MEAN AND VARIANCE OF FUNCTION OF A RANDOM VARIABLE

Definition: A function of a random variable is a rule that transforms a point from a sample space into a real number line.

If X and Y are random variables on the sample space S, then Y is said to be function of X provided Y can be represented by Y(S) = ϕ (X(S)for every s$\in$ S. Some real valued function of a real variable: 2X+t, X^2+3X and tX are some example of function of X.

Definition: Let X and Y be random variable on the same space S with Y = ϕ (X), then the mean of function of X_n is given by:

E(Y) = E(ϕ (X) = $\sum_{i=1}^{n} \phi(xi)$ f(xi), where f(xi) is the probability mass function of X.

Example 1: An unbiased green octahedral die is tossed. If Y denote twice the number appears and (i) Z = 3 + Y (ii) Z = $^1/_2$Y + 1, find E(Z)

(i) Here, S = {1, 2, 3, - - -, 8}

Y(S) = {2, 4, 6, - - -, 16}

The distribution of Y is given below

yi	2	4	6	8	10	12	14	16
f(yi)	$^1/_8$	$^1/_8$	$^1/_8$	$^1/_8$	$^1/_8$	$^1/_8$	$^1/_8$	$^1/_8$
3 + yi	5	7	9	11	13	15	17	19
$^1/_2$yi + 1	2	3	4	5	6	7	8	9

Let ϕ (Y) = 3+ yi :. Z = ϕ (Y) where Z and Y are r.vs on the same sample space.

E(Z) = E(ϕ (Y) = $\Sigma\,\phi$ (yi)f(yi) = Σ (3 + yi)f(yi)

= 5 x $^1/_8$ + 7 x $^1/_8$ + - - - + $^{19}/_8$ = 12 (ii)Given that ϕ (Y) = $^1/_2$Y + 1

$$E(Z) = E(\phi (Y) = E (^1/_2 + 1)$$
$$= \Sigma \ (^1/_2 + 1) f(yi)$$
$$= {}^1/_8 (2 + 3 + 4 + - - - +9) = {}^{44}/_8 = 5.5$$

Exercise: A fair coin is tossed twice. If X denote the number of heads appearing in the sample space S, find the distribution of Z. Given that Y = ϕ (X), find E(Y), if (a) ϕ (X) = $X^2 - 1$, (b) ϕ (X) = 2+3X. Find the value of Y of (a) and (b) above.

Xi	0	1	2
xi^2-1	-1	0	3
$(xi2-1)^2$	1	0	9

CONTINOUS RANDOM VARIABLE

Definition: A random variable X is said to be continuous if the set of its positive values are contained in an interval, i.e X $\in$ {a, b} for $x_1, x_2, - - -, x_n \in$ X.

Definition 2: A continuous random variable can also be defined as that random variable which takes on a non-countably infinite number of values.

Definition 3: If X is a continuous random variable, a function f(x) is said to be continuous probability density function if it satisfies the following conditions:

(1) $F(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x) dx = 1.$

Definition 4: If X lies between a and b i.e. if a< X< b, the probability that X lies between a and b is defined by $P(a< X< b) = \int_a^b f(x) dx$

Note: If X is continuous, we shall assume unless otherwise stated that:

$P(a\leq X \leq b) = P(a\leq X < b) = P(a< X\leq b).$

Definition 5: If X is a continuous random variable, the distribution function F(X) is given by

F(X) = P(X≤ x) = P(-∞ ≤ X ≤ x) = $\int_{-\infty}^{x} f(u)du$.

It should be noted that F(X) is a monotically increasing function.

Example: if X is a continuous random variable with pdf

$$F(X) = \begin{cases} KX/8, & 0 < x < 4 \\ 0, & \text{otherwise} \end{cases}$$

Find (i) the constant K, (ii) P(1<x<3) (iii) P(-2<x<2) and (iv) P(-3 < x < 5).

