 = \{\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5},\ldots\}$ has only the accumulation point **1**, which itself is not an element of P_2.

(f) The point set $P_3 = \{1,\frac{5}{4},\frac{11}{9},\frac{19}{16},\frac{29}{25},\ldots\}$ for which the rule of formation is $(n^2 + n - 1)/n^2$, $n = 1,2,3,\ldots$, has the accumulation point 1, which, in this case, is also an element of the set P_3.

(g) The point set $P_4 = \left\{\frac{+1}{2},\frac{-2}{3},\frac{+3}{4},\frac{-4}{5},+\ldots\right\}$ has two accumulation points 1 and – 1, neither of which themselves belong to the set P_4.

3. Finite sets cannot contain accumulation points. In the case of infinite sets, however, the Bolzano-Weierstrass* theorem states:

Each bounded infinite point set has at least one accumulation point.

A linear point set is said to be *bounded* if it lies entirely interior to a closed interval. The proof of the Bolzano-Weierstrass theorem is proved very easily by a process using *nested intervals*.

The bisecting point of the interval in which the bounded infinite set lies divides the set into two subsets, of which at least one of them contains an infinite number of points. Again, the bisecting point of this infinite subset divides it into two new subsets, of which at least one contains an infinite number of points. Continuing the bisection in this way, eventually one arrives at an arbitrarily small subset that contains an infinite number of points. This proves the theorem.

The set of all lattice points of a straight line is infinite and yet has no accumulation point, but it is not bounded. In the foregoing examples, the point sets P_1, P_2, P_3, and P_4 had accumulation points. However, they were bounded. (Do not conclude, however, that an infinite set must necessarily be bounded in order to contain accumulation points.)

4. If in any neighborhood of point P, no matter how small, there is a non-denumerable infinity of points of a point set M, then the point P is called a *condensation point*. Every condensation point is therefore an accumulation point, but not every accumulation point is a condensation point.

Examples:

The set of all real points contains only condensation points.

The set of all irrational points contains only condensation points.

The set of all rational points contains only accumulation points; no one of them is a condensation point.

Exercises

1. By suitable selection of point sets, show that $(1 + \eta)\cdot n = 1 + \eta$.
2. In the same manner as in Exercise 1, show that $(\lambda + 1)n = \lambda + 1$.
3. What accumulation points are contained in the set of all solutions to the equation

$$x^2 - \frac{n+3}{2n}x + \frac{n+1}{2n^2} = 0$$

if n takes on the set of all natural numbers?
4. Give examples of sets that have no accumulation points.
5. Prove: *every non-denumerable point set has at least one condensation point*. Hint: (a) If the set is bounded, then the proof is the same as for the Bolzano-Weierstrass theorem, with the word “infinite” replaced by “non-

denumerable.” (b) If the set is not bounded, then it is separable into a denumerable number of subsets, of which at least one contain a non-denumerable set of points. Then apply the method of proof (a) to this non-denumerable subset.

XIV. Closed, Dense and Perfect Sets

1. If all the accumulation points of a point set M are themselves contained as elements in the set M, the set M is said to be *closed*.

Examples:

(a) The point set $P_1 = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ is not closed. Its accumulation point 0 does not itself belong to the set.

(b) The point set $P_2 = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\}$ is not closed. Its accumulation point 1, does not itself belong to the set.

(c) The point set $P_3 = \{1, \frac{5}{4}, \frac{11}{9}, \frac{19}{16}, \frac{29}{25}, \frac{41}{36}, \dots\}$ is closed. Its accumulation point 1 belongs to the set.

(d) The point set $P_4 = \left\{\frac{+1}{2}, \frac{-2}{3}, \frac{+2}{4}, \frac{-4}{5}, \dots\right\}$ not closed; in contrast, the set $P_5 = \left\{+1, -1, \frac{+1}{2}, \frac{-2}{3}, \frac{+3}{4}, \frac{-4}{5}, \dots\right\}$ is closed.

(e) The point set of a closed interval is closed.

(f) The point set of an open interval is not closed, for the point set in the real interval (a,b) has a and b as accumulation points which themselves, however, do not belong to the point set.

2. The accumulation points of a set M can be collected into a new set M'. This set M' is called the *derivative** of the set M. It is possible for M' to be an empty set, e.g. if M is a finite set, or the set of all lattice points of a straight line. The fact that M is a closed set can now be defined by the relation $M' \subseteq M$, i.e., the derivative M' is a subset of M.

3. *The derivative of any point set is a closed set*.

Proof: let M' be a derivative of M. Assume that P is an accumulation point of M'. Then there are within the neighborhood of P an infinite number of points of M', which are accumulation points of M. In the neighborhood of P we thus have an infinite number of points of M. This means that P is also an accumulation point of M and must therefore belong to M'. Thus M' contains its accumulation points and is closed. Thus for every set if, the relation $M'' \subseteq M'$ always holds, where M'' is the derivative of M'.

4. A set is called *dense*, when between any two points of M there is at least one other point of M. This means that in a dense set M there is no point which has a *next* neighboring point.

Examples:

(a) The set of all real (or rational) points of a straight line in a closed or open interval, is dense.

(b) The point set $P_2 = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$ is not dense.

(c) A dense point set does not necessarily consist only of accumulation points. The point set $M = \{0,1),2,(3,4\}$ is dense, for the point 2 has no immediate neighboring point in the set M. This is because the boundaries 1 and 3 have been excluded. However 2 is not an accumulation point of M. Again 1 and 3 are accumulation points, but they do not belong to M. Hence M is dense, but not closed.

5. A dense point set M can only be similar to a dense point set. For, suppose M is dense and a non-dense point set N contains two neighboring points. Then if M is similar to N, the two points m_1 and m_2 of M, which are paired with two neighboring points n_1 and n_2 of N, must be two neighboring points. Then M would not be dense. Hence M can only be similar to a dense point set.

6. All dense denumerable point sets that have no first or last element are similar to each other. As we have previously learned, their ordinal-type is η. If two ordered sets are similar, then both sets have either a first (or last) element, or neither set has such elements. From this it follows that all dense denumerable point sets must belong to one of the ordinal-types η, $1 + \eta$, $\eta + 1$, or $1 + \eta + 1$.

7. If every point of a point set M is also an accumulation point of M, then M is a subset of M', $M \subseteq M'$, and we say the set M is *dense-in-itself*.

Examples:

(a) The point set of every closed or open interval is dense-in-itself. Further, the set of all points (or of all rational points) of a straight line is dense-in-itself.

(b) The point set $M = \{0,1),2,(3,4\}$ is dense, but not dense-in-itself.

(c) The closed point set $P = \{0,1,2,3\}$ is dense-in-itself, since every one of its points is an accumulation point. It is not dense, however, since 1 and 2 are neighboring points of the set P, and between them there is no further point of P.

8. If a point set is closed and dense-in-itself, it is called *perfect.* Perfect sets contain all their accumulation points; and every one of their points is an accumulation point. In this case we have $M' \subseteq M$ and $M \subseteq M'$ and hence $M = M'$. A perfect set is identical with its derivative.

Examples:

(a) The set of all points of a straight line or of a closed interval is perfect.

(b) The set of all rational points of a straight line or of a closed interval is dense, also dense-in-itself, but not closed and therefore not perfect. It does not contain the non-denumerable infinite set of its irrational accumulation points. In the neighborhood of every irrational point there are infinitely many rational points. This is evident if we think of

$$1.4 < 1.41 < 1.414 < 1.4142 < \sqrt{2} < 1.4143 < 1.415 < 1.42 < 1.5.$$

(c) A finite set is always closed, but never dense, dense-in-itself, or perfect, for it contains no accumulation points. Its derivative is the null-set.

(d) Every perfect linear point set has the cardinal number **c.**

Exercises

1. Is the point set {0.1,0.01,0.001,…} closed?
2. What is the derivative of the set of all rational points of the interval (1,2) ?
3. What is the derivative of the set of all rational points of the interval 1,2 ?
4. Is the set of all rational points in the interval 2,3 closed?
5. Is the set of all rational points in the interval 2,3 dense?
6. Is the set of all rational points in the interval 2,3 dense-in-itself?
7. Is the derivative of the set of all rational points in the interval 2,3 closed? Is it perfect?
8. Is the derivative of every infinite set a perfect set?

XV. Continuous Sets

1. In 1872, Dedekind introduced a concept of “cut” that can be applied to ordered point sets (and therefore to ordered sets of real numbers). If a point set M is divided into two non-empty subsets M_1 and M_2, this partitioning is called a *cut* in the set M and is designated by $M_1|M_2$. In making the cut, we shall agree that all points of M_1 shall lie to the left of all points of M_2. Then

$M = M_1 \cup M_2$ (ordered union) and $M_1 \cap M_2 = 0$ (the intersection of M_1 and M_2 is empty).

2. In making a cut, the following cases can arise:

(a) M_1 possesses a last point and M_2 a first point. This type of cut is called a *jump* in the set M. Between the last point of M_1 and the first point of M_2 there is no point of M. At the jump, M is not dense.

