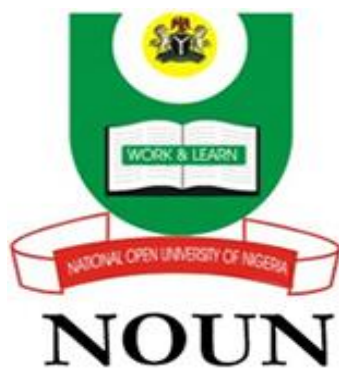




NATIONAL OPEN UNIVERSITY OF NIGERIA

ADVANCED MATHEMATICAL ECONOMICS

ECO 459

FACULTY OF SOCIAL SCIENCES

COURSE GUIDE

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Introduction

Welcome to ECO: 459 ADVANCED MATHEMATICAL ECONOMICS.

ECO 459: Advanced Mathematical Economics is a two-credit and one-semester undergraduate course for Economics student. The course is made up of thirteen units spread across fifteen lectures weeks. This course guide gives you an insight to Advanced Mathematical Economics

in a broader way and how to study the make use and apply mathematical analysis in economics. It tells you about the course materials and how you can work your way through these materials. It suggests some general guidelines for the amount of time required of you on each unit in order to achieve the course aims and objectives successfully. Answers to your tutor marked assignments (TMAs) are therein already.

Course Content

This course is basically on Advanced Mathematical Economics because as you are aspiring to become an economist, you must be able to apply mathematical techniques to economics problems. The topics covered include linear algebraic function, Calculus I: Differentiation, Calculus II: Integration and Differential analysis

Course Aims

The aims of this course is to give you in-depth understanding of the economics as regards

- Fundamental concept and calculation of probability distribution
- To familiarize students with the knowledge of hypotheses testing
- To stimulate student's knowledge on sampling theory
- To make the students to understand the statistical calculation of t test, f test and chi-square analysis.
- To expose the students to analysis of simple linear regression analysis
- To ensure that the students know how to apply simple linear regression to economics situations.
- To make the students to be to interpret simple linear regression analysis result.

Course Objectives

To achieve the aims of this course, there are overall objectives which the course is out to achieve though, there are set out objectives for each unit. The unit objectives are included at the beginning of a unit; you should read them before you start working through the unit. You may want to refer to them during your study of the unit to check on your progress. You should always look at the unit objectives after completing a unit. This is to assist the students in accomplishing the tasks entailed in this course. In this way, you can be sure you have done what was required of you by the unit. The objectives serves as study guides, such that student could know if he is able to grab the knowledge of each unit through the sets of objectives in each one. At the end of the course period, the students are expected to be able to:

- Understand the history of linear algebra
- know how to calculate linear equation
- also know meaning and how to calculate exponential equation
- Understand the rudimentary of simultaneous equation
- know meaning and how to calculate Elimination and Substitution methods of simultaneous equation.
- understand the meaning of sequence and series
- understand and know how to calculate Arithmetic and Geometric progression.
- also know the meaning and how to calculate sum of Arithmetic and Geometric progression and sum to infinity of a Geometric progression.
- Understand the meaning of differentiation and know how to calculate differentiation problems such as differentiation from the first principle, a constant and polynomials.
- Understand other techniques of differentiation such as product rule, quotient rule and the implicit function.
- Understand how to calculate exponential, logarithmic and trigonometric functions
- Differentiate $\sin x$, $\cos x$ and $\tan x$
- Know how to apply differentiation to economics problem.
- Understand how to calculate definite and indefinite integral
- Know how to work through integration by a constant and polynomial
- Understand how to calculate integration of exponential function
- Know how to calculate integration of trigonometric function

- Understand how to calculate integration by substitution and by parts
- Understand how to calculate integration by substitution and by parts
- Know how to apply integration to economic problems
- Understand how to calculate differential equation and separation of variables
- Know how to apply partial integration and integrating factor
- Understand how to apply differential equation to economics
- Know how to apply the general formula of a difference equations
- Understand how to calculate lagged income determination model, cobweb model and the Harrod Dormar model
- Know the meaning and types of Optimization problems
- Understand how to calculate Optimization problems, Unrestricted Optimization and Constraint Optimization
- Know the meaning of dynamic Economics, scope, concepts and limitations.
- Understand how to calculate dynamic equation

Working Through The Course

To successfully complete this course, you are required to read the study units, referenced books and other materials on the course.

Each unit contains self-assessment exercises called Student Assessment Exercises (SAE). At some points in the course, you will be required to submit assignments for assessment purposes. At the end of the course there is a final examination. This course should take about 15weeks to complete and some components of the course are outlined under the course material subsection.

Course Material

The major component of the course, What you have to do and how you should allocate your time to each unit in order to complete the course successfully on time are listed follows:

1. Course guide
2. Study unit
3. Textbook
4. Assignment file
5. Presentation schedule

Study Unit

There are 13 units in this course which should be studied carefully and diligently.

MODULE ONE: LINEAR ALGEBRAIC FUNCTION

- UNIT 1 Linear Algebra**
- UNIT 2 Simultaneous Equation**
- UNIT 3 Sequence and Series**

MODULE TWO: CALCULUS I: DIFFERENTIATION

- UNIT 1 DIFFERENTIATION**
- UNIT 2 Other Techniques of Differentiation: the Composite Function**
- UNIT 3 Differentiation of Exponential Logarithmic and Trigonometric Functions**

MODULE 3 CALCULUS II: INTEGRATION

- Unit 1 Integration**
- Unit 2 Integration of Exponential and Trigonometric**
- Unit 3 Integration by Substitution and by Parts**

MODULE FOUR DIFFERENTIAL ANALYSIS

- UNIT 1 Differential Equation**
- UNIT 2 Difference Equation**
- UNIT 3 Optimization**
- UNIT 4 Dynamics Analysis**

Each study unit will take at least two hours, and it include the introduction, objective, main content, self-assessment exercise, conclusion, summary and reference. Other areas border on the Tutor-Marked Assessment (TMA) questions. Some of the self-assessment exercise will necessitate discussion, brainstorming and argument with some of your colleges. You are advised to do so in order to understand and get acquainted with historical economic event as well as notable periods.

There are also textbooks under the reference and other (on-line and off-line) resources for further reading. They are meant to give you additional information if only you can lay your hands on any of them. You are required to study the materials; practice the self-assessment

exercise and tutor-marked assignment (TMA) questions for greater and in-depth understanding of the course. By doing so, the stated learning objectives of the course would have been achieved.

Textbook and References

For further reading and more detailed information about the course, the following materials are recommended:

Adewale, A. O., (2014)., Introduction to Advanced Mathematics, 2nd edition, Meltor Press Limited, Lagos Nigeria.

Adesanya, A. A., (2014). Introduction to Linear Algebra, 1st edition, Melting Point Publication limited, Lagos Nigeria.

Adeyanju, D. F., (2008)., Introduction to Basic Mathematics, 1st edition, Tinder Publication limited, Lagos Nigeria.

Babatunde, K. L., (2016)., Economics and Mathematics, a broader perspectives, 1st edition, BTT Press Limited, Nigeria

Cecilia, F. E., (2015)., Introduction to Mathematics, Economics and its Application, 2nd edition, Messer Press Limited, Lagos, Nigeria.

Chantyl, M. I., (2012)., Intoduction to Mathematical Economics, 1st edition, Messer Press Limited, Lagos, Nigeria.

Dalt, M. A., (2014)., Basic Mathematics, an Empirical Analysis 1st edition, Tinder Publication limited, Lagos Nigeria.

Dowling, E. T., (1984)., Introduction to Mathematical Economics, 2nd edition, Macgaw-Hill Publication Limited.

Faraday, M. I., (2013)., Impact of Mathematics on Economics Analysis, vol 4, pg 28-39, Mac Publication Limited

George, S. R., (2009)., Introduction to Mathematical Economics, 2nd edition, DTT Publication Limited.

Jelius, M. D., (2010)., Introduction to Mathematical Economics, 1st

edition, Top world Publication Limited.

Sander, E. T., & Mergor, S. G., (2011)., Introduction to Basic Mathematics, 1st edition, Mill world Publication limited, Lagos Nigeria.

Sogunro, Y., & Abdurasid, A., & Adewuyi, B., (1998)., Basic Business Mathematics, Concepts and Applications, volume 2, Sogunro Publication Limited, Lagos State Polytechnic, Isolo, Lagos.

Talbert, J. P., & Godman, A., Ogum, G. E. O., (1984)., Additional Mathematics for West Africa, Longman Group UK Limited.

Assignment File

Assignment files and marking scheme will be made available to you. This file presents you with details of the work you must submit to your tutor for marking. The marks you obtain from these assignments shall form part of your final mark for this course. Additional information on assignments will be found in the assignment file and later in this Course Guide in the section on assessment.

There are four assignments in this course. The four course assignments will cover:

Assignment 1 - All TMAs' question in Units 1 – 4 (Module 1 and 2)

Assignment 2 - All TMAs' question in Units 5 – 7 (Module 2 and 3)

Assignment 3 - All TMAs' question in Units 8 – 10 (Module 3 and 4)

Assignment 4 - All TMAs' question in Unit 11 – 13 (Module 4).

Presentation Schedule

The presentation schedule included in your course materials gives you the important dates for this year for the completion of tutor-marking assignments and attending tutorials. Remember, you are required to

submit all your assignments by due date. You should guide against falling behind in your work.

Assessment

There are two types of the assessment of the course. First are the tutor-marked assignments; second, there is a written examination.

In attempting the assignments, you are expected to apply information, knowledge and techniques gathered during the course. The assignments must be submitted to your tutor for formal Assessment in accordance with the deadlines stated in the Presentation Schedule and the Assignments File. The work you submit to your tutor for assessment will count for 30 % of your total course mark.

At the end of the course, you will need to sit for a final written examination of three hours' duration. This examination will also count for 70% of your total course mark.

Tutor-Marked Assignments (TMAs)

There are four tutor-marked assignments in this course. You will submit all the assignments. You are encouraged to work all the questions thoroughly. The TMAs constitute 30% of the total score.

Assignment questions for the units in this course are contained in the Assignment File. You will be able to complete your assignments from the information and materials contained in your set books, reading and study units. However, it is desirable that you demonstrate that you have read and researched more widely than the required minimum. You should use other references to have a broad viewpoint of the subject and also to give you a deeper understanding of the subject.

When you have completed each assignment, send it, together with a TMA form, to your tutor. Make sure that each assignment reaches your tutor on or before the deadline given in the Presentation File. If for any reason, you cannot complete your work on time, contact your tutor before the assignment is due to discuss the possibility of an extension. Extensions will not be granted after the due date unless there are exceptional circumstances.

Final Examination and Grading

The final examination will be of three hours' duration and have a value of 70% of the total course grade. The examination will consist of questions which reflect the types of self-assessment practice exercises and tutor-marked problems you have previously encountered. All areas of the course will be assessed

Revise the entire course material using the time between finishing the last unit in the module and that of sitting for the final examination to. You might find it useful to review your self-assessment exercises, tutor-marked assignments and comments on them before the examination. The final examination covers information from all parts of the course.

Course Marking Scheme

The Table presented below indicates the total marks (100%) allocation.

Assignment	Marks
Assignments (Best three assignments out of four that is marked)	30%
Final Examination	70%
Total	100%

Course Overview

The Table presented below indicates the units, number of weeks and assignments to be taken by you to successfully complete the course, Statistics for Economist (ECO 254).

Units	Title of Work	Week's Activities	Assessment (end of unit)
	Course Guide		
Module 1 LINEAR ALGEBRAIC/EXPONENTIAL FUNCTION			
1	Linear Algebra	Week 1&	Assignment 1

		2	
2	Simultaneous Equation	Week 3 & 4	Assignment 1
3	Sequence and Series	Week 5	Assignment 1
Module 2 CALCULUS I: DIFFERENTIATION			
1	Differentiation	Week 6	Assignment 1
2	Other Techniques of Differentiation: the Composite Function	Week 7	Assignment 2
3	Differentiation of Exponential Logarithmic and Trigonometric Functions	Week 8	Assignment 2
Module 3 CALCULUS II: INTEGRATION			
1	Integration	Week 9	Assignment 2
2	Integration of Exponential and Trigonometric	Week 10	Assignment 3
3	Integration by Substitution and by Parts	Week 11	Assignment 3
Module 4 DIFFERENTIAL ANALYSIS			
1	Differential Equation	Week 12	Assignment 3
2	Difference Equation	Week 13	Assignment 4
3	Optimization	Week 14	Assignment 4
4.	Dynamics Analysis	Week 15	Assignment 4
	Total	15 Weeks	

How To Get The Most From This Course

In distance learning the study units replace the university lecturer. This is one of the great advantages of distance learning; you can read and work through specially designed study materials at your own pace and at a time and place that suit you best.

Think of it as reading the lecture instead of listening to a lecturer. In the same way that a lecturer might set you some reading to do, the study units tell you when to read your books or other material, and when to embark on discussion with your colleagues. Just as a lecturer might give you an in-class exercise, your study units provides exercises for you to do at appropriate points.

Each of the study units follows a common format. The first item is an introduction to the subject matter of the unit and how a particular unit is integrated with the other units and the course as a whole. Next is a set of learning objectives. These objectives let you know what you should be able to do by the time you have completed the unit.

You should use these objectives to guide your study. When you have finished the unit you must go back and check whether you have achieved the objectives. If you make a habit of doing this you will significantly improve your chances of passing the course and getting the best grade.

The main body of the unit guides you through the required reading from other sources. This will usually be either from your set books or from a readings section. Some units require you to undertake practical overview of historical events. You will be directed when you need to embark on discussion and guided through the tasks you must do.

The purpose of the practical overview of some certain historical economic issues are in twofold. First, it will enhance your understanding of the material in the unit. Second, it will give you practical experience and skills to evaluate economic arguments, and understand the roles of history in guiding current economic policies and debates outside your studies. In any event, most of the critical thinking skills you will develop during studying are applicable in normal working practice, so it is important that you encounter them during your studies.

Self-assessments are interspersed throughout the units, and answers are given at the ends of the units. Working through these tests will help you to achieve the objectives of the unit and prepare you for the assignments and the examination. You should do each self-assessment exercises as

you come to it in the study unit. Also, ensure to master some major historical dates and events during the course of studying the material.

The following is a practical strategy for working through the course. If you run into any trouble, consult your tutor. Remember that your tutor's job is to help you. When you need help, don't hesitate to call and ask your tutor to provide it.

1. Read this Course Guide thoroughly.
2. Organize a study schedule. Refer to the 'Course overview' for more details. Note the time you are expected to spend on each unit and how the assignments relate to the units. Important information, e.g. details of your tutorials, and the date of the first day of the semester is available from study centre. You need to gather together all this information in one place, such as your diary or a wall calendar. Whatever method you choose to use, you should decide on and write in your own dates for working through each unit.
3. Once you have created your own study schedule, do everything you can to stick to it. The major reason that students fail is that they get behind with their course work. If you get into difficulties with your schedule, please let your tutor know before it is too late for help.
4. Turn to Unit 1 and read the introduction and the objectives for the unit.
5. Assemble the study materials. Information about what you need for a unit is given in the 'Overview' at the beginning of each unit. You will also need both the study unit you are working on and one of your set books on your desk at the same time.
6. Work through the unit. The content of the unit itself has been arranged to provide a sequence for you to follow. As you work through the unit you will be instructed to read sections from your set books or other articles. Use the unit to guide your reading.
7. Up-to-date course information will be continuously delivered to you at the study centre.
8. Work before the relevant due date (about 4 weeks before due dates), get the Assignment File for the next required assignment. Keep in mind that you will learn a lot by doing the assignments carefully. They have been designed to help you meet the objectives of the course and, therefore, will help you pass the exam. Submit all assignments no later than the due date.

9. Review the objectives for each study unit to confirm that you have achieved them. If you feel unsure about any of the objectives, review the study material or consult your tutor.
10. When you are confident that you have achieved a unit's objectives, you can then start on the next unit. Proceed unit by unit through the course and try to pace your study so that you keep yourself on schedule.
11. When you have submitted an assignment to your tutor for marking do not wait for it return before starting on the next units. Keep to your schedule. When the assignment is returned, pay particular attention to your tutor's comments, both on the tutor-marked assignment form and also written on the assignment. Consult your tutor as soon as possible if you have any questions or problems.
12. After completing the last unit, review the course and prepare yourself for the final examination. Check that you have achieved the unit objectives (listed at the beginning of each unit) and the course objectives (listed in this Course Guide).

Tutors and Tutorials

There are some hours of tutorials (2-hours sessions) provided in support of this course. You will be notified of the dates, times and location of these tutorials. Together with the name and phone number of your tutor, as soon as you are allocated a tutorial group.

Your tutor will mark and comment on your assignments, keep a close watch on your progress and on any difficulties you might encounter, and provide assistance to you during the course. You must mail your tutor-marked assignments to your tutor well before the due date (at least two working days are required). They will be marked by your tutor and returned to you as soon as possible.

Do not hesitate to contact your tutor by telephone, e-mail, or discussion board if you need help. The following might be circumstances in which you would find help necessary. Contact your tutor if.

- You do not understand any part of the study units or the assigned readings
- You have difficulty with the self-assessment exercises
- You have a question or problem with an assignment, with your tutor's comments on an assignment or with the grading of an assignment.

You should try your best to attend the tutorials. This is the only chance to have face to face contact with your tutor and to ask questions which are answered instantly. You can raise any problem encountered in the course of your study. To gain the maximum benefit from course tutorials, prepare a question list before attending them. You will learn a lot from participating in discussions actively.

Summary

The course, Advanced Mathematical Economics (ECO 459), expose you to the analysis of linear Algebra/exponential function and you will also be introduced to Linear Algebra, Simultaneous Equation, Sequence and Series. This course also gives you insight into Calculus I: differentiation in other to know some of the techniques of Differentiation, Other Techniques of Differentiation: the Composite Function, Differentiation of Exponential Logarithmic and Trigonometric Functions. Thereafter it shall enlighten you about Calculus II: integration which gives an insight to the techniques of integration, Integration of Exponential and Trigonometric, Integration by Substitution and by Parts. Finally, the calculation of Differential analysis was also examined to enable you understand more about Differential Equation, Difference Equation, Optimization and Dynamics Analysis.

On successful completion of the course, you would have developed critical thinking skills with the material necessary for efficient and effective discussion on Advanced Mathematical Economics: Linear Algebra/exponential function, calculus I: differentiation, calculus II: integration and Differential analysis.

However, to gain a lot from the course please try to apply anything you learn in the course to term papers writing in other economics courses. We wish you success with the course and hope that you will find it fascinating and handy.

MODULE ONE: LINEAR ALGEBRAIC FUNCTION

UNIT 1 Linear Algebra

UNIT 2 Simultaneous Equation

UNIT 3 Sequence and Series

CONTENTS

0.0	Introduction
1.0	Objectives
2.0	Main Content
3.1	History of Linear Algebra
3.2	Linear Equation
3.2.1	One Variable
3.2.2	Two Variable
3.3	Exponential Equation
4.0	Conclusion
5.0	Summary
6.0	Tutor-Marked Assignment
7.0	References/Further Readings

UNIT 1 LINEAR ALGEBRA

1.0 INTRODUCTION

Linear algebra is the branch of mathematics concerning vector space and linear mappings between such spaces. It includes the study of lines, planes and subspaces, but it can also be said that linear algebra is concerned with some of the properties common to all the vector spaces. Linear algebra is also central to both pure and applied mathematics. For instance, abstract algebra is when we relax the axioms of a vector space, leading to a number of generalizations. However, functional analysis studies the infinite – dimensional version of the theory of vector spaces.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- understand the history of linear algebra
- know how to calculate linear equation
- also know meaning and how to calculate exponential equation

3.0 MAIN CONTENT

3.1 History of Linear Algebra

The study of linear algebra first emerges from the study of determination that were used in solving different systems of linear equations. It should then be noted that Leibniz in 1663 and Gabriel Cramer derived Cramer's rule for solving the theory of linear systems by using Gaussian elimination.

Matrix algebra was also first to emerge in England in the mid-1800. The theory of extension was published by Hermann Grassmann which is now been called today as linear algebra.

3.2 Linear Equation

A linear equation is called an algebraic equation in which each term is either a constant or the product of a constant is a single variable. An example of a linear equation with only one variable, say x can be written as $ax + b = 0$, where a and b are constants and $a \neq 0$. In the equation above the constants may be numbers parameters or linear functions of parameters. However, the differences between the variables and parameters may depend on the problem.

Let us take a look at three variable linear equations $ax + by + cz + d = 0$, where a, b, c and d are constants and a, b and c are now zero. Linear equations can then occur frequently in most areas of mathematics and especially in applied mathematics.

Let also take a look at equations with exponents greater than one are non-linear and an example of such non-linear of two variables is $axy^tb = 0$ where a and b are constants and $a \neq 0$. It also has two variables x and y and is non-linear because the sum of the exponents of the variables in the first term axy is two.

Let us start our calculation with one variables;

3.2.1 One Variable

$$ax = b \text{ _____ (1)}$$

If $a = 0$, then when $b = 0$, every number is a solution, but if $b \neq 0$ there is no solution.

Example 1

Given $2x = 8$

$$x = \frac{8}{2}$$

$$x = 4$$
$$x = 4.$$

Example 2

Given $16x = 32$

$$x = \frac{32}{16}$$

$$x = 2.$$

Example 3

Given $16m = Q$

$$m = \frac{Q}{16}.$$

3.2.2 Two Variable

For two variable; $y = mx + b$, where m and b are constants or parameter.

Example 4

Given

$$y = 2x + 4m. \text{ find } m \text{ in terms of } y.$$

Solution

$$y = 2x + 4m$$

$$y - 2x + 4m \text{ (make } m \text{ the subject of the formula)}$$

Divide both sides by 4

$$\frac{y - 2x}{4} = \frac{4m}{4}.$$

$$m = \frac{y - 2x}{4}$$

Example 5

$$2x + 8 = -6x + 4$$

Find x

Solution

Collect like terms of x together and constant together

$$2x + 6x = 4 - 8$$

$$8x = -4$$

$$x = -\frac{4}{8} \frac{1}{2}$$

$$x = -1/2.$$

3.3 Exponential Equation

An exponential equation is one in which a variable occurs in the exponent for example

$5^{2x+3} = 5^{3x-1}$, when both sides of the equation have the same base, the exponents on either side are equal by the property $x^a = x^b$ then $a = b$.

However, if $x^\circ = 1$.

Example 6

$$\text{Given } 2^{4x} \times 4^{2x-1} = 16^x \times 32^{x-2}$$

Find x .

$$2^{4x} \times 4^{2x-1} = 16^x \times 32^{x-2}.$$

$$2^{4x} \times 4^{2(2x-1)} = 2^{4(x)} \times 2^{5(x-2)}.$$

*from the law of indices when $x^a \times x^b = x^{a+b}$

* applied the law of indices; we have:

$$2^{4x} + 2(2x - 1) = 2^{4(x)+5(x-2)}$$

From the law of exponential equation, if the base are equal, then we equate the powers.

$$4x + 2(2x - 1) = 4(x) + 5(x - 2)$$

$$4x + 4x - 2 = 4x + 5x - 10$$

Collect likes terms together.

$$4x + 4x - 4x - 5x = -10 + 2$$

$$8x - 9x = -8$$

$$-x = -8.$$

Multiply both sides by negative.

$$-(-x = -8).$$

$$x = 8.$$

Example 7

Given

$$5^{6x-1} \times 25^x = 125^{2x+1} \times 625^{-x}$$

Find x

$$5^{6x-1} \times 5^{5(x)} = 5^{3(2x+1)} \times 5^{4(-x)}$$

$$5^{6x-1} + 5(x) = 5^{3(2x+1)+4(-x)}$$

Since base are equal equate their powers.

$$6x - 1 + 5x = 3(2x + 1 + 4(-x))$$

$$6x + 5x = 6x + 3 - 4x$$

Collect likes terms

$$6x + 5x - 6x + 4x = 3 + 1$$

$$11x - 2x = 4$$

$$9x = 4$$

$$x = \frac{4}{9}.$$

Self-Assessment Exercise

Given

$$2^{2x-1} \times 16^{3x} = 4^{x+4} \times 32^{-2x}$$

Find x

4.0 CONCLUSION

In this unit we can conclude that linear algebra is the study of linear sets of equations and their transformation properties. Linear algebra allows the analysis of rotations in space, least squares fitting, solution of coupled differential equations, determination of a circle passing through three given points, as well as many other problems in mathematics, physics, and engineering. Confusingly, linear algebra is not actually an algebra in the technical sense of the word algebra.

5.0 SUMMARY

In this unit, we have discuss extensively on linear algebra such as the history of linear algebra and the calculation of linear equation of one and two variables while exponential equation was also examined as a follow up to extension of linear algebra.

6.0 TUTOR-MARKED ASSIGNMENT

1. $x + 2 = 10$, find x .
2. $2x + y = 15$, find x in terms of y .
3. $6^{x+1} \times 36^{5x+2} = 6^{-x+1} \times 216^{2x}$, find x .
4. $3^{10x} \times 9^{3x+1} = 27^{x+1} \times 81^{-x+8}$ find x .

7.0 REFERENCES/FURTHER READINGS

Sander, E. T., & Mergor, S. G., (2011),. Introduction to Basic Mathematics, 1st edition, Mill world Publication limited, Lagos Nigeria.

UNIT 2 SIMULTANEOUS LINEAR EQUATION

CONTENTS

- 1.0 Introduction
- 2.0 Objectives
- 3.0 Main Content
 - 3.1 Elimination Method
 - 3.2 Substitution Method
- 4.0 Conclusion
- 5.0 Summary
- 6.0 Tutor-Marked Assignment
- 7.0 References/Further Readings

1.0 INTRODUCTION

Simultaneous linear equation is also known as a system of equations or an equation system is called a finite set of equations for which common solutions are sought. However, an equation system is usually classified in the same manner as single equations. There are three types of simultaneous equations namely: Elimination Method, Substitution Method and Graphical Method. However, in this unit the scope of this course material only is the Elimination method and the Substitution method will be used.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- understand the rudimentary of simultaneous equation
- know meaning and how to calculate Elimination and Substitution methods of simultaneous equation.

3.0 MAIN CONTENT

3.1 Elimination Method

One way of solving a linear system is to use the elimination method. In the elimination method you either add or subtract the equations to get an equation in one variable.