Solution

(i) K = 1, $f(x) = \begin{cases} KX/8, & 0 < x < 4 \\ 0, & \text{otherwise} \end{cases}$

(ii) $\int_{1}^{3} f(x)dx = \int_{1}^{3} \frac{x}{8}\, dx$, $\frac{1}{8}\left[\frac{x^2}{2}\right]_1^3 = \frac{8}{16} = \frac{1}{2}$

(iii) P(-2<x<2) = P(0<x<2) since -2<x<0 is outside the function condition

$$\int_{0}^{2} f(x)dx = \frac{1}{4}.$$

(iv) P(-3<x<5) = P(0<x<4) = 1.

RELATIONSHIP BETWEEN THE PROBABILITY DENSITY FUNCTION F(X) AND PROBABILITY DISTRIBUTION FUNCTION F(X)

(i) If F(X) is a distribution function of a continuous random variable X,

F(X) = P(X≤ x) = $\int_{-\infty}^{x} f(u)du$ where F(X) is a probability density function.

(ii) If f(x) is a probability density function and F(X) is the corresponding distribution function of a continuous random variable X.

F(x) = dF(X)/dx i.e.

The derivative of the probability distribution function is the density function.

Example: Given that f(x) = $\begin{cases} tx^3 & 0<x<4 \\ 0, \text{ otherwise} \end{cases}$

Probability density function of a continuous random variable X. finds (a) th constant t and (b) the distribution function.

Solution

(a) Since f(x) is a pdf i.e. $\int_0^4 f(x)dx = 1$

$t = {}^1/_{64}$.

$F(x) = {}^1/_{64}\, x^3 \quad 0 < x < 4$.

(b) By definition of distribution function F(X)

(c) $F(X) = P(X<x) = \int f(u)du \qquad f(u) = 1/64u^3,\ 0<x<4$

(d) $F(X) = P(X<x) = \int_{-\infty}^{x} \frac{1}{64} u^3 du$

Using the condition given for f(x) above, if x<0, then

F(X) = 0, if 0<x<4, then

(e) $F(X) = \int_0^x f(u)du = \int_0^x \frac{1}{64} u^3 du$

$= [u^4/_{256}]^4_0 = x^4/_{256}$

(f) If $x \geq 4$, then $F(x) = \int_4^0 f(u)du + \int_0^4 f(x) = \int_0^4 \frac{1}{64} u^3 du + \int_4^x 0\, du$

$= \left.\frac{u4}{256}\right|_0^4 + 0 = 1$

Hence the refund probability distribution function is

$F(X) = \begin{cases} 0 \quad x^4/_{256} & x<0 \\ & 0<x<4, \\ 1 & x \geq 4. \end{cases}$

Example: use the above distribution function to find $P(1 \leq x \leq 2)$

Solution: $P(1 \leq x \leq 2) = P(x \leq 2) - P(x \leq 1) = F(2) - F(1) = 2^4/_{256} - 1^4/_{256} = {}^{15}/_{256}$

Exercise: The distribution function of a random variable X is given by

F(X) = $\begin{cases} x^2/4 & x \leq 0 \\ & 0 \leq x \leq 2 \\ 1, & x \geq 2 \end{cases}$

Find the pdf and use your result to find P(1<x<2)

Solution

f(x) = dF(X)/dx = d(0)/dx + d($x^2/4$)/dx + d(1)/dx

f(x) $\begin{cases} x/2 & 0 \leq x \leq 2 \ = 0 + x/2 + 0 = x/2 \\ 0, \text{ elsewhere} \end{cases}$

P(1<x<2) = $\int_1^2 \frac{x}{2} dx = [x^2/4]^2_1 = 1 - \frac{1}{4} = \frac{3}{4} = 0.75$

MEAN AND VARIANCE OF A CONTINUOUS RANDOM VARIABLE

Definition 1: For a continuous r.v. X, having density function F(X) the expectation of X is defined as E(X) = $\int_{-\infty}^{\infty} Xf(X)$dx. As earlier mentioned the expectation of X is often called the mean of X and is denoted by the μ_x or μ when the particular r.v. is understood.