(b) M_1 possesses a last point, but M_2 has no first point.

(c) M_1 possesses no last point, but M_2 has a first point. In cases (b) and (c) we call the cut *continuous*.

(d) M_1 possesses no last point and M_2 has no first point. This type of cut is called a *gap*.

In all the dense sets thus far considered no jump is possible. A jump assumes the existence of two neighboring points.

Examples:

(a) The non-dense point set {0,1,2,3} has the jump 1|2. [Case (a)]

(b) In the set R of all rational points of the interval 1,3, the point "2" generates a continuous cut. The rule for making the partition is either

$$R_1 = \{\langle 1,2 \rangle\}, \quad R_2 = \{(2,3\rangle\}. \quad \text{[Case } (b)\text{]}, \quad \text{or}$$

$$R_1 = \{\langle 1,2)\}, \quad R_2 = \{\langle 2,3\rangle\}. \quad \text{[Case } (c)\text{]}$$

In the first partitioning "*2*" is the last point of R_1 and R_2 has no first point. In the second case R_1 has no last point and "2" is the first point of R_2.

(c) In the same set R, the point $\sqrt{2}$ which does not belong to R, generates a non-continuous cut, a gap. In this case, R_1 has no last and R_2 no first point.

3. We call a point set *continuous* if every cut in the set is continuous. A continuous set has neither jumps or gaps. The set of all rational points has no jumps and is a dense set. But it has gaps, in fact, an infinite number of gaps. Indeed, the dense denumerable set of all rational points displays a non-denumerable set of gaps. As was shown in the case of $\sqrt{2}$, the irrational points fill in the gaps. The rational points fill the number scale densely but no where continuously. This is a surprising result because our vague intuition could easily lead to a contradiction if the delusive appearance were not clarified by the theory of sets.

4. The one-to-one correspondence between the point set of a straight line and a set of numbers was first made possible by the introduction of irrational numbers. If we limit ourselves to rational numbers only, then every number corresponds to a point on the straight line (a rational point), but not conversely. For example, if the diagonal of a square with unit edge is laid off from the origin 0, the point which is the end point of the diagonal is without a corresponding number unless one introduces $\sqrt{2}$ as an irrational number that fills the gap $r_{1_i} <$ $\sqrt{2} < r_{2k}$ in the cut $R_1|R_2$

This particular gap in the set of rational numbers can be represented as follows. To the first subset R_1 belong all the positive rational numbers m/n for which $m^2/n^2 < 2$; to the subset R_2 belong all m/n for which $m^2/n^2 > \mathbf{2}$.

5. The previous theorem, that all dense denumerable sets which have no first or last element are similar to each other enables us to completely characterize the ordinal-type η. (η is the ordinal-type of the set of all rational points of a straight line or of an open interval.) These properties are:

(a) The ordered set R is denumerable.
(b) It is dense.
(c) It contains no first and no last point.

Cantor has proved that these three properties completely determine the ordinal-type η.

6. The ordinal-type of the linear continuum, bounded on both sides, (for example the point set 0,1) is symbolized by θ. Cantor has shown that θ can be completely characterized by the properties:

(a) The ordered point set C contains a dense denumerable subset R, so that between any two points of C there lies at least one point of R.

(b) The set C is continuous.
(c) The set C contains a first and a last point.

Since θ is represented by a closed point set, the ordinal-types $\theta + \theta$, $\theta + \theta + \theta$, $\theta \cdot n$, and $\theta \cdot \omega$ are all different. As an example, the point set {1,2,3,4} that generates the jump 2|3 has the ordinal-type $\theta + \theta$

7. The ordinal-type of the unbounded linear continuum (or the set of all points of a straight line or an open interval) is symbolized by λ. It can be completely described by the properties:

(a) The ordered point set $\bar{C}$ contains a dense denumerable subset R, so that between any two points of C there lies at least one point of R.
(b) $\bar{C}$ is continuous.
(c) $\bar{C}$ contains no first and no last element. Thus $1 + \lambda + 1 = \theta$; further $\lambda + \lambda + \lambda = \lambda$.

Examples:

(a) $[\{``1" \cup (1,2) \cup ``2"\}] = [\{\langle 1,2 \rangle\}] = \theta.$

(b) $[\{(1,2) \cup \text{“}2\text{”} \cup (2,3)\}] = [\{(1,3)\}] = \lambda$.

The ordinal type $\lambda + \lambda$ is different from λ, for $\lambda + \lambda$ can be represented by the point set $\{(1,2) \cup (2,3)\}$, which is not everywhere continuous, but has the gap 2. In contrast, $\eta + \eta = \eta$ for the set of all rational points in the interval (1,3) with the point 2 removed is still dense, unbounded, and denumerable.

8. The ordering properties that characterize a set as closed, dense, dense-in-itself, perfect or continuous naturally apply to all sets similar to it.

Exercises

1. Prove that $\sqrt{2}$ is not a rational number, that is, that it cannot be the quotient of two relatively prime numbers m/n.
2. Describe the cut in the set of all rational numbers generated by $\sqrt{5}$
3. What type of cut is produced by the point $\sqrt[3]{6}$ in (a) the set of natural numbers? (b) the set of all rational numbers? (c) the set of all real numbers?

XVI. Range and Continuity of Functions

1. The function $y = f(x)$ has been defined as a single-valued mapping of a set Y onto a set X. If this correspondence is one-to-one, that is, for each value y of Y, a value x of X is uniquely determined, the function is called *univalent*. If x is the independent variable, a definite set of values, X, is assumed; $f(x)$ is defined over this set X.

Examples:

(a) Let the function be given by $y - f(x) = x^2$. Let the domain of X be defined as the set of all real numbers. Then Y is the set of all positive real numbers. The function however is not univalent: every element of X is paired with a definite value of Y, but not conversely. If $y = f(x) - x^2$ is defined only for the set X of positive real numbers, then the set Y has the same values as before, but now we have a univalent function.

(b) Let $y = f(x) = x!$ be defined over the set $X = \{1,2,3,\ldots\}$. This univalent function maps the set X onto the set $\bar{Y} = \{1,2,6,24,120,\ldots\}$.

(c) If heated water is brought into a cooler environment and for the next hour the temperature of the water is read at ten-minute intervals, one obtains a set of tabular values, for example:

t (min.):	0	10	20	30	40	50	60
T (°C.):	71.2	45.6	32.8	26.4	23.2	21.6	20.8

This table defines the univalent function $T = f(t)$ over the set $t = \{0,10,\ldots,60\}$ onto the set of values $T = \{71.2,45.6,\ldots,20.8\}$.

2. Every real function defined in this manner can be represented by a curve (or set of points) because it is possible to map all ordered pairs of real numbers with the points of a plane (Cartesian coordinates). Figure 20 shows the function $y = x^2$; Figure 21, the function $y = x!$ where $X = \{1,2,3,\ldots\}$; and Figure 22, the function $T = f(t)$.

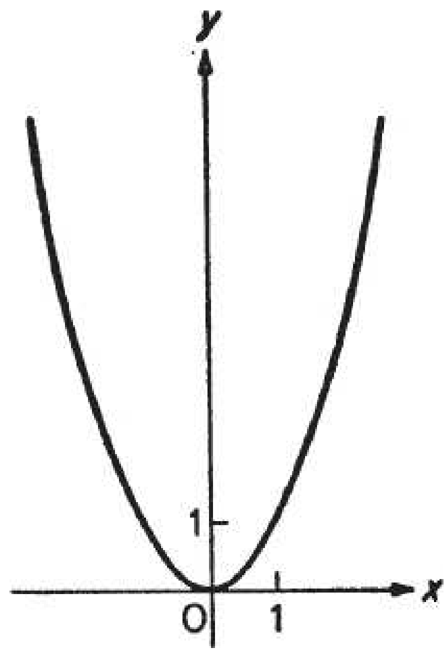


Figure 20. The function $y = x^2$.

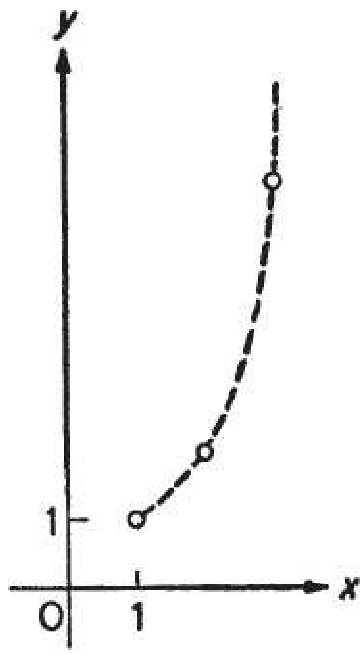


Figure 21. The function $y = x!$ $X = \{1,2,3,...\}$.

