Note that the given P.D.E is non-linear and of standard type I i.e it is of form $f(p,q)=0$.

Hence, it's complete solution is of the form

$$z = ax + by + c.$$

such that $f(a,b) = 0$ equals $3a^2 - 2b^2 - 4ab$

i.e $b = \dfrac{-4a \pm \sqrt{16a^2 + 24a^2}}{4}$

$$= a\left(\frac{-1 \pm \sqrt{10}}{2}\right)$$

Hence, the desire complete solution is

$$z = a\left[x + y\left(\frac{-1 \pm \sqrt{10}}{2}\right)\right] + c$$

Example 8

Solve $p^2 - q^2 = 1$

Solution

Here $F(p,q) \Rightarrow p^2 - q^2 - 1 = 0$. therefore

$f(a,b) = a^2 - b^2 - 1 = 0$ and $b^2 = a^2 - 1$

which implies that

$$b = \pm (a^2 - 1)^{1/2}$$

Therefore, a complete solution (Taking $b = (a^2-1)^{1/2}$)

$$z = ax + [(a^2-1)^{1/2}]y + c$$

Note that must have taking $b = -(a^2-1)^{1/2}$ and obtained as a complete solution

$$z = ax - (a^2-1)^{1/2}y + c$$

Example 9

Solve $pq + p + q = 0$

Solution

~~solution~~ Note that the given P.D.E is non-linear and is of the $f(a,b) =$

Also $f(a,b) = ab + a + b = 0$

And: $(a+1)b + a = 0$: which implies that

$$b = \frac{-a}{a+1}$$

Solve $p^2 + q^2 = 9$.

soln

$f(p,q) \Rightarrow p^2+q^2-9=0$

Also; $f(a,b) = a^2+b^2-9=0$

which implies ~~$b^2+b^2-9=0$~~ $b^2 = 9-a^2$

$\Rightarrow b = \pm\sqrt{9-a^2}$

Therefore Complete solution is

$z = ax + \left(\sqrt{9-a^2}\right)y + c$ Or

$z = ax - \left(\sqrt{9-a^2}\right)y + c$

TYPE II

This type of equation is Analogous to the extended Clairaut's (equ)ation which is

$$z = ax+by+f(a,b) \quad \text{———} \quad (14)$$

(see (7) module 1).

In particular, a general form of this type of non-linear P.D.E

$$z = px+qy+f(p,q) \quad \text{———} \quad (15)$$

whose complete solution is (14).

Example 11

solve $z = px+py+p^2q^2$

soln

Note that the given P.D.E is of the (type II) i.e. $[(z = px+qy+f(p,q))$

Hence a Complete solution is

$$z = ax+by+a^2b^2$$

Differentiating the above Complete solution w.r.t a and b we obtain

$0 = x+2ab^2$ and

$0 = y+2a^2b$

from $0 = x+2ab^2$ we obtain

$$-\frac{x}{2b^2} = a$$

which when combined with $0 = y + 2ab^2$ yields

$$0 = y + 2b\left(\frac{x^2}{4b^4}\right)$$

$$\Rightarrow 0 = 1 + \frac{x^2}{2b^3}$$

$$\Rightarrow b^3 = \frac{-x^2}{2y}$$

Therefore $b = -\sqrt[3]{\frac{x^2}{2y}}$

Also;

from $0 = y + 2a^2b$, we have

$$-\frac{y}{2a^2} = b \quad \text{———} \quad (*)$$

Substituting $(*)$ into $0 = x + 2ab^2$ yields

$$0 = x + \frac{2ay^2}{4a^4}$$

i.e $0 = x + \frac{y^2}{2a^3}$

i.e $0 = 2a^3x + y^2$

$$\Rightarrow a^3 = \frac{-y^2}{2x}$$

$$\Rightarrow a = -\sqrt[3]{\frac{y^2}{2x}}$$

Hence

The singular solution is

$$z = -x\sqrt[3]{\frac{y^2}{2x}} - y\sqrt[3]{\frac{x^2}{2y}} + \sqrt[3]{\frac{x^2y^2}{16}}$$

15

Example 12

Obtain

(i) A complete solution and hence

(ii) The singular solution of the following non-linear P.D.E of order 1

$$z = px + qy + 3p^{1/3}q^{1/3}$$

Solution

$= 9x + by + 3a \ldots$

To obtain the associated (Corresponding Singular solution) we different complete solution w.r.t a and b and obtain.

$$x + a^{-2/3} b^{1/3} = 0$$

And

$$y + a^{1/3} b^{-2/3} = 0 \quad \text{respectively}$$

Then,

$$ax + by = -2a^{1/3} b^{1/3},$$

$$xy = a^{1/3} b^{1/3}$$

And; substituting in the Complete solution, we obtain the following si solution

$$z = -2a^{1/3} b^{1/3} + 3a^{1/3} b^{1/3} = a^{1/3} b^{1/3} = 1/xy \text{ or}$$

$$xyz = 1$$

Example 1.13

Solve $z = px + qy + p^2 + pq + q^2$

Solution

The given P.D.E is

i) Non-linear

ii) of order one (1)

iii) of the type II [i.e it is of the form $z = px + qy + f(p, q)$]

Hence, a Complete solution of the P.D.E is;

$$z = ax + by + a^2 + ab + b^2$$

Now,

Differentiating the complete solution w.r.t a and b, we have

$$x + 2a + b = 0,$$

$$y + 2a + 2b = 0$$

Solving, To obtain $a = \frac{(y - 2x)}{3}$; $b = \frac{(x - 2y)}{3}$ and substituting into the Complete solution.

the Singular solution is $3z = xy - x^2 - y^2$ (students should verify this)

TYPE III

This is a non-linear P.D.E of the form $f(z, p, q) = 0$.