Definition 2: For a continuous random variable X, mean of X is defined as μ = E(X) = $\int_{-\infty}^{\infty} Xf(X)$dx and V(X) = E[(X- μ)^2] = E(X^2) - μ^2, where E(X2) = $\int_{-\infty}^{\infty} \mathbf{X2} f(x)$dx.

Example: A continuous r.v. X has probability density function given by

F(X) = $\begin{cases} de^{-3x} & x>0 \\ 0, & \text{otherwise} \end{cases}$

Find the constant d and mean of X.

Solution: $\int de^{-3x}$ dx = 1 = $\left| \frac{de^{-3x}}{-3} \right|_0^{\infty}$

$0 - (-d/3) = 1$, $d = 3$

Hence,

$$F(X) = \begin{cases} 3e^{-3x} & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Mean of X = E(X) = $\int_0^{\infty} Xf(X)\,dx$ (integration by part)

$$= \int_0^{\infty} X(3e^{-3x})dx$$

$$\left.\frac{3\text{-}xe^{-3x}}{3}\right|_0^{\infty} - \left.\frac{e^{-3x}}{9}\right|_0^{\infty}$$

$E(X) = {}^1/_3$

Example: A continuous random variable has a probability density function

$$f(x) = \begin{cases} {}^x/_3 + C & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Find (a) the constant C, (b) the mean and variance X and (c) the standard deviation of X.

Solution

$\int ({}^x/_3 + C)dx = \frac{x^2}{3} + CX^2 \Big|_0 = 1 \qquad C = 1/6$

$$\text{Hence, } F(X) = \begin{cases} {}^x/_3 + 1/6 & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Mean of X = E(X) = $\int_0^{2} Xf(X)\,dx = 11/9$

Variance of X = $16/9 - (11/9)^2 = 23/81$

$$\text{S.D} = \sqrt{var(X)} = \frac{\sqrt{23}}{81} = 0.53.$$

Exercise

(1) Show that the following are probability density function (pdf)

(a) $f_1(x) = \ell^{-x}$ $x>0$

(b) $f_2(x) = 2\ell^{-2x}$ $x>0$

(c) $f(x) = (\theta + 1) f_1(x) - \theta\ f_2(x)$ $0 < \theta < 1$

(2) Following is a constant K so that the following is a pdf

$F(x) = Kx^2$ $-k<x<k$

$\int Xf(X)\,dx = 1$

$\int_{-k}^{k} k\, x^2 = 1$

$\left.\frac{x^3}{3k}\right|_{-k}^{k} = \frac{k^3 + k^3}{3} = \frac{2k^4}{3}$

(3) A fair coin is tossed until a head appears. Let X denotes the number of tossed refined.

(a) Find the density function of X.

(b) Find the mean and variance of X.

DISCRETE DISTRIBUTION: BERNOULLI DISTRIBUTION

Given an experiment whose outcomes can be classified into 1 or 2 classes for example pass or fail, on and off, head or tail. If we let x = 1 for a success and x = 0 for a failure, then the pdf or pmf is given by

$$p(X) = \begin{cases} 1, \text{ with prob } p \\ 2, \text{ with prob } 1-p \end{cases}$$

$p(x) = p^x (1-p)^{1-x}$

A random variable with the above probability mass function is said to be a Bernoulli random variable and the probability distribution is called a Bernoulli distribution.

$E(X) = p$

$V(X) = pq.$

BINOMIAL DISTRIBUTION

This distribution deals with repeated and independent trials of an experiment with two outcomes resulting in success or failure, 0 or 1, true or false, yes or no. If the interest is on the no of successes and not in the order in which they occur, the probability of exactly X successes in n repeated trials is given by

$$f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0, 1, 2, ---, n \\ 0, & \text{otherwise} \end{cases}$$

p is the prob of success and q = 1-p is the prob of failure and x is the number of successes in repeated trials. F(x) gives the pdf of binomial distribution.