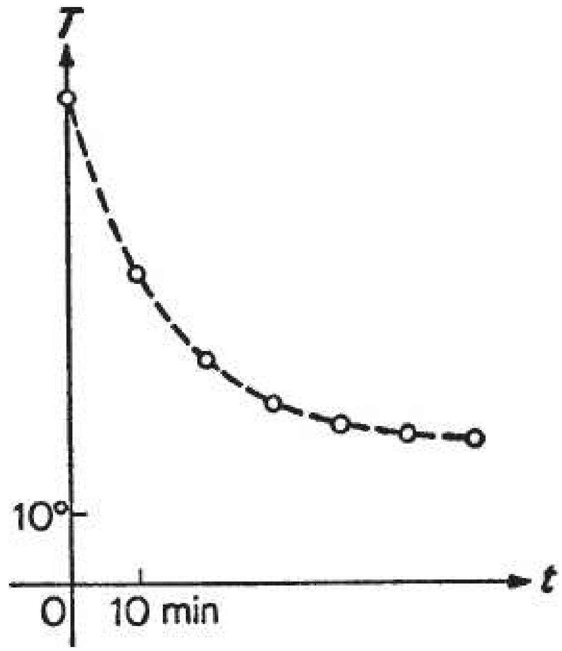


Figure 22. The measure sequence $T = f(t)$.

3. If for a function $y = f(x)$, which is defined over a set X, the corresponding set X lies entirely within an open or closed interval, (a,b) or (a,b), we say the function is *bounded*; that a is its *lower bound*, and b its *upper bound*. If the bounds a and b are values of the set Y (a closed interval) we call a and b the *greatest lower* and the *least upper bounds*, respectively. The length of the interval $s = |b-a|$ is called the *range** of the function $y = f(x)$ over X.

Examples:

(a) The function $y = f(x) = \sin x$ has, by definition, the domain X = {all real numbers}. Then $Y = \{-1, +1)\}$, and the function is bounded. Its bounds (greatest lower and least upper) are "-1" and "+1." Its range is 2. (See Figure 23.)

(b) Let the function $y = 1/(x^2 + 1)$ be defined for the set X of all real numbers. Then the set of values Y is $Y = \{(0,1\}$. It has

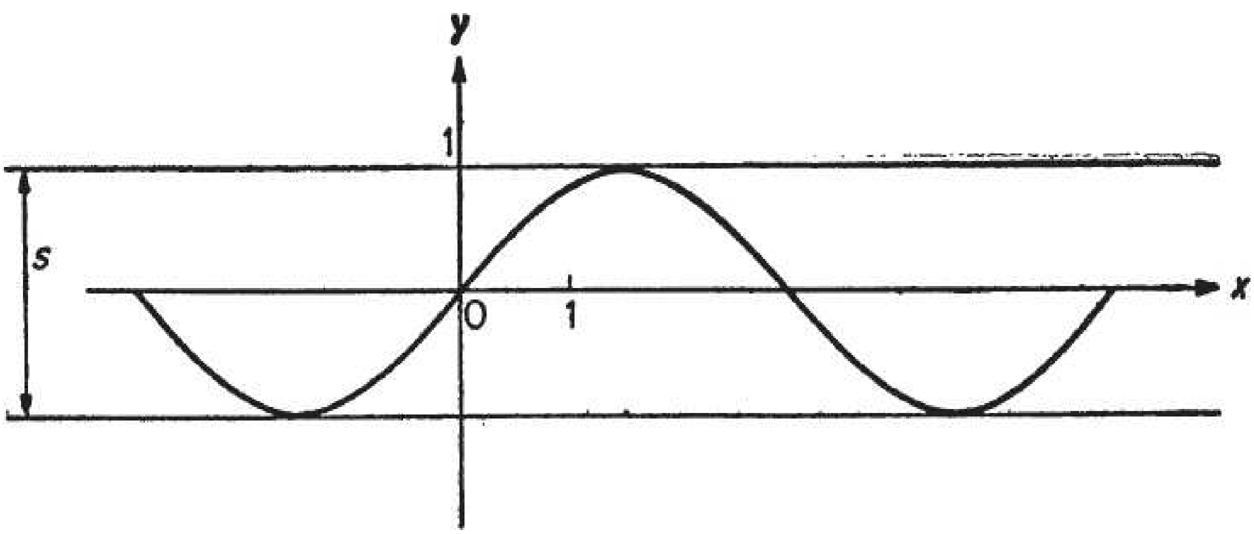


Figure 23. The bounded function $y = \sin x$.

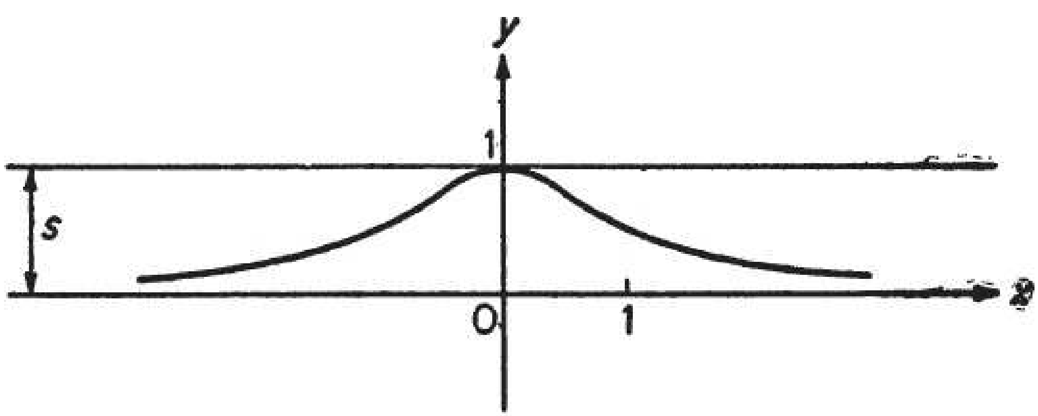


Figure 24. The bounded function $y = \dfrac{1}{(x^2 + 1)}$.

the bounds "*0*" and "1." The range is $s = 1$ (Figure 24). In both examples (a) and (b) the function is not inversely single-valued.

4. A function is said to be *continuous* at x_1 if in every neighborhood of x_1 (that is, $x_1 - \varepsilon < x < x_1 + \varepsilon$), the range of values of Y can be made as small as one desires ($s < \delta$) by the choice of a sufficiently small value of ε. Or: a function is continuous at x_1 if at x_1 the range of values of $f(x)$ becomes zero: $\lim_{x \to x_1} s = 0$. This definition is the same as one given earlier where Δy appeared in place of s.

Examples:

(a) Let the function $y - f(x)$ be defined as follows: over the interval $X_1 = \{0,1\}$, $f_1(x) = x$ and over the interval $X_2 = \{1,2\}$, $f_2(x) = x + 1$. At the point where $x\subset = 1$, the function is discontinuous, for $\lim_{x \to x_1} f_1(x) = 1$ and $\lim_{x \to x_1} f_2(x) = 2$; $\lim_{x \to x_1} s = 1$. (See Figure 25.)

(b) Let the function $y = (x^2 - 1)/(x - 1)$ have the domain of definition $X = \{0,2\}$. At the point $x_1 = 1$ it is indeterminate. If we agree at the start, $f(x_1) = 3$ at $x_1 = 1$, then the function is discontinuous. Then $\lim_{x \to x_1} s = 1$. If however, we agree that the value of the function at the point $x_1 = 1$ shall have the limiting value $f(x_1) = \lim_{x \to x_1} f(x) = 2$, then the discontinuity is removed. Following Riemann* such a discontinuity (Figure 26) is called "removable" (hebbare).

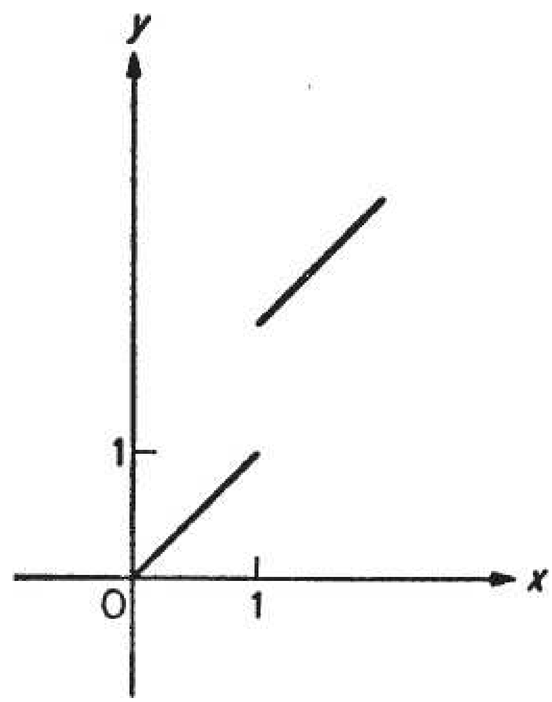


Figure 25. The function $y = f(x)$ is discontinuous at $x_1 = 1$.