When the coefficients of one variable are opposites you add the equations to eliminate a variable and when the coefficients of one variable are equal you subtract the equations to eliminate a variable.

Example 1

Given

$$\begin{array}{l} 3x + 4x = 6 \text{_____} (1) \\ 5y - 4x = 10 \text{_____} (2) \end{array}$$

From the equation above, we can eliminate the x-variable by subtracting equation two from equation one.

$$\begin{array}{r} 3y + 4x = 6 \\ \underline{5y - 4x = 10} \\ 8y = 16 \\ y = \frac{16}{8} \\ y = 2 \end{array}$$

We can then substitute the value of y in any of the original equation to find the value of x.

$$3y + 4x = 6$$

$$3(2) + 4x = 6$$

$$6 + 4x = 6 \Rightarrow 4x = 6 - 6$$

$$4x = 0 \Rightarrow x = \frac{0}{4}$$

$$x = 0.$$

Therefore, the solution of the linear system is $(x, y) \Rightarrow (0, 2)$

Example 2

Given

$$6x + 2y = 18 \text{_____} (i)$$

$$10x + 8y = 44 \text{_____} (ii)$$

To eliminate x or y it depends on you. So if you want to eliminate x, you will use the coefficient of x in equation one to multiply equation two (That is 6) while the coefficient of x in equation two is to multiply equation one (that is 10).

$$\begin{array}{r|l}
 10 & 6x + 2y = 180 \text{ (i)} \\
 6 & 10x + 8y = 44 \text{ (ii)} \\
 \hline
 & \begin{array}{l}
 \cancel{60x} \quad 6x + 20y = 180 \text{ (iii)} \\
 - \\
 \cancel{60x} \quad 10x + 48y = 264 \text{ (iv)}
 \end{array}
 \end{array}$$

Then subtract equation (iv) from (iii)

$$-28y = 84$$

$$y = \underline{84}$$

$$y = 3.$$

The substitute the value of $y = 3$ into equation (iv) or (iii) to get the value of x .

$$60x + 20y = 180$$

$$60x + 20(3) = 180$$

$$60x + 60 = 180$$

$$60x = 180 - 60$$

$$60x = 120$$

$$x = \frac{120}{60}$$

$$x = 2.$$

3.2 Substitution Method

Substitution method is an algebraic method used to find an exact solution of a system of equations. Suppose there are two variables in the equation x and y . to use the substitution method, find the value of x and y from any of the given equations.

Then we can substitute the value we obtain of the given variable in the other equation such that a single variable equation is obtained and solved. The following steps should be followed to solve substitution method:

Step 1

Write any of the equations to get the value of x or y.

Step 2

Substitute that value of x or y in the other equation.

Step 3

Solve the obtained single variable equation.

Example 3

Given :

$$x + y = 2 \text{_____} (i)$$

$$2x + 2y = 6 \text{_____} (ii)$$

From equation $x + y = 2 \text{_____} (i)$

Make x the subject of the formula

$$x = 2 - y \text{_____} (iii)$$

The substitution x in equation (ii)

$$2x + 3y = 6 \text{_____} (ii)$$

$$2(2 - y) + 3y = 6$$

$$4 - 2y + 3y = 6$$

Collect likes terms

$$-2y + 3y = 6 - 4$$

$$y = 2.$$

Now, substitute $y = 2$ in equation (iii)

$$x = 2 - y \text{_____} (iv)$$

$$x = 2 - 2$$

$$x = 0.$$

$$\therefore (x, y) = (0, 2)$$

Let us take some few examples on the substitution method of simultaneous equation.

Example 4

$$\text{Given } 2x - 3y = 1 \text{_____} (i)$$

$$4x - 2y = 2 \text{_____} (ii)$$

If you remember in example three (3) above we make use of equation (i) by making x the subject of the formula.

But, let us now make use of equation (ii) in this example 4.

$$4x - 2y = 2 \text{_____} (ii)$$

Recall that you are to decide whether you want to get the value of x or y first. The values you want to get first determine the equation you are going to have.

Let us get the value of x first here, so that we can make y the subject of the formula in equation (iii).

$$\text{So, from equation (ii) } 4x + 2y = 2$$

$$2y = 2 - 4x$$

$$y = \frac{2 - 4x}{2} \text{_____} (iii)$$

Now let us substitute for y in equation (i)

$$2y - 3y = 1 \text{_____} (i)$$

$$2x - 3\left(\frac{2 - 4x}{2}\right) = 1$$

$$\frac{2x}{1} - \frac{6 + 12x}{2} = 1$$

$$\text{L.C.M} = 2.$$

Multiply through by 2

$$\frac{2x - 2x}{1} - 2x \left(\frac{6 + 12x}{2} \right) = 2x1$$

$$4x - 6 + 12x = 2$$

$$4x + 12x = 2 + 6$$

$$16x = 8$$

$$x = \frac{8}{16} \Rightarrow n = \frac{1}{2}.$$

Substitute for x in equation (iii)

$$\left(\frac{2 - 4x}{2} \right) = 1 \text{ (iv)}$$

$$y = \frac{2 - 4\left(\frac{1}{2}\right)}{2} = 1 \Rightarrow \frac{2 - 2}{2} = \frac{0}{2}.$$

$$y = 0.$$

$$(x, y) \Rightarrow \left(\frac{1}{2}, 0 \right)$$

Self Assessment Exercise

Given $x - 2y = 3$ (i)

$2x - 4y =$ (ii)

Find the value of x and y.

4.0 CONCLUSION

In this unit, we can conclude that simultaneous equation is a set of two or more equations, each containing two or more variables whose values can simultaneously satisfy both or all the equations in the set, the number of variables being equal to or less than the number of equations in the set.

5.0 SUMMARY

In this unit, we have discuss extensively on Elimination and substitution methods of simultaneous equation and different examples in this unit has shown that either of the two methods are good and easy to use in solving problems of simultaneous equation.

6.0 TUTOR-MARKED ASSIGNMENT

1. Given $x + y = -10$ _____ (i)

$$2y - 4x = -3$$
_____ (ii)

Find x and y using

- a. Elimination method.
- b. Substitution method.

2. Given $2a - 2b = 5$ _____ (i)

$$a + b = -6$$
_____ (ii)

Find a and b using

- a. Elimination method.
- b. Substitution method.

3. Given $2y + y - 13 = 4x + 3y + 3$ _____ (i)

$$11 - 10y + 8x = -x + 2y - 4$$
_____ (ii)

Find x and y using

- a. Elimination method.
- b. Substitution method.

7.0 REFERENCES/FURTHER READINGS

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UNIT 3 SEQUENCES AND SEIRES

CONTENTS

- 1.0 Introduction
- 2.0 Objectives
- 3.0 Main Content
 - 3.1 Sequences as a Function
 - 3.2 Arithmetic Progression (A.P)
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- 4.0 Conclusion
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1.0 INTRODUCCION

A sequence sometimes called a progression is an ordered list is known as 'elements' or the 'items' of the sequence. It can also be a set of numbers arranged in a definite order according to some definite rule or a function whose domain is the N of natural numbers.

However, a series is what you get when you add up all the terms of a sequences, where the addition and the resulting value are called the sum or the summation. For example 1, 2, 3, 4 is a sequence, with terms 1, 2, 3, and 4, the corresponding series is the sum $1 + 2 + 3 + 4$ and the value of the series is 10. Sequence maybe named or referred to by an upper case letter such as 'A' or 'S'. The terms of a sequence are usually named something like ' a_i ' or a_n with the subscripted letter 'I' or 'n' being the 'index' or the counter. Therefore the second term of a sequence might be named " a_2 " which is pronounced 'ay – sun two and a_{12} will be pronounced as the twelfth term.

The sequence can also be written in terms of its terms. For example, the sequence of terms ' a_i ' with the index running from $i = 1$ to $i = \infty$ can be written as:

$$\{a_n\}_{n=1}^{\infty}$$

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- understand the meaning of sequence and series
- understand and know how to calculate Arithmetic and Geometric progression.
- also know the meaning and how to calculate sum of Arithmetic and Geometric progression and sum to infinity of a Geometric progression.

3.0 MAIN CONTENTS

3.1 Sequences as a Function

Let us take an example of $T_n = 3n - 1$, where T_n is the n th term and $n = 1, 2, 3, \dots$, this gives us a formula for finding any required term. However, a sequence $T_1, T_2, T_3, \dots, T_n$, is the set of images given by the function $T_n = f(n)$ of the position integers say 1, 2, 3, $\dots$, n . sequence may be finite or infinite. When we talk about the finite sequence, n will have an upper value and T_n will be the last term of the sequence. If it is an infinite sequence, there is no last term.

Example 1

Write down the first 5th terms of each sequences if:

a. $T_n = 3n$
 b. $T_n = (-3)^n$.

a. $T_n = 1 - 3n$

First term $\Rightarrow n = 1 \Rightarrow T_1 = 1 - 3(1)$
 $= 1 - 3 = -2$

Second term $\Rightarrow n = 2 \Rightarrow T_2 = 1 - 3(2)$
 $= 1 - 6 = -5$

Third term $\Rightarrow n = 3 \Rightarrow T_3 = 1 - 3(3)$
 $= 1 - 9 = -8$

Fourth term $\Rightarrow n = 4 \Rightarrow T_4 = 1 - 3(4)$
 $= 1 - 12 = -11$

Fifth term $\Rightarrow n = 5 \Rightarrow T_5 = 1 - 3(5)$
 $= 1 - 15 = -14$

b. $T_n = (-3)^n$.

$$\text{First term} \Rightarrow n = 1 = T_1 = (-3)^1 = -3 = -3$$

$$\text{Second term} \Rightarrow n = 2, = T_2 = (-3)^2 = 9$$

$$\text{Third term} \Rightarrow n = 3, = T_3 = (-3)^3 = -27$$

$$\text{Fourth term} \Rightarrow n = 4, = T_4 = (-3)^4 = 81$$

$$\text{Fifth term} \Rightarrow n = 5, = T_5 = (-3)^5 = -243.$$

Example 1

Given the n th term of a sequence is given, write down the first terms.

$$1^{\text{st}} \text{ term} = a$$

$$2^{\text{nd}} \text{ term} = a + d$$

$$3^{\text{rd}} \text{ term} = a + 2d$$

$$4^{\text{th}} \text{ term} = a + 3d.$$

You will be surprise how we arrived at the above. Can we say it has answered the question? No.

$$N\text{th term of a sequence is } T_n = a + (n - 1)d.$$

$$\text{First term} \Rightarrow T_1 = a + (1 - 1)d$$

$$T_1 = a + (0)d$$

$$T_1 = a + 0 \Rightarrow a.$$

$$\text{First term} \Rightarrow T_2 = a + (2 - 1)d$$

$$T_2 = a + (1)d$$

$$T_2 = a + d$$

$$\text{Third term} \Rightarrow T_3 = a + (3 - 1)d$$

$$T_3 = a + (2)d \Rightarrow a + 2d$$

$$\text{Fourth term} \Rightarrow T_4 = a + (4 - 1)d$$

$$T_4 = a + (3)d$$

$$= a + 3d.$$

Where a is the first term and d is the common difference.

Therefore, the first 4 term are $a, a + 2d, a + 3d, a + 4d$.

3.2 Arithmetic Progression (AP)

In mathematics, an arithmetic progression (AP) or arithmetic sequence is a sequence of numbers such that the difference between the

consecutive terms is constant. Let us take a look at this example, say 2, 4, 6, 8, 10, ... is an arithmetic progression with a common difference of 2.

The n th term of a AP is given as:

$$T_n a + (n - 1)d$$

Where T_n nth term

a = first term

n = n^{th} of term

d = common difference.

Example 2

What is the 15th term of the sequence $-3, 2, 4, \dots$?

$$a = -3, d = 2 - (-3) = 2 + 3 = 5$$

$$n = 15.$$

$$T_{15} = a + (n - 1)d \Rightarrow -3 + (15 - 1)5$$

$$T_{15} \Rightarrow -3 + (14)5 = -3 + 70 = 69.$$

Example 3

Find a formula for the n th term of the AP.

12, 5, 2, ...

Solution

$$a = 12, d = 5 - 12 = -9$$

$$T_n = a + (n - 1)d$$

$$T_n = 12 + (n - 1) - 9 \Rightarrow 12 + (-7n + 7)$$

$$T_n = 12 - 7n + 7$$

$$T_n = 12 + 7 - 7n \Rightarrow T_n = 19 - 7n.$$

Example 4

The 6th and 13th term of a A.P are 0 and 14 respectively. Find the A.P

$$T_n = a + (n - 1)d$$

$$n = 6, T_6 = a + (n - 1)d = 0. \text{_____} (i)$$

$$n = 13, T_{13} = a + (n - 1)d = 14. \text{_____} (ii)$$

Find equation (i) $a + (6 - 1)d = 0$

$$a + 5d = 0 \text{_____} (iii)$$

From equation (ii) $a + (13 - 1)d = 14$

$$a + 12d = 14 \text{_____} (iv)$$

Combine equation (iii) and (iv)

$$a + 5d = 0 \text{_____} (iii)$$

$$a + 12d = 14 \text{_____} (iv)$$

Subtract equation (iii) from (iv).

$$a + 5d = 0$$

$$\underline{-a + 12d = 14}$$

$$-7d = -14$$

$$d = \frac{-14}{-7}$$

$$d = 2.$$

Substitute for d in equation (iii)

$$a + 5d = 0 \text{ -----} (iii)$$

$$a + 5(2) = 0$$

$$a + 10 = 0$$

$$a = -10.$$

First term (a) = -10 , and common difference (d) = 2

3.3 Geometric Progression (G.P)

A geometric Progression is also known as a geometric sequence, is a sequence of numbers where each term after the first is found by multiplying the previous one by a fixed, non-zero numbers called the common ration. Let us consider the following 2, 6, 18, 54,..... is a geometric progression with common ratio 3.

The general term of a geometric progression (G.P) is $a, ar, ar^2, ar^3, ar^4, \dots$

Where $r \neq 0$ is the common ration and a is the first term.

Therefore the nth term of a G.P is given as $T_n = Gr^{n-1}$

Example 4

Find (a) the 10th term of the given 128, by 32,.....(b) a formula for the nth term.

Solution

a. The sequence is a G. P as $r = \frac{64}{128} = \frac{32}{64} = \frac{1}{2}$

$$a = 128.$$

$$T_{10} = ar^{n-1} \Rightarrow T_{10} = 128 \left(\frac{1}{2} \right)^{10-1}$$

$$T_{10} = 128 \left(\frac{1}{2} \right)^9 \Rightarrow 128 \times \left(\frac{1}{2} \right)^9 = \frac{128}{2^9} = \frac{2^9}{2^9} a = 128.$$

$$\Rightarrow 2^7 \div 2^9 = 2^{7-9}$$

$$\Rightarrow 2^{-2} = \frac{1}{2^2} = \frac{1}{4}.$$

b. $T_n = ar^{n-1}$

$$T_n = 128 \times \left(\frac{1}{2} \right)^{n-1} \Rightarrow \frac{2^7}{2^{n-1}} = 2^{8-1}.$$

Example 5

The 4th and 8th terms of a G.P are 24 and $\frac{8}{27}$ respectively. Find the two possible values of a and r.

4th term

$$T_n = ar^{n-1} \Rightarrow T_4 = ar^{4-1} = 24$$

$$ar^3 = 24 \text{_____} (i)$$

8th term

$$T_n = ar^{n-1} \Rightarrow T_8 = ar^{8-1} = \frac{8}{27}$$

$$ar^7 = \frac{8}{27} \text{_____} (ii)$$

Combine equation (i) and (ii)

$$ar^3 = 24 \text{_____} (i)$$

$$ar^7 = \frac{8}{27} \text{_____} (ii)$$

Divide equation (ii) by (i)

$$\frac{ar^7}{ar^3} = \frac{8}{24} \div 24 \Rightarrow ar^{7-3} \frac{\cancel{8}}{24} \times \frac{1}{\cancel{24}^3}$$

$$r^4 = \frac{1}{81}$$

$$r^4 = 81^{-1}$$

$$r^4 = 3^{-4}.$$

Multiply the powers by 14

$$(r^4)^{1/4} = (3^{-4})^{1/4}$$

$$r = 3^{-1} = 1/3$$

Substitute r in equation (i)

$$ar^3 = 24$$

$$3\left(\frac{1}{3}\right)^3 = 24$$

$$a\left(\frac{1}{27}\right) = 24$$

$$\frac{a}{27} = 24$$

$$a = 24 \times 27$$

$$a = 648$$

$$a = 648, r = \frac{1}{3}.$$

3.4 Sum of A.P.

The formula for the sum of the nth terms of an A.P is given as:

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

Where S_n = sm of an A.P

a = first term

n = no of term

d = common difference

Example 6

Find the sum of the first 20 terms of $2 + 5 + 8 + \dots$

$$a = 2, d = 3, n = 20$$

$$S_{20} = \frac{20}{2}[2(2) + (20 - 1)3]$$

$$= 10[4 + (19)3] \Rightarrow 10[4 + 57] \Rightarrow 10[61]$$

$$S_{20} = 610.$$

Example 7

Find the sum of the first 10th terms of $2 + 4 + 6 + 8, \dots$

$$a = 2, d = 2, n = 10$$

$$S_{10} = \frac{10}{2} [2(2) + (10 - 1)2] \Rightarrow 5[4 + (9)2]$$

$$\Rightarrow 5(4 + 18) \Rightarrow S_{10} = 5[22] = 110.$$

Example 8

The sum of 8 terms of an A.P is 12 and the sum of 16 terms is 56. Find the A.P.

Solution

8th term: Using $S_n = \frac{n}{2} [2a + (n - 1)d]$

$$S_8 = \frac{8}{2} [2a + (8 - 1)d] = 12$$

$$4[2a + 7d] = 12$$

$$8a + 28d = 12$$

Divide through by 4.

$$\frac{8a}{4} + \frac{28d}{4} = \frac{12}{4}$$

$$2a + 7d = 3 \text{ _____ (i)}$$

16 terms, using $S_n = \frac{n}{2} [2a + (n - 1)d]$

$$S_{16} = \frac{16}{2} [2a + (16 - 1)d] = 56$$

$$8(2a + 17d) = 56$$

$$16a + 136d = 56$$

Divide through by 8

$$\frac{16a}{8} + \frac{136d}{8} = \frac{56}{8}$$

$$2a + 17d = 7 \text{ (ii)}$$

Combine equation (i) and (ii)

$$2a + 7d = 3 \text{ (i)}$$

$$2a + 17d = 7 \text{ (ii)}$$

Subtract equation (ii) from (i)

$$2a + 7d = 3 \text{ (i)}$$

$$-2a + 17d = 7 \text{ (ii)}$$

$$-10d = -5$$

$$d = \frac{-5}{-10}$$

$$d = 1/2$$

Substitute for d in equation (i)

$$2a + 7\left(\frac{1}{2}\right) = 3$$

$$\frac{2a}{1} + \frac{7}{2} = \frac{3}{1}$$

$$L.C.M = 2$$

Multiply through by 2

$$2 \times \frac{2a}{1} + 2 \times \frac{7}{2} = 2 \times \frac{3}{1}$$

$$4a + 7 = 6$$

$$4a + 6 = 7$$

$$4a = -1$$

$$a = -14.$$

3.5 Sum of a G.P

The formula for sum of the first terms of a G.P is given as:

$$S_n = \frac{a(1 - r^n)}{1 - r} \Rightarrow \text{this is when } r < 1$$

$$S_n = \frac{a(r^n - 1)}{r - 1} \Rightarrow \text{this is when } r > 1$$

Example 8

Find the sum of the first 10 terms of $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \dots$

$$a = 1/8, r = \frac{1}{4} \div \frac{1}{8} = \frac{1}{4} \times \frac{8}{1} = 2, n = 10.$$

Since $r > 1$, we use:

$$S_n = \frac{a(r^n - 1)}{r - 1} = S_{10} = \frac{1/8 (2^{10} - 1)}{2 - 1}$$

$$\frac{\frac{1}{8} (1024 - 1)}{1} = \frac{1}{8} (1023) = \frac{1023}{8}$$

$$= 127 \frac{7}{8}.$$

3.6 Sum to Infinity of a G.P

The sum to infinity of a G.P is the formula for sum to infinity of G.P is given as:

$$S_{\infty} = \frac{a}{r-1} \text{ when } r < 1$$

$$S_{\infty} = \frac{a}{r-1} \text{ when } r > 1$$

Example 9

To what values does the sum of $20 + 12 + \frac{36}{5} + \dots$ tend as $n \rightarrow \infty$?

$$a = 20, r = \frac{12}{20} = \frac{3}{5}$$

Since $r < 1$

We have

$$S_{\infty} = \frac{20}{\frac{13}{15}} = \frac{20}{\frac{5-3}{5}}$$

$$s_{\infty} = \frac{20}{\frac{2}{5}} = 20 \div \frac{2}{5}$$

$$= 10 \cancel{20} \times \frac{5}{\cancel{2}}$$

$$= 50.$$

Self-Assessment Exercise

To what values does the sum of $10 + 12 + \frac{26}{5} + \dots$ tend as $n \rightarrow \infty$?

4.0 CONCLUSION

In this unit, we can conclude that a sequence is an enumerated collection of objects in which repetitions are allowed. Like a set, it contains members (also called elements, or terms). However, the number of elements (possibly infinite) is called the length of the sequence. Unlike a set, order matters, and exactly the same elements can appear multiple times at different positions in the sequence.

5.0 SUMMARY

In this unit, we have been able to discuss sequences as a function, arithmetic and geometric progression in full detail, sum of arithmetic and geometric progression and sum to infinity of a geometric progression. Base on the all the examples in this unit, justice has been done to the topic sequence and series.

6.0 TUTOR-MARKED ASSIGNMENT

1. The 5th and 10th term of A.P are -12 and -27 respectively. Find the A.P and its 15th term.
2. Which term of the A.P, 2, 5, 8, ... is 44?
3. The 1st and 7th term of a G.P are $40\frac{1}{2}$ and $\frac{1}{8}$ respectively. Find the 2nd term.
4. The sum of the first 10 terms of A.P is 15 and the sum of the next 10 terms is 215. Find the A.P.
5. In a G.P the product of the 2nd and 4th terms is double the 5th terms and the sum of the first 4 terms is 80. Find the G.P.

7.0 REFERENCES/FURTHER READINGS

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MODULE TWO: CALCULUS I

UNIT 1 DIFFERENTIATION

UNIT 2 Other Techniques of Differentiation: the Composite Function

UNIT 3 Differentiation of Exponential Logarithmic and Trigonometric Functions

UNIT 1: DIFFERENTIATION

CONTENTS

- 1.0 Introduction
- 2.0 Objectives
- 3.0 Main Content
 - 3.1. Differentiation
 - 3.1.1. Differentiation from the first principle
 - 3.1.2. Differentiation of a constant
 - 3.1.3. Differentiation of Polynomials
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1.0 INTRODUCTION

Calculus was a branch of mathematics which was largely established by Newton and Leibnitz in the 17th century. It involves a very important ideas and techniques. Calculus is divided into two parts: Differentiation and integration.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Understand the meaning of differentiation and know how to calculate differentiation problems such as differentiation from the first principle, a constant and polynomials.

3.0 MAIN CONTENTS

3.1 Differentiation

Before we can start our discussion on differentiation, we will first take a look at what is GRADIENTS of a straight line. However, the gradient of a straight line is the ratio of increase in y to increase in x in going from one point to another on the line. Let us take an example below to explain further.

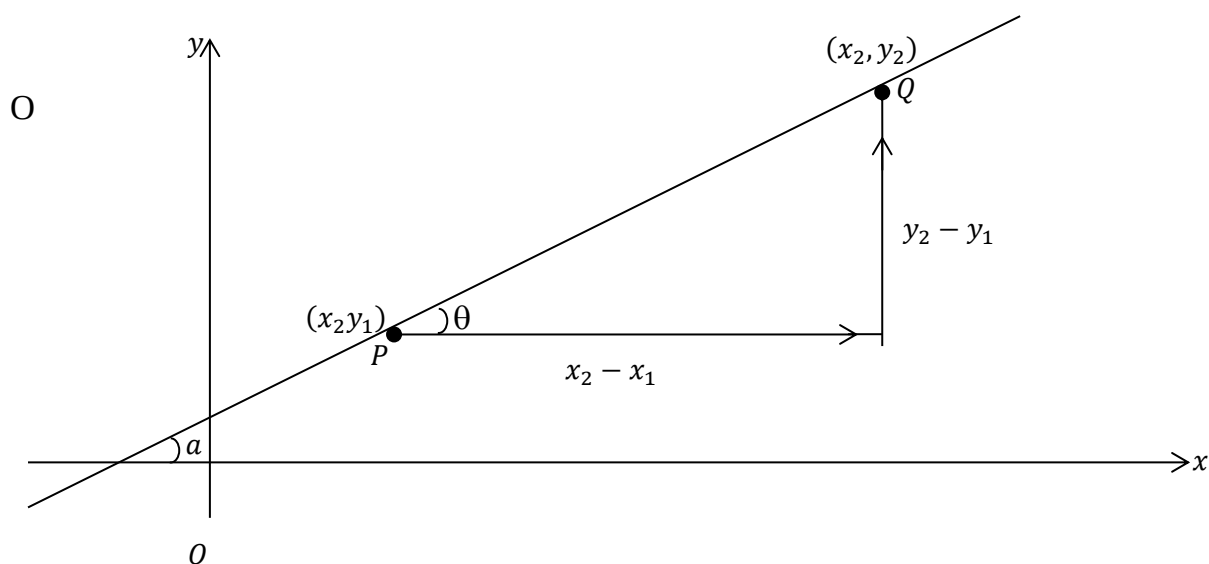


Figure1: Showing Gradient of a straight line.

From the graph above, if (x_1, y_1) , (x_2, y_2) are two points on a line, the gradient will be

$\frac{y_2 - y_1}{x_2 - x_1}$ which is equal to the definition of gradients given above. The gradient will of course be constant all along the line and also equal to the tangent of the angle ' θ ' made with the positive direction of the x -axis.

Therefore we can say that a gradient function will be change in y over change in x.