PROPERTIES OF BINOMIAL DISTRIBUTION

(i) It has n independent trials

(ii) It has constant prob of success p and prob of failure q = 1-p from trial to trial.

(iii) There is assigned prob to non-occurrence of events.

(iv) The mean (μ) = np and the variance (δ^2) = npq.

(v) Each trial can result in one of only two possible outcomes called success or failure.

NB: $\binom{n}{x} = \frac{n!}{x!(n-x)!}$

Given that X is a random variable with Binomial Distribution

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, ---, n.$$

The mean of the random variable X with Binomial Distribution given above is

E(X) = np

V(X) = npq

Application: If X has a binomial distribution with n = 5 and p = $^1/_2$. Find (i) the P(X=1) (ii) P(X>1).

1. $\binom{5}{1}$ $(^1/_2)5$
2. 1 - $\binom{5}{0}$ $(^1/_2)5$ + $\binom{5}{1}$ $(^1/_2)5$

Example: Assume that boys and girls are equally likely to be born. What is the prob of there being?

(1) No boys in a family of 3 children

(2) Only one boy in a family of 3 children.

(3) At least one boy in a family of 3 children.

POISSON DISTRIBUTION

A random variable X is said to have a poisson distribution if the

$$f(x) = \frac{\pi^x \ell^{-\pi}}{x!}, \quad x = 0, 1, 2, --- \quad \pi > 0$$

The poisson distribution provides a realistic model for many random phenomena. Since the values of a poisson random variable are the non-negative integers, any random phenomenon for which a count of some sort is of interest is a candidate for modeling by assuming a poisson distribution. Such a count might be the no of fatal traffic accidents per week in a given state, the no of radioactive particle emission's per unit of time, the no of telephone calls per hour coming into the switch board of a large business, the no of organizations per unit volume of some fluid, the no of defects per unit of some material.

PROPERTIES OF POISSON DISTRIBUTION

(1) Random variable X assumes integer values.

(2) The population average or rate is known.

(3) There is assigned probability to non-occurrence of events.

(4) The mean (μ) and the variance (δ^2) are equal.

$E(X) = \pi$

$V(X) = \pi$

Example: Suppose that the average number of telephone calls arriving at the switch board of a small corporation is 30 calls per hour.

(i) What is the prob that no calls will arrive in a 3 - minute period?

(ii) What is the prob that more than five calls will arrive in a 5 – minute period?

Solution

Let assume X is call arriving at any time has a poisson distribution assume that time is measured in minutes; then 30 calls per hour is equivalent to 0.5 calls per minute, so the mean rate of occurrence is 0.5 per minute.

(i) p(no calls in 3-minutes period) = $\ell^{-vt} = \ell^{-0.5(3)} = 0.223$.

(ii) p(more than 5 calls in 5-minutes interval) = $\sum_{k=6}^{\infty} \frac{\ell^{-vt} (vt)^x}{x!}$

$$= \sum_{k=6}^{\infty} \frac{\ell^{-(0.5)(5)} (2.5)^x}{x!} = 0.42$$

or $1 - p(X \leq 5) = 1 - \left\{ \sum_{x=0}^{5} \frac{\ell^{-(2.5)} (2.5)^x}{x!} \right\}$

Exercise: Suppose that flaws in plywood occur at random with an average of one flaw per 50 square feet.

(i) What is the prob that a 4 fool by 8 fool sheet will have no flaws?