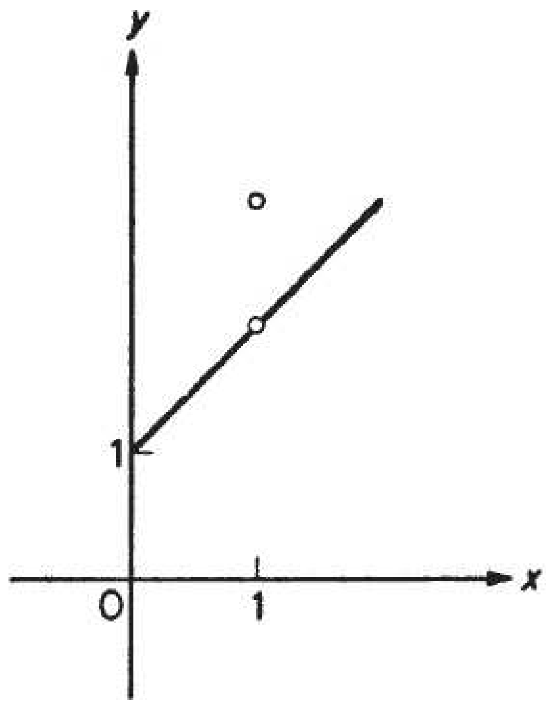


Figure 26. The function $y = \frac{(x^2 - 1)}{(x - 1)}$ has an open discontinuity at $x_1 = 1$.

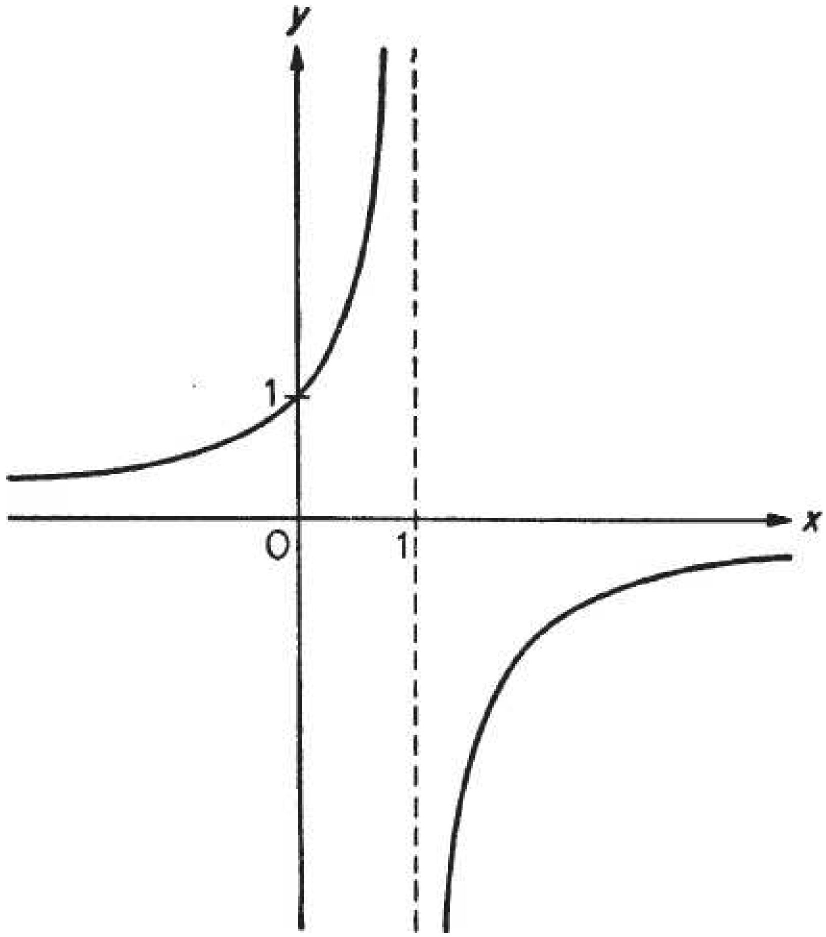


Figure 27. The function $y = 1/(1 - x)$ has a pole at $x_1 = 1$.

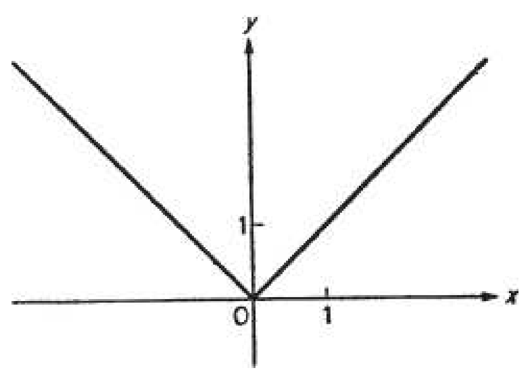


Figure 28. The function $y - |x|$ is continuous everywhere but not differentiable at $x_1 = 0$.

5. The following theorems hold for continuous functions:

Every junction continuous in a closed interval which assumes two values a and b, also assumes every value between a and b.

Every function continuous in a closed interval is bounded. Examples:

Examples;

(a) The function $y = y = \frac{1}{1 - x}$ is not bounded in the interval $X = \{0,2\}$. It is not everywhere continuous. At $x_1 = 1$ it has a pole. (See Figure 27).

(b) The function $y - |x|$ is continuous for all real values of x. (See Figure 28.) However, at the point $x_1 = 0$, $y = |x|$ is, as is well-known, not differentiable.

Exercises

1. Show that the function shown in Figure 22 can also be defined by the functional relation $T = 20 + 51.2 \cdot 2^{-i/10}$.

2. Is the function $y = \text{In } (x^2 + 2)/(x^2 + 1)$ bounded? $X = \{\text{all real numbers}\}$.

3. Determine the range of the function $y - x^2$ for the interval $x_1 - \varepsilon, x_1 + \varepsilon)$ if $x_1 = 1$ and (a) $\varepsilon = 0.2$? (b) $\varepsilon = 0.02$? (c) $\varepsilon = 0.001$?

4. At the point $x_1 = 3$, what value of ε must be selected for the function $y = 5x - 8$ so that the range s becomes less than 0.01?

5. Is the function $y = (x^2 - 3x + 2)/(x^2 - 1)$ everywhere continuous?

6. Is the quotient of two continuous functions also a continuous function?

7. Can the [a]body-temperature curve” be used as an example of a continuous function?

*Bernard Bolzano (1781–1848); Karl Th. W. Weierstrass (1815–97).

*The derivative is also called a “derived set.”

*Editor’s note: The word “range” is usually applied to the *set of values assumed by y*, the dependent variable.

*Bernhard Riemann (1826–66).

6

CONCLUSION

XVI. The Paradoxes of Set Theory

1. Starting with a consideration of finite sets, we took several steps forward into an introductory study of the infinite and began to compute with transfinite numbers. We became aware of the fact that we can count and compute beyond the infinite and do this with that complete definiteness which is characteristic of all mathematical definitions and operations.

2. We then studied point sets through a sort of set-theoretic microscope and, since we viewed them in infinite enlargement, sharp differences emerged in these sets. We now know that a dense set, as well as a set dense-in-itself can still have an infinite number of gaps.

3. In set theory we have entered a field of inquiry which today places the various areas of mathematical study on a firmer foundation, and which has also enriched them. In this penetrating creation of the human mind we made wide use of the "freedom" to bring to light new concepts. These concepts are simple and clear even though their abundance demands careful distinctions.

4. But in our freedom of creation, we were not so careless as to allow paradoxes to appear. Such paradoxes arise easily when the construction of new sets is unrestricted; for example, when sets themselves are permitted to occur as elements of new sets. The paradoxes that have been uncovered in the theory of sets have proved to be a deterrent to the program of the theory toward its present acceptance. A final clarification of all these problems does not exist to this day. In our previous work we have deliberately avoided definitions that would have led to paradoxes. Now, in closing, we shall investigate just a few of these paradoxes.

5. Suppose we construct the set N of all sets, each of which does not contain itself as an element. That there is a set which does not contain itself as an element is known to us, for we have almost always worked with sets in which the elements were not sets, but things, numbers, or points. The only exception was the set *U(M)* of all subsets of a set *M*. Its members are only sets. But *U(M)* did not contain itself as a set. Its most inclusive member was the set *M*, the improper subset, and here we have

$$|U(M)| > |M|$$

Now one can also say: "construct the set *J* of all sets, each of which contains itself as an element." As yet such a set has never been found in the field of mathematics, and it is possible that there is no such set. If there is one, it would be ruled out of mathematics since Cantor's definition demands that in collecting elements into a whole (a set), the set must be something new, different from its elements.*

Nevertheless, let us assume that there is a set of all sets each of which contains itself as an element. This set itself is then included in *J.* Now any arbitrary set of all sets belongs either to *N* or to *J.* (*N* is the set of all sets each of which does not contain itself as an element.) We now ask where the set *N* falls in this classification, in *N* or in *J*?

Assumption I. The set of all sets each of which does not contain itself as a set belongs to *N.*

Then *N* must contain itself as an element. This contradicts the definition of *N.* Therefore *N* cannot be in *N.*

Assumption II. N is an element of *J*.

All elements of *J*, however, are sets each of which contains itself as an element. By definition, *N* cannot be in *J*.