Note: Differentiation is also called rate of change. The process of finding the gradient function $\frac{dy}{dx}$ or $f'(x)$ is called differentiation and $\frac{dy}{dx}$ will now be called the derivation of y with respect to x and it also sometimes called the differential coefficient y. in the course of our study, we shall use wrt for 'with respect to'. Has a wider meaning to $\frac{dy}{dx}$ which will gradually emerge and this is the idea of instantaneous rate of change $\frac{dy}{dx}$ measures the rate of change of the quantity 'y' as compared with x.

Example 1

If x^2 , $\frac{dy}{dx} = 2x$, we can then say that the rate of change of y as compared with x is $2x$. But it must be noted that this rate of change is not constant but varies with the value of x, bound together by the relation $y = x^2$. It is important here to get this idea that $\frac{dy}{dx}$ measures the rate of change of y as compared with x, firmly fixed in one's mind. It was this good idea which made differentiation (the basis of calculus) one of the great inventions of mathematics and this has help us to be able to deal with changing quantities and measures their rates of change accurately.

3.1.1 Differentiation from the First Principle

Example 1

Differentiate $3x^2 - x + 5$ with respect to x from the first principle.

Solution

Let $y = 3x^2 - x + 5$, an increment from x to $x + dx$ produce corresponding increment in y to $y + dy$ (that is a place of x in y put $x + dx$).

$$\begin{aligned}\text{Therefore, } y + dy &= 3(x + dx)^2 - (x + dx) + 5 \\ dy &= 3(x + dx)^2 - (x + dx) + 5 - y\end{aligned}$$

$$dy = 3(x^2 + 2xdx + dx^2) - (x + dx) + 5 - (3x^2 - 2x - 5).$$

$$dy = 3x^2 + 6xdx + 3dx^2 - x - dx + 5 - 3x^2 + 2x - 5$$

$$dy = 6xdx + 3dx^2 - dx$$

$$\text{Slope } \frac{dy}{dx} = \frac{(6xdx + 3dx - 1)dx}{dx} = 6x + 3dx - 1.$$

$$\text{As } dx > 0, \frac{dy}{dx} = \frac{dy}{dx}, \text{ therefore } \frac{dy}{dx} = 6x + 0 - 1 = 6x - 1$$

Example 2

Differentiate $\frac{1}{x^2}$ with respect to x from principle.

Note here that any small increase of dx in x produces a corresponding increase of dy in y .

$$y + dy = \frac{1}{(x + dx)^2}$$

$$dy = \frac{1}{(x + dx)^2} - y = \frac{1}{(x + dx)^2} - \frac{1}{x^2}$$

$$dy = \frac{x^2 - (x + dx)^2}{(x + dx)^2 x^2} = \frac{x^2 - (x^2 + 2xdx + dx^2)}{(x + dx)^2 x^2}$$

$$dy = \frac{x^2 - x^2 - 2xdx - dx^2}{(x + dx)^2 (x^2)} = \frac{-2xdx - dx^2}{(x + dx)^2 (x^2)}$$

$$\frac{dy}{dx} = \frac{-2xdx - dx^2}{(x + dx)^2 (x^2) dx} = \frac{-(2x + dx)dx}{(x + dx)^2 (x^2) dx}$$

$$\frac{dy}{dx} = \frac{(2x + dx)}{(x + dx)^2 (x^2)}$$

$$\text{As } dx \rightarrow 0, \frac{dy}{dx} = \frac{dy}{dx} = \frac{-2x + 0}{(x + 0)^2 (x^2)} = \frac{-2x}{x^4}$$

$$\frac{dy}{dx} = \frac{-2}{x^3} = -2x^{-3}.$$

Generally if $y = ax^n$.

$$\frac{dy}{dx} = nax^{n-1} \text{ where } a \text{ is a constant}$$

3.1.2 Differentiation of a Constant

If $y = k$ where K is constant, the $\frac{dy}{dx} = 0$. This means that the differentiation of a coefficient of a constant is zero.

Example 1

$Y = 10$, differentiate wrt x

$$\frac{dy}{dx} = 0 \times 10^{0-1} = 0 \times 10x^{-1} = 0$$

Hence if $y = 10$, $\frac{dy}{dx} = 0$

Example 2

If $y = 2x^2$, $\frac{dy}{dx} = 2 \times 2 \times x^{2-1} = 4x^1 = 4x$

Example 3

If $y = \frac{1}{x}$, $\frac{dy}{dx} = x^{-1-1} = -x^{-2} = \frac{-1}{x^2}$

Example 4

If $y = 3x^2 - 2x + 6$, $\frac{dy}{dx} = 2 \times 3x^{2-1} - 1 \times 2x^{1-1} + 0$.
 $= 6x^1 - 2x^0 = 6x - 2$.

Example 5

If $ax^3 + bx^2 + bx + c$, dy

$$\begin{aligned}\frac{dy}{dx} &= 3ax^{3-1} + 2bx^{2-1} + bx^{1-1} + 0 \\ &= 3ax^2 + 2bx^1 + bx^0 \\ &= 3ax^2 + 2bx + bx^1 \\ &= 3ax^2 + 2bx + b.\end{aligned}$$

Example 6

Given $y = 10x - 7 + x^2$

$$\begin{aligned}\frac{dy}{dx} &= 10x^{1-1} - 0 + 2x^{2-1} \\ &= 10 \times 1 - 0 + 2x^1 \\ &= 10 + 2x.\end{aligned}$$

3.1.3 Differentiation of Polynomials

Given that $y = ax + br + cw$, where a , b and c are constant and u , v and w are function of x .

$$\boxed{\frac{dy}{dx} = a \frac{du}{dx} + b \frac{dv}{dx} + c \frac{dw}{dx}}$$

Example 1

Differentiate the following wrt x .

$$\begin{aligned}y &= 3x^4 - 2x^2 - \frac{1}{x} + 8 = 3x^4 - 4x^3 + 2x^2 - x^{-1} + 8. \\ \frac{dy}{dx} &= 3 \frac{d}{dx}(x^4) - 4 \frac{d}{dx}(x^3) + 2 \frac{d}{dx}(x^2) - \frac{d}{dx}(x^{-1}) + \frac{d}{dx}(8). \\ &= 3(4x^3) - 4(3x^2) + 2(2x) - (-x^{-2}) + 0 \\ &= 12x^3 - 12x^2 + 4x + x^{-2} \\ &= 12x^3 - 12x^2 + 4x + \frac{1}{x^2}.\end{aligned}$$

Example 2

$$\begin{aligned}y &= \frac{(2x-1)(3x+1)}{2x^2} = \frac{6x^2 - x - 1}{2x^2} \\ &= \frac{6x^2}{2x^2} - \frac{x}{2x^2} - \frac{1}{2x^2} \\ y &= 3 \frac{-1}{2x} - \frac{1}{2x^2} = 3 - \frac{x^{-1}}{2} - \frac{x^{-2}}{2} \\ \frac{dy}{dx} &= 0 - \frac{-x^{-2}}{2} - \frac{2x^{-3}}{2x} \\ &= 0 + \frac{x^{-2}}{2} + \frac{2x^{-3}}{2x} = \frac{1}{2x^2} + \frac{2}{2x^3}\end{aligned}$$

$$= \frac{1}{2x^2} + \frac{2}{2x^3} = \frac{1}{2x^2} - \frac{1}{x^3}$$

$$y = \sqrt[2]{x} - \frac{3}{\sqrt{x}} = 2x^{\frac{1}{2}} - \frac{3}{x^{\frac{1}{2}}}$$

$$= 2x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = 2\left(\frac{1}{2}\right)x^{-\frac{1}{2}} - 3\left(\frac{-1}{2}\right)x^{-\frac{3}{2}}$$

$$= \frac{2}{2}x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}}$$

$$x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}} = \frac{1}{x^{\frac{1}{2}}} + \frac{3}{2x^{\frac{3}{2}}}$$

$$= \frac{1}{\sqrt{x}} + \frac{3}{2\sqrt{(5x)^3}}$$

Example 4

Given:

$$y = 500 + 4x + 2x^2 - 10x^3 - 12x^4 \text{ find } \frac{dy}{dx}$$

$$\frac{dy}{dx} = 0 - 4 + 4x - 30x^2 - 48x^3$$

$$\frac{dy}{dx} = 4 + 4x - 30x^2 - 48x^3$$

Self-Assessment Exercise

Given:

$$y = 40 + 2x + 3x^2 - 8x^3 - 7x^4 \text{ find } \frac{dy}{dx}$$

4.0 CONCLUSION

In this unit, we can conclude that differentiation is a process of finding a function that outputs the rate of change of one variable with respect to

another variable. Informally, in this unit, we can assume that we are tracking the position of a car on a two-lane road with no passing lanes. However, if we assuming the car never pulls off the road, we can abstractly study the car's position by assigning it a variable, x . Since the car's position changes as the time changes, we say that x is dependent on time, or $x = f(t)$. This tell us where the car is at each specific time. Finally, we can then say that differentiation gives us a function dx/dt which represents the car's speed that is the rate of change of its position with respect to time.

5.0 SUMMARY

In this unit, we have been able to discuss on the meaning of differentiation, calculation of differentiation from the first principle, differentiation of a constant and differentiation of polynomials and the examples discussed in this unit will help you to be vast in calculus called differentiation.

6.0 TUTOR-MARKED ASSIGNMENT

1. Differentiate $y = 10x^5 - 6x^2 + 2x^2 - x^2 + 3$ wrt x .
2. Differentiate $y = 11a^2 + 4a - 6$ wrt x .
3. Differentiate $y = 2x^2$ from the first principle
4. Differentiate $y = x + 2$ from the first principle
5. Differentiate $y = 2\sqrt{x} - \frac{4}{\sqrt{x}}$.

7.0 REFERENCES/FURTHER READINGS

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UNIT 2 OTHER TECHNIQUES OF DIFFERENTIATION: THE COMPOSITE FUNCTION

CONTENTS

- 1.0 Introduction
- 2.0 Objectives
- 3.0 Main Contents
 - 3.1 Examples of Composite Function
 - 3.2 Other techniques of differentiation: the Product Rule
 - 3.3 Other techniques of differentiation: the Quotient Rule
 - 3.4 Other techniques of differentiation: the implicit function

1.0 INTRODUCTION

Composite function is also called function of a function or the chain rule. That is if $y = u$ and u is a function of say x .

Therefore the formular for composite function is:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

So the formular above is used in calculating various problem under composite function of differentiation.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Understand other techniques of differentiation such as product rule, quotient rule and the implicit function.

3.0 MAIN CONTENT

3.1 Examples of Composite Function

Example 1

Differentiate wrt x $y = \sqrt{2x + 5}$

Let us take $u = 2x + 5$ and y becomes $y = u^{\frac{1}{2}}$.

$$\frac{dy}{dx} = 2 \text{ and } \frac{dy}{dx} = \frac{1}{2} u^{-\frac{1}{2}}$$

Therefore from the composite function we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ \frac{dy}{dx} &= \frac{1}{2} u^{-\frac{1}{2}} \times 2 = u^{-\frac{1}{2}} = \frac{1}{\sqrt{2x + 5}} \end{aligned}$$

Example 2

$$\frac{2}{(3x^2 - x + 7)^3}$$

Take $u = 3x^2 - x + 7$ and $y = \frac{2}{u^3} = 2u^{-3}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = -6u^{-4} \text{ and } \frac{dy}{du} = 6x - 1$$

$$\frac{dy}{dx} = -6u^{-4} \times (6x - 1), \text{ Therefore,}$$

$$\frac{dy}{dx} = \frac{6(6x - 1)}{(3x^2 - x + 7)^4}$$

Example 3

$$\frac{1}{\sqrt{3x - 2}} = \frac{1}{(3x^2 - 7)^{\frac{1}{2}}} (3x - 2)^{-\frac{1}{2}}$$

Let $u = 3x - 7$ and $y = (3x - 4)^{-\frac{1}{2}}$

$$\frac{dy}{dx} = 3, \frac{dy}{du} = -\frac{1}{2}u^{-\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{-1}{2}u^{-\frac{3}{2}} \times 3 = \frac{-3}{2}u^{-\frac{3}{2}}$$

$$= \frac{-3}{2}(3x-4)^{-\frac{3}{2}} = -\frac{3}{2(3x-4)^{\frac{3}{2}}}$$

$$= \frac{-3}{2\sqrt{(3x-4)^3}}$$

Example 4

$$y = (2 + \sqrt{n})^3$$

$$\text{Let } y = (2 + \sqrt{n})^3 = (2 + x^{\frac{1}{2}})^3$$

$$\text{Let } u = 2 + x^{\frac{1}{2}} \text{ and } y = u^3$$

$$\frac{dy}{du} = 3u^2 \text{ and } \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$$

$$\text{Therefore } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$3u^2 \times \frac{1}{2}x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}}u^2 = \frac{3u^3}{2x^{-\frac{1}{2}}}$$

$$\text{This becomes } = \frac{3(1 + \sqrt{x})^2}{2x^{\frac{1}{2}}}$$

$$= \frac{3(1 + \sqrt{x})^2}{2\sqrt{x}}$$

3.2 Other Techniques of Differentiation: The Product Rule

Given $y = (2x - 1)^3(x^3 + 7)$ is the product of two expressions $(2x - 1)^3$ and $(x^3 + 7)$. However, to find $\frac{dy}{dx}$, we can try to expand the product but that will take a greater time and cumbersome to do. So we can just differentiate each expression separately, so we now see how $\frac{dy}{dx}$ can be found from the two derivatives.

Take for example here, let $y = uv$, where u and v are each function of x . Therefore uv is the product of two functions. For the example given above u will be $(2x - 1)^3$, and v will be $(x^3 + 7)$.

In conclusion the formula for the product rule is given as:

$$\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$$

Example 1

Differentiate $y = (3x - 4)(x^2 + 8)$ wrt x

Take u as $3x - 4$ and $v = x^2 + 8$

Then $\frac{du}{dx} = 3$ and $\frac{dv}{dx} = 2x$

$$\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$$

$$\frac{dy}{dx} = (x^2 + 8) 3 + (3x - 4) 2x$$

$$= 3x^2 + 24 + (6x^2 - 8x)$$

$$= 3x^2 + 24 + 6x^2 - 8x$$

$$\frac{dy}{dx} = 3x^2 + 6x^2 - 8x + 24$$

$$\frac{dy}{dx} = 9x^2 - 8x + 24$$

Example 2

Differentiate $y = x^2(2x - 5)^4$ wrt x .

Let $u = x^2$ and $v = (2x - 5)^4$

$$\frac{dy}{dx} \text{ and } v \frac{du}{dx} + u \frac{dv}{dx}$$

$$\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$$

$$\frac{dy}{dx} = 2x(2x - 5)^3 + x^2[4(2x - 5)^3]$$

$$\begin{aligned} \frac{dy}{dx} &= 2x(2x - 5)^3(2x - 5 + 4x) \\ &= 2x(2x - 5)^3(6x - 5) \end{aligned}$$

Example 3

$$y = 3x^2(2x - 5)^4$$

Let $u = 3x^2$ and $v = (2x - 5)^4$

$$\begin{aligned} \frac{dy}{dx} &= 6x \text{ and } \frac{dv}{dx} = 4(2x - 5)^3 \times 2 \\ &= 8(2x - 5)^3 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= v \frac{du}{dx} + u \frac{dv}{dx} = [(2x - 5)^4 \times 6x] + 3x^2[8(2x - 5)^3] \\ &= 6x(2x - 5)^3[(2x - 5) + 4x] \\ &= 6x(2x - 5)^3(6x - 5). \end{aligned}$$

Example 4

Differentiate $y = \sqrt{(x + 1)}(x + 3)^2$

Let $u = (x + 1)^{-\frac{1}{2}}$ and $v = (x + 3)^2$

$$\begin{aligned} \frac{du}{dx} &= \frac{1}{2}(x + 1)^{-\frac{1}{2}} \times 1, \frac{dv}{dx} = 2(x + 3)^1 \times 1 = 2(x + 3) \\ &\frac{1}{2}(x + 1)^{-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} &= v \frac{du}{dx} + u \frac{dv}{dx} = (x+3)^2 \left[\frac{1}{2} (x+1)^{-\frac{1}{2}} \right] + (x+1)^{\frac{1}{2}} [2(x+3)] \\
&= \left[\frac{1}{2} (x+3)^2 \times \frac{1}{(x+1)^{\frac{1}{2}}} \right] + 2(x+3)(x+1)^{\frac{1}{2}} \\
&= \left[\frac{1}{2} \frac{(x+1)^2}{(x+1)^{\frac{1}{2}}} \right] + [(2x+6)(x+1)^{\frac{1}{2}}] \\
&= \frac{(x+1)^2 + 2(x+1)^{\frac{1}{2}}(2x+6)(x+1)^{\frac{1}{2}}}{2(x+1)^{\frac{1}{2}}} \\
&= \frac{(x^2 + 6x + 9) + 2(x+1)(x+1)(2x+6)}{2(x+1)^{\frac{1}{2}}} \\
&= \frac{x^2 + 6x + 9 + 2(2x^2 + 2x + 6 - 6)}{2(x+1)^{\frac{1}{2}}} \\
&= \frac{x^2 + 6x + 9 + 4x^2 + 16x + 12}{2(x+1)^{\frac{1}{2}}} \\
&= \frac{5x^2 + 22x + 21}{2(x+1)^{\frac{1}{2}}}
\end{aligned}$$

3.3 Other Techniques of Differentiation: The Quotient Rule

Suppose $y = \frac{u}{v}$ where U and V are each function of x

Then $y = uv^{-1}$ and we can then make use of the product rule to find

$$\frac{dy}{dx}.$$

Therefore

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

Note that you should not use the quotient rule where the denominator is a single term.

Example 1

$$\text{Differentiate } y = \frac{x^2}{\sqrt{x+1}}$$

$$\text{Let } u = x^2 \text{ and } \frac{du}{dx} = 2x$$

$$v = \sqrt{x+1} = (x+1)^{\frac{1}{2}} \\ \frac{dv}{dx} = \frac{1}{2}(x+1)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x+1}} \text{ and } v^2 = (x+1)$$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{2}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\sqrt{x+1} \times 2x - \frac{x^2}{2\sqrt{x+1}}}{(x+1)} \\ &= \frac{2\sqrt{x+1} \times \sqrt{x+1} \times 2x - x^2}{2\sqrt{x+1}(x+1)} \end{aligned}$$

Now let us multiply the numerator and denominator by $2\sqrt{x+1}$

$$\frac{dy}{dx} = \frac{4x(x+1) - x^2}{2(x+1)^{1/2}} = \frac{x(3x+4)}{2(x+1)}$$

Example 2

$$y = \frac{2x+1}{3x-2}$$

$$\text{Let } u = 2x+1 \text{ and } v = 3x-2$$

$$\frac{du}{dx} = 2 \text{ and } \frac{dv}{dx} = 3$$

$$\frac{du}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(3x-2)(2) - (2x+1)(3)}{(3x-2)^2}$$

$$= \frac{6x - 5 - 6x - 3}{(3x - 2)^2}$$

$$= \frac{-7}{(3x - 2)^2}.$$

Example 3

$$y = \sqrt{\frac{x-1}{x+1}}$$

Applying the surd rule $\sqrt{\frac{x-1}{x+1}} = \frac{\sqrt{x-1}}{\sqrt{x+1}}$

Therefore $\frac{\sqrt{x-1}}{\sqrt{x+1}} = \frac{(x-1)^{1/2}}{(x+1)^{1/2}}$

Let $u = (x-1)^{1/2}$ and $v = (x+1)^{1/2}$

$$\frac{du}{dx} = \frac{1}{2}(x-1)^{-1/2} \frac{dv}{dx} = \frac{1}{2}(x+1)^{-1/2}$$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$= \frac{[x+1]^{1/2} \left[\frac{1}{2}(x-1)^{-1/2} \right] - [x+1]^{1/2} \left[\frac{1}{2}(x+1)^{-1/2} \right]}{[(x+1)^{1/2}]^2}$$

$$= \left[\frac{\frac{(x+1)^{1/2}}{2(x-1)} - \frac{(x-1)^{1/2}}{2(x+1)^{1/2}}}{((x+1)^{1/2})^2} \right] = \left(\frac{\frac{(x+1)^{1/2}}{2(x-1)} - \frac{(x-1)^{1/2}}{2(x+1)^{1/2}}}{(x+1)} \right)$$

$$= \left(\frac{(x+1)^{1/2}}{2(x-1)} - \frac{(x-1)^{1/2}}{2(x+1)^{1/2}} \right) \div (x+1)$$

$$= \left(\frac{(x+1)^{1/2}(x+1)^{1/2}}{2(x-1)^{1/2}} - \frac{(x-1)^{1/2}(x-1)^{1/2}}{(x+1)^{1/2}} \right) \div (x+1)$$

$$\begin{aligned}
&= \frac{(x+1) - (x-1)}{2(x-1)^{1/2}(x+1)^{1/2}} \times \frac{1}{x+1} \\
&= \frac{x+1-x+1}{2(x-1)^{1/2}(x+1)^{1/2}(x+1)^1} \\
&= \frac{2}{2(x-1)^{1/2}(x+1)^{1/2}(x+1)^1} \\
&= \frac{1}{(x-1)^{1/2}(x+1)^{3/2}}
\end{aligned}$$

3.4 Other Techniques of Differentiation: The Implicit Function

Differentiation by implicit functions is differentiating with respect to a term by term throughout, but using the product rule.

Example 1

Find $\frac{dy}{dx}$ if $x^2y - 7x = 3$

To differentiate x^2y consider it as uv and use the product rule, then the derivative of x^2y is:

$$\begin{aligned}
&x^2 \frac{dy}{dx} + y \times 2x \\
&\left(u \frac{dv}{dx} + v \frac{du}{dx} \right)
\end{aligned}$$

Therefore $x^2y - 7x = 3$, $x^2 \frac{dy}{dx} + y2x - 7 = 0$.

$$\frac{dy}{dx} : \frac{dy}{dx} = \frac{7 - 2xy}{x^2}.$$

Example 2

Find the equation of the tangents where $x = 2$ on the curve $x^2 + y^2 - 2x + y = 6$

Let us find $\frac{dy}{dx}$ by differentiating term by term.

We then have $2x + 2y \frac{dy}{dx} - 2 + \frac{dy}{dx} = 0$

$$\frac{dy}{dx} \Rightarrow \frac{dy}{dx} (2y + 1) = 2 - 2x$$

$$\frac{dy}{dx} = \frac{2 - 2x}{2y + 1}.$$

Let us now substitute $x = 2$ in the original equation of the curve:

We have $4 + y^2 - 4 + y = 6$, which can be written as $y^2 + y - 6 = 0$

$$\text{either } (y + 3)(y - 2) = 0$$

$$y = -3 \text{ or } 2.$$

Therefore, we concluded that there are two points on the curve where $x = 2$; $(2, -3)$ and $(2, 2)$.

Example 3

If $x^3 + y^3 = 3xy$ find $\frac{dy}{dx}$

$$\frac{d}{dx}(x^3) \frac{d}{dx}(y^3) = \left[3y \frac{d}{dx}(x) + 3x \frac{d}{dx}(y) + xy \frac{d}{dx}(3) \right]$$

$$3x^2 + \frac{d}{dx}(y^3) \frac{dy}{dx} = 3y(1) + 3x \frac{dy}{dx} + xy(0)$$

At this point it should be noted that we cannot differentiate y^s , so we have

$$\frac{d}{dx}(y^3) = \frac{d}{dx}(y^3) \cdot \frac{dy}{dx}$$

$$\Rightarrow 3x^2 + 3y^2 \frac{dy}{dx} = 3y + 3x \frac{dy}{dx} + 0$$

$$3y^2 \frac{dy}{dx} - 3x \frac{dy}{dx} = 3y - 3x^2$$

$$(3y^2 - 3x) \frac{dy}{dx} = 3(y - x^2)$$

$$\frac{dy}{dx} = \frac{3(y - x^2)}{3y^2 - 3x} = \frac{3(y - x^2)}{3(y^2 - x)}$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

Self-Assessment Exercise

Given:

If $x^3 + y^3 = 3xy$ find $\frac{dy}{dx}$

4.0 CONCLUSION

In this unit, we can conclude that other techniques of differentiation has given rise to higher level of differentiation that may need special formula or techniques in solving the problem. So, is at this end that this unit will be more useful for you as a student to be able to differentiate a composite function from an implicit function in one hand and also to differentiate a product rule from a quotient rule.

5.0 SUMMARY

In this unit, we have been able to discuss on the extensively on other techniques of differentiation such as composite function, implicit function, the product rule and quotient rule and each of these techniques has been dealt it by treating examples for you to understand how to use the techniques in solving problems.

6.0 TUTOR-MARKED ASSIGNMENT

1. Differentiate $y = \frac{x+1}{2x-2}$ wrt x.
2. Differentiate $y = \sqrt{\frac{x-1}{x+2}}$.
3. if $y = (x+1)^2(2x+1)$ differentiate.
4. Given $x^3 + y^3 = 4xy$ find $\frac{dy}{dx}$

7.0 REFERENCES/FURTHER READINGS

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UNIT 3 DIFFERENTIATION OF EXPONENTIAL, LOGARITHMIC AND TRIGONOMETRIC FUNCTIONS

CONTENTS

- 1.0 Introduction
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- 3.0 Main Contents
 - 3.1 Differentiation of exponential function
 - 3.2 Differentiation of Logarithmic functions
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 - 3.3.1. Derivatives of Sin x
 - 3.3.2. Derivatives of Cos x
 - 3.3.3. Derivatives of Tan x
 - 3.3.4. Worked examples
 - 3.4 Application of differentiation to Economics Problem

1.0 INTRODUCTION

Differentiations of exponential, logarithmic and trigonometric functions are further application of differentiation in advanced mathematics.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Understand how to calculate exponential, logarithmic and trigonometric functions
- Differentiate Sin x, Cos x and Tan x
- Know how to apply differentiation to economics problem.