A typical example is

$z = p^2 + q^2$ (x and y do not appear explicitly)

Let us assume that $z = G(x + ay) = G(u)$

where a is an arbitrary constant

Then; $p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{du}{dx} = \frac{dz}{du}$, and

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{du}{dy} = a\frac{dz}{du}$$

When these are substituted into the given differential equation, we obtain ordinary differential equation of first-order.

$$\left(f(z), \frac{dz}{du}, a\frac{dz}{du}\right) = 0$$

whose solution is the required complete solution

Example 17

Solve $1 + p^2 = qz$

solution

Note that the given non-linear P.D.E is of type III (x and y do not appear explicitly)

Hence, we assume $z = G(x + ay) = G(u)$

Then, $\frac{\partial z}{\partial x} = \frac{dz}{du}\frac{du}{dx}$; $\frac{\partial z}{\partial y} = \frac{dz}{du}\frac{du}{dy}$

$p = \frac{dz}{du}$, $q = a\frac{dz}{du}$, And the given equation becomes

$$\left(\frac{dz}{du}\right)^2 - az\frac{dz}{du} + 1 = 0$$

or

$$\frac{dz}{az - \sqrt{a^2z^2 - 4}} = \frac{1}{2}du$$

Rationalizing the left member of the latter equation, we obtain whose solution is (Student should show these as an assignment)

$$\tfrac{1}{2}az^2 + \tfrac{1}{a}\left[\frac{az}{2}\sqrt{a^2z^2-4} - 2\ln\left(az+\sqrt{a^2z^2-4}\right)\right] = 2(u+b)$$

A complete solution is then,

$$a^2z^2 + az\sqrt{a^2z^2-4} - 4\ln\left(az+\sqrt{a^2z^2-4}\right) = 4a(x+ay+b)$$

Note that

$$a^2z^2 + az\sqrt{a^2z^2-4} + 4\ln\left(az+\sqrt{a^2z^2-4}\right) = 4a(x+ay+b)$$ is obtained

from $\frac{az}{dz} + \sqrt{a^2z^2-4} = \tfrac{1}{2}du$ is also a complete solution

Example 15

Solve

$$4(1+z^3) = 9z^4pq$$

Solution

Again the given non-linear P.D.E is of type (Type III).

Hence; we assume $z = g(x+ay) = g(u)$ Then,

$$p = \frac{dz}{du}, \quad q = a\frac{dz}{du}$$

And the given equation becomes

$$4(1+z^3) = 9az^4\left(\frac{dz}{du}\right)^2 \quad \text{or}$$

$$\frac{3\sqrt{az^2}\,dz}{\sqrt{1+z^3}} = 2du$$

Integrating, we have

$$\sqrt{a(1+z^3)} = u+b$$

And, a complete solution is

$$a(1+z^3) = (x+ay+b)^2$$

Using the results of differentiating these w.r.t a and b, we have;

$$1+z^3 = 2(x+ay+b)y \quad \text{And,}$$

$$2(x+ay+b) = 0$$ and the singular solution is

$$z^3+1 = 0$$

TYPE IV

The equations in this class are of the form $f(x,p) = f_2(y,q)$ is Non-l P.D.E involving only x, y, p and q explicitly (But not z). In such a way that they (x, y, p, q) are seperated, so that x and p are together in one group while y and q form the other group. Thus P.D.E's are solved by putting $f_1(x,p) = f_2(y,q) = a$ (arbitrary constant) ——

These eqn (17) give p and q.

In particular, solving (17) we obtain $p = G_1(x,a)$ & $q = G_2(y,a)$ ——

Since z is a $f(x,y)$ we have

$dz = pdx + qdy$, which in view of (18) yields

$$dz = G_1(x,a)dx + G_2(y,a)dy \quad \text{—— (19)}$$

Hence;

Integrating (19) we have

$$z = \int G_1(x,a)dx + \int G_2(y,a)dy + b \quad \text{—— (20)}$$

Containing two arbitrary constants and this is the required complete soln

Example 16

Solve $q = 2yp^2$

Solution

The given non-linear P.D.E of order one can be re-written as $p^2 = \frac{q}{2y}$

And these is of the form

$$f_1(x,p) = f_2(y,q)$$

putting $f_1(x,p) = p^2 = a^2$ (say) and

$$f_2(y,q) = q = a^2$$

Solving these eqns we have;