Both of these assumptions lead to contradictions. This is known as RusselPs paradox (1903).*

The set of all sets each of which does not contain itself as an element thus becomes an inadmissable concept, which is excluded from the theory of sets to avoid contradictions.

6. As an intuitive analogy to RusselPs paradox we have the story of that poor village barber who shaved all the villagers (and only those) who did not shave themselves. What about the barber himself? If he shaves himself, he is a self-shaver, and he ought not to shave himself. If he does not shave himself, then he belongs to that set the members of which he must shave. No matter what he does, he will be inconsistent.

7. Entirely similar is the situation of the liar who admits, "I lie." If he really lies, then the expression is true, and he does not lie. If, however, he does not lie, then, the expression is untrue, and he lies.

8. The "set of all conceivable sets" is obviously the most inclusive set there can be. It must, a priori, have the greatest cardinal number. Yet the set of its subsets has a greater cardinal number and is therefore "more inclusive."

The expression "set of all sets" leads to contradictions and for this reason it is excluded in our theory of sets. This "set of all sets" is, to be sure, a set that contains itself as an element. But this set, as we have already inferred, is in contradiction to Cantor's definition of set.

"No totality can contain members which are defined *only* by means of that totality and are therefore dependent on that totality." (Russell)

9. The Italian mathematician, Burali-Forti† likewise, in 1897 pointed out the paradox that: "The set of all ordinal numbers has a larger ordinal number than the largest number in the set of all ordinal numbers." Likewise, the "set of all cardinal numbers," as can be shown, has a greater cardinal number than the greatest cardinal number contained in the set.

10. This assortment of paradoxes must suffice here. If we exclude the hazardous concepts "set of all sets," "set of all ordinal numbers," etc., the contradictions will not arise. These concepts are selfcontradictory, and under the strong axiomatic foundation of the theory of sets, given first by Zermelo in 1908, there is no place for them.

XVII. Formalism and Intuitionalism

1. Mathematics, and especially its foundations, can be considered from different viewpoints. Two quite different points of view are *formalism* and *intuitionalism.* (There are rather extreme cases, and we shall not go into others.)

2. The paradoxes of set theory were the direct cause of the outbreak of a strong disagreement between formalists and intuitionalists. However, the disagreement was concerned not only with these paradoxes, but penetrated deeply into mathematical thought.

3. We are already familiar with formalism, whose most outstanding advocate is David Hilbert. Cantor's set theory is an extension of formalistic thinking. The characteristics of formalism are:

(a) *The axiomatic method.* At the start of mathematics, a system of independent fundamental statements called axioms are postulated. They are (we trust) complete and free of contradiction. From these fundamental theorems, further theorems are deduced by logical procedures. The objects of thought about which statements are made in these theorems are "objects" without "meaning," e.g. "numbers" and "symbols without content." That we represent the geometrical constructs of our spatial perception in terms of "point," "line," "plane," etc., is possible, but it is not necessary.

Examples:

1. Consider the parallel axiom: in a given plane, through a point A outside a straight line a, one and only one straight line can be drawn that does not intersect a.

2. As a first theorem demonstrated from this postulate, we learn: the sum of the angles of a triangle is 180°.

(b) *Existence as freedom from contradiction.* Not only the axioms, but also the concepts must be free from contradiction.

For every concept, the law of identity must hold, i.e.,

$$a = a \qquad \text{(I)}$$

$$\left.\begin{array}{l}\text{If } a = b\text{, then we cannot have } a \neq b.\\ \text{If } a \neq b\text{, then we cannot have } a = b.\end{array}\right\} \qquad \text{(II)}$$

For the formalist, mathematical existence of a concept is synonymous with freedom from contradiction.

Examples:

The "set of all sets" is a concept filled with contradictions, therefore, a nonexistent concept. In this case, it is true that both (I): $M = M$ and (II): $M \neq M$ (since $M \subset M$).

(c) *The law of the excluded middle*. Either $a = b$ or $a \neq b$; a third possibility does not exist. The formalist makes repeated use of this method of reasoning for proving an impossibility (indirect method of proof).

Examples:

1. There is no rational number m/n for which $m/n = \sqrt{2}$ hence the opposite is ture, $\sqrt{2} \neq m/n$.

2. The application of the second diagonal process: the set of real numbers is not denumerable, because the assumption of denumerability leads to a contradiction.

3. The proof of the equivalence theorem.

4. The proof of the Bolzano-Weierstrass theorem.

(d) *The decidability of every mathematical problem*. To the early formalist every mathematical problem was decidable, even if at the time it had not been decided. According to Hilbert, every mathematician surely shares "the conviction that every definite mathematical problem must necessarily be capable of a rigorous settlement—through the use of pure thought."

Examples:

1. The number 299,909 is either a prime number, or it is not a prime number. In a finite number of steps it can be proved prime.

2. There are either a finite or an infinite number of prime number twins. (So far this is unsolved.) Twin primes are two prime successive odd numbers, as (11, 13); (17, 19).

3. Every whole number can be represented as the sum of two prime numbers. (Theorem of Goldbach,* which is as yet unsolved.)

4. There is at least one number triad (x,y,z) for which $x^n + y^n = z^n$, $n = 3,4,5,...$, or there is no such number triad. (Theorem of Fermat,* which is as yet unsolved.)

5. There are either finitely or infinitely many "perfect numbers." A *perfect number* is equal to the sum of all its (proper) factors. Some perfect numbers are 6 = 1 + 2 + 3; 28 = 1 + 2 + 4 + 7 + 14; 496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248. Euclid, in his *Elements* proved that every number

$$Z = (2^{n+1} - 1) \cdot 2^n$$

is perfect if $2^{n+1} - 1$ is a prime number. Whether or not there are infinitely many prime numbers having the form $2^{n+1} - 1$ is at present unknown. For example $2^{127} - 1$ is a 39-digit prime number. Euler has proved that there can be no perfect even numbers other than those constructed according to Euclid's law. The question of whether or not there are odd perfect numbers is still unanswered.

4. One of the primary objectives of elementary mathematical instruction has been to learn the mastery of this formalistic mathematics. Formalism, as we have seen, is not a mechanical thought- lacking procedure, but is on the contrary the high and difficult art of making abstractions and logical deductions. In this sense, all mathematicians are more or less formalists.

5. The strength of the formalistic method, its rigor, and its elegance appeared to give mathematics an immovable permanence, until about the year 1900, when intuitionalism broke this firm foundation wide open.

6. The intuitionalists received their name by viewing the natural numbers as "original-intuition" (a compelling inner feeling of knowing these numbers). They consider the natural numbers as something originally given, for which no further foundation is necessary. (For the formalists, the natural numbers and the arithmetic operations upon them require proof of being free of contradiction.) As a forerunner of the intuitionalists we mention Kronecker† who expressed their fundamental viewpoint in his well-known expression, “The natural numbers were made by God, all else is the work of man.” The intuitionalists have now focused sharp criticism on “the work of man.” Among the leading intuitionalists are Hermann Weyl* and especially Jan Brouwer,† who because of his extreme position calls himself a neo-intuitionalist.

7. Characteristics of intuitionalism are:

(a) *The original-intuitiveness of the natural numbers*.

(b) *Mathematical existence as constructibility.*

To accept freedom from contradiction as existence, as is claimed by the formalists, is impossible for the intuitionalists. They see in freedom from contradiction only a game to be played with empty words. For intuitionalists, “Mathematics is more a performance than a study.” Objectivity must rise above the method. Mathematical thought is pure construction.

“Construction” means starting from simple objects into whose nature we have insight and, by using a finite number of steps, producing something else. What is incapable of such construction is valueless, e.g. every mere existence statement of the form “There are ....” Such an existence statement is, according to Weyl, “A sheet of paper which shows the existence of a treasure, without, however, revealing in what place it lies.”

Examples:

1. The intuitionalists consider the well-ordering theorem valueless, because it merely proves the existence of “well-ordering” without showing a way in which well-ordering can be achieved.

2. What is the significance of having “proved” the existence of transcendental numbers, if, in an individual case, it is not possible to decide whether a given real number is transcendental or algebraic?

How frivolous has been the treatment of concepts such as “ordering,” “rule,” etc., that are not capable of constructible verification!

Examples:

1. In the decimal number $\sqrt{2}$ = 1.4142 ..., let us replace every fourth digit after the decimal point by the digit 5. Is the number thus constructed, y = 1.4145 ... 5 ... 5 ..., algebraic or transcendental? To this day we do not know the answer!

2. The digits $a_1, a_2, a_3, ..., a_i, ...$ in the decimal number $r = 0.\, a_1a_2a_3... a_i...$ can be assigned by the digits in the number π = 3.14159 ... in the following manner.

First separate into groups of ten the digits after the decimal point in the decimal numeral for π. If the *ith* group consists of ten equal digits, all 7's, let $a_i = 1$, otherwise let $a_i = 0$. It is now impossible, in a finite number of steps, to determine whether $r = 0$, or $r \neq 0$. The formalists say: "There is such a number *r*, for it is determined uniquely and without contradiction by the one-to-one correspondence between its digits and groups of digits of π." The intuitionalists say: "At no time do we know whether *r* is equal to zero or *r* is different from zero. We cannot construct *r.* The number *r* has no mathematical existence."