(ii) At most one flaw

Solution

(a) P(no flaw) = $\ell^{-1/50\,(32)} = \ell^{-(.64)} = .527$

(b) P(at most one flaw) = $\ell^{-.64} + .64\,\ell^{-.64} = .865$

GEOMETRIC DISTRIBUTION

Suppose that independent trials each having a prob of success is performed until a success occur. If we let X be the no of trials required then,

$F(x) = p(1-p)^{x-1}$, x = 1, 2, - - - - - -

X is said to have a Geometric Random Variable with parameter p.

x-1 ⇒ failure, x ⇒ no of trials for the first success. The sample size is not fixed.

PROPERTIES OF GEOMETRIC DISTRIBUTION

(1) There is a sequence of independent trials (outcome of one trial does not depends on another).

(2) Only two outcomes e.g. success or failure are possible at each trial.

(3) There is a constant prob. of success at each trial.

(4) X is the no of trials for the 1st success to appear (different from that of Binomial distribution).

Show that f(x) is a pdf.

$\sum_{x=1}^{\infty} p(1-p)_{x-1} = p/1-q = p/p = 1.$

$E(X) = \sum_{x=1}^{\infty} X\,pq_{x-1} = p+2pq+3pq^2+ - - -$

$p(1+2q+3q^2 + - - -)$

$p(1-p)^{-2} = p/(1-p)^2 = p/p^2\ 1/p$

E(X2) = EX(X-1) + E(X)

$\sum_{x=1}^{\infty} X(X-1)pq_{x-1}$

$= 0 + 2pq + 6pq^2 + 12pq^3 + 20pq^4$

$= 2pq(1 + 3q + 6q^2 + 10q^3 + - - -)$

$= 2pq(1 + 3q + 3.2q2 + 5.2q2 - - -)$

$= 2pq(1-q)^{-3} = 2pq/(1-q)^3 = 2\not{p}q/p^{\not{3}\ 2} = 2q/p^2$

$E(X2) = 2q/p^2 + 1/p$

$V(X) = 2q/p^2 + 1/p - 1/p^2$

$= \frac{2q + p - 1}{P^2} \quad = \frac{2q - q}{p^2} \quad = q/p^2$

Example: A container has 6 white balls and 4 black balls. Balls are randomly selected one at a time until a black ball is picked. Assure that each ball selected is replaced before the next one selected. What is the prob that

(1) Exactly 3 draws are needed

(2) Not more than two draws are needed.

Solution

$F(x) = p(1-p)^{x-1}$, x = 1, 2, - - -

X is the no of draws for the 1st black ball. P = 0.4

$P(x) = (0.4)(0.6)^{x-1}$, x = 1, 2, - - -

(i) $P(3) = (0.4)(0.6)^2 = 0.144$

(ii) $P(1) + p(2) = 0.4 + 0.4 \times 0.6 = 0.64$

UNIFORM DISTRIBUTION

This is the distribution in which all the possible values have equal probability.

E.g. losing a fair coin; let x = 0 or 1 for head or tail respectively then $p(x) = {}^1/_2$, x = 0, 1. Throwing a die let x be the outcome on the die x = {1, 2, 3, 4, 5, 6}

P(x) ${}^1/_6$. When x = 1 - - - 6., it is an example of uniform distribution

P(X=x) = 1/n, where x = 1, 2, - - - n.

Definition: a discrete random variable X taking values 1, 2, 3, - - -, n such that p(X=x) = 1/n, x= 1, 2, - - -, n has a discrete uniform distribution.