3.0 MAIN CONTENT

3.1 Differentiation of Exponential Function

Differentiate of e^x

$$\text{If } y = e^{-x}; \frac{dy}{dx} = -e^{-x} \left(\text{Note here that } e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \right)$$

$$\text{However if } y = e^{ax} = \frac{dy}{dx} = ae^{ax}$$

Example 1

Differentiate $y = e^{5-2x}$ wrt x

$$\text{Let } u = 5 - 2x$$

$$y = e^u$$

$$\frac{dy}{du} = e^u, \text{ therefore } \frac{dy}{dx} = -2$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u \times (-2) = -2e^u = -2e^{5-2x}.$$

Example 2

If $y = e^{0.2x^2+5x+6}$ Differentiate wrt x .

$$\text{Let } u = e^{0.2x^2+5x+6}$$

$$\frac{dy}{dx} = 0.4x + 5$$

$$y = e^u \Rightarrow \frac{dy}{dx} = e^u.$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = e^u(0.4x + 5)$$

$$(0.4x + 5)e^u = (0.4x + 5)e^{0.2x^2+5x+6}.$$

Example 3

Differentiate $y = e^{\sqrt{x}}$ wrt x .

We can re-write as $y = e^{x^{1/2}}$

Let $u = x^{\frac{1}{2}}$

$$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

But $y = e^u$ and $dy/du = e^u$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} = \frac{du}{dx} = e^u \times \frac{1}{2\sqrt{x}} = \frac{e^u}{2\sqrt{x}} \\ &= \frac{e^{x^{1/2}}}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}\end{aligned}$$

Example 4

Differentiate $y = 5^x e^{x^5}$ wrt x .

Let $u = 5^x$ and $v = e^{x^5}$

$$\frac{du}{dx} = 5^x \log_e 5 \text{ and } \frac{dv}{dx} = 5x^4 e^{x^5}$$

$$\begin{aligned}\frac{dy}{dx} &= v \frac{du}{dx} + u \frac{dv}{dx} \\ &= e^{x^5} \left(5^x \log_e 5 \right) + 5^x (5x^4 e^{x^5}) \\ &= 5^x e^{x^5} \left(\log_e 5 + 5x^4 \right)\end{aligned}$$

3.2 Differentiation of Logarithmic Functions

Let us take a look at $y = \log_e x$

$$\Rightarrow x = e^y, \text{ then } \frac{dx}{dy} = e^y$$

$$\Rightarrow \frac{1}{\frac{dx}{dy}} = \frac{1}{e^y} \Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$$\text{Therefore } \frac{d}{dx} \left(\log_e x \right) = \frac{1}{x}.$$

if $y = \log_e^x$, then $\frac{dy}{dx} = \frac{1}{x}$.

Also note that $\log_e^x = \ln x$ and if the log involve a different base say a , by usual method, it can be change to base e .

Example 1

If $y = \log_a^x$

$\Rightarrow a^y = x$.

$\log_e a^y = \log_e^x$ (i.e. taking log of both sides to base e).

Therefore: $y \log_e a = \log_e^x$.

$y = \log_a^x = \log_e^x$

$y = \frac{\log_e^x}{\log_e a} = \log_e^x \times \frac{1}{\log_e a}$

but $\log_e a = \frac{1}{\log_a e}$ (Note: that is when the base change)

Therefore $y = \log_e^x \times \log_a e$

$\frac{dy}{dx} = \frac{1}{x} \times \log_a e$

Special case: if $y = \log_{10}^x$

$\frac{dy}{dx} = \frac{1}{x} \times \log_{10} e = \frac{1}{x} \times 0.4343$

Note that $\frac{1}{\log_a e} = \frac{1}{\frac{1}{\log_e a}} = \log_a e$

$\log_a e = \frac{1}{\log_e a}$ (why is this so?)

$$\text{Let } \log_e a = P$$

$$e^P = a \text{ (take log of both sides to base } a)$$

$$\log_a e^P = \log_e a$$

$$P \log_a e = 1$$

$$P = \frac{1}{\log_a e}$$

$$\text{Since } P = \log_e a$$

$$\text{Therefore } \log_e a = \frac{1}{\log_a e}.$$

Example 2

$$\text{Differentiate } y = \log_e x^2$$

$$\text{Let } u = x^2, \frac{dy}{dx} = 2x$$

$$y = \log_e u, \frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times 2x = \frac{2x}{u} = \frac{2x}{x^2} = \frac{2}{x}.$$

Example 3

$$\text{Differentiate } \log_e^{(3x^2+5x-10)}$$

$$\text{Let } u = 3x^2 + 5x - 10, \frac{du}{dx} = 6x + 5$$

$$\Rightarrow y = \log_e u, \frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times (6x + 5)$$

$$\Rightarrow \frac{(6x + 5)}{u} = \frac{(6x + 5)}{(3x^2 + 5x + 10)}.$$

Example 4

$$y = \log_e (2x^2 + x)^{\frac{1}{4}}$$

$$\text{let } u = (2x^2 + x)^{\frac{1}{4}}$$

$$\frac{dy}{dx} = (2x^2 + x)^{-\frac{3}{4}} \times (6x^2 + 1) = \frac{(6x^2 + 1)}{4(2x^3 + x)^{\frac{3}{4}}}$$

$$y = \log_e^u, \frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times \frac{6x^2 + 1}{4(2x^3 - x)^{\frac{3}{4}}}$$

$$= \frac{1}{(2x^3 + x)^{\frac{1}{4}}} \times \frac{(6x^2 + 1)}{4(2x^3 - x)^{\frac{3}{4}}}$$

$$\frac{dy}{dx} = \frac{6x^2 + 1}{4(2x^3 + x)^{\frac{1}{4}}(2x^3 + x)^{\frac{3}{4}}}$$

$$= \frac{6x^2 + 1}{4(2x^3 + x)^{\frac{1}{4} + \frac{3}{4}}} = \frac{6x^2 + 1}{4(2x^3 + x)}$$

3.3 Differentiation of Trigonometric Functions

From general mathematic recall the following properties.

$$1. \quad \operatorname{cosec} \emptyset = \frac{1}{\sin \emptyset} \quad 2. \quad \sec \emptyset = \frac{1}{\cos \emptyset}$$

$$3. \quad \cot \emptyset = \frac{1}{\tan \emptyset} = \frac{\cos \emptyset}{\sin \emptyset} \quad 4. \quad \sin^2 \emptyset + \cos^2 \emptyset = 1$$

$$5. \quad 1 + \tan^2 \theta = \sec^2 \theta \qquad 6. \quad 1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$$

$$7. \quad \sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$8. \quad \sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$9. \quad \cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$10. \quad \cos(A - B) = \cos A \cos B + \sin A \sin B$$

We can go on and on. So let us now go straight to the derivatives of trigonometric function.

3.3.1 Derivatives of $\sin x$

$$\frac{d}{dx}(\sin x) = \cos x$$

Where x is in radians.

Example 1

Differentiate wrt x $\sin 3x$

Let $y = \sin 3x$ and let us take $u = 3x$

Then $y = \sin u$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}, \text{ but } \frac{dy}{du} = \cos u, \frac{du}{dx} = 3$$

$$\frac{dy}{dx} = \cos u \times 3 = 3 \cos 3x.$$

Example 2

Differentiate $y = \sin(2x - 4)$ wrt x .

$$\frac{dy}{dx} = \cos(2x - 4) \text{ (Note here that we are differentiating Sin to get Cos)}$$

Therefore we differentiate $2x - 4$ to get 2.

$$\frac{dy}{dx} = \cos(2x - 4) \times 2 = 2 \cos(2x - 4)$$

3.2.3 Derivation of $\cos x$

$$\frac{d\cos}{dx} = -\sin x \text{ where } x \text{ is in radians}$$

Example 1

Differentiate $\cos 5x$, wrt x .

Let $y = \cos 5x$

$$\frac{dy}{dx} = -\sin 5x \times 5 = -5 \sin 5x.$$

3.3.3 Derivation of $\tan x$

$$\frac{d}{dx} \tan x = \sec^2 x \text{ where } x \text{ is in radians}$$

Differentiate $y = \tan 4x$ wrt x

Let $u = 4x$, then $y = \tan u$

$$\frac{du}{dx} = 4, \quad \frac{dy}{du} = \sec^2 u$$

$$\begin{aligned} dy/dx &= \frac{dy}{du} \times \frac{du}{dx} = \sec^2 u (4) \\ &= 4 \sec^2 u \\ &= 4 \sec^2 u \cdot 4x \end{aligned}$$

3.3.4 Worked Example

1. Differentiate $y = \sin(7x^2 - 3x + 1)$ wrt x .

Let $u = 7x^2 - 3x + 1$

$$y = \sin u, \quad \frac{dy}{du} = \cos u.$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos u \times (14x - 3)$$

$$\frac{dy}{dx} = (14x - 3) \cos u = (14x - 3) \cos(7x^2 - 3x + 1)$$

$$\frac{d}{dx} [\sin(7x^2 - 3x + 1)] = (14x - 3) \cos(7x^2 - 3x + 1).$$

2. Differentiate $y = \sin 3x$ wrt x .

$$\text{Let } u = x \text{ and } v = \sin 3x$$

$$\frac{dy}{dx} = 1 \text{ and } \frac{dv}{dx} = 3 \cos 3x$$

$$\begin{aligned} \frac{dy}{dx} &= v \frac{du}{dx} + u \frac{dv}{dx} = \sin 3x(1) + x(3 \cos 3x) \\ &= \sin 3x + 3x \cos 3x. \end{aligned}$$

3. Differentiate $y = \sin^2(x^2)$

$$\text{Let } u = x^2 \Rightarrow y = \sin^2 u$$

$$\frac{dy}{du} = 2 \sin u \cos u.$$

$$\frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 2 \sin u \cos(2x)$$

$$4x \sin u \cos u$$

$$= 4x \sin x^2 \cos x^2.$$

4. Differentiate $y = \sin x^\circ$ wrt x .

$$\text{Note } x^\circ = \frac{\pi}{180} x \text{ radian}$$

$$y = \sin\left(\frac{\pi}{180} x\right) \text{ radian,}$$

$$\text{Let } u = \frac{\pi}{180} x, \text{ then } y = \sin u, \frac{dy}{du} = \cos u$$

$$\frac{du}{dx} = \frac{\pi}{180}$$

$$\frac{dy}{dx} = \frac{\pi}{180} \cos\left(\frac{\pi}{180}x\right) = \frac{\pi}{180} \cos x^\circ$$

5. Differentiate $y = \sin \sqrt{x}$ wrt x .

$$\text{Let } u = \sqrt{x} = x^{\frac{1}{2}}, \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$y = \sin u, \frac{dy}{dx} = \cos u$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos u \frac{1}{2\sqrt{x}} = \frac{\cos u}{2\sqrt{x}} = \frac{\cos \sqrt{x}}{2\sqrt{x}}.$$

6. Differentiate $y = \cos 7x$

$$\text{Let } u = 7x \text{ and } \frac{dy}{dx} = 7$$

$$y = \cos u \text{ and } dy/dx = -\sin u.$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\sin u \times 7 = -7 \sin u.$$

$$= 7 \sin 7x.$$

7. Differentiate $y = \cos(4x^2 - 5x + 11)$

$$\text{Let } u = 4x^2 - 5x + 11, \frac{du}{dx} = 8x - 5$$

$$y = \cos u \text{ and } \frac{du}{dx} = 8x - 5$$

$$y = \cos u \text{ and } \frac{dy}{du} = -\sin u.$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \sin u (8x - 5) = -(8x - 5) \sin u$$

$$= -(8x - 5) \sin(4x^2 - 5x + 11)$$

$$= (5 - 8x) \sin(4x^2 - 5x + 11).$$

8. $y = \cos\left(\frac{\pi}{6} - 5x\right)$

$$\text{Let } u = \frac{\pi}{6} - 5x \Rightarrow \frac{du}{dx} = -5$$

$$y = \cos u \Rightarrow \frac{dy}{du} = -\sin u$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\sin u (-5) = 5 \sin u$$

$$\frac{dy}{dx} = 5 \sin \left(\frac{\pi}{6} - 5x \right)$$

9. $y = \sqrt{\cos x}$

$$\text{Let } u = \cos x, \frac{du}{dx} = -\sin x$$

$$y = \sqrt{u} = u^{\frac{1}{2}} \Rightarrow dy/du = \frac{1}{2} u^{-\frac{1}{2}}$$

$$\frac{1}{\sqrt[2]{u}} = \frac{1}{\sqrt[2]{\cos x}}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{1}{\sqrt[2]{\cos x}} \times (-\sin x) = \frac{\sin x}{\sqrt[2]{\cos x}} \end{aligned}$$

10. if $y = \cos 2x = \frac{\cos 2x}{\sin 2x}$, Differentiate wrt x.

$$\text{Let } u = \cos 2x \text{ and } v = \sin 2x$$

$$\frac{du}{dx} = -2 \sin 2x \text{ and } \frac{dv}{dx} = 2 \cos 2x$$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$= \frac{\sin 2x (-2 \sin 2x) - \cos 2x (2 \cos 2x)}{\sin^2 2x}$$

$$= \frac{-2 \sin^2 2x - 2 \cos^2 2x}{\sin^2 2x} = \frac{-2(\sin^2 2x + \cos^2 2x)}{\sin^2 2x}$$

$$= \frac{-2(1)}{\sin^2 2x} = -2 \operatorname{cosec}^2 2x$$

11. Differentiate $y = \tan x^2$

$$\text{Let } u = x^2 \text{ and } \frac{du}{dx} = 2x$$

$$\text{Then } y = \tan u \text{ and } \frac{dy}{dx} = \sec^2 u$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} = \sec^2 u (2x) \\ &= 2x \sec^2 x^2. \end{aligned}$$

12. $y = \frac{\cos x}{e^{x^2}}$

$$\text{Let } u = \cos x \text{ and } v = e^{x^2}$$

$$\frac{du}{dx} = -\sin x \text{ and } \frac{dv}{dx} = 2x e^{x^2}$$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$\frac{e^{x^2}(-\sin x) - \cos x(2x e^{x^2})}{(e^{x^2})^2}$$

$$= \frac{-e^{x^2} \sin x - 2x e^{x^2} \cos x}{(e^{x^2})^2}$$

$$= \frac{-e^{x^2}(\sin x + 2x \cos x)}{(e^{x^2})^2}$$

$$= \frac{-(\sin x + 2x \cos x)}{(e^{x^2})^2}$$

$$= \frac{-(\sin x + 2x \cos x)}{e^{x^2}}$$

$$= -e^{x^2}(\sin x + 2x \cos x).$$

3.4 Application of Differentiation To Economics Problem

1. A firm has analysed its operating conditions prices and cost have developed the following functions Revenue (R) = $400 - 4q^2$ (per thousands) and cost (c) = $q^2 + 10q + 30$ (per thousand naira) where q is the number of units produced and sold.

If the firm wishes to maximize profit

- a. What quantity should be sold and at what price.
- b. What will be the amount of profit.

Solution

Profit is maximized when $MC = MR$.

$$R = 400q - 4q^2, \frac{dR}{dq} = 400 - 8q.$$

$$C = q^2 + 10q + 30, \frac{dc}{dq} = 2q + 10$$

Then since $MR = MC \Rightarrow 400 - 8q = 2q + 10$

$$400 - 10 = 2q + 8q$$

$$390 = 10q$$

$$q = \frac{390}{10} = 39 = 39000 \text{ units.}$$

To maximized profit the firm should be able to sell 39 units of it products.

$$R = 400q - 4q^2$$

$$\text{When } q = 39, R = 400(39) - 4(39)^2$$

$$\Rightarrow R = 15600 - 6084 = 9516 = \text{N}9516000.$$

$$\begin{aligned} \text{Hence the price for each unit} &= \frac{9516000}{39000} \\ &= \text{N}244 \end{aligned}$$

To maximize profit, the firm should sell each unit of its product at the rate of ~~N~~244.

c. Profit = Revenue – Cost.

$$\begin{aligned} Pr &= 400q - 4q^2 - (q^2 + 10q + 30) \\ &= 400q - 4q^2 - q^2 - 10q - 30 \\ &= 390q - 5q^2 - 30 \\ &= 390(39) - 5(39)^2 - 30 \\ &= 15210 - 7605 - 30 = \text{N}7575 \text{ per thousand} \\ &= \text{N}757500. \end{aligned}$$

∴ In maximizing profit, the firm will realize ~~N~~757500 as per profit.

2. The demand for a product is given by $q = 72 - 3p$.

- Determine the revenue function
- What is the revenue when price is ~~N~~2.
- What is the marginal revenue and at what price is the marginal revenue equal to zero.
- What is the maximum revenue.

Solution

i. Revenue = $P \times q = (72 - 3P)P = 72P - 3P^2$
Revenue function is $72P - 3P^2$

ii. When price is ~~N~~2, $R = 72(2) - 3(2)^2$
 $= 144 - 12 = \text{N}132$

iii. Marginal revenue (MR) = $\frac{dR}{dq} = \frac{d}{dq}(72P - 3P^2)$

$$MR = 72 - 6P.$$

When $MR = 0$, $\Rightarrow 72 - 6P = 0$
 $72 - 6P$

$$P = \frac{72}{6}$$

$$P = \text{N}12.$$

The marginal revenue is $72 - 6P$ and when the marginal revenue is zero the price will be ~~N~~12.

iv. $R = 72P - 3P^2$
Maximum revenue = $72(12) - 3(12)^2$

$$= 864 - 432 = \text{N}432.$$

3. The demand function for a product is given by $P = 200 - 4Q$ and the supply function for the same product is given by $P = 5Q - 40$.
- Determine the equilibrium price and quantity.
 - Find the elasticity of demand function when $P = \text{N}20$.
 - Find the elasticity of supply at $P = \text{N}25$.

Solution

1. For equilibrium to be attained; demand = supply
- $$200 - 4Q = 5Q - 40$$
- $$-4Q - 5Q = -40 - 200$$
- $$-9Q = -240$$
- $$Q = \frac{-240}{-9} = \frac{80}{3}$$
- $$Q = \text{N}26.67.$$

To determine the equilibrium quantity; substitute for Q in $200 - 4Q$.

$$Q = 200 - 4\left(\frac{80}{3}\right)$$

$$Q = 200 - \frac{320}{3} = 200 - 106.67$$

$$Q = 93.33.$$

- ii. The elasticity = $\frac{P}{Q} \left(\frac{dQ}{dP} \right)$

However the demand function is given by $q = 200 - 4P$

$$\therefore dQ/dP = -4.$$

$$\text{Elasticity of demand} = \frac{P}{200 - 4P} \times (-4)$$

$$= \frac{4P}{200 - 4P}$$

$$\text{When } P = 20, \text{ elasticity of demand} = \frac{-4(20)}{200 - 4(20)}$$

$$\frac{-80}{200-80} = \frac{-80}{200-120} = \frac{-2}{3}$$

Elasticity of demand = $\frac{-2}{3}$ (inelastic demand).

iii. Supply equation is $q = 5P - 40$

$$\frac{dq}{dP} = 5$$

Elasticity of supply = $\frac{P}{q} \left(\frac{dq}{dP} \right) \Rightarrow \frac{P}{5P-40} (5)$

$$E^s = \frac{P}{5P-40}$$

If P is given to 25 : $\frac{5(25)}{5(25)-40} = \frac{125}{125-40}$

$$\Rightarrow \frac{125}{85} = \frac{25}{17} = 1.47.$$

$E^s = 1.47$ (Elastic Supply).

4.(A) The demand and supply function for a product are respectively given as $P = 500 - 4q$ and $P = 200 + 5q$, where P is the price and q is the demand in unit.

- i. Determine the elasticity of the demand
- ii. The elasticity of supply
- iii. The equilibrium price and quantity

- (B)i. Using the information in (a) above to find the revenue function and hence determine the revenue for a sales of 20 units .
- ii. If the cost function for the same product $c = 20 + 10q$, then determine the profit for a sales of 30 units.

Solution

i. Demand function is given as $P = 500 - 4q$.

$$\frac{dP}{dq} = -4, \text{ but } \frac{dq}{dP} = \frac{1}{-4}.$$

$$\begin{aligned}\text{Elasticity of demand} &= \frac{P}{q} \times \frac{dq}{dP} \\ \Rightarrow \frac{500 - 4q}{4q} &= \frac{4q - 500}{4q} = \frac{4(q - 125)}{4q} \\ E_d &= \frac{q - 125}{q}.\end{aligned}$$

ii. Supply functions is $P = 200 + 5q$

$$\frac{dq}{dP} = 5 \text{ therefore we can then have } \frac{dP}{dq} = \frac{1}{5}.$$

iii. At equilibrium $500 - 4q = -200 + 5q$

$$\begin{aligned}500 + 200 &= 5q + 4q \\ 700 &= 9q \\ q &= 700/9 = 77.78 = 78.\end{aligned}$$

Therefore $E_q = 78$ units.

So if $q = 77.78$ (Note don't use approximation figure).

$$P = 500 - 4\left(\frac{700}{9}\right)$$

$$\begin{aligned}P &= 500 - \frac{2800}{9} = \frac{4500 - 2800}{9} \\ &= \frac{1700}{9} = 188.89\end{aligned}$$

Equilibrium price $E_p = \text{N}188.89$.

(C)i. Revenue (R) = $P \times q = (500 - 4q) \times q$.

$$R = 500q - 4q^2.$$

However the revenue function is $500q - 4q^2$.

When $q = 20$, $R = 500(20) - 4(20)^2$

$$= 1000 - 1600 = \text{N}8400.$$

$$\begin{aligned}
 \text{ii. Profit} &= \text{Revenue} - \text{Cost} \\
 &= 500q - 4q^2 - (20 + 10q) \\
 &= 500q - 4q^2 - 20 - 10q \\
 &= 490q - 4q^2 - 20
 \end{aligned}$$

When $q = 30$.

$$\begin{aligned}
 P &= 490(30) - 4(30)^2 - 20 \\
 &= 14700 - 3600 - 20 = \text{N}11080.
 \end{aligned}$$

We can then conclude that for the sales of 30 units the profit is N11080.

Self-Assessment Exercise

- 4.(A) The demand and supply function for a product are respectively given as $P = 700 - 8q$ and $P = 100 + 3q$, where P is the price and q is the demand in unit.
- Determine the elasticity of the demand
 - The elasticity of supply
 - The equilibrium price and quantity
- (B)i. Using the information in (a) above to find the revenue function and hence determine the revenue for a sales of 10 units .
- If the cost function for the same product $c = 30 + 20q$, then determine the profit for a sales of 20 units.

4.0 CONCLUSION

In this unit, we can conclude that differentiation of trigonometric functions takes a look at derivative of trigonometric functions such as Sin, Cos and Tan respectively. However, application of differentiation to economics analysis also give rise to day to day problems we face with economic activities.

5.0 SUMMARY

In this unit, we have been able to discuss on the extensively on differentiation of exponential function, logarithmic functions and trigonometric function. More so, derivatives of Sin x , Cos x , Tan x and

application of differentiation to economics problems was also examined, therefore, you should be able to have a wider knowledge of this unit.

6.0 TUTOR-MARKED ASSIGNMENT

1. Differentiate $y = \frac{\sin x}{2 + \cos x}$.
2. Differentiate $y = \frac{\sin x - \cos x}{\sin x + \cos x}$.
3. Differentiate $y = \cos\left(\frac{\pi}{8} - 7x\right)$
4. The demand function for a product is given by $P = 10 - 5q$ while the supply function for the same product is given by $P = 4q - 8$. The total cost function for the same product is $TC = 210 - 81q + 2q^2$. Determine:
 - i. The equilibrium quantity.
 - ii. The break even point for the product.
 - iii. The point of maximum profit for the product and the maximum profits made.
 - iv. The number of units that will maximize cost.
 - v. The elasticity of demand and supply for the product at prices of N5 and N16 respectively.
 - vi. The interpretation of the result of (V) above.

7.0 REFERENCES/FURTHER READINGS

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MODULE 3 CALCULUS II: INTEGRATION

Unit 1	Integration
Unit 2	Integration of Exponential and Trigonometric
Unit 3	Integration by Substitution and by Parts

UNIT 1 INTEGRATION

CONTENTS

1.0	Introduction
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1.0 INTRODUCTION

Integration is also known to as ante-differentiation. In mathematics, an integral assigns numbers to functions in a way that can describe displacement, area, volume and other concepts what arise by combining infinitesimal data. Integration is one of the two main operations of calculus, with its inverse, differentiation, being the other. Given a function f of a real variable x and an interval $[a, b]$ of the real line, the definite integral.

$\int_a^b f(x)dx$. This is defined informally as the signed area of the region in the xy – plane that is bounded by the graph of f, the x-axis and the vertical lines $x = a$ and $x = b$. the area above the x-axis adds to the total and that below the x-axis subtracts from the total.