(c) *Forbidding the use of the law of the excluded middle*. To the intuitionalist, the law of the excluded middle is an unfounded prejudice. According to him, this law, taken from classical logic, arose from abstraction from the study of finite sets. One gives this law, without proper authority, an a priori validity for infinite sets. The intuitionalists go on to say that one of the two relations

$$a = b \quad \text{and} \quad a \neq b$$

need not always be true (as the formalists maintain), but that instead there is yet a third possibility, namely:

(d) *The undecidability of mathematical problems.* The intuitionalists believe that not all problems are solvable. To this day, however, no undecidable problem of *mathematics* has become known, although many numerical problems are still undecidable.

Examples:

1. Is $2^{1024} + 1$ a prime number? We do not know. This question, however, is answerable for both the formalists and the intuitionalists. In a finite number of steps (finite construction) the question can be solved, even if it needs a hundred years of computation. (Electronic computers may reduce this time tremendously.)

2. Has the equation $x^n + y^n = z^n$, $n = 3,4,5,...$ a non-zero integral solution for *x, y*, and *z?* The intuitionalists believe three answers are possible:

(a) There is at least one whole number triplet *x, y*, *z.*

(b) There is no number triplet *x*, *y*, *z.*

(c) The question is undecidable.

8. The rigorous criticism of intuitionalism makes the following demands on the present theory of sets:

(a) The removal of everything beyond the denumerable. The continuum is then no longer an actual infinity, but only a "medium of creation," namely, the set of all unending number sequences (decimal numbers) whose digits can be selected at will.

(b) Elimination of the equivalence theorem.

(c) Elimination of the Bolzano-Weierstrass theorem.

(d) Elimination of the well-ordering theorem.

(e) Elimination of the theorem of Cantor: $|U(M)| > |M|$.

(f) Renunciation of the Dedekind method of cuts, and much more.

9. Intuitionalism has not merely created difficulties for the classical formalistic mathematics. Inasmuch as it has forced critical rigor, it has also revitalized the formalistic field. For a time, the conflict of the two ideologies was violent. Cantor wrote: "There is inherent here to a certain measure, a question of power. ... It is asked, 'Which ideas are the stronger, more embracing, and fruitful ones, Kronecker's or mine?' Time only will decide the outcome of our battle."

And how did the battle turn out? From the quiet distance of several decades and thousands of kilometers, a mathematician and historian of our day gives us his judgement.*

"It was a battle of life and death between Hilbert's formalism and Brouwer's intuitionism for the possession of mathematics. It does not seem to have occurred to either combatant that while he was engaged in trying to exterminate his enemy, some ragged camp follower might make off with the prize; or that it might not make the slightest difference to mathematics whether the battle for him, was won, lost, or drawn."

There probably will always be formalists and intuitionalists. Indeed, the real mathematician will follow the paths of both. This polarity, productive of so much tension, will become a source of new mathematical creation.

10. "Let us be glad that we can enter into the theory of sets, into that paradise from which no one can drive us out." (Hilbert) Since it has opened our eyes to the "levels" of infinity, has it not taught us something that borders on the "miraculous"? Because the infinite has been acquired through mathematical analysis, the "miraculous" loses some of its incomprehensibility.

We close with a word from Stevin*, the man who, through his development of computation with decimal numbers,† created the implement without which the theory of sets could not have developed its methods of proof, namely, the representation of every real number as an infinite decimal number.

Stevin gave a motto to his work (an expression of his emotion after discovering the law of the parallelogram of forces for an inclined plane) a statement that likewise applies without qualification to set theory: "*A miracle, and yet no miracle!*"

*In this connection, the following method of reasoning is also prohibited as being paradoxical: What is a set? Answer: An abstract concept. But the set of all abstract concepts contains itself as an element (an abstract concept).

*Bertrand Russell (born 1872).

†Cesare Burali-Forti (1861–1931).

*Christian Goldbach (1690–1764).

*Pierre de Fermat (1601–65).

†Leopold Kronecker (1823–91).

*Hermann Weyl (1885–1955).

†Luitzen Egbertus Jan Brouwer (born 1882).

*E. T. Bell, *The Development of Mathematics*, 1st ed. McGraw-Hill, New York, 1945.

*Simon Stevin (1548–1620).

†De thiende, Dutch, the tenth (part), 1585; la disme, French, 1634.

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APPENDIX

XVIII. Glossary of Definitions and Theorems

Page 4. A *set* is a collection of definite distinct objects of our perception of or our thought. These objects are called the *elements* of the set. (Cantor)

Page 6. Two sets are *equal* if they contain the same elements.

Page 7. A set N is a *subset* of a set M if every element of N is also an element of M.

Page 8. If N is a *proper subset* of M, then the set of elements of M that do not belong to N is called the *complement set R to N* over M.

Page 9. The *union* of two sets is the set of all elements each of which is contained in at least one of the two sets.

Page 9. The *intersection* of two sets is the set of all elements each of which is contained in both sets.

Page 13. Two sets, M and N, are *equivalent* to each other if the elements of M can be ordered in one-to-one correspondence with those of N.

Page 18. If there is no proper subset of M that is equivalent to ikf, then M is a *finite* set.

If there is a proper subset of M that is equivalent to M, then M is an *infinite (transfinite)* set. (Dedekind)

Page 20. The set of prime numbers is denumerable. The set of all whole numbers is denumerable. The set of all rational numbers is denumerable.

Page 23. The set of all algebraic numbers is denumerable.

Page 28. The set of all real numbers is non-denumerable.

Page 34. The set of all real transcendental numbers is non-de-numerable and has the cardinal number c.

Page 36. The set of all real functions in the interval $0 < x < 1$ has a greater cardinal number than the continuum.

Page 53. The set of all real continuous functions has the cardinal number c.

Page 37. Given any infinite set, there exists a set having a greater cardinal number than the given set.

Page 37. The set of all subsets of a set M always has a greater cardinal number than the set M itself.

Page 41. If M is equivalent to a subset N_1 of N and N is equivalent to a subset M_1 of M, then M and N are equivalent sets (equivalence theorem).

Page 57. An ordered set M is *similar* to an ordered set N, if the elements of M and N can be placed in one-to-one correspondence in such a manner than when for any two elements m_1 and m_2 of M, the relation $m_1 \prec m_2$ holds, then for the corresponding elements n_1 and n_2 of N the relation $n_1 \prec n_2$ also holds.

Page 58. If a set M is equivalent to an ordered set N, then M can be so ordered that M and N become similar sets.

Page 64. Every finite set is well-ordered.

Page 65. Every set can be well-ordered (well-ordering theorem).

Page 71. Every bounded infinite point set has at least one accumulation point (theorem of Bolzano-Weierstrass).

XIX. Brief Historical Outline

1. *The actual infinite*; *the theory of sets:*

Bernhard Bolzano, born October 5, 1781, in Prague, died December 18, 1848, in Prague; was a preacher and professor of theology in Prague. A forerunner of Cantor, his work *The Paradoxes of the Infinite* appeared posthumously in 1851.

Georg Cantor, born March 3, 1845, in Petersburg, the son of a merchant from Copenhagen, died January 6, 1918, in Halle; emigrated to Germany with his family when he was 11 years of age. He studied in Darmstadt, Gottingen, and Berlin. In Berlin he was a student of Kronecker and Weierstrass. He was professor at Halle, where from 1878 he published frequently on the theory of sets.

Ernst Zermelo, born July 27, 1871, died may 21, 1953, in Freiburg; proved the well-ordering theorem in 1904.

Cesare Burali-Forti (1861/1931). In 1897 he proposed the well-known paradox named after him.

Bertrand Russell, born May 18, 1872, in Chapstow; professor at Cambridge; in 1903 proposed the well-known paradox named after him. His *Introduction to Mathematical Philosophy* appeared in 1923.

2. *Intuitionalism and formalism*:

Leopold Kronecker, born December 7, 1823, in Liegnitz, died December 29, 1891, in Berlin; teacher of Cantor and forerunner of intuitionalism. He said "The integers were made by God, all else is the work of man." He rejected his student Cantor's theory of sets.

Hermann Weyl, born November 9,1885, in Elmshorn, died December 1955, in Geneva. He was professor at Princeton and among other works wrote: *The Continuum* (1918), *Concerning the New Crisis in Foundations of Mathematics* (1921), *Philosophy of Mathematics and Natural Sciences* (1926).

Luitzen Egbertus Jan Brouwer, born February 27,1881, in Overschie, Netherlands; professor at Amsterdam. He rejected Cantor's theory of sets and turned against Hilbert's formalistic methods. Among other things he wrote: *Concerning the Foundations of Knowledge* (1907), *Intuitionalism and Formalism* (1912), *Intuitionalistic Theory of Sets* (1919).