Show that p(X) is a pmf

$E(X) = \Sigma\, Xf(X) = \Sigma\, X1/n = \bar{X}$.

$$\frac{E(X)}{n} = \frac{(1 + 2 + 3 + - - - + n)}{n} = n\frac{(1+n)}{2n} = n\frac{+ 1}{2}$$

$$E(X^2) = \Sigma\, X^2 1/n = \frac{1}{n}\ (1 + 4 + 9 + - - -) = \frac{n(n + 1)(2n + 1)}{6n} = \frac{(n + 1)(2n + 1)}{6}$$

$$V(X) = E(X2) – [E(X)]2 = \frac{(n + 1)(2n + 1)}{6} - \left[\frac{n + 1}{2}\right]^2$$

$$= \frac{2n^2 + 3n + 1}{6} – n\frac{2 + 2n + 1}{4}$$

$$= \frac{4n^2 + 6n + 2 – 3n^2 – 6n – 3}{12}$$

$$= \frac{n^2 – 1}{12} = \frac{(n – 1) (n – 2)}{12}$$

CONTINUOUS DISTRIBUTION

Uniform distribution: A random variable X is said to have a uniform distribution over the interval (a, b) if the prob density function is given by

$$F(X) = \frac{1}{b – a} , a < x < b$$
$$0, \text{otherwise.}$$

Show that it is a pdf.

$E(X) = \frac{b + a}{2}$, $V(X) = \frac{(b – a)^2}{12}$

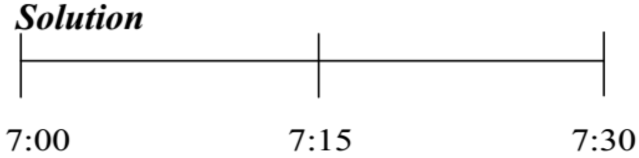
Where $b^3 – b^2 = (b – a) (b^2 + ab + a^2)$.

Example: Buses arrive at a shop at 15 mins interval starting from 7am. If a passenger arrives at the b/stop at a time that is uniformly distributed between 7 and 7:30 am. Find that the passenger waits

(a) Less than 5 mins for a bus

(b) More than 10 mins for a bus.

Solution



(a) $P(7{:}10 < x < 7{:}15) + p(7{:}25 < x < 7{:}30)$

$10 < x < 15 \qquad 25 < x < 30$

Uniform between (0, 30) a = 0, b = 30,

$$F(X) = \frac{1}{30-0} = \frac{1}{30}$$

$$\int_{10}^{15} \frac{1}{30}\,dx + \int_{25}^{30} \frac{1}{30}\,dx$$

$$\left.\frac{x}{30}\right|_{10}^{15} + \left.\frac{x}{30}\right|_{25}^{30}$$

$$\frac{(15-10)}{30} + \frac{(30-25)}{30} = \frac{5+5}{30} = \frac{10}{30} = {}^{1}/_{3}$$

(b) $p(7{:}00 < x < 7{:}05) + p(7{:}15 < x < 7{:}20)$

$0 < x < 5 \qquad 15 < x < 20$

$$\int_{0}^{5} \frac{1}{30}\,dx + \int_{15}^{20} \frac{1}{30}\,dx$$

$$\left.\frac{x}{30}\right|_{0}^{5} + \left.\frac{x}{30}\right|_{15}^{20}$$

$= 5/30 + 5/30 = 10/30 = {}^{1}/_{3}$

NORMAL DISTRIBUTION

A random variable X is said to have a normal distribution if the

$$F(X) = \frac{1}{\delta\sqrt{2\pi}\, e^{-1/2\,\delta^{2}}}, \qquad -\infty < X < \infty$$

$$-\infty < \mu < \infty$$

$$\delta^{2} > 0$$

The mean is μ and the variance is δ^{2}. The standard form is $\mu = 0$, and $\delta^{2} = 1$. If $X \sim N(0, 1)$, then X has a standard normal distribution.

PROPERTIES OF NORMAL DISTRIBUTION

(1) The curve is symmetrical about the vertical axis through the mean.

(2) It has a bell shape

(3) The mean, mode and median coincide at the point μ .

(4) δ^2 determine the shape of the curve, when δ^2 is large, the curve tend to be flat and peaked when δ^2 is small.