However, integration is the reverse of differentiation, and the term integral may also refer to the related notion of the anti derivative, a function F whose derivative is the given function f, therefore it is called indefinite integral and written as:

$$F(x) = \int f(x)dx$$

Furthermore, the principles of integration were formulated independently by Isaac Newton and Gottfried Leibniz in the late 17th century, who thought of the integral as an infinite sum of rectangles of infinitesimal width.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Understand how to calculate definite and indefinite integral
- Know how to work through integration by a constant and polynomial

3.0 MAIN CONTENT

3.1 Worked Examples on Indefinite Integral

Example of integration

- (1) Integrate $x^{2/3}$ wrt. x

$$\int x^{2/3} dx = \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + c = \frac{x^{5/3}}{5/3} + c = \frac{3x^{5/3}}{5} + c$$

- (2) Integrate $5x$ wrt. X

$$\int 5x dx = \int 5x^1 dx = \frac{5x^{1+1}}{1+1} + c = \frac{5x^2}{2} + c$$

- (3) Integrate $\sqrt[3]{x}$ wrt. X

$$\begin{aligned}\int \sqrt[3]{x} dx &= \int 9x^{1/2} dx = \frac{9x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{9x^{3/2}}{3/2} + c \\ &= \frac{2(9x^{3/2})}{3} + c = 6x^{3/2} + c \text{ or } \sqrt[6]{x^3} + c\end{aligned}$$

(4) Integrate $\frac{4}{\sqrt[3]{x}}$ wrt x

$$\begin{aligned}\int \frac{4}{\sqrt[3]{x}} dx &= \int \frac{4}{x^{\frac{1}{3}}} dx = \int 4x^{-\frac{1}{3}} dx = \frac{4x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + c \\ &= \frac{4x^{2/3}}{2/3} + c = \frac{3(4x^{2/3})}{2} = 6x^{2/3} + c \\ &\text{or } 6\left(\sqrt[3]{x^2}\right) + c.\end{aligned}$$

(5) Integrate $\frac{x^3}{\sqrt{x}}$ wrt x

$$\begin{aligned}\int \frac{x^3}{\sqrt{x}} dx &= \int \frac{x^3}{x^{1/2}} dx = \int_{1/2}^{x^{3.5}x^{-1/2}dx} = \int_{7/2}^{x^{5/2}dx} \\ &= \frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + c = \frac{x}{\frac{7}{2}} + c = \frac{2x}{7} + c.\end{aligned}$$

3.2 Integration of a Constant

Let G be a constant, then $\int G dx = Gx + c$ where c is the arbitrary constant introduced since it is an indefinite integral. It should be noted here that G can be written as Gx^0 if the integration is with respect to x.

$$\begin{aligned}\text{Then: } \int G dx &= \int Gx^0 dx = \frac{Gx^{0+1}}{0+1} + c. \\ &= \frac{Gx^1}{1} + c = Gx + c.\end{aligned}$$

Let us take an example on this analysis.

$$1. \quad \text{Integrate } \int 8dx$$

$$\int 8dx = \int 8x^0 dx = 8 \frac{x^{0+1}}{0+1} + c = 8x + c.$$

$$2. \quad \int -2dx$$

$$\int -2dx = -2x + c$$

$$3. \quad \int 10dx$$

$$\int 10dx = 10x + c.$$

3.3 Integration of a Polynomial

If a given integral function is a polynomial, it can be performed by integrating term by term, with the constant factor brought outside the integral sign.

$$1. \quad \int (x^3 dx - 5^2 + 13x - 11)dx$$

$$= \int x^3 dx - \int 5x^2 dx + \int 13x dx - \int 11 dx.$$

$$= \int x^3 dx - 5 \int x^2 dx + 13 \int x dx - 11 \int dx$$

$$= \frac{x^4}{4} - \frac{5x^3}{3} + \frac{13x^2}{2} - 11x + c.$$

$$2. \quad = \int \left(\frac{3x^4 + 5x - 4}{x^3} \right) dx$$

$$= \int \left(\frac{3x^4}{x^3} + \frac{5x}{x^3} - \frac{4}{x^3} \right) dx = \int (3x + 5x^{-2} - 4x^{-3}) dx$$

$$= 3 \int x dx + 5 \int x^{-2} dx - 4 \int x^{-3} dx$$

$$= 3 \frac{x^2}{2} + \frac{5x^{-1}}{-1} - \frac{4x^{-2}}{-2} + c$$

$$= \frac{3}{2}x^2 + \frac{5}{x} - \frac{2}{x^2} + c.$$

$$3. \int \sqrt{x}(1+x)dx$$

$$= \int x^{1/2}(1+x)dx = \int (x^{1/2} + x^{1/2} \cdot x)dx$$

$$= \int (x^{1/2} + x^{3/2})dx = \frac{x^{3/2}}{3/2} + \frac{x^{5/2}}{\frac{5}{2}} + c$$

$$= \frac{2}{3}x^{3/2} + \frac{2}{5}x^{5/2} + c.$$

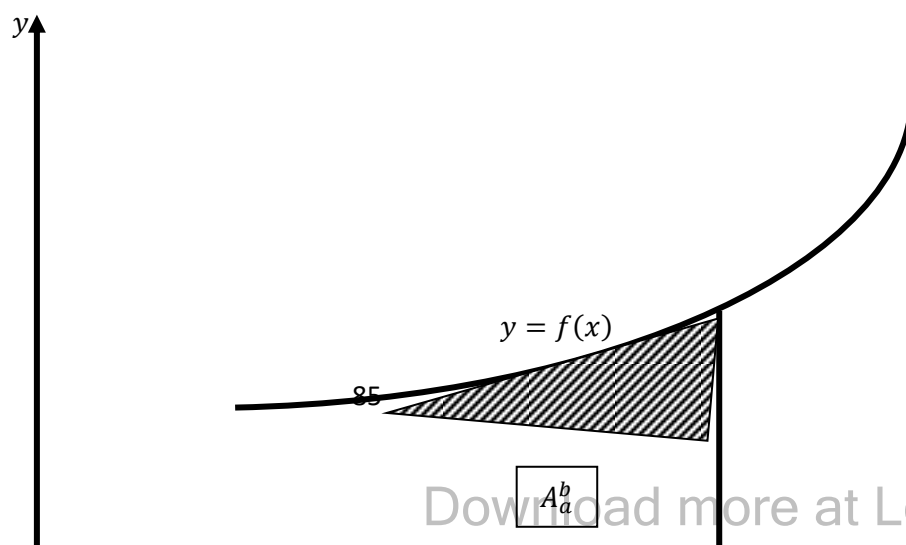
3.4 Definite Integral

Definite integral is when a numeric result as it is the case in the calculation of total revenue between two output levels when the marginal revenue is given, the expression is termed a definite integral and it is written as $\int_a^b ydx$.

The expression above shows that, 'a' is a definite value called lower limit or lower bound and 'b' is also a definite value called upper limit or upper bound. In majority of the cases 'b' maybe greater than a especially when it is applied to business or economic problems.

Let us consider the curve $y = f(x)$, $y > 0$ in the range of $x = a$ to $x = b$ shown in figure 2 below:

The area A between the curve and the x-axis is given by: $A = \int f(x)dx + c = g(x) + c$, say, when $x = a$, the area = c.



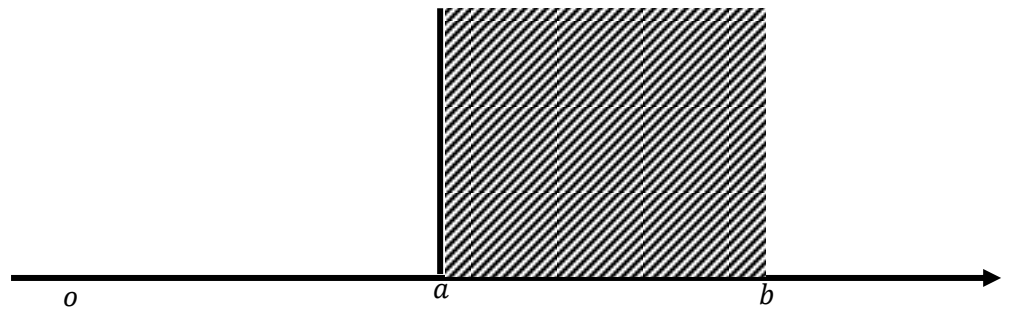


Figure showing the curve $y = f(x)$.

Therefore $g = (a) + c$, or $c = -g(a)$

Then $A_a^x = g(x) - g(a)$ and So $A_a^b = g(b) - g(a)$

$= (\text{Value of integral when } x = b) - (\text{Value of integral when } x = a)$

When it can be written as $\int_a^b f(x)dx$ and this is called the definite integral of $f(x)$

Example 1

$$\begin{aligned} \text{Evaluate } \int_1^2 x^2 dx &= \left(\frac{x^3}{3} + c \right)_1^2 \\ &= \left(\frac{2^3}{3} + c \right) - \left(\frac{1^3}{3} + c \right) \Rightarrow \left(\frac{8}{3} + c \right) - \left(\frac{1}{3} + c \right) = \frac{8}{3} + c - \frac{1}{3} - c = \frac{7}{3}. \end{aligned}$$

Example 2

$$\begin{aligned} &\int_{-1}^1 \frac{(x+1)(x^2-2x+2)}{x^2} dx \\ &= \int_{-1}^1 \frac{(x^3 - 2x^2 + 2x + x^2 - 2x + 2)}{x^2} dx \\ &= \int_{-1}^1 \left(\frac{x^3 - x^2 + 2}{x^2} \right) dx = \int_{-1}^1 \left(x + 1 + \frac{2}{x^2} \right) dx \\ &= \int_{-1}^1 (x - 1 + 2x^{-2}) dx = \left(\frac{x^2}{2} - x + \frac{2x^{-1}}{-1} \right) \end{aligned}$$

(You should note here that the constant c has not been included since it will cancel out later).

$$\begin{aligned}
&= \left[\frac{x^2}{2} - x - \frac{2}{x} \right]_{-1}^1 = \left[\frac{1^2}{2} - 1 - \frac{2}{1} \right] - \left[\frac{(-1)^2}{2} - (-1) - \frac{-2}{-1} \right] \\
&= \left(\frac{1}{2} - 1 - 2 \right) - \left(\frac{1}{2} + 1 + 2 \right) = -2\frac{1}{2} - 3\frac{1}{2} = -6.
\end{aligned}$$

Example 3

$$\begin{aligned}
\int_1^{0.5} \frac{2}{x^2} dx &= 2 \left[\frac{x^{-1}}{-1} \right]_1^{0.5} = -2 \left[\frac{1}{x} \right]_1^{0.5} \\
&= -2 \left[\frac{1}{0.5} - \frac{1}{1} \right] = -2(2 - 1) = -2(1) = -2.
\end{aligned}$$

Self-Assessment Exercise

$$\text{Evaluate } \int_2^3 2x^2 dx = \left(\frac{x^2}{2} + c \right)_1^2$$

4.0 CONCLUSION

In this unit, we can conclude that integration is seen as a branch of mathematics concerned with the theory and applications (as in the determination of lengths, areas, and volumes and in the solution of differential equations) of integrals and integration. However, we also concludes in this unit with the calculation of definite and indefinite integral of integration.

5.0 SUMMARY

In this unit, we have been able to discuss on the topic integration where we look at definite and indefinite integral. We also look through the calculation of integration of a constant and polynomial.

6.0 TUTOR-MARKED ASSIGNMENT

$$1. \int_a^{2a} (2^2 + y - 2) dy$$

$$2. \int_0^{1/2} (2x^2 + x^2 + 4) dx$$

$$3. \int \left(x + \frac{1}{x} \right)^2 dx$$

$$4. \int \frac{(1 - 3x)^2}{\sqrt{3x}} dx.$$

7.0 REFERENCES/FURTHER READINGS

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UNIT 2 INTEGRATION OF EXPONENTIAL AND TRIGONOMETRIC FUNCTION

CONTENTS

- 1.0 Introduction
- 2.0 Objectives
- 3.0 Main Content
 - 3.1 Worked Examples of Exponential Function
 - 3.2 Integration of Trigonometric Function
- 4.0 Conclusion
- 5.0 Summary
- 6.0 Tutor-Marked Assignment
- 7.0 References/Further Readings

1.0 INTRODUCTION

Like we did in module two of this course material where we perform differentiation of exponential and trigonometric function, the same will also be performed here.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Understand how to calculate integration of exponential function
- Know how to calculate integration of trigonometric function

3.0 MAIN CONTENT

3.1 Worked Examples of Integration of Exponential Function

$$1. \int e^{5x+2} \Rightarrow \text{let } u = 5x + 2 \Rightarrow \frac{du}{dx} = 5$$

$$dx = \frac{du}{5}$$

$$\text{Therefore } \int e^{5x+2} dx = \int e^u \frac{du}{5}$$

$$\begin{aligned} &= \frac{1}{5} \int e^u du = \frac{1}{5} e^u + c \\ \Rightarrow \frac{1}{5} \int e^u du &= \frac{1}{5} e^u + c \Rightarrow \frac{1}{5} e^{5x+2} + c. \end{aligned}$$

$$2. \int (e^{3x-2} - e^{-2x+3}) dx \Rightarrow \int e^{3x-2} dx - \int e^{-2x+3} dx.$$

$$\text{Let } P = 3x - 2 \text{ and } U = -2x + 3$$

$$\frac{dp}{dx} = 3 \text{ and } \frac{du}{dx} = -2$$

$$dx = \frac{dp}{3} \text{ and } dx = -\frac{du}{2}$$

$$\Rightarrow \int e^p \frac{dp}{3} - \int e^u - \frac{du}{2} \Rightarrow \frac{1}{3} \int e^p dp - \left[-\frac{1}{2} \int e^u du \right]$$

$$\Rightarrow \frac{1}{3} e^p + c - \left(-\frac{1}{2} e^u + c \right) \Rightarrow \frac{1}{3} e^p + \cancel{c} + \frac{1}{2} e^u - \cancel{c}$$

$$\Rightarrow \frac{1}{3} e^p + \frac{1}{2} e^u \Rightarrow \frac{1}{3} e^{3x-2} + \frac{1}{2} e^{-2x+3} + c.$$

$$3. \int (3x^2 + 1)e^{x^3+x+2} dx$$

$$\Rightarrow \int 3x^2 + 1 dx \cdot \int e^{x^3+x+2} dx$$

$$\Rightarrow \text{Let } u = x^3 + x + 2 \Rightarrow \frac{du}{dx} = 3x^2 + 1$$

$$\Rightarrow dx = \frac{du}{3x^2 + 1} \Rightarrow \int (3x^2 + 1) \cdot e^u \cdot \frac{du}{3x^2 + 1}$$

$$\Rightarrow e^{x^3+x+2} + c.$$

$$4. \int_0^1 \frac{1}{\sqrt{x}} e^{\sqrt{x}} dx$$

$$\Rightarrow \int_0^1 e^{-3u} dx \Rightarrow \text{let } u = -3x \Rightarrow \frac{du}{dx} = -3$$

$$dx = \frac{du}{-3} \Rightarrow \int_0^1 e^u \cdot \frac{du}{-3}$$

$$\Rightarrow \frac{1}{-3} \int_0^1 e^u du = -\frac{1}{3} [e^u + c]_0^1$$

$$\Rightarrow -\frac{1}{3} (e^1 + c) - [e^0 + c] \Rightarrow -\frac{1}{3} (e + c - 1 - c) \Rightarrow -\frac{1}{3} (e - 1)$$

Let us recall that euler's constant = 2.718.

Therefore we can substitute for the value of e in the above equation.

$$\Rightarrow -\frac{1}{3} (2.718 - 1) \Rightarrow -\frac{1}{3} (1.718)$$

$$\Rightarrow -\frac{1.718}{3} = -0.573.$$

$$5. \int \frac{x^2 dx}{2x^3 + 3}$$

$$\text{Let } u = 2x^3 + 3, \frac{du}{dx} = 6x^2 \Rightarrow dx = \frac{du}{6x^2}$$

$$\int \frac{x^2 dx}{2x^3 + 3} = \int \frac{x^2}{u} \cdot \frac{du}{6x^2}$$

$$\int \frac{1}{6} du = \frac{1}{6} \int \frac{1}{u} du$$

$$\frac{1}{6} \log_e u + c = \frac{1}{6} \log_e (2x^3 + 3) + c$$

$$= \frac{1}{6} \ln(2x^3 + 3) + c \Rightarrow$$

Recall from logarithmic rule:

$$\log_e^x = \ln x.$$

3.2 Integration of Trigonometric Function

Let us remember that from our knowledge of differentiation, the derivative of $\sin x$ is $\cos x$ while the derivatives of $\cos x$ is $-\sin x$.

Let us remember that $\sin x$ integration is the opposite of differentiation, therefore the integral of $\cos x$ is $\sin x$, and that of $\sin x$ is $-\cos x$.

For example $\int \sin x dx = -\cos x + c$ and $\int \cos x dx = \sin x + c$ where c is an arbitrary constant.

Example 1

Integrate $\cos 4x$ wrt x .

$$1. \int \cos 4x dx$$

$$\text{Let } u = 4x, \frac{du}{dx} = 4 \Rightarrow dx = \frac{du}{4}$$

$$\therefore \int \cos 4x dx = \int \cos u \frac{du}{4} \Rightarrow$$

$$\Rightarrow \frac{1}{4} \int \cos u du.$$

$$\Rightarrow \frac{1}{4} \sin u + c = \frac{1}{4} \sin 4x + c$$

$$2. \int \sin 7x dx$$

$$\text{Let } u = 7x, \frac{du}{dx} = 7 \Rightarrow dx = \frac{du}{7}$$

$$\therefore \int \sin 7x dx = \int \sin u \frac{du}{7} = \frac{1}{7} \int -\cos u du$$

$$= \frac{1}{7}(-\cos u) + c = \frac{-1}{7}\cos u + c$$

$$= \frac{1}{7}\cos 7x + c.$$

$$3. \int \sin(5x + 2) dx$$

$$\text{Let } u = 5x + 2, \frac{du}{dx} = 5 \Rightarrow dx = \frac{du}{5}$$

$$\therefore \int \sin(5x + 2) dx = \int \sin u \cdot \frac{du}{5}$$

$$\Rightarrow \frac{1}{5} \int \sin u du \Rightarrow \frac{1}{5}(-\cos u) + c$$

$$\Rightarrow -\frac{1}{5}\cos(5x + 2) + c.$$

$$4. \int \left(5x^4 + \cos 4x - \frac{1}{x} \right) dx$$

$$= \int 5x^4 dx + \int \cos 4x dx - \int \frac{1}{x} dx$$

$$= \frac{5x^5}{5} + \frac{1}{4}\sin 4x - \log_e^x + c$$

$$= x^5 + \frac{1}{4}\sin 4x - \ln x + c.$$

$$5. \int [e^{3x} - 3 \cos(6x - 1)] dx$$

$$\int e^{3x} dx - \int 3 \cos(6x - 1) dx$$

$$= \frac{1}{3}e^{3x} dx - 3 \left[\frac{1}{4}\sin(6x - 1) \right] + c$$

$$= \frac{1}{3}e^{3x} - \frac{3}{4}\sin(6x + 1) + c.$$

$$\begin{aligned}
6. \quad & \int \tan x dx \Rightarrow \int \frac{\sin x dx}{\cos x} \\
& \Rightarrow \text{Let } u = \cos x, \frac{du}{dx} = -\sin x \\
& \Rightarrow dx = \frac{du}{-\sin x} \\
& \therefore \int \frac{\sin x dx}{\cos x} = \int \frac{\sin x}{u} \cdot \frac{du}{-\sin x} \\
& = -\int \frac{du}{u} = -\log_e u + c = -\ln \cos x + c \\
& = -\ln \cos x + c
\end{aligned}$$

$$7. \quad \int 3x^2 \sin x^3 dx$$

$$\text{Let } u = x^3, \frac{du}{dx} = 3x^2 \Rightarrow dx = \frac{du}{3x^2}$$

$$\begin{aligned}
\therefore \int 3x^2 \sin x^3 dx &= \int 3x^2 \sin u \cdot \frac{du}{3x^2} \\
&= \int \sin u du = -\cos u + c.
\end{aligned}$$

$$8. \quad \int e^{2 \cos 3x} \sin 3x dx$$

$$\text{Let } u = 2 \cos 3x, \frac{du}{dx} = -6 \sin 3x$$

$$\Rightarrow dx = \frac{du}{-6 \sin 3x}$$

$$\therefore \int e^{2 \cos 3x} \sin 3x dx = \int e^u \sin 3x \cdot \frac{du}{-6 \sin 3x}$$

$$= -\frac{1}{6} \int e^u du = -\frac{1}{6} e^u + c$$

$$= -\frac{1}{6} e^{2 \cos 3x} + c.$$

Self-Assessment Exercise

$$\int \tan 2x dx \Rightarrow \int \frac{\sin 2x dx}{\cos 2x}$$

4.0 CONCLUSION

In this unit, we can conclude that integration of exponential function and trigonometric function are a family of integrals involving trigonometric functions.

5.0 SUMMARY

In this unit, we have been able to discuss on the topic integration of exponential function and trigonometric function and a lot of examples were examined in this unit and I think this will go a long way for your understanding of this topic.

6.0 TUTOR-MARKED ASSIGNMENT

1. $\int [e^2x - 4 \cos(2x - 7)]dx$

2. $\int \sin 4x dx$

3. $\int 3x^2 \sin x^3 dx$

7.0 REFERENCES/FURTHER READINGS

- Sogunro, Y., & Abdurasid, A., & Adewuyi, B., (1998)., Basic Business Mathematics, Concepts and Applications, volume 2, Sogunro Publication Limited, Lagos State Polytechnic, Isolo, Lagos.
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UNIT 3 INTEGRATION BY SUBSTITUTION AND BY PARTS

CONTENTS

- 1.0 Introduction
- 2.0 Objectives
- 3.0 Main Content
 - 3.1 Integration by Parts
 - 3.2 Integration by Parts
- 4.0 Conclusion
- 5.0 Summary
- 6.0 Tutor-Marked Assignment
- 7.0 Reference/Further Readings

1.0 INTRODUCTION

In integration some of the function cannot be easily integrated unless the independent variables x is definitely changed to another variable u and given rise to the relationship between x and u to be known.

Integration by substitution is used to change from one integral to another that is easier to solve. The process of integrating by substitution is basically the process of applying the chain rule, but in reverse.

However, integration by parts or partial integration is a theorem that relates the integral of a product of functions to the integral of their

derivatives and antiderivatives. Normally we use it to transform the antiderivative of a product of functions into an antiderivatives for which a solution can be easily found.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Understand how to calculate integration by substitution and by parts

3.0 MAIN CONTENT

3.1 Integration by Substitution

Let us look at two composed functions f and g that are continuous over given interval, let $u = g(x)$ and $du = g'(x)dx$. Such that $\int f(g(x))g'(x)dx = \int f(u)du = f(u) = f(g(x))$.

First you determine the substitution $u = g(x)$ and differentiate it to obtain $du = g'(x)$ and dx .

You then substitute u and du into the integral, complete the integration and replace the original functions using the substitutions.

Let us now takes some examples to illustrate our analysis on integration by substitution:

$$1. \quad \int 3\sqrt{4y-6}dy = \int 3(4y-6)^{\frac{1}{2}}dy$$

$$\text{Let } u = 4y - 6, \frac{du}{dy} = 4 \Rightarrow dy = \frac{du}{4}$$

$$\therefore \int 3\sqrt{4y-6}, dy = 3 \int \frac{u^{\frac{1}{2}}du}{4}$$

$$\Rightarrow \frac{3}{4} \int u^{\frac{1}{2}} du$$

$$= \frac{3}{4} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{6}{12} u^{\frac{3}{2}} + c = \frac{1}{2} u^{\frac{3}{2}} + c$$

$$\text{but } u = 4y - 1.$$

$$\therefore \int \sqrt{4y-6} dy = \frac{1}{2}(4y-6)^{\frac{3}{2}} + c.$$

$$2. \quad \int \frac{du}{(3x-4)^2}$$

$$\therefore \text{Let } u = 3x - 4, \frac{du}{dx} = 3$$

$$dx = \frac{du}{3}$$

$$\int \frac{dx}{(3x-4)^2} = \int \frac{1}{2} \cdot \frac{du}{3} = \frac{1}{3} \int u^{-2} du$$

$$\Rightarrow \frac{1}{3} \left[\frac{u^{-1}}{-1} \right] + c = \frac{-1}{3} \left(\frac{1}{u} \right) + c$$

$$\Rightarrow -\frac{1}{3} \left[\frac{1}{3x-5} \right] + c$$

$$= \frac{-1}{9x-15} + c = \frac{1}{15-9x} + c.$$

$$3. \quad \int_0^1 \frac{e^{2x} dx}{3e^{2x} + 5}$$

$$\text{Let } u = 3e^{2x} + 5, \frac{du}{dx} = 6e^{2x} \Rightarrow dx = \frac{du}{6e^{2x}}$$

$$\therefore \int_0^1 \frac{e^{2x} dx}{3e^{2x} + 5} = \int_0^1 \frac{e^{2x}}{u} \cdot \frac{du}{6e^{2x}}$$

$$dx = \frac{du}{6e^{2x}}$$

$$= \int_0^1 \frac{1}{6u} du = \frac{1}{6} \int_0^1 \frac{1}{u} du$$

$$= \frac{1}{6} \left(\log_e u \right)_0^1 = \frac{1}{6} [\log_e (3e^{2x} + 5)]$$

$$= \frac{1}{6} [\log_e (3e^2 + 5) - \log_e (3 + 5)]$$

$$= \frac{1}{6} [\log_e(22.167 + 5) - \log_e^8]$$

$$= \frac{1}{6} [\log_e(27.167) - \log_e^8]$$

$$= \frac{1}{6} (3.3020 - 2.0794)$$

$$= \frac{1}{6} (1.2226) = 0.203766.$$

$$4. \quad \int \frac{x^4}{(2x^5 - 14)^7} dx$$

$$\text{Let } u = 2x^5 - 14, \frac{du}{dx} = 10x^4$$

$$\Rightarrow dx = \frac{du}{10x^4}$$

$$\begin{aligned} \int \frac{x^4}{(2x^5 - 11)^7} dx &= \int \frac{x^4}{u^7} \cdot \frac{du}{10x^4} \\ &= \frac{1}{10} \int \frac{1}{u^7} = \frac{1}{10} \int u^{-7} du \\ &= \frac{1}{10} \left[\frac{u^{-6}}{-6} + c \right] = \frac{-1}{60} u^{-6} + c \\ &= \frac{1}{10} (2x^5 - 11)^{-6} + c \\ &= \frac{-}{60(2x^5 - 11)^6} + c. \end{aligned}$$

$$5. \quad \int (x^2 + 2x + 3) \left(\frac{1}{3}x^3 + x^2 + 3x - 7 \right)^6 dx$$

$$\text{Let } u = \frac{1}{3}x^3 + x^2 + 3x + 7, \frac{du}{dx} = x^2 + 2x + 3$$

$$dx = \frac{du}{x^2 + 2x + 3}$$

$$\begin{aligned}
&= \int (x^2 + 2x + 3) \left(\frac{1}{3}x^3 + x^2 + 3x - 7 \right)^6 dx \\
&= \int (x^2 + 2x + 3) u^6 \cdot \frac{du}{(x^2 + 2x + 3)} \\
&= \int u^6 du = \frac{u^7}{7} + c \\
&= \frac{1}{7} \left(\frac{1}{3}x^3 + x^2 + 3x - 7 \right)^7 + c.
\end{aligned}$$

$$6. \quad \int \tan 2x dx$$

$$= \int \frac{\sin 2x dx}{\cos 2x}$$

$$\text{Let } u = \cos 2x, \frac{du}{dx} = -2 \sin 2x$$

$$\Rightarrow dx = -\frac{du}{2 \sin 2x}$$

$$\therefore \int \frac{\sin 2x dx}{\cos 2x} = \int \frac{\sin 2x}{u} \cdot \frac{du}{-2 \sin 2x}$$

$$= \frac{1}{-2} \int \frac{1}{u} du = -\frac{1}{2} \log_e u + c$$

$$= -\frac{1}{2} \log_e^{\cos 2x} + c$$

3.2 Integration by Parts

In calculus and more generally in mathematical analysis, Integration by parts or partial integration is a theorem that relates the integral of a product of functions to the integral of their derivative and antiderivatives. It is used to transform the antiderivatives of a product of functions into an antiderivative for which a solution can be easily found.