David Hilbert, born January 23, 1861, in Königsburg, died February 14, 1932, in Göttingen. He brought the axiomatic method to completion. Among his works were *Foundations of Geometry* (1899) and *Foundations of Mathematics* (1928). He was an opponent of the intuitionalists. He wrote, "I must again bring to mathematics the voice of indisputable truth that it seems to be losing through the paradoxes of the theory of sets: indeed, I think that this is possible with complete maintenance of all of its possessions. The method that I employ, is no other than the axiomatic method."

XXI. Bibliography

For those who care to study further in the theory of sets, the following references will be found an extension and deepening of the present volume, yet to too difficult.

Cantor, George, *Contributions to the Founding of the Theory of Transfinite Numbers.* Dover Publications, Inc., New York. These original papers of Cantor (1895, 1897) were translated from the German by Philip E. B. Jourdain, who provides an 82 page historical and explanatory introduction. Cantor's work is somewhat abstract and not well adapted for first reading in the subject.

Duren, W. L., et al., *Universal Mathematics—Part II.* Tulane University Book Store, New Orleans, 1955. A presentation of set theory from a contemporary point of view. It treats largely of finite sets and Boolean algebra. In addition, it gives the finite combinatorial background used in the present book.

Fraenkel, Abraham A. *Abstract Set Theory.* North Holland Publishing Co., Amsterdam, 1953. A more rigorous and thorough treatment of the material in this volume. An excellent follow-up for the interested reader. It contains the most comprehensive bibliography of all writings on set theory up to 1950 (128 pages).

Kamke, E. *Theory of Sets.* Dover Publications, Inc., New York. Translated from the German by F. Bagemihl, this is a presentation in which the degree of difficulty is between that of the present volume and Frankel's Abstract Set Theory.

Kemeny, J. G., Snell, J. L., and Thompson, G. L. *Introduction to Finite Mathematics.* Prentice-Hall, Inc., Englewood Cliffs, N. J. This is a modern approach to finite sets and operations with sets. The applications are to other than the physical sciences, are close to experience, and furnish good background for later study of infinite sets and limiting processes. It should be read along with the present volume.

Weyl, Hermann, *Philosophy of Mathematics and the Natural Sciences.* Princeton University Press. A good account of the philosophical problems in mathematics (Metamathematics) including that of formalism and intuitionalism.

For those who can read German and French the following references are of value.

Bachmann, Heinz, *Transfinite Zahlen.* Springer-Verlag, Berlin. The most recent point of view on set theory and a most comprehensive bibliography of works from 1950 to 1955. A first treatment of arithmetic of cardinal numbers without use of the axiom of choice.

Bourbaki, N., *Theorie des Ensembles. Livre I.* Herman & C^{ie}, Paris. The first and very compact volume by a group of mathematicians (called Bourbaki) on the rigorous foundation of analysis. A very difficult book.

Hausdorff, F., *Mengenlehre*, 3rd rev. ed., Dover Publications, Inc., New York. This is a comprehensive treatment with much descriptive and explanatory material.

XXII. Glossary of Symbols

$a = b$ *a* is equal to *b*.
$a \neq b$ *a* is not equal to *b*.
$a < b$ *a* is less than *b*.
$a > b$ *a* is greater than *b*.
$M = \{1,2,3,\ldots,n\}$ Finite set with the elements: 1, 2, 3, ... , *n*
$M = \{1,2,3,\ldots\}$ Infinite set with the elements: 1, 2, 3, ...

$a \varepsilon M$ *a* is an element of *M*.

$b \notin$ M *b* is not an element of *M*.

$M = N$ The sets *M* and *N* are equal. They contain the same elements.

$N \subset M$ *N* is a proper subset of *M*.

$N \subseteq M$ *N* is a (proper or improper) subset of *M*.

$R = M — N = \bar{R}$ The complementary set *R* to *N* over *M* contains only those elements of *M* that do not belongs to the subset *N*.

$S = M \cup N$ The union *S* contains those elements that are contained either in *N* or in *M*, or in *M* and *N* simultaneously.

$D = M \cap N$ The intersection contains only those elements that are contained both in *M* and in *N*.

$M \sim N$ *M* is equivalent to *N*. *M* can be mapped on *N*.

$M \nsim N$ *M* is not equivalent to *N*.

$|M| = m$ Cardinal number (power) of *M*

a Cardinal number of a denumerable set

c Cardinal number of the continuum

$M \times N$ The cross product set (Cartesian product) contains the elements:

$$(m_1,n_1),\ (m_1,n_2)\ \ldots\ (m_2,n_1),\ (m_2,n_2)\ \ldots.$$

$$|M \times N| = \mathfrak{m} \cdot \mathfrak{n}.$$

N/M The covering set *N/M* (*N* is covered by *M*) has the cardinal number $|N/M| = m^n$.

U(*M*) The power set of *M* is the set of all subsets of *M*.

$$|U(M)| = 2^{\mathfrak{m}} > \mathfrak{m}.$$

$a \prec$ *a* comes before *b*.

$a \succ$ *a* comes after *b*.

$M \simeq N$ *M* and *N* are similar and have equal ordinal-types.

ω Ordinal-type (ordinal number) of the set {1,2,3,}.

*ω Ordinal-type of the set {... 3,2,1}.

η Ordinal-type of the set of all rational points of a straight line.

λ Ordinal-type of the unbounded linear continuum.

θ Ordinal-type of the bounded linear continuum.

μ = [*M*] Ordinal-type of the ordered set *M*.

a, *b* Closed interval (including *a* and *b*).

(*a,b*) Open interval (excluding *a* and *b*).

a,*b*); (*a,b* Half-open interval.

$a|b$ Jump. Between a and b there are no elements.

ANSWERS TO EXERCISES

Page 7

1. Teachers, students, flowers, chairs, etc. **2**. No. The lattice points (±1,±1) still belong to *K*, but they are no longer elements of *Q*. **3**. No. It contains the lattice points (±2,±3) and (±3,±2). Make a drawing. **4**. $\left\{\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{9}, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}, \frac{3}{4}, \ldots, \frac{8}{9}\right\}$. The set contains 27 elements. **5**. The inverses of the eight unit fractions $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{9}$ are not improper fractions, but whole numbers.

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1. $2^4 = 16$ subsets.{}, {2}, {3}, {5}, {7}, {2,3}, {2,5}, {2,7}, {3,5}, {3,7}, {5,7}, {2,3,5}, {2,3,7}, {2,5,7}, {3,5,7}, {2,3,5,7}. **2**. The assertion follows from the definition. **3**. Yes. It follows from the definitions of the operations. **4**. $\binom{10}{5} = 252$. Within each squad there are in addition 5! = 120 different permutations of positions. **5**. An improper subset, for all seniors must take the examinations.

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1. Feet and shoes; eyes and ears; etc. Perhaps books and book covers. **2**. 4! = 24, e.g.:

$$\begin{array}{llll} a \leftrightarrow a & a \leftrightarrow a & a \leftrightarrow a & \text{etc.} \\ b \leftrightarrow b & b \leftrightarrow b & b \leftrightarrow c & \\ c \leftrightarrow c & c \leftrightarrow d & c \leftrightarrow b & \\ d \leftrightarrow d & d \leftrightarrow c & d \leftrightarrow d & \end{array}$$

3. (a) $K \subset Q$; (b) $K; \cap Q = K$; (c) $|Q — K| = 12$; (d) Yes, even equal. Make a drawing. **4**. Only (a), (b), (c), and (e). **5**. In the domain of the definition they must be in one-to-one correspondence. **6**. $|K| = |P| = 21$. **7**. If in the pairing for the purposes of dancing, no one is left unpaired, then $D \sim H$. **8**. Let us hope so. **9**. $y = 2x$. **10**. Yes; $N \sim G$.

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1. Coordinate the set of points through parallel projection by drawing parallels t.o the third side.

2. $z = 10^{n-1}$;

n	1	2	3
z	1	10	100

. 3. Consider Figures 29, 30, and 31. 4. $n!$

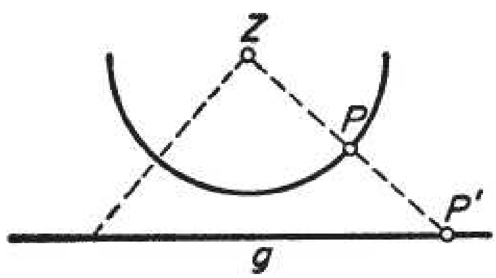

Figure 29. Mapping a semicircle on a straight line.

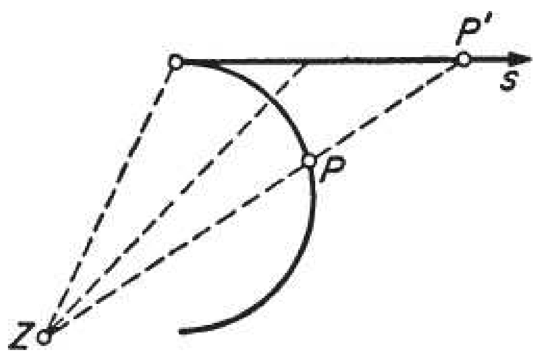

Figure 30. Mapping a semicircle on a ray.