(5) The total area under the curve and above the horizontal axis is equal to 1.

$$\int f(x)dx = 1$$

CUMMULATIVE DISTRIBUTION FUNCTION

$P(X<x) = p(-\infty < x < x) = \int_{-\infty}^{x} f(x)dx$

This integral cannot be evaluated explicitly, however, the value of integral has been extensively tabulated for $\mu = 0, \delta^2 = 1$.

i.e. for the standard normal (standardization) if $X \sim N(\mu, \delta^2)$

then $Z = \dfrac{x-\mu}{\delta} \sim N(0, 1)$

If $x \sim N(\mu, \delta^2)$

$$P(-\infty < x < a) = p\left[\frac{-\infty - \mu}{\delta} \le \frac{x-\mu}{\delta} \le \frac{a-\mu}{\delta}\right]$$

$$= p\left[-\infty < Z < \frac{a-\mu}{\delta}\right]$$

$= p(a < x < b)$

$$= \left[\frac{a-\mu}{\delta} < \frac{x-\mu}{\delta} < \frac{b-\mu}{\delta}\right]$$

$$= p\left[\frac{a-\mu}{\delta} < Z < \frac{b-\mu}{\delta}\right]$$

$= \Phi(b) - \Phi(a)$

Example: find the prob that $p(1.5 < Z < 2.3)$

Solution: $\Phi(2.3) - \Phi(1.5)$

$0.9893 - 0.9332$

$= 0.0561$

Exercise: find the prob of

(a) $P(0.1 < Z < 2.03)$

(b) $P(-1 < Z < 2)$

Example: The score in a test is distributed as normal with mean 50 and standard deviation 10.

(a) What is the prob of obtaining the score less than 55.

(b) A score between 45 and 55.

Solution:

(a) $P(X < 55) = p\left[\frac{a-\mu}{\delta} < \frac{55-50}{10}\right]$

$= p(Z < 0.5) = 0.6915$

(b) $P(45 < x < 55) = p\left[\frac{45-50}{10} < Z < \frac{55-50}{10}\right]$

$= p(-0.5 < Z < 0.5)$

$= \Phi(0.5) - \Phi(-0.5)$

$= 0.6915 - 0.3085$

$= 0.3830$

Example: The mean and standard deviation of the weight of boys in a group are known to be 55kg and 2kg respectively. Find the proportion of boys whose weight are:

(i) Between 50kg and 60kg

(ii) More than 58kg

(iii) Less than 52kg.

Solution

(i) P[(50 – 55)/2 < Z < (60 – 55)/2]

$= P(-2.5 < Z < 2.5) = \Phi(2.5) - \Phi(-2.5)$

$= 0.9938 – 0.0062 = 0.9876$

(ii) $P(X > 58) = 1 - P(X \leq 58)$

$= 1 – 0.9332$

$= 0.0668.$

(iii) $P(X < 52) = P(Z < -1.5) = 0.0668$

Using the same example but put standard deviation to be 4, obtain the weight limit that will contain:

(i) 68% (ii) 95%

Solution

0.68 = 1- 0.68 = 0.32/2 = 0.16, 1- 0.16 = 0.84

P(-1 < Z < 1)

$$Z = \frac{x - \mu}{\delta}$$

$x - \mu = Z\delta$

$x = Z\delta + \mu$

$x = -1(4) + 55$

$= 51$

:. The weight limit is between 51 and 59.

(ii) 95% Z(-1.96 < Z < 1.96)

$x = -1.96(4) + 55 = 47.16$

$x = 1.96(4) + 55 = 62.84$

Exercise: the mean and the standard deviation in a mathematical test is 70 and 10 respectively. Find the score in standard units of student receiving (i) 85 (ii) 40 marks.