For example

If $u = u(x)$ and $du = u^1(x)dx$ while $v = v(x)$ and $dv = v^1(x)dx$, therefore integration by parts can be written as:

$$\int_a^b u(x) v'(x) dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x) dx$$

Or let us consider the derivation of a product say 'uv' where u and v are each a function of x.

$$\frac{d}{dx}(uv) = v \frac{du}{dx} + u \frac{dv}{dx}$$

Integrating with respect to x we have

$$uv = \int \left(v \frac{du}{dx} + u \frac{dv}{dx} \right) dx$$

$$uv = \int v \frac{du}{dx} dx + \int u \frac{dv}{dx} dx$$

$$uv = \int v du + \int u dv$$

$$\text{Therefore: } \int u dv = uv - \int v du.$$

Note that the part to be used here is the dv and it must be readily and also $\int v du$ should not be complex than the normal $\int u dv$.

Example 1

$$\text{Evaluate } \int 2xe^x dx$$

Using integration by parts:

$$\text{Let } u = 2x \text{ and } dv = e^x dx$$

$$\text{Then } \frac{du}{dx} = 2 \Rightarrow du = 2dx$$

$$\text{Therefore } \int dv = \int e^x dx = v = e^x$$

$$\text{Recall that } \int v dv = uv - \int v du$$

$$\int 2xe^x dx = 2xe^x - \int e^x 2dx$$

$$= 2xe^x - 2 \int e^x dx$$

$$= 2xe^x - 2e^{x+c}$$

$$= 2xe^x(x-1) + c.$$

$$2. \quad \int x^2 e^{-3x} dx$$

Using integration by parts

Let $u = x^2$ and $dv = e^{-3x} dx$

$$\frac{du}{dx} = 2x \text{ and } du = 2x dx$$

$$= \int dv = \int e^{-3x} dx \Rightarrow v = \frac{-1}{3} e^{-3x}$$

$$= \int v dv = uv - \int v du$$

$$\int x^2 e^{-3x} dx = x^2 \left(-\frac{1}{3} e^{-3x} \right) - \int -\frac{1}{3} e^{-3x} \cdot 2x dx$$

$$= -\frac{1}{3} x^2 e^{-3x} + \frac{2}{3} \int x e^{-3x} dx$$

$$= -\frac{1}{3} x^2 e^{-3x} + \frac{2}{3} I$$

(Note Here that $I = \int x e^{-3x}$, We cannot integrate I easily unless by parts)

Let us now solve I by parts

Let $w = x$ and $dz = e^{-3x} dx$

$$\frac{dw}{dx} = 1 \Rightarrow dw = dx$$

$$\text{Therefore: } \int dz = \int e^{-3x} dx \Rightarrow z = \frac{-1}{3} e^{-3x}$$

$$\int w dz = wz - \int z dw$$

$$\text{so } \Rightarrow \int x e^{-3x} dx = x \left(\frac{-1}{3} e^{-3x} \right) - \int -\frac{1}{3} e^{-3x} dx$$

$$= \frac{-1}{3} x e^{-3x} + \frac{1}{3} \int e^{-3x} dx$$

$$= \frac{-1}{3} x e^{-3x} + \frac{1}{3} \left(\frac{-1}{3} e^{-3x} \right)$$

$$\text{Therefore } I = \int x e^{-3x} dx = \frac{-1}{3} x e^{-3x} - \frac{1}{9} e^{-3x}$$

$$\Rightarrow \int x^2 e^{-3x} dx = \frac{-1}{3} x^2 e^{-3x} + \frac{2}{3} I = \frac{-1}{3} x^2 e^{-3x} + \frac{2}{3} \left(\frac{-1}{3} x e^{-3x} - \frac{1}{9} e^{-3x} \right) + c$$

$$= \frac{-1}{3} x^2 e^{-3x} - \frac{2}{9} x e^{-3x} - \frac{2}{27} e^{-3x} + c$$

$$= \frac{-1}{3} e^{-3x} \left(x + \frac{2}{3} x + \frac{2}{9} \right) + c$$

$$3. \quad \int x^2 \ln x dx$$

$$\text{Let } u = \ln x \text{ and } dv = x^2 dx$$

$$\frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{dx}{x}$$

$$\text{Therefore } \int dv = \int x^2 dx \Rightarrow v = \frac{x^3}{3}$$

$$\text{By parts } \int u dv = uv - \int v du$$

$$\int x^2 \ln x dx = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \cdot \frac{dx}{x}$$

$$= \int x^2 \ln \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \cdot \frac{dx}{x}$$

$$= \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 dx$$

$$\begin{aligned}
&= \frac{1}{3}x^3 \ln x - \frac{1}{3}\left(\frac{x^3}{3}\right) + c \\
&= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + c \\
&= \frac{1}{3}x^3 \left(\ln x - \frac{1}{3}\right) + c \\
&= \frac{1}{3}x^3 \left(3 \ln \frac{x-1}{3}\right) + c \\
&= \frac{1}{9}x^3(3 \ln x - 1) + c
\end{aligned}$$

Self-Assessment Exercise

Evaluate $\int 2xe^x dx$

4.0 CONCLUSION

In this unit, we can conclude that integration by parts is a method for evaluating a difficult integral and we should note that when the integral is a product of functions and the formula will definitely moves the product out of the equation so that the integral can be solved easily while integration by substitution can also be called U-substitution and is a method for finding integrals using some fundamental theorem of calculus.

5.0 SUMMARY

In this unit, we have been able to discuss on the topic integration by substitution and by parts and various examples were used to teach how the two integration work.

6.0 TUTOR-MARKED ASSIGNMENT

1. Evaluate $3 \int (x^3 + 3x)(x^2 + 1)e^{x^3+3x} dx$
2. Evaluate $\int x \sin x dx$
3. $\int e^x \sin x dx$
4. $\int_{n/4}^{n/3} \tan^2 x \sec^2 x dx$

5. $\int 2x \sin x^2 dx.$

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UNIT 4 APPLICATION OF INTEGRATION TO ECONOMICS PROBLEM

CONTENTS

- 1.0 Introduction
- 2.0 Objectives
- 3.0 Main Content
 - 3.1 Analysis of Total Cost and Marginal Cost to Economics Problem
 - 3.2 Application of integration to Economics Problem using Total cost and Marginal cost Analysis
- 4.0 Conclusion
- 5.0 Summary
- 6.0 Tutor-Marked Assignment
- 7.0 Reference/Further Readings

1.0 INTRODUCTION

From our previous discussion on differentiation, you will have seen how apply differentiation to economics analysis?

However, the same approach will also be adopted here under application of integration to economics.

Under the analysis of differentiation, when we differentiate total cost we get marginal cost, but when the marginal costs of these functions are given, they can be integrated to obtain the original functions differentiated. It should also be noted that when the marginal cost functions is given, it integrated with respect to the quantity produced to obtain the total cost function. Sometime we can get additional information which it is called initial condition and this is required so as to get a unique total cost function.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Understand how to calculate integration by substitution and by parts
- Know how to apply integration to economic problems

3.0 MAIN CONTENT

3.1 Analysis of Total Cost and Marginal Cost Using

Integration

As a student of mathematics for Economics, you should note that the constant of integration is in terms of a fixed cost or initial overhead and when there is no production, the total cost will be equal to the fixed cost. Let us take an example of total cost of producing q units is given by:

$C = F(q)$, to get marginal cost (MC).

$$C(q) = \frac{dc}{dq} = F^1(q).$$

To get the total cost function, given the marginal cost $C = F(q)$.

$$\text{Therefore TC} = \int F^1(q) dq = dq = F(q) + K$$

Where K is a constant of integration.

It is however the same thing when given the marginal revenue function with respect to quantity demanded to get the revenue function, the

integration constant established will be zero value due to the fact that revenue depend on sales and where there is no sales revenue will be zero. This makes the constant too to be zero unless otherwise. Thus, if the total revenue of demanding q units is given by the function $R = F(q)$ therefore, the marginal revenue is $R(q) = \frac{dR}{dq} = F(q)$.

Hence the total revenue function, given marginal revenue becomes:

$$R(q) = F(q)$$

$$R(q) = \int F^1(q) dq = R$$

Where C is constant and when $q = 0, R(0) = R \Rightarrow 0$

$$R = f(q).$$

We can also integrated the marginal profit function to get the total product function, integrate marginal price function to get price function and also integrate marginal prosperity to consume function to get the total marginal consumption function.

3.2 Application of integration to Economics Problem using Total Cost and Marginal Cost Analysis

1. The marginal cost of a trader has been found to be $MC = 3q^2 + 8q + 400$. Determine the total variable cost of producing 100 units of the trade's product.

Solution

$$MC = 3q^2 + 8q + 400$$

$$\text{Total cost (TC)} = \int MC dq = (4q^2 + 8q + 400) dq$$

$$TC = \frac{3q^3}{3} + \frac{8q^2}{2} + 400q + K.$$

$$TC = q^3 + 4q^2 + 400q + K.$$

$$\text{But } TC = TVC + FC$$

Where TVC the total variable is cost and F is the fixed cost. Hence $TVC = 3q^3 + 4q^2 + 400q$ and $FC = K$ when $q = 100$.

$$\begin{aligned}
TVC &= 100^3 + 4(100)^2 + 400(100) \\
&= 1,000,000 + 4(10,000) + 40,000 \\
&= 1,000,000 + 40,000 + 40,000 \\
&= 1,080,000.
\end{aligned}$$

Therefore the total variable cost of producing 100 units of the farmer's product is ~~₦~~1,080,000.

2. If the fixed cost of manufacturing a product is ~~₦~~10,000 and the marginal cost at x units of output is ~~₦~~ $(60 + 2.5x)$. Find:
- The function for the total cost of manufacturing x units.
 - The total cost of 200 units.

Solution

Let us start by representing total cost as C .

$$\text{Marginal cost (MC)} = \frac{dc}{dx} = 60 + 2.5x$$

Integrating with respect to x .

$$\int dc = \int (60 + 2.5x) dx$$

$$C = 60x + \frac{2.5x^2}{2} + K$$

Where k is the constant = fixed cost = 10,000.

$$\text{Therefore } C = 60x + 1.25x^2 + 10,000$$

When $x = 200$

$$\begin{aligned}
(200) &= 60(200) + 1.25(200)^2 + 10,000 \\
&= 12,000 + 1.25(40,000) + 10,000 \\
&= 12,000 + 50,000 + 10,000 \\
&= \del{₦}61,200.
\end{aligned}$$

Therefore, the cost of producing 200 units is ~~N~~61,200

3. The marginal profit of producing x units per day is given by:

$$P(x) = 200 - 0.4x$$

$P(0) = 0$ where $P(x)$ is the profit in naira. Calculate the profit function and the profit of producing 10 units.

Solution

$$P(x) = 200 - 0.4x$$

To calculate the profit function we need to integrate the marginal profit function with respect to x .

$$P(x) = \int (200 - 0.4x) dx$$

$$P(x) = 200x - \frac{0.4x^2}{2} + k$$

Where k is the constant

$$P(x) = 200x - 0.2x^2 + k$$

$$\text{since } p(0) = 0 \Rightarrow 0 = 200(0) - 0.2(0)^2 + k$$

$\Rightarrow P(x) = 200x - 0.2x^2$ is the profit function.

$$\text{When } x = 10, p(10) = 200(10) - 0.2(10)^2$$

$$= 2000 - 0.2(100)$$

$$= 2000 - 20$$

$$= \text{N}1980$$

The profit on production of 10 units is ~~N~~1960

4. If the marginal revenue function for a commodity is $(6q^2 - 12q + 4)$ naira per units when the level of production is q units, determine the total revenue function and find the total revenue when 40 units are sold.

Solution

Marginal revenue = MR .

$$MR = \frac{dR}{dq} = 6q^2 - 12q + 2$$

$$\int dR = \int (6q^2 - 12q + 2) dq$$

$$R = \frac{6q^3}{3} - \frac{12q^2}{2} + 2q + k.$$

$$R = 2q^3 - 6q^2 + 2q + k$$

Where k is a constant.

When $q = 0, R = 0$ that is when there is no sales, there will be no revenue.

Therefore $k = 0$

$R = 2q^3 - 6q^2 + 2q \Rightarrow$ This is the total revenue function.

When $q = 40$

$$R = 2(40)^3 - 6(40)^2 + 2(40)$$

$$R = 2(64000) - 6(1600) + 80$$

$$R = 128000 - 9600 + 80$$

$$R = \text{N}198,400$$

The revenue on sales of 40 units is ~~N~~198,400.

5. The marginal propensity to consume (in billions of naira) is given as: $\frac{dc}{dx} = 0.8 + \frac{0.4}{2x^{-1/2}}$, where c is the consumption and x is the income.
- Find the consumption function given that when income is zero consumption is 20 billion naira.
 - Find the income when consumption is put at ~~N~~60 billion.
 - Determine the consumption when income is put at ~~N~~50 billion.
 - Determine also the saving function when income is zero.

$$MPC = \frac{dc}{dx} = 0.8 + \frac{0.4}{2x^{-1/2}}$$

$$= 0.8 + 0.8x^{-\frac{1}{2}}$$

$$C = \int (0.8 + 0.8x^{-\frac{1}{2}}) dx$$

$$= 0.8x + 0.8x^{-\frac{1}{2}+1} + k$$

$$C = 0.8x + \frac{0.8x^{\frac{1}{2}}}{\frac{1}{2}} + k$$

$$C = 0.8x + 1.6x^{\frac{1}{2}} + k$$

When $x = 0, C = 20$

$$20 = 0.8(0) + 1.6(0)^{\frac{1}{2}} + k$$

$$20 = k.$$

$$\therefore k = 20$$

$$C = 0.8x + 1.6x^{\frac{1}{2}} + 20$$

Where C is the consumption.

$$\text{ii. } C = 0.8x + 1.6x^{\frac{1}{2}} + 20$$

When $C = 50$

$$\Rightarrow 50 = 0.8x + 1.6x^{\frac{1}{2}} + 20$$

$$50 - 20 = 0.8x + 1.6x^{\frac{1}{2}}$$

$$30 = 0.8x + 1.6x^{\frac{1}{2}}$$

$$0.8x + 1.6x^{\frac{1}{2}} - 30 = 0$$

Multiply through by 10

$$8x + 1.6x^{\frac{1}{2}} - 300 = 0.$$

Let $a = x \Rightarrow x = a$.

$$\text{Hence } 8a^2 + 16a - 300 = 0$$

$$a = \frac{a = 8, b = 16, c = -300}{-b \pm \frac{\sqrt{b^2 - 4ac}}{2a}}$$

$$a = -16 \pm \frac{\sqrt{16^2 - 4(8)(-300)}}{2(8)}$$

$$= -16 \pm \frac{\sqrt{256 - (-9600)}}{16}$$

$$= -16 \pm \frac{\sqrt{9856}}{16}$$

$$= \frac{-16 \pm 31.40}{16}$$

$$a = \frac{-16 \pm 31.40}{16} \text{ or } = \frac{-16 - 31.40}{16}$$

$$a = \frac{15.4}{16} \text{ or } = \frac{47.4}{16}$$

$$a = 0.96 \text{ or } 2.96.$$

From the result, we take the positive value that is $a = 0.96$

But $a = 0.96$

$$a = x^{\frac{1}{2}} \Rightarrow x^2 = 0.96$$

Square both sides

$$\left(x^{\frac{1}{2}}\right)^2 = (0.96)^2$$

$$x = 0.9216; x = 0.92$$

Multiply by 100

$$x = 0.92 \times 100$$

$$x = 92.$$

So, when the consumption is at ~~N~~60 billion, income will be ~~N~~92 billion.

$$\text{iii. } C = 0.8x + 1.6x^{\frac{1}{2}} + 20$$

When $x = 50$

$$C = 0.8(50) + 1.6(50)^{\frac{1}{2}} + 20$$

$$= 40 + 16(\sqrt{50}) + 20$$

$$= 40 + 16(7.07) + 20$$

$$= 40 + 113.12 + 20$$

$$= 173.12.$$

When income is at ~~N~~50 billion, consumption will be ~~N~~173.12 billion.

iv. Recall that marginal propensity to save $MPS = 1 - MPC$.

$$MPS = 1 - \left(0.8 + \frac{0.4}{2x^{-1/2}} \right)$$

$$= 0.2 - \frac{0.4}{2x^{1/2}} = 0.2 - 0.8x^{1/2}$$

$$\int = \int (0.2 - 0.8x^{1/2}) dx$$

$$= 0.2x - \frac{0.8x^{\frac{1}{2}}}{\frac{1}{2}} + k$$

Where $k = \text{constant}$

$$S = 0.2x - 1.6x^{\frac{1}{2}} + k$$

$$k = 0.$$

Therefore saving function:

$$S = 0.2x - 1.6x^{\frac{1}{2}}$$

Self-Assessment Exercise

The marginal profit of producing x units per day is given by:

$$P(x) = 400 - 0.3x$$

$P(0) = 0$ where $P(x)$ is the profit in naira. Calculate the profit function and the profit of producing 22 units.

4.0 CONCLUSION

In this unit, we can conclude that the application of integration to economics problem using the total cost and marginal cost will enable the students to be able to analyze and make prediction/conclusion on certain situation cost analysis.

5.0 SUMMARY

In this unit, we have been able to discuss on the topic analysis of total cost and marginal cost using integration and the application of integration to Economics problem using both the total and marginal cost analysis.

6.0 TUTOR-MARKED ASSIGNMENT

1. The demand for a product is given by $P = 400 - 80q$ and its supply is given by $P = 200 - 4q$.
 - i. Determined the equilibrium quantity and price.
 - ii. Determined the consumers' surplus
 - iii. Find the producers' surplus.
2. The expenditure and the income rates per day from an undertaking are given as follows:

$E^1(t) = a + bt$ and $I^1(t) = c + dt$ respectively in naira (Note a, b, c and d are constants. Determine:

- i. The optional time to terminate the project.
 - ii. The profit at this time.
 - iii. Determine the maximum profit given that $a = 20, b = 0.45, c = 50, d = 0.30$.
3. It is known that the population of a community changes at a rate proportional to the population. If the population in 1940 was 40,000 and in 1950 it was 44,000. Find the equation for the

population at the time t , where t is the number of years after 1940. What is the expected population in 1960? What would be the population by the year 2000 and 2010 respectively (Assume that $\log_e^{1.2} = 0.18$).

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MODULE FOUR DIFFERENTIAL ANALYSIS

- UNIT 1 Differential Equation**
- UNIT 2 Difference Equation**
- UNIT 3 Optimization**
- UNIT 4 Dynamics Analysis**

CONTENTS

- 1.0 Introduction
- 2.0 Objectives
- 3.0 Main Content
 - 3.1. Worked examples on differential equation
 - 3.2. General formula for first-order linear differential equation
 - 3.3. Exact differential equation and partial integration
 - 3.4. Integrating factor
 - 3.4.1. Rules for integrating factor
 - 3.5. Separation of Variables
 - 3.6. Economic Applications
- 4.0 Conclusion
- 5.0 Summary
- 6.0 Tutor-Marked Assignment

UNIT ONE DIFFERENTIAL EQUATION

Introduction

A differential equation is any equation which derivate, either ordinary derivatives or partial derivatives. However, a differential equation containing one or more functions of one independent variable and its derivatives. Various differentials and functions become related to each other via equations, and thus a differential equation is a result that describes dynamically changing phenomena, evolution and variation. Furthermore, quantities are defined as the rate of change of other quantities such as derivatives of displacement with respect to time or gradients of quantities which is how they enter differential equations.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Understand how to calculate differential equation and separation of variables
- Know how to apply partial integration and integrating factor
- Understand how to apply differential equation to economics

3.0 MAIN CONTENT

3.1 Work Examples of Differential Equation

Ex1: To solve the differential equation $y'(t) = 7$ for all the functions $y(t)$ which satisfy the equation.

$$y'(t) = \int 7 dt = 7 + c_1$$

Here we integrate both sides of the equations to find the integrals.

$$\text{Therefore } y(t) = \int (7 dt = 7 + c_1) dt = 3.5t^2 + c_1t + c$$

The above solution is called a general solution which indicates that when c is unspecified, a differential equation has an infinite number of possible solutions. If c is specified, then the differential equation has a particular or definite solution which alone gives rise to all the possible solution.

Ex.2: The order and degree of a differential equations is given below:

1. $\frac{dy}{dt} = 2x + 6 \Rightarrow$ This is a first-order, first degree
2. $\frac{dy}{dt} = 3x + 10 \Rightarrow$ This is also a first-order, first degree
3. $\left(\frac{dy}{dt}\right)^4 - 7t^5 = 0 \Rightarrow$
 $\Rightarrow$ This is a first order, fourth – degree.
4. $\left(\frac{dy}{dt}\right)^4 - 11t^5 = 0 \Rightarrow$ This is also a first order, fourth – degree.
5. $\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^3 + 2x^5 = 0 \Rightarrow$ This is a second order, first degree.
6. $\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^3 + 6x^2 = 0 \Rightarrow$ This is a second order, first degree.
7. $\left(\frac{d^2y}{dt^2}\right)^7 + \left(\frac{d^3y}{dt^3}\right)^5 + 75y \Rightarrow$ This is a third, fifth – degree.
8. $\left(\frac{d^2y}{dt^2}\right)^7 + \left(\frac{d^3y}{dt^3}\right)^5 + 15y \Rightarrow$ This is a third, fifth – degree.

3.2 General Formular For First – Order Linear Differential Equation

In the first-order linear differential equations of 5ay dy/dt, where y must be of the first degree and there may no product y(dy/dt).

Let us take a look at the equation below:

$$\frac{dy}{dt} + vy = z$$

Where v and z maybe constants or functions of time. Therefore the formular for a general solution is given as:

$$y(t) = e^{-\int v dt} \left(A + z e^{-\int v dt} dt \right) \quad (1)$$

Let us called the above equation one, where A is an arbitrary constant. But it should be noted here that a solution is composed of two parts: $e^{-\int v dt} A$.

This is called the complementary function and $e^{-\int v dt} dt$ is called the particular integral. However, the particular integral y_p equals the

intertemporal equilibrium level of $y(t)$ while the complementary function y_c is the derivation from the equilibrium. In this analysis, we should take note that for $y(t)$ to be stable dynamically, y_c must tend to zero as t tends to infinity, which means K in the analysis of e^{kt} must be negative. Finally, we can check the solution of a differential equation by differentiation.

Ex 1: Find the general solution for the differential equation $dy/dt + 4y = 12$ given that $v = 4$ and $z = 12$.

SOLUTION

The equation becomes:

$$y(t) = e^{-\int v dt} \left(A + \int 12e^{-\int 4 dt} dt \right)$$

Let then set $\int 4 dt = 4t + c$, but c is always ignored and subsumed under A .

The equation becomes:

$$y(t) = e^{-4t} \left(A + \int 12e^{-4t} dt \right)$$

After integrating the remaining integral we have: $12e^{-4t} dt = 3e^{-4t} + c$.

When we ignored the constant again we have: $y(t) = e^{-4t}(A + 3e^{-4t}) = Ae^{-4t} + 3$.

Recall $e^{-4t}e^{4t} = e^0 = 1$ and as t tends to infinity, $y_c = Ae^{-4t} \Rightarrow 0$ and $y(t)$ tends to $y_p = 3$ and we can say here that the intertemporal equilibrium level $y(t)$ is dynamically stable.

Ex 2: Given $dy/dt + 3t^2y = t^2$.

Where $v = +3t^2$ and $z = t^2$. Find the general solution

$$y(t) = e^{-\int 3t^2 dt} \left(A + \int t^2 e^{\int 3t^2 dt} dt \right)$$

Let us now integrate the exponents $\int 3t^2 dt = t^3$. Then let us substitute back in the above equation.

$$y(t) = e^{-\int 3t^2 dt} \left(A + \int t^2 e^{\int 3t^2 dt} dt \right)$$

Then, we integrate the remaining integral:

Let $u = t^3, du/dt = 3t^2$ and $dt = du/3t^2$

$$\int t^2 e^{\int t^3} dt = \int t^2 e^u \frac{du}{3t^2} = \frac{1}{3} \int e^u du = \frac{1}{3} e^u = \frac{1}{3} e^{t^3}.$$

Therefore let us substitute into $y(t) = e^{-t^3} (A + \int t^2 e^{t^3} dt)$

We have:

$$y(t) = e^{-t^3} \left(A + \frac{1}{3} e^{t^3} \right) = A e^{-t^3} + \frac{1}{3}$$

Therefore we can calculate that $t \rightarrow \infty$ (infinity), $y_c = A e^{-t^3} \rightarrow 0$ and $y(t)$ tends to $1/3$. We say here that the equilibrium is dynamically stable.

Ex 3: Given $\frac{dy}{dt} = 3y$

$$\frac{dy}{dt} - 3y = 0.$$

From the equation above $v = -3, z = 0$

$$y(t) = e^{-\int -3dt} \left(A + \int 0 e^{\int -3dt} dt \right)$$

Let substitute $\int -3dt = -3t$ and $y(t) = e^{3t} (A + \int 0 dt) = A e^{3t}$, where $t = 0, y = 2$. Therefore $2 = A e^{3(0)} \Rightarrow A = 2$.

3.3 Exact Differential Equations and Partial Integration

When there is more than one independent variable, such as $f(y, t)$ where $M = dF/dy$ and $N = dF/dt$, then the total differentiation is given as: $dF(y, t) = Mdy + Ndt$ _____(1)

From the above equation, F is a function of more than one independent variable, M and N are partial derivatives and equation one above is called a partial differential equation and in a situation where the differential is set equal to zero, so that $Mdy + Ndt = 0$, this is called an exact differential equation because the left side exactly equals the differential of the primitive function $F(y, t)$. Moreover, if we have an exact differential equation $\frac{dM}{dt}$ must equal to $\frac{dN}{dy}$, that is $\partial^2 F(dt/dy) = \partial^2 F(dy/dt)$.