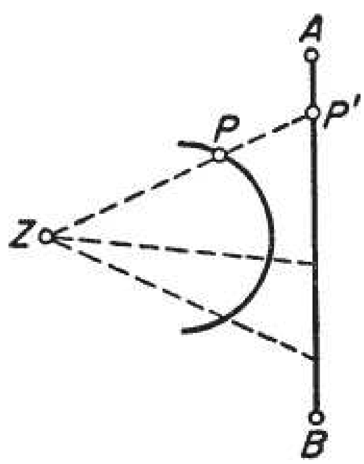

Figure 31. Mapping a semicircle on a segment.

1. a!; to compute a! see Exercise 2, page 15. **2**. Yes. Construct the arithmetic mean $(r_1 + r_2)/2$. **3**. To the algebraic numbers listed on page 25, the height five will give in addition the following numbers:

$$\pm\tfrac{1}{3};\quad \pm 3;\quad \pm\sqrt{\pm\tfrac{1}{2}};\quad \pm\sqrt{\pm 2};\quad \pm\tfrac{1}{2}\pm\sqrt{\tfrac{1}{4}\pm 1}.$$

4. Sin 7°30´ is an algebraic number for $\sin 7°30' = +\sqrt{\tfrac{1}{2}(1 - \cos 15°)}$ and $\cos 15° = \sqrt{\tfrac{1}{2}(1 + \cos 30°)}$ and $\cos 30° = \sqrt{3}/2$. In fact, sinα° is always an algebraic number, if α is rational. On the contrary, sin x (x = radian measure), where x is rational, is always a transcendental number.

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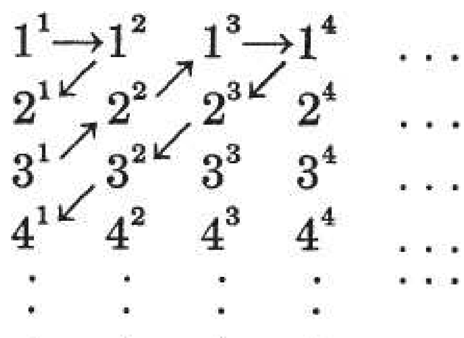
1. Prove it as in Section VI, paragraph 7. (The set of points in the interior of a cube.) **2**. $z = x + iy$. The cardinal number of the set of all real number pairs (x,y) is c. Therefore, the cardinal number of the set of all complex numbers is c. **3**. a. **4**. Yes. **5**. The cardinal number of all powers m^n is that of the set of all pairs of natural numbers in the plane. One can also prove this by the diagonal process:



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1. The cardinal number of all pairs of real numbers c_1 and c_2 is c. **2**. The circle set is the set of all points equivalent to the three-dimensional space, therefore it has the cardinal number c. Consider that a, b, and r are Cartesian coordinates of space points. **3**. $1 < 2 < 3 < 4 < \ldots < a < c < f$.

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1. Yes. If M and N have the same finite cardinal number, neither of the sets is equivalent to a proper subset of the other. **2**. For example, choose $U_1 = \{3,7,11,\ldots\}$ and $G_1 = \{2,6,10,\ldots\}$. Then $U_1 \subset U$, $G_1 \subset G$, and $U_1 \sim G$: the ordering is obtained by $u_1 = 2g - 1$. $G_1 \sim U$: the ordering is obtained by $g_1 = 2u$. **3**. The subset EF of AB is mapped on CD by a central projection from Z.$_1$

The subset *GH* of *CD* is mapped on *AB* by a central projection from Z_2.

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1. Yes. **2**. $a! = c$; for $2^a \leqq a^a \leqq a^a$ and $2^a = a^a = c$. **3**. Always.

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1. $|N/M| = 3^{12} = 531{,}441$. **2**. $0 < y < 1$. **3**. $Y = \{-1, +1\}$. **4**. $|N/M| = 6^{16} = 2{,}821{,}109{,}907{,}456$. **5**. $1^m = 1$. All *m* places of *M* are covered with a single element. The covering set has the cardinal number 1. $m^1 = m$. A place can be covered by *m* different element. There are *m* coverings. The covering set has the cardinal number nt. **6**. The set *X*, ($|X| = c$) is covered by the set *Y* ($|Y| = a$). This gives $a^c = f$.

7.$m^p - n^p = m\text{-}m\text{-}m \ldots n\text{-}n\text{-}n \ldots = m\text{-}n\text{-}m\text{-}n\text{-}m\text{-}n \ldots = (m\text{-}n)^p$, since the commutative law holds.

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1.
$$(1 + \omega) + ({}^*\omega + 1) + \omega = \omega + {}^*\omega + \omega$$
$$= [\{a,1,3,5,\ldots,6,4,2,b,c_1,c_2,c_3,\ldots\}]$$

2. $1 + \eta + 1$

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1. The well-ordered sets are $Z, Z_1 Z_2, Z_3$.

2. $[Z] = \omega$ is an ordinal number;
$[\bar{Z}] = {}^*\omega + \omega$
$[Z_1] = \omega + \omega = \omega \cdot 2$ is an ordinal number;
$[Z_2] = \omega + \omega = \omega \cdot 2$ is an ordinal number;
$[Z_3] = \omega + \omega + 1 = \omega \cdot 2 + 1$ is an ordinal number.

3. No. **4**. Yes. The section belonging to the last element of the second set (having the ordinal number $\mu + 1$) is similar to the first set (having the ordinal number μ). **5**. Yes, in infinitely many ways, e.g.,

$$\begin{array}{l}\{\ldots,-2,-1,0,+1,+2,\ldots\};\\ \qquad\ \ \updownarrow\quad \updownarrow\ \updownarrow\quad \updownarrow\quad \updownarrow\\ \{\ldots,-3,-2,-1,0,+1,\ldots\}.\end{array}$$

6. No. Sets of the ordinal type $\eta + 1$ have no first element. Sets of the type $1 + \mu$ have subsets without a first element (e.g., the subset that arises from removing the first element).

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1. {1 $\cup$ rational points of (1,2)} $\cup$ {2 $\cup$ rational points of (2,3)} $\cup$ +... $\cup$ {n $\cup$ rational points of (n,n + 1)} = {1 $\cup$ rational points of (1, n + 1)}.
2. {(0,1) $\cup$ 1} $\cup$ {(1,2) $\cup$ 2} $\cup$ ... $\cup$ {(n − 1, n) $\cup$ n} = {(0,n) $\cup$ n}.
3. $x_1 = (n + 1)/2n$, $x_2 = 1/n$. Accumulation points are $\frac{1}{2}$, 0.
4. All finite sets have no accumulation points. Some unbounded infinite sets have no accumulation points, e.g., {1,2,3,4,...}.

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1. No. It does not contain its accumulation point "0." **2**. The set of all points in the interval 1,2. **3**. The set of all points in the interval 1,2. **4**. No. It does not contain irrational points of the interval that are accumulation points of the rational points. **5**. Yes. Compare with Exercise 2, page 26. **6**. Yes. Every point is an accumulation point. **7**. Yes. Yes. The derivative of any arbitrary selected set is closed. The derivative of the given set is the interval 2,3. This interval is closed; it is dense-in-itself, hence also perfect. **8**. No. It is merely closed, e.g., M = {0.1,0.01,0.001,...}. Then M^1 = {0}, 0 is no accumulation point of M^1.

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1. Assume $\sqrt{2} = m/n$, then $m^2/n^2 = 2$ or $m^2 = 2n^2$. It follows from this that m must be an even number. Let $m = 2k$ (k a natural number). Then $m_2 = 4k_2 = 2n^2$ or $n^2 = 2k^2$, from which it follows that n must be an even number. This contradicts the premise that m and n are relatively prime. **2**. $\sqrt{5}$ is that gap which separates all the rational numbers m/n for which $m^2/n^2 < 5$ from those rational numbers for which $m^2/n^2 > 5$.
3. (a) A jump: 1|2; (b) A gap; (c) A continuous cut.

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1. All measurable values satisfy the given equation. **2**. Yes. Y = {(0, In 2}. **3**. Let $s = 4x_1 \cdot \epsilon = 4\epsilon$, then (a) $s = 0.8$; (b) $s = 0.08$; (c) $s = 0.004$. **4**. $s = 10 \cdot \epsilon < 0.001$. **5**. $s = 4$. **6**. No. At x_1 = —1 there is a pole. At = +1 there is an (indeterminate) removable discontinuity. The condition is $y \text{ (at } x_2 = +1) = \lim_{x \to 1} y = -\frac{1}{2}$. **7**. $y = f(x)/g(x)$ where $f(x)$ and $g(x)$ are everywhere continuous, is also everywhere continuous, except for $g(x) = 0$. **8**. Yes, if we assume that between two measured body temperatures every intermediate value is taken on, i.e., the temperature does not rise by discrete quantities. (Newton: "Nature does not make leaps.")

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