$p^2 = a^2$ and hence $p = \frac{dz}{dx} = a$, so that

$$z = ax + c.$$

And when $q = \frac{dz}{dy} = 2a^2y$, we have

$$z = a^2y^2 + c$$

Hence; a complete integral is

$z = ax + a^2y^2 + b$ (where b is an arbitrary constant)

$f_1(x, p) = p - x^2 = a$ and

$f_2(y, q) = q + y^2 = a$

Hence we have

$p = a + x^2$

$q = a - y^2$

Integrating $dz = p\,dx + q\,dy = (a+x^2)dx + (a-y^2)dy$

Thence required complete solution is

$$z = ax + \frac{x^3}{3} + ay - \frac{y^3}{3} + b$$

Example 18

Solve $\sqrt{p} - \sqrt{q} + 3x = 0$

Solution

Set $\sqrt{p} + 3x = a$ and $\sqrt{q} = 0$, then we have

$p = (a-3x)^2$ and

$q = a^2$

a complete solution is $z = \int p\,dx + \int q\,dy + b = \int (a-3x)^2 dx + a^2 \int dy + b$

i.e $z = -\frac{1}{9}(a-3x)^3 + a^2 y + b$

Example 19:

Solve $q = -px + p^2$

Solution

Set $p^2 - px = a$ and $q = a$ Then

$p = \frac{1}{2}\left(x + \sqrt{x^2 + 4a}\right)$

a complete solution is

$$z = \frac{1}{2}\int\left(x + \sqrt{x^2+4a}\right)dx + a\int dy + b$$

Or

$$z = \frac{1}{4}\left[\left(x^2 + x\sqrt{x^2+4a}\right) a \ln\left(x + \sqrt{x^2+4a}\right) + ay + b\right]$$

CHARPIT's METHOD

Consider the non-linear partial differential equation

$f(x, y, z, p, q) = 0$ ——— (21)

Firstly, note that (21) is not one of the four types disscussed above.
Also since z depends both x and y, we have

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = p\,dx + q\,dy \quad \text{——— (22)}$$

Forming Charpit's auxilliary equation (proof omitted)

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial z}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

We may obtain a relation

$$f(p, q) = 0 \quad \text{——— (24)}$$

between p and q. Equations (21) and (24) will yields p and q which when substituted in (22) give the required solution.

Example 21

Solve $2zx - px^2 - 2qxy + pq = 0$

Soln

Charpit's auxilliary equations are

$$\frac{dp}{2z - 2qy} = \frac{dq}{0} = \frac{dz}{px^2 + 2xyq - 2pq} = \frac{dx}{x^2 - q} = \frac{dy}{2x - p}$$

Hence $dq = 0$ gives $q = a$ (constant).

Putting $q = a$ in the given p.d.e we have $p = \frac{2x(z - ay)}{x^2 - a}$

Now substituting value of p and q in $dz = p\,dx + q\,dy$ we have

$$dz = \frac{2x(z - ay)}{x^2 - a}dx + a\,dy.$$

Or

$$\frac{dz - a\,dy}{z - ay} = \frac{2x}{x^2 - a}dx.$$

Which gives on Integration

$$z - ay = c(x^2 - a)$$

i.e $\quad z = ay + c(x^2 - a).$

Example 22

Solve $q = -xp + p^2$

$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = -p$, $\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} = 0$ and $-\left(p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q}\right)$

$= 2p^2 + xp + q.$

Hence Charpit's auxilliary equation are (6.4)

$\frac{dp}{-p} = \frac{dy}{1}$, we have $\ln p = -y + \ln a$ or $p = ae^{-y}$

Using the given differential equation

$q = -xp + p^2 = -axe^{-y} + a^2e^{-2y}$

Then $dz = pdx + qdy$ becomes.

$dz = ae^{-y}dx + (-axe^{-y} + a^2e^{-2y})dy$

Integration yields

$z = axe^{-y} - \frac{1}{2}a^2e^{-2y} + b$

23/07/2018. M:3

CLASSIFICATION OF P.D.E, REDUCTION TO CANONICAL OR NORMAL FOR

~~RIEMANN METHOD~~

3.1 Classification of Partial differential equations of second order

Consider a general partial differential equation of second order for function of two independent variables x and y in the form ~~$R_r + S_s$~~

$$R_r + S_s + T_t + f(x, y, z, p, q) = 0 \quad \text{——— (1)}$$

where R, S, T are continuous functions of x and y only possessing partial derivatives defined in domain D on the $x-y$ plane.

Note again that $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$ and $t = \frac{\partial^2 z}{\partial y^2}$.

Then (1) is said to be:

i) hyperbolic at a point (x, y) in domain D if $S^2 - 4RT > 0$

ii) Parabolic at a point (x, y) in domain D if $S^2 - 4RT = 0$

iii) Eliptic at a point (x, y) in domain D if $S^2 - 4RT < 0$

Observe that the type of (1) is determined solely by its principal $(R_r + S_s + T_t)$ which involves the highest order derivatives of z)