If we want to perform a solution of an exact differential equation, it calls for successive integration with respect to one independent variable at a time while we hold the other independent variables constant.

Ex1: Solve the exact non linear differential equation

$$(6yt + 9y^2)dy + (3y^2 + 8t)dt = 0$$

Step 1: See whether the equation is an exact differential equation

$$\text{Let } M = 6yt + 9y^2 \text{ and } N = 3y^2 + 8t$$

$$\frac{dM}{dt} = 6y \text{ and } \frac{dN}{dy} = 6y.$$

Therefore $\frac{dM}{dt} \neq \frac{dN}{dy}$, so the equation is not an exact differential equation.

Step 2: So, the equation is a partial derivative, so we will integrate M partially with respect to y by treating t as a constant and try to add a different/new function say $Q(t)$ for any additive term of t which would have been eliminated by the original differentiation with respect to y . Note in this process that dy is replaces dy in partial integration.

$$F(y, t) = \int (6yt + 6yt + 9y^2)dy + Q(t) = 3y^2t + 3y^2 + Q(t)$$

$$3y^2t + 3y^2 + Q(t)$$

$$\frac{dF}{dt} = 3y^2 + Q(t)$$

$$\text{Since } \frac{dF}{dt} = N \text{ and } N = 3y^2 + 8t$$

$$\text{Substitute } dF/dt = 3y^2 + 8t$$

$$3y^2 + 8t = 3y^2 + Q(t)$$

$$\text{But } Q(t) = 8t.$$

Let us now integrate $Q(t)$ with respect to t :

$$Q(t) = \int Q'(t)dt = \int 8tdt = 4t^2$$

$$\text{Therefore: } F(y, t) = 3y^2t + 3y^2t4t^2 + c$$

Ex 2: Given $(12y + 7t + 6)dy + (7y + 4t - 9)dt = 0$

$\frac{dM}{dt} = 7 = \frac{dN}{dy}$, so this is an exact differential equation.

$$F(y, t) = \int (12y + 7t + 6)dy + Q(t) = 6y^2 + 7yt + 6y + Q(t)$$

$$dF/dt = 7y + Q'(t),$$

$$\frac{dF}{dt} = N = 7y + 4t - 9$$

$$\text{Therefore: } 7y + Q'(t) = 7y + 4t - 9$$

$$Q'(t) = 4t - 9$$

$$Q(t) = \int (4t - 9)dt = 2t^2 - 9t$$

$$F(y, t) = 6y^2 + 7t + 6y + 2t^2 - 9t + c.$$

3.4 Integrating Factors

In this analysis, we are talking about some exact differential equation that are not exact, because we have seen in unit 3.2 above that not all differential equations are exact but we can make some to be exact by means of an integrating factor and it is called a multiplier that permits the equation to be integrated.

Ex 1: Testing the nonlinear differential equation $5ytdy + (5y^2 + 8t)dt = 0$ shows that it is not exact. With $M = 5yt$ and $N = 5y^2 + 8t$, $dm/dt = 5y \neq dN/dy = 10y$.

We then multiply by an integrating factor of t , which makes it exact:

$$5yt^2 + dy + (5y^2t + 8t^2) = 0. \text{ From this equation } dm/dt = 10yt = dN/dy.$$

3.4.1 Rules for the Integrating Factor

The two rules below will help us to find the integrating factor for a nonlinear first-order differential equation, and if such factor exists. Assuming $dm/dt \neq dN/dy$.

Rule One: If $\frac{1}{N} \left(\frac{dM}{dt} - \frac{dN}{dy} \right) = f(y)$ alone,

then $e^{\int f(y)dy}$ is an integrating factor.

Rule Two: If $\frac{1}{M} \left(\frac{dN}{dy} - \frac{dM}{dt} \right) = g(t)$ alone,

then $e^{\int g(t)dt}$ is an integrating factor.

Ex 2: To illustrate the rule above, find the integrating factor where

$$5ytdy + (5y^2 + 8t)dt = 0 \quad M = 5yt \quad N = 5y^2 + 8t$$

$$\frac{dM}{dt} = 5y \neq \frac{dN}{dy} = 10y$$

Let us now apply rule one:

$$\frac{1}{5y^2 + 8t} (5y - 10y) = \frac{-5y}{5y^2 + 8t} \Rightarrow$$

Which is not a function of y alone and will not supply an integrating factor for the equation. Therefore we apply the second rule:

$$\frac{1}{5yt} (10y - 5y) = \frac{5y}{5yt} = \frac{1}{t} \Rightarrow$$

Which is a function of t alone and the integrating factor, therefore is

$$e^{\int (1/t)dt} = e^{\ln t} = t.$$

Ex 3: Given $\frac{dy}{dt} = \frac{y}{t} : \left(\frac{1}{ty} \right)$

Let us rearrange the equation:

$$tdy = ydt : tdy - ydt = 0$$

$$1. \quad dM/dt = 1 \neq dN/dy = -1,$$

Multiplying by $1/(ty)$

$$\frac{dy}{y} - \frac{dt}{t} = 0$$

Therefore $dM/dt = 1 \neq dN/dy = -1$ and since neither of the function contains the variables with respect to which it is being partially differentiated.

2. $F(y, t) = \int \frac{1}{y} (dytQ(t) = \ln y + Q(t))$
3. $dF/dt = Q^1(t)$, but $dF/dt = N = -1/t$, so $Q^1(t) = -1/t$
4. $Q(t) = \int -\frac{1}{t} dt = -\ln t$
5. $F(y, t) = \ln y - \ln t + C$ which can be expressed in different ways. However, since C is an arbitrary constant, we can then write $\ln y - \ln t = c$, thus:

$\ln(y/t) = c$ and the expression of each side of the equation as exponents of e and reality that $e^{\ln x} = x$

$$e^{\ln(y/t)} = e^c$$

$$\frac{y}{t} = e^c \text{ or } y = te^c$$

3.5 Separation of Variables

In this case, we will try to perform some mathematical analysis of separating variables in an equation. But solution of nonlinear first-degree differential equations is complex and a first-order first degree differential equation is one in which the highest derivative is the first derivative dy/dt and that derivative is raised to power of 1. It is nonlinear if it contains a product of y and dy/dt or y raised to a power other than 1. Therefore if equation is written in the form of separated variables such that $M(y)dy + N(t)dt = 0$, where M and N are function of y and t above, the equation can then be solved by ordinary integration.

Ex 1: Give the nonlinear differential equation

$$t^2 dy + y^3 dt = 0.$$

Where $M \neq F(y)$ and $N \neq F(t)$

Let us the multiply $1/(t^2 y^3)$ by

$$t^2 dy + y^3 dt = 0 \text{ to separate the variables:}$$

$$\text{we have: } \frac{1}{y^3} dy + \frac{1}{t^3} dt = 0$$

Let us now integrate the separated variable,

$$\int y^{-3} dy + \int t^{-2} dt = \frac{-1}{2} y^{-2} - t^{-1} + c$$

$$F(y, t) = \frac{-1}{2} y^{-2} - t^{-1} + c$$

$$= \frac{-1}{2y^2} - \frac{-1}{2} + c$$

Ex 2: Let us separate the variable

$$ydy = -5tdt - ydy + 5tdt = 0$$

Now let integrate each of the term separately

$$\frac{y^2}{2} + \frac{5t^2}{2} = c_1$$

$$y^2 + 5t^2 = 2c_1$$

$$y^2 + 5t^2 = c.$$

3.6 Economic Applications

In analyzing application of differential equation in economics, some worked examples will be examined.

Ex 1: Find the demand function $Q = F(p)$ if point elasticity ϵ is -1 for all P greater than 0.

Solution

$$\begin{aligned}\epsilon = \frac{dQ}{dP} &= -\frac{(5P + 2P^2)}{Q} \times \frac{Q}{P} \\ &= -(5 + 2P)\end{aligned}$$

We can now separate the variables

$$dQ + (5P + 2P^2)dP = 0$$

Let us then integrate $Q + 5P + P^2 = C$

$$Q = -P^2 - 5P + C$$

Given $Q = 500$ and $P = 10$

Substitute for Q and P in the equation above

$$500 = -100 - 50 + c$$

$$C = 650$$

$$\therefore Q = 650 - 5P - P^2$$

2) Find the demand function $Q = f(P)$

If $\epsilon = -k$, a Constant.

$$\epsilon = \frac{dQ}{dP} \times \frac{P}{Q} = -k$$

$$\text{Therefore } \frac{dQ}{dP} = \frac{KQ}{P}$$

Let us separate the variables:

$$\frac{dQ}{Q} + \frac{K}{P} dP = 0$$

$$\ln Q + K \ln P = C$$

$$QP^K = C.$$

$$\text{Therefore } Q = cP^{-K}$$

3) Find the demand function

$$Q = f(P) \text{ If } \epsilon = -(5P + 2P^2)/Q$$

and $Q = 500$ when $P = 10$.

$$\epsilon = \frac{dQ}{dP} \times \frac{P}{Q} = \frac{-(5P + 2P^2)}{Q}$$

$$\begin{aligned} \frac{dQ}{dP} &= -\frac{(5P + 2P^2)}{Q} \times \frac{Q}{P} \\ &= -(5 + 2P) \end{aligned}$$

We can now separate the variables

$$dQ + (5P + 2P^2)dP = 0$$

Let us then integrate $Q + 5P + P^2 = C$

$$Q = -P^2 - 5P + C$$

Given $Q = 500$ and $P = 10$.

Substitute for Q and P in the equation above

$$500 = -100 - 50 + c$$

$$C = 650$$

$$\therefore Q = 650 - 5P - P^2$$

4. The Solow model examines equilibrium growth paths with full employment of both capital and labour. However, base on the assumption that:

- a. Output is a linearly homogeneous function of capital and labour exhibiting constant returns to scale.
- b. A constant proportion s of output is saved and invested.
- c. The supply of labour is growing at a constant rate r .

Therefore assumption a to c gives $y = F(K, L)$, $\frac{dK}{dt} = k = sy$ and $L = L_0 e^{rt}$

You are required to derive the differential equation in terms of the single variable K/L which serves as the basis of the model.

Solution

Let us substitute for 4 into $\frac{dK}{dt} = k = sy$

We then have $\frac{dK}{dt} = sf(K, L)$ _____(i)

Substituting L in $L = L_0 e^{rt}$ into equation (1) above.

$$\frac{dK}{dt} = sf(K, L_0 e^{rt}) \text{ _____(ii)}$$

This is the time path capital function (dK/dt) must follow for full employment of a growing labour force.

To then convert to a function of

K/L , let $Z = k/L$, then $K = ZL$

Let us now make use of $L = L_0 e^{rt}$

$$\therefore K = ZL_0 e^{rt} \text{ (iii)}$$

Take the derivate of the equations above and use the product rule since z is a function of t .

$$\frac{dK}{dt} = z(rL_0 e^{rt}) + L_0 e^{rt} \frac{dz}{dt}$$

$$\Rightarrow \left(zr + \frac{dz}{dt} \right) L_0 e^{rt} \text{ (iv)}$$

Let us now equated equation (ii) and equation (iv)

$$Sf(K, L_0 e^{rt}) = \left(zr + \frac{dz}{dt} \right) L_0 e^{rt} \text{ (v)}$$

From equation (v) above, we can see that the left hand side is a linearly homogenous production function, we may divide both inputs by $L_0 e^{rt}$ and multiply the function itself by $L_0 e^{rt}$ without changing its value. Thus:

$$Sf(K, L_0 e^{rt}) = sL_0 e^{rt} f \frac{k}{L_0 e^{rt}} \times 1 \text{ (vi)}$$

Substituting equation (vi) into equation (v) and dividing both into sides by $L_0 e^{rt}$.

$$sf \left(\frac{k}{L_0 e^{rt}} \times 1 \right) = zr + \frac{dz}{dt} \text{ (vii)}$$

We can then substitute z for $k/L_0 e^{rt}$ and substituting zr from both sides

$$\frac{dz}{dt} Sf(z, 1) - zr$$

Which is a differential equation in terms of the single variable z and two parameters r and s , where $z = k/L$, r = the rate of growth of the labour force and s = the savings rate.

Self-Assessment Exercise

Find the demand function $Q = F(p)$ if point elasticity ϵ is -2 for all P greater than 0.

4.0 CONCLUSION

In this unit, we can conclude that a differential equation is a mathematical equation that relates some function with its derivatives. In applications, the functions usually represent physical quantities, the derivatives represent their rates of change, and the equation defines a relationship between the two.

5.0 SUMMARY

In this unit, we have been able to discuss on some examples on differential equation, general formula for first order linear differential equation, exact differential equations and partial integration, integrating factor, rules for the integrating factor, separation of variables and economic applications.

6.0 TUTOR-MARKED ASSIGNMENT

Solve the following differential equations:

1. Given $(12y^2t^2 + 10y)dy + (18y^3t)dt = 0$
2. Given $8tyy^1 = -(3t^2t + 4y^2)$
3. Given $60t^2y^1 = -(12t^3 + 20y^3)$

Use the integrating factor provided in parentheses to solve the following equation:

4. $6tdy + 12ydt = 0$ (t)
5. $t^2dy + 3y + dt = 0$ (t)

6. Solve the equation using the procedure for separating variable:

$$\frac{dy}{dt} = \frac{-8t}{y}$$

7. Assume that the demand for money is for transaction purposes only. Thus,

$$M_d = KP(t)Q$$

Where K is constant, P is the price level and Q is real output. Assume $M_s = M_d$ and is exogenously determined by monetary authorities. If inflation or the rate of change of prices is proportional to excess demand for goods in society and from walras law, an excess demand for goods is the same thing as an excess supply of money, so that

$\frac{dP(t)}{dt} = b(M_s - M_d)$. Find the stability conditions, when real output Q is constant.

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UNIT TWO Differences Equations

CONTENTS

- 1.0 Introduction
- 2.0 Objectives
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 - 3.1. General formula differences equations
 - 3.2. Lagged income determination model
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1.0. Introduction

A difference equation is a mathematical equation that relates some function with its derivatives. In applications, the functions usually represent physical quantities, the derivatives represent their rates of change, and the equation defines a relationship between the two.

However it's also shows a relationship between a dependent variable and a lagged independent variable or variables which changes at discrete intervals of time for example $I_t = F(y_{t-1})$ where I and Y are measured at the end of each year. But it should be noted here that the order of a difference equation is determined by the greatest number of periods lagged. For instance a first-order difference equation express a time lag of one period while a second-order two periods; therefore the change in y as t changes from t to t+1 is called the first difference of y.

$$\frac{\Delta y}{\Delta t} = \Delta y_t = y_{t+1} - y_t$$

Where Δ is an operator replacing d/dt that is used to measure continuous change in differential equations.

However, let us take the order of a difference equation:

$$I_t = a(y_{t+1} - y_{t-2}) \Rightarrow \text{order 2}$$

$$Q_s = x + y P_{t,1} \Rightarrow \text{order 1}$$

$$y_{t+1} + 3 - 9y_{t+2} + 2y_{t+1} + 6y_t = 8 \Rightarrow \text{order 3}$$

$$\Delta y_t = 5y_t \Rightarrow \text{order 1}$$

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Know how to apply the general formula of a difference equations
- Understand how to calculate lagged income determination model, cobweb model and the Harrod Dormar model

3.0 Main Content

3.1 General Formula Differences Equations

Assuming we have a first-order difference equation where the linear that is all the variables are raised to the first power and there are no cross products.

$$y_t = by_{t-1} + a \quad (1)$$

Where b and a are constants, the general formula therefore for a definite solution to this problem is given as:

$$y_t = \left(y_0 - \frac{a}{1-b}\right)b^t + \frac{a}{1-b}$$

When $b \neq 1$

$$y_r = y_0 + at \text{ when } b = 1$$

Ex 1: Consider the difference equation $y_t = 7y_{t-1} + 16$ and $y_0 = 5$.

In the equation $b = -7$ and $a = 16$.

Solution

Since $b \neq 1$ we have: using the general formula

$$y_t = \left(5 - \frac{16}{1+7}\right)(-7)^t + \frac{16}{1+7}$$

$$= 3(-7)^t + 2$$

$$y_0 = 3(-7)^0 + 2 = 5, \text{ since } (-7)^0 = 1$$

$$y_r = 3(-7)^1 + 2 = -19$$

Then let us substitute $y_1 = -19$ for y_t and $y_0 = 5$ for y_{t-1} in the original equation.

$$-19 = -7(1 = 5) + 16 = -35 + 16.$$

3.2 Lagged Income Determination Model

In our normal income determination model in macroeconomics: there may be no lags in the model but here we assume that consumption is a function of the previous period's income, that is:

$$c_t = c_0 + c y_{t-1}, y_t = c_t + I_t$$

Where $I_t = I_0$, thus,

$$y_t = c_0 + c y_{t-1} + I_0$$

$$y_t = c y_{t-1} + c_0 + I_0$$

Where $b = c$ and $a = c_0 + I_0$. Let us substitute these values in the general formula we discuss in Unit One. Moreover, since the marginal propensity to consume c cannot equal to 1 and assuming $y_t = y_0$ at $t = 0$

$$y_t = \left(y_0 \frac{c_0 + I_0}{1 - c} \right) (c)^t + \frac{c_0 + I_0}{1 - c}$$

The stability of the time path thus depends on c . but we should recall that $0 < mpc < 1$, and $|c| < 1$ and the time path will converge.

Since $c > 0$, there will be no Oscillations. The equilibrium is stable and as

$t \rightarrow \infty, y_t \rightarrow (c_0 + I_0)/(1 - c)$, which is the intertemporal equilibrium level of income.

Ex 1: Given $y_t = c_t + I_t$,

$c_t = 200 + 0.9y_{t-1}, I_t = 100$ and $y_0 = 4500$, solve for y .

$$y_t = 200 + 0.9y_{t-1} + 100 = 0.9y_{t-1} + 300$$

$$y_t = \left(4500 - \frac{300}{1 - 0.9} \right) (0.9)^t + \frac{300}{1 - 0.9}$$

$$= 1500(0.9)^t + 300$$

In conclusion, with $|0.9| < 1$, the time path converge with $0.9 > 0$ and there is no oscillation. Thus, y_t is dynamically stable. As $t \rightarrow \infty$, the first term on the right hand side goes to zero and y_t approaches the intertemporal equilibrium level of income that is $300/(1 - 0.9) = 3000$.

3.3 The Cobweb Model

This is an economic model that explains why prices might be subject to periodic fluctuations in certain types of markets. It also describes the cyclical supply and demand in a market where the amount produced

must be chosen before prices are observed. The model is based on a time lag between supply and demand decisions.

Agricultural markets are a context where the cobweb model might apply, since there is a lag between planting and harvesting. Suppose for example that as a result of unexpectedly bad weather, farmers go to market with an unusually small crop of strawberries.

This shortage, equivalent to a leftward shift in the market's supply curve, the result is high prices. If the farmers expect these high process conditions to continue, then in the following year, they will raise their producing of strawberries relative to other crops.

Therefore, when they go to market the supply will be high, resulting in low prices. If they then expect low prices to continue, they will decrease their production of strawberries for the next year, which will result to a high price again. Therefore many products, such as that of agricultural commodities discussed earlier which are planted a year before marketing current supply depends on last year's price. This poses interesting stability question.

If $Q_d = c + bP_t$ and $Q_s = g + hP_{t-1}$

$$c + bP_t = Q_s = g + hP_{t-1}$$

$$bP_t = hP_{t-1} + g - c$$

divide through by b.

$$P_t = \frac{h}{b}P_{t-1} + \frac{g - c}{b}$$

Therefore since $b < 0$ and $h > 0$ under normal demand and supply conditions

$h/b \neq 1$ using

$$P_t = \left[P_0 - \frac{(g - c)/b}{1 - h/b} \right] \left(\frac{h}{b} \right)^t + \frac{(g - c)/b}{1 - h/b}$$

$$P_t = \left(P_0 - \frac{g - c}{b - h} \right) \left(\frac{h}{b} \right)^t + \frac{g - c}{b - h} \quad (1)$$

At equilibrium $P_t = P_{t-1}$

Then, let us substitute P_e for P_t and P_{t-1} in $c + bP_t = g + hP_{t-1}$

Which becomes $P_t = \frac{h}{b}P_{t-1} + \frac{g-c}{b}$

Therefore $P_e = \frac{g-c}{b-h}$

Then substitute in equation (1) above:

$$P_t = (P_0 - P_e) \left(\frac{h}{b}\right)^t + P_e$$

3.4 The Harrod Domar Model

The Harrod-Domar model is a classical Keynesian model of economic growth. It is used in development economics to explain an economy's growth rate in terms of the levels of saving and productivity of capital. It suggests that there is no natural reason for an economy to have balanced growth. The model was developed independently by Roy .F. Harrod in 1939, and Evsey Domar in 1946, although a similar model had been proposed by Gustav Cassel in 1924. The Harrod-Domar model was precursor to the exogenous growth model.

The Harrod model attempts to explain the dynamics of growth in the economy. It assumes:

$$S_t = sy_t$$

Where s is a constant equal to both the MPS and APs. It also assumes the acceleration principle that is investment is proportional to the rate of change of national income overtime.

$$I_t = a(y_t - y_{t-1})$$

Where a is a constant equal to both the marginal and average capital-output ratios. In equilibrium: $I_t = S_t$, therefore $a(y_t - y_{t-1}) = sy_t$, $(a - s)y_t = a y_{t-1}$

Dividing through by $a - s$ to conform to $y_t = [a/(a - s)]y_{t-1}$. Since $a/(a - s) \neq 1$.

$$y_t = (y_0 - 0) \left(\frac{a}{a-s}\right)^t + 0 = \left(\frac{a}{a-s}\right)^t y_0.$$

The stability of the time path thus depends on $a|(a - s)$. Since a = the capital-output ratio, which is normally larger than 1 and since s = MPS which is larger than 0 and less than 1, the base $a|(a - s)$ will be larger than 0 and usually larger than

1. Therefore, y_t is explosive but nonoscillating. Income will expand indefinitely, which means it has no bounds.

Ex 1: The warranted rate of growth that is the path the economy must follow to have equilibrium between saving and investment each year can be found as follows in the Harrod model.

$$y_t = \left(\frac{a}{a-s}\right) y_0 \text{-----} (a)$$

The rate of growth G between the periods is defined as

$$G = \frac{y_t - y_0}{y_0} \text{-----} (b)$$

Substituting y_1 in equation (b)

$$G = \frac{[a|(a-s)]y_0 - y_0}{y_0}$$

$$G = \frac{[a|(a-s) - 1]y_0}{y_0}$$

$$G = \frac{a}{a-s} - 1 = \frac{a}{a-s} - \frac{a-s}{a-s}$$

$$= \frac{s}{a-s}$$

The warranted rate of growth is $G_w = \frac{s}{a-s}$

Ex 2: Assume that the marginal propensity to save in the Harrod model above question is 0.12 and the capital output ratio is 2.12. To find y_t we have:

$$y_t = \left(\frac{a}{a-s}\right)^t y_0$$

$$y_t = \left(\frac{2.12}{2.12 - 0.12} \right)^t y_0 - y_t = (1.06)^t y_0$$

The warranted rate of growth is

$$G_w = \frac{s}{a - s} = \frac{0.12}{2.12 - 0.12}$$

$$G_w = \frac{0.12}{2}$$

$$G_w 0.06.$$

Self-Assessment Exercise

Assume that the marginal propensity to save in the Harrod model above question is 0.34 and the capital output ratio is 4.28. To find y_t we have:

$$y_t = \left(\frac{a}{a - s} \right)^t y_0$$

4.0 CONCLUSION

In this unit, we can conclude that a difference equation, also called recurrence equation, is an equation that defines a sequence recursively: each term of the sequence is defined as a function of the previous.

5.0 SUMMARY

In this unit, we have been able to discuss on the general formula of difference equations, lagged income determination model, the cobweb model and the Harrod-Domar model.

6.0 TUTOR-MARKED ASSIGNMENT

1(a) Solve the difference equation given below:

$$y_t = 6y_{t-1}$$

(b) $y_t = \frac{1}{8}y_{t-1}$

(c) $y_t = \frac{1}{4}y_{t-1} + 60$ and $y_0 = 8$

2. Calculate the income given the following

(a) Given $c_t = 400 + 0.6y_t + 0.35y_{t-1}$

$$I_t = 240 + 0.15y_{t-1} \text{ and } y_0 = 700$$

(b) Given $c_t = 300 + 0.5y_t + 0.4y_{t-1}$

$$I_t = 200 + 0.2y_{t-1} \text{ and } y_0 = 6500$$

(c) Given $c_t = 200 + 0.5y_t$,

$$I_t = 3(y_t - y_{t-1}) \text{ and } y_0 = 10000$$

3. For the data given below, determine

a. the market price P_t in any time period

b. the equilibrium price P_e and

c. The stability of the time path.

$$Qd = 180 - 0.75P_t$$

$$st = 30 + 0.3P_{t-1}, P_0 = 220.$$

7.0 REFERENCES/FURTHER READINGS

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UNIT THREE: OPTIMIZATION

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1.0. Introduction

Economics is both a positive as well as a normative social science. It helps in explaining why the various economic units behave the way they do, it guides the decision makers regarding what it must do given its objectives and constraints, if any. For example, positive economics would offer rationalization as to why a particular household buys a particular consumption basket or why a producer produces a given quantity of its output.

In contrast, normative economics would help in determining the most appropriate consumption basket for the household, given its objective that is maximization of utility or satisfaction and the budget (expenditure or income) constraint and in determining the most appropriate production level for the entrepreneur, given its objective of profit maximization and the constraints such as technology, raw material and availability. Therefore these roles of economics are referred to as the one dealing with economic optimization.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Know the meaning and types of Optimization problems
- Understand how to calculate Optimization problems, Unrestricted Optimization and Constraint Optimization

3.0 MAIN CONTENT

3.1 Meaning and Types of Optimization Problems

Optimization deals with the determination of extreme values which could be maximum or minimum for the goal (objective) variable. The goal variable could be just one unique or more multiple. For example, a private firm might pursue profit maximization as it is single goal. If so, the optimization technique must determine the value of the variables,

which are under the firm's control, called choice variables, so that they ensure the achievement of maximum possible profit to the firm. Moreover, a manufacturing company might aim at minimizing its average cost of production as the sole objective. In that case, the role of optimization techniques would be to find out the values of the variables under the firm's control that are consistent with the minimum possible value for its average cost. Thus if project depends only on the level of production, the profit maximizing problem is a single variable optimization problem. More so, if the profit depends on the level of output as well as the advertisement budget or in addition on some other variables as well, there is a multiple variable optimization.

The optimization problem facing a decision – making unit is further classified into unconstrained and constrained problems. In the former, the decision maker optimizes subject to no constraint, internal or external. In the latter, it has one or more constraints also called side conditions, imposed either by itself (Internal) or by outside agencies (external) such as government and or market conditions. An example of unconstrained optimization would be one where the firm aims at the maximum possible profit subject to one or more constraints such as a fixed production cost budget, a fixed quantity of output to be availability of scarce raw – materials in fixed quantity, employment of minimum number of unskilled labor etc.

The constrained optimization problems are further classified into equality and inequality constrained problems. For examples, a profit-maximizing firm might be required to produce a specific quantity of output of all of its multiple products, or if it is single product firm, it might be faced with a fixed production cost budget or a fixed quantity of a particular scarce raw-material. Under these conditions, the optimizing firm must strictly adhere to the given number of the constrained variable. An example of optimization subject is inequality constraints would be the one where the firm is seeking maximum possible profits but there is a fixed quantity of skilled labour available in the market: the firm can employ all the skilled labour, any quantity less than it but it no more.

3.2 Worked Example on Optimization

1) Given $P = 18000 - 2000k$

$C = 2000$ per unit

Required

- i. Find the revenue function
- ii. Cost function
- iii. Profit Maximization output
- iv. Marginal cost.

- i) Revenue function (where $q = x$)

$$R = pxq$$

$$R = (18000 - 2000x)R$$

$$R = 18000 - 2000R^2$$

- ii) Cost function $(\pi) = TR - TC$

Where TR = total revenue and TC = total cost.

$$= (18000x - 2000x^2) - 2000x$$

$$= 18000x - 2000x - 2000x^2$$

$$= 16000x - 2000x^2$$

$$MR = \frac{dTR}{dx} = 18000 - 4000x$$

$$MC = \frac{dTR}{dx} = 2000$$

- iii) Profit maximization output

$$MR = MC.$$

$$2000 = 18000 - 4000x$$

$$4000x = 18000 - 2000$$

$$x = 4.$$

- 2) Given $P = 120 - 22q$

$$C = 4q^2 - 40q^2 + 120q + 2$$

Required

- (a) Find profit maximization level of output.
- (b) Show that there is profit making.
- (c) Calculate the price
- (d) Calculate Total Revenue
- (e) Calculate Total Cost
- (f) Calculate Marginal cost.

(1) Profit Maximization $MC = MR$

$$TR = pxq$$

$$= (120 - 22q)q$$

$$TR = 120q - 22q^2$$

$$MR = \frac{dTR}{dq} = 120 - 44q$$

$$MC = \frac{dc}{dq} = 120 - 44q$$

$$MC = \frac{dc}{dq} = 12q^2 - 80q + 120$$

But $MC = MR$

$$12q^2 - 80q + 120 = 120 - 44q$$

$$12q^2 - 80q + 44q = 120 - 120 = 0$$

$$12q^2 - 36q = 0$$

$$12q^2(q - 3) = 0$$

$$12q^2 = 0 \text{ or } q - 3 = 0$$

$$q = 0/12 \text{ or } q = 3$$

ii) Profit $(\pi) = TR - TC$

$$\pi = (120q - 22q^2) - (4q^3 - 40q^2 + 120q + 2)$$

$$\pi = 120q - 22q^2 - 4q^3 + 40q^2 + 120q - 2$$

$$\pi = 120q - 120q - 22q^2 + 40q^2 + 4q^3 - 2$$

$$\pi = 180q^2 - 4q^3 - 2$$

Set $q = 3$

$$\pi = 180(3)^2 - 4(3)^3 - 2$$

$$\pi = 180(9) - 4(27) - 2$$

$$\pi = 1510$$

iii) Price = $120 - 22q$

$$= 120 - 22(3)$$

$$= 120 - 66$$

$$= 54.$$

iv) Total Revenue = $p \times q$

$$= (120 - 22q)q$$

$$= 120 - 22q$$

$$= 120(3) - 22(3)^2$$

$$= 340 - 198$$

$$= 142.$$

v) Total Cost

$$TC = 4q^3 - 40q^2 + 120q + 2$$

$$TC = 4(3)^3 - 40(3)^2 + 120(3) + 2$$

$$= 4(27) - 40(9) + 360 + 2$$

$$= 180 - 360 + 360 + 2$$

$$= 180 + 2$$

$$TC = 110.$$

$$MC = \frac{dc}{dx} = 12q^2 - 80q + 120$$

$$= 12(3)^3 - 80(3) + 120$$

$$= 180 - 240 + 120$$

$$= 180 - 120$$

$$= -12.$$

3.3. Unrestricted Optimization

$$y = f(x_1, x_2)$$

$$(1) \quad y = -x^2 + 4x + 7$$

Find (i) Stationary Point.

(ii) Check whether it is Minima or maxima.

$$(1) \quad f(x) = -2x + 4 = 0$$

$$-2x + 4 = 0$$

$$-2x = -4$$

$$x = 4/2$$

$$x = 2.$$

Stationary Point = 2

$$f_{11}(x)$$

= -2 (Note that we differentiate the above equation).

(2) Given $y = x^3 - 12x^2 + 36x + 8$, calculate the minimization level.

$$f_1(x) = \frac{3x^2}{3} - \frac{24x}{3} + \frac{36}{3} = 0$$

$$f(x) = x^2 - 8x + 12 = 0$$

$$x^2 - 8x + 12 = 0$$

$$(x - 6)(x - 2) = 0$$

$$x - 6 = 0 \text{ or } x - 2 = 0$$

$$x = 6 \text{ or } x = 2$$

$$f_{11}(x) = 6x - 24$$

$$6x = 24$$

$$x = \frac{24}{6}$$

$$x = 4(\text{Minimization})$$

(3) Given $y = x^3 - 12x^2 + 36x + 8$ calculate the point of inflexion

$$f_1(x) = 3x^2 - 24x + 36$$

$$f_{11}(x) = 6x - 24$$

$$f_{111}(x) = 6(\text{point of inflexion})$$

$$f_{1111}(x) = 0.$$

3.4. Constraint Optimization

(4) Given Utility Function $U = xy$ and is subject to $(st)y = P_1X_1 + P_2y$.

Where $P_n = \text{N}2, P_y = \text{N}5$, Find x and y , where $y = 100$

$$U = xy \quad \text{Let } P_1 = x \quad P_2 = y$$

$$S + 100 = 2x + 5y$$

$$Z = xy = \lambda(100 - 2x - 5y)$$

$$\frac{dt}{dx} = y - 2\lambda = 0 \text{_____} (i)$$

$$\frac{dt}{dx} = x - 5\lambda = 0 \text{_____} (ii)$$

$$\frac{dZ}{d\lambda} = 100 - 2x - 5y = 0 \text{_____} (iii)$$

From equation (i) $y - 2\lambda = 0$

$$y = 2\lambda$$

$$\lambda = \frac{y}{2} \text{_____} (iv)$$

From equation (ii) $x - 5\lambda = 0$

$$x = 5\lambda$$

$$\lambda = x/5 \text{_____} (v)$$

Equate $\lambda = \lambda$

$$\frac{y}{2} = \frac{x}{5}$$

$$y = \frac{2x}{5} \text{ (vi)}$$

From equation (iii) $100 - 2x + 5y = 0$

Substitute for y in equation (iii)

$$100 - 2x - 5\left(\frac{2x}{5}\right) = 0$$

$$100 - 2x - 2x = 0$$

$$100 - 4x = 0$$

$$x = 100/4$$

$$x = 25.$$

Substitute for x in equation _____ (vi)

$$y = \frac{2x}{5} \text{ (vi)}$$

$$y = \frac{2(25)}{5} = \frac{50}{5} = 10$$

$$y = 10, x = 25$$

$$(5) \quad U = q_1^{1.5} q_2 \text{ and } U = x^{1.5} y$$

$$\text{St } 100 - 100 = 3q_1 + 4q_2$$

$$Z = \text{Let } q_1 = x \text{ and } q_2 = y$$

$$Z = y^{1.5} y + \lambda(100 - 3x - 4y)$$

$$\frac{dZ}{dx} = 1.5x^{0.5}y - 3\lambda = 0 \text{ (i)}$$

$$\frac{dZ}{dx} = x^{1.5} - 4\lambda = 0 \text{ (ii)}$$

$$\frac{dZ}{d\lambda} = 100 - 3x - 4y = 0 \text{ (iii)}$$

From (1) $1.5x^{0.5}y = 3\lambda$

$$\lambda = \frac{1.5x^{0.5}}{y} \div 3$$

$$\lambda = \frac{1.5x^{0.5}}{3} y \text{ (iv)}$$

From (iv) $x^{1.5} - 4\lambda = 0$

$$x^{1.5} = 4\lambda$$

$$\lambda = \frac{x^{1.5}}{4} \text{ (v)}$$

Equate $\lambda = \lambda$

$$\frac{1.5x^{0.5}y}{3} = \frac{x^{1.5}}{4}$$

$$(1.5x4)x^{0.5}y = 3x^{1.5}$$

$$\frac{6x^{0.5}y}{3} = \frac{3x^{1.5}}{3}$$

$$2x^{0.5}y = x^{1.5}$$

$$y = \frac{x^{1.5}}{2x^{0.5}} = \frac{1}{2}x^{1.5-0.5}$$

$$= \frac{1}{2}x^1$$

$$= 0.5x^1$$

$$y = 0.5x$$

$$OR = y = \frac{x}{2} \text{ (vi)}$$

Put equation (vi) into (iii)

$$100 - 3x - 4y = 0$$

$$100 = 3x + 4y$$

$$100 = 3x + 4\left(\frac{x}{2}\right)$$

$$100 = 3x + 2x$$

$$100 = 5x$$

$$100 = 100/5$$

$$x = 20.$$

Put x in equation (vi)

$$y = \frac{x}{2} = 20/2$$

$$y = 10$$

$$x = 20, y = 10.$$

3. Given $Q = AK^\alpha L^B$ (180 quant)

s. t $C^\circ = rk + WL$ (150 cost),

Calculate the Output Maximization.

$$Z = AK^\alpha L^B + \lambda(C^\circ - rk - wL)$$

$$\frac{dz}{dk} = \alpha AK^{\alpha-1}L^B - \lambda r = 0.$$

$$= \frac{\alpha (AK^\alpha L^B)}{k} - \lambda r = 0$$

$$\alpha \frac{Q}{k} - \lambda r = 0 \text{ (i)}$$

$$\frac{dZ}{dL} = \beta AK^{\alpha-1}L^{B-1} - \lambda w = 0$$

$$\frac{\beta AK^\alpha L^B}{L} - \lambda w = 0$$

$$\beta \frac{Q}{L} - \lambda w = 0 \text{ (ii)}$$

$$\frac{dL}{dx} = C^\circ - rk - wL = 0 \text{ (iii)}$$

In equation $\frac{\alpha Q}{k} = \lambda r$

$$\lambda = \frac{\alpha Q}{rk}$$

In equation (ii) $\frac{\beta Q}{L} = \lambda w$

$$\lambda = \beta Q / wL$$

If $\lambda = \lambda$

$$\frac{\alpha Q}{rk} = \frac{\beta Q}{wL}$$

$$\frac{\alpha}{\beta} = \frac{rk}{wL}$$

Self-Assessment Exercise

Given Utility Function $U = xy$ and is subject to $(st)y = P_1X_1 + P_2y$.

Where $P_n = \text{N}3$, $P_y = \text{N}4$, Find x and y, where y = 200

4.0 CONCLUSION

In this unit, we can conclude that optimization means finding an alternative with the most cost effective or highest achievable performance under the given constraints, by maximizing desired factors and minimizing undesired ones. In comparison, maximization means trying to attain the highest or maximum result or outcome without regard to cost or expense. Practice of optimization is restricted by the lack of full information, and the lack of time to evaluate what information is available.

5.0 SUMMARY

In this unit, we have been able to discuss on the meaning and types of Optimization problems, worked examples on Optimization and also to be able to differentiate Unrestricted Optimization from a Constraint Optimization.

6.0 TUTOR-MARKED ASSIGNMENT

1. Given Utility Function $U = xy$ and is subject to $(st)y = P_1X_1 + P_2y$.

Where $P_n = \text{N}2, P_y = \text{8}$, Find x and y , where $y = 300$

2. Given $y = x^3 - 10x^2 + 18x + 4$, calculate the minimization level

3. $y = -x^2 + 2x + 4$

Find (i) Stationary Point.

(ii) Check whether it is Minima or maxima.

4. Given $y = x^3 - 8x^2 + 10x + 3$ calculate the point of inflexion

5. $U = q_1^{1.5}q_2$ and $U = x^{1.5}y$

St $-100 = 4q_1 + 8q_2$

7.0 REFERENCES/FURTHER READINGS

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UNIT FOUR DYNAMICS ANALYSIS

CONTENTS

- 1.0 Introduction
- 2.0 Objectives
- 3.0 Main Content
 - 3.1. Concept of dynamic Economics
 - 3.2. Scope and importance of dynamics Economics
 - 3.3. Limitations of dynamics

3.4.	Dynamic Equation
4.0	Conclusion
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6.0	Tutor-Marked Assignment
8.0	Reference/Further Readings

1.0. Introduction

Since the end of the nineteenth century, economic analysis has been fairly rigidly compartmentalized into statics and dynamics. Various, sometimes conflicting definitions of these terms have appeared in the literature; some have based the distinction on the nature of the subject matter studied, while others have emphasized the difference in analytic approach. Utilizing the first of these viewpoints, we may distinguish between a stationary economic phenomenon that is, one that does not change with the passage of time and a developing or changing phenomenon. But no matter which type of phenomenon we study, we may focus upon it an analytic apparatus which we describe as dynamic; that is, one that takes explicit account of the role of the passage of time in the structure of the subject; or we can subject it to a static analysis, which deals with its mechanism at a given moment and abstracts from the effect of past events on the present and future. For example, an investigation of the determination of the level of employment at a given moment and its dependence on current consumption, investment, and governmental demand may be considered static in character, but a discussion of the same problem that considers how the relationship between today's supply of capital equipment and its growth affects tomorrow's investment demand and how this process can generate a time path of investment demand can be considered a dynamic analysis.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Know the meaning of dynamic Economics, scope, concepts and limitations.
- Understand how to calculate dynamic equation

3.0 MAIN CONTENT

3.1. Concept of Dynamic Economics

The concept of dynamics is derived from Physics. It refers to a state where there is a change such as movement. However, tides of the sea, a bird flying in the sky are examples of dynamics. But the word 'dynamic' has a different meaning in economics. It is known from the dynamics study that there is movement in statics also but this movement is certain, regular and expected. While dynamics refers to that movement which is uncertain, unexpected and irregular.

Moreover, let us look at an aeroplane flying in the sky, we say it is in a dynamic state only if its direction, height and speed are uncertain. We know from day to day experience that fluctuations occur in the economy quite often. And it is not possible to make correct predictions about such fluctuations. The concept of dynamics is nearer to reality. In dynamic economics we study the economic variables like consumption function, income and investment in a dynamic state.

However, according to Prof. Harrod, "Economic dynamics is the study of an economy in which rates of output are changing." In the real world, economic variables like population, capital, techniques of production, fashions, habits, etc. do not change at a constant rate. The rate of change is different at different times. For example, the population of a country may increase at a rate of 3% in the first year; 5% in the second year and 7% in the third year, if the other economic variables change at unequal rates, the rate of output will also change at different times. In a dynamic state, there is uncertainty of every change. So, it is not possible to make correct predictions.

According to Prof. Hicks, "Economic dynamics refers to that part of economic theory in which all quantities must be dated." From Prof. Hicks's definition, we come to know that time element occupies great importance in dynamic economics. Here economic variables are related to different points of time. According to Baumol, "Economic dynamics is the study of economic phenomenon in preceding and succeeding events."

Comparative economic statics does not show the path of change of the old to new equilibrium. But in dynamic economics we also study the path of change or the movement towards equilibrium.

Recently the concept of dynamics has been applied to the economy as a whole, Prof Clark has pointed out the following features of a dynamic economy:

- (i) In a dynamic economy, population grows;

- (ii) Quantity of capital grows;
- (iii) Modes of production improve;
- (iv) Industrial institutions undergo changes. Inefficient organizations are replaced by efficient organisations.
- (v) Habits of the people, fashions and customs change, as wants of the people increase.

We can conclude by saying that dynamic economics relates to a dynamic economy where uncertainty and expectations play their part.

3.2. Scope and Importance of Dynamic Economics:

Dynamic economics is becoming more and more popular since 1925. Though the principles advocated by Clark and Aftalian were dynamic in nature yet their main purpose was to explain the business fluctuations. After 1925, dynamic economics became popular not only in business fluctuations but also in the determination of income and growth models.

The following points explain the scope and importance of dynamic economics:

1. Study of Time Element:

Time element occupies an important role in dynamic economics. Economic problems concerning continuous change of economic variables and path of change can be studied only in dynamic economics.

2. Trade Cycles:

Theories of trade cycles have been advocated only through the introduction of dynamic economics. Theories of trade cycles are based on dynamic economics as they refer to the fluctuations of the different time periods.

3. Basis of many Economic Theories:

Dynamic economics has an important place in economics because many economic theories are based on it. For example, saving and investment theory, theory of interest, effect of time element in price determination, etc. are based on dynamic economics.

4. More Flexible Approach:

Dynamic analysis is more flexible. Models regarding the possibilities of economic change can be developed in dynamic analysis. That is why it has been found a useful mode of study. Dynamic economics is also useful in solving the problems of economic planning, economic growth and trade cycles.

5. Realistic Approach:

Dynamic economic analysis is nearer to the reality. In a real world, economic variables like national income, consumption, etc. change irregularly and uncertainly. Moreover, economic variables of the previous period also affect the present economy. And time element occupies an important role in economic analysis.

3.3. Limitations of Dynamic Economics:

Dynamic economic analysis has its shortcomings too. It is difficult to understand.

Its main limitations are the following:

1. Complex Approach:

Dynamic economic analysis is a complex approach for the study of economic variables because it is based on time element. To find solutions of various problems, we have to make use of mathematics and economics which is beyond the understanding of a common man.

2. Not Fully Developed:

Many economists like Samuelson and Harrod, have developed dynamic approach of economic analysis. They have developed their theories through dynamic analysis. But this mode of economic analysis has not been fully developed. The reason is that factors affecting economic variables change very soon. Dynamic approach is not developing at the speed at which economic factors change.

3.4 General Analysis

Given $Y_t - aY_{t+1} + C$ _____ (1)

To write equation (i) in form of homogeneous equation

$$Y_t - aY_{t+1} = 0 \left\{ \begin{array}{l} \text{Neglecting Constant} \\ C \end{array} \right\}$$

Let $Ab^{t-1} = Y_{t-1}$

$$\text{Let } Ab^t = Y_t$$

$$\text{Therefore } Ab^t - aAb^{t-1} = 0$$

$$Ab^{t-1}(b - a) = 0$$

$$\text{Either } Ab^{t-1} = 0 \text{ or } b - a = 0$$

$$b = a \text{ (i)}$$

However: $Ab^t = A(a)^t$ – This is what is called complimentary series which is called Y_c

$$Y_c = Ab^t$$

$$Y_c = A(a)^t \text{ – Complementary Series}$$

$$Y_p = \text{is called particular integra}$$

$$\text{Let } Y_t = k = Y_{t-1}$$

$$\text{From equation (1)}$$

$$Y_t - aY_{t-1} + C = 0$$

$$K - aK + C = 0$$

$$K - aK = -C$$

$$K(1 - a) = -C$$

$$K = \frac{-C}{1 - a}$$

3.5 Dynamic Equation

From the analysis in 3.1 above:

$$Y_t = Y_c + Y_p$$

$$Y_t = A(a)^t + \frac{-C}{1 - a}$$

$$\text{at time } (t) = 0$$

$$Y_0 = A(a)^0 + \frac{-C}{1-a}$$

$$A = Y_0 - \left(\frac{-C}{1-a} \right)$$

$$A = Y_0 - \frac{-C}{1-a}$$

Example:

Given $Y_t - 2Y_{t-1} = 2$

$Y_t - 2Y_{t-1} = 0$, find A^t at $Y_0 = 5$

Assume $Ab^t = Y_t$

$$Ab^{t-1} = Y_{t-1}$$

$$Ab^t - 2Ab^{t-1} = 0$$

$$Ab^{t-1}(b - 2) = 0$$

Either $Ab^{t-1} = 0$ or $b - 2 = 0$

$$b = 2$$

But $Y_c = Ab^t$

$$Y_c = A(2)^t$$

Therefore Let $Y_t = K = Y_{t-1}$

or $Ab^t = K = Ab^{t-1}$

From equation (1)

$$Y_t - 2Y_{t-1} = 2$$

$$K - 2K = 2$$

$$K - (1 - 2) = 2$$

$$K - (-1) = 2$$

$$-K = 2$$

$K = -2 \rightarrow$ This is what is called the particular Integra (Y_p).

Therefore, the dynamic equation

$$Y_t = Y_c + Y_p$$

$$\therefore Y_t = A(2)^t + (-2)$$

$$Y_t = A(2)^t - 2$$

at time (t) = 0

$$Y_0 = A(2)^0 - 2$$

$$Y_0 = A \times 1 - 2$$

$$Y_0 = A - 2$$

But $Y_0 = 5$

$$A = Y_0 - 2$$

$$A = 5 + 2$$

$$A = 7$$

Finally $Ab^t = (17)(2)^t = (14)^t$

Ex 2: If $Qd = 18 - 3P_t$ _____ (i)

$$Qs = -3 + 4P_{t-1}$$
_____ (ii)

Find (i) Intertemporal equilibrium

(ii) Is there Stability?

At equilibrium $Qd = Qs$

$$18 - 3P_t = -3 + 4P_{t-1}$$

$$18 + 3 = 4P_{t-1} + 3P_t$$

$$21 = 4P_{t-1} + 3P_t$$

Re-arranged

$$4P_{t-1} + 3P_t = 21$$

$$3P_t + 4P_{t-1} = 21 \text{_____} (iii)$$

$$\text{Let } Ab^t = P_t$$

$$Ab^{t-1} = P_{t-1}$$

$$3Ab^t + 4Ab^{t-1} = 0 \{\text{Note: Neglecting the constant}\}$$

$$Ab^{t-1}(3b + 4) = 0$$

$$\text{Either } Ab^{t-1} = 0 \text{ or } 3b + 4 = 0$$

$$3b = -4$$

$$b = -4/3$$

$$\text{However, } Y_c = Ab^t$$

$$Y_c = A \left(\frac{-4}{3} \right)^t$$

If $b = 0 \rightarrow$ it means there is stability.

If $b > 1 \Rightarrow$ it means there is explosion that is explosive economy.

If $b < -1$ or $0 \Rightarrow$ it means there is damped situation

$$Y_p = P_t = K = P_{t-1}$$

From equation (iii)

$$3P_t + 4P_{t-1} = 21$$

$$3K + 4K = 21$$

$$7K = 21$$

$$K = 21/7 = 3$$

$$\therefore Y_p = K = 3.$$

The Dynamic Equation:

$$Y_t = Y_c + Y_p$$

$$Y_t = A \left(-4/3 \right)^t - 3$$

at time $t = 0$

$$Y_0 = A \left(-4/3 \right)^0 + 3$$

$$Y_0 = A \times 1 + 3$$

$$Y_0 = A + 3$$

$$\text{Therefore: } Qd = 18 - 3P_t$$

$$= 18 - 3Y_t$$

$$Qd = 18 - 3 \left[\left(-4/3 \right)^t + 3 \right]$$

$$Qs = -3 + 4P_{t-1}$$

$$= -3 + 4Y_{t-1}$$

$$= -3 + 4 \left[A \left(-4/3 \right)^t + 3 \right] - 1$$

at time $(t) = 0$

$$= 18 - 3 \left[A \left(-4/3 \right)^0 + 3 \right]$$

$$= 18 - 3(A \times 1 + 3)$$

$$= 18 - 3(A + 3)$$

$$= 18 - 3A - 9$$

$$= -3A + 18 - 9$$

$$Qd = -3A + 9.$$

$$Q_s = -3 + 4 \left[A \left(-4/3 \right)^0 + 3 \right] - 1$$

$$= -3 + 4(A \times 1 + 3) - 1$$

$$= -3 + 4(A + 3) - 1$$

$$= -3 + 4(-A - 3)$$

$$= -3 - 4A - 12$$

$$= -4A - 3 - 12$$

$$Q_s = -4A - 15$$

At equilibrium $Q_d = Q_s$.

$$= 3A + 9 = -4A - 15$$

$$= -3A + 4A = -15 - 9$$

$$A = -24.$$

$$Q_d = 18 - 3P_t$$

$$= 18 - 3 + 4 \left[A \left(-4/3 \right)^t + 3 \right]$$

$$= 18 - 3 \left[A \left(-4/3 \right)^t + 3 \right].$$

Self-Assessment Exercise

If $Q_d = 12 - 4P_t$ _____ (i)

$Q_s = -2 + 7P_{t-1}$ _____ (ii)

Find (i) Intertemporal equilibrium

(ii) Is there Stability?

4.0 CONCLUSION

In this unit, we can conclude that adynamic analysisis the changes in an economic system over time, particularly those reflected in the behavior of markets, businesses, and the general economy.

5.0 SUMMARY

In this unit, we have been able to discuss on the concept of dynamic economics, scope and importance of dynamics economics, limitations of dynamics economics, general analysis of dynamics and dynamics equation.

6.0 TUTOR-MARKED ASSIGNMENT

1. Given $Y_t - 3Y_{t-1} = 3$

$Y_t - 3 = 0$, find A^t at $Y_0 = 8$

2. Given $Y_t - 4Y_{t-1} = 9$

$Y_t - 7Y_{t-1} = 0$, find A^t at $Y_0 = 10$

3.

$Y_{t+2} + Y_{t+1} - 2Y_t = C$

$Y_{t+2} + Y_{t+1} - 2Y_t = 12$

Find whether it is suitable or unsuitable

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