Solution

(i) Z = (85 – 70)/10 = 1.5 (ii) Z = (40-70)/10 = -3

(ii) Find the mark corresponding to standard score (a) 0, (b) 1.70 (c) 1.15

$x = \mu + Z\delta$

(a) $x = 70 + 0\delta = 70$

(b) x = 70 + 1.70(10) = 87

(c) x = 70+ 1.15(10) = 81.5

(3) A particular storage battery lasts on the average five years. If the battery lives are normally distributed with standard deviation 0.5 years. Find the prob that the given battery will

(1) Last less than 6.4 years = 0.9974

(2) More than 5.3 years

(3) Between 4.8 years and 6 years = 0.6541

BIVARIATE PROBABILITY DISTRIBUTION

Let f(x,y) be the pdf of two r.v. X and Y. A pdf or a distribution function called a joint pdf or a joint distribution when more than one variable is involved. Thus f(x,y) is the joint (bivariate) pdf of the r.v. X and Y.

Consider the event $a < x < b$. This event can occur when and only when the event $a < x < b, -\infty < y < \infty$ occurs, i.e. $p(a < x < b, -\infty < y < \infty)$

$$= \int_a^b \int_{-\infty}^{\infty} f(x,y)\,dy\,dx \qquad \text{- - - (1)}$$

$P(a < x < y, -\infty < y < \infty)$

$$= \sum_{x=a}^{b} \sum_{-\infty}^{\infty} f(x,y) \qquad \text{- - - (2)}$$

Note that each of equation (1) and (2) is a function of x above $f_1(x)$.

$f_1(x)$ is called the marginal p.d.f.

$p(-\infty < x < \infty, -\infty < y < \infty) = \int_{\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx$

$$= \sum_x \sum_y f(x,y) = 1$$

If X and Y are independent then f(x,y) = f(x) f(y).

But if dependent f(x,y) = f(x/y)f(y) or f(y/x)f(x)

Example: Let the joint pdf of X and Y be

$F(x,y) = \frac{x+y}{21}$ x = 1, 2, 3.
y = 1, 2

0,   otherwise

$P(x=3) = f_1(3)$

$= f(3,1) + f(3,2)$

$P(x=3,y=1,2) =$

$\sum_{x=3} \sum_{y=1}^{2} f(x,y)$ 4/21 + 5/21 = 9/21 = 3/7

$P(y=2) = f_2(2)$

$= f(1, 2) + f(2, 2) + f(3, 2)$

$= \frac{1+2}{21} + \frac{2+2}{21} + \frac{3+2}{21} = \frac{12}{21}$

$f_1(x) = \sum_{y=1}^{3} \frac{X+Y}{21}$

$= \frac{x+1}{21} + \frac{x+2}{21} = \frac{2x+3}{21}$

:. The marginal p.d.f. of X

$f_1(x) = \sum_{y=1}^{3} \frac{X+Y}{21} = \frac{1+y}{21} + \frac{2+y}{21} + \frac{3+y}{21} = \frac{6+3y}{21} = \frac{3(2+y)}{21} = \frac{2+y}{21}$ 7

Example: The joint density function of x and y is given as

$F(x,y) = 2(x+y-2xy), 0 < x < 1$

$0 < y < 1$

Find the marginal of X and Y

Solution

F(x) = $\int_{Ry} f(x, y)dy$

= $\int_0^1 s(x + y - 2xy)dy$

= $2 \left| \frac{xy}{2} + \frac{y^2}{2} - \frac{2xy^2}{2} \right|_0^1$

= 2 (x + ½ - x) = 1

F(y) = 1

:. $F_1(x)$ and $f_1(y)$ are uniformly distributed

CONDITIONAL PROBABILITY DISTRIBUTION FUNCTION

Let X and Y denote r.v. of discrete type which have the distribution p.d.f f(x, y).

Let $f_1(x)$ and $f_1(y)$ denote the marginal p.d.f. of X and Y respectively.

$f(Y/X) = \frac{f(x, y)}{f(x)}$, provided f(x) > 0

Also,

$f(X/Y) = \frac{f(x, y)}{f(y)}$, provided f(y) > 0

Example: In the previous example

$f(x,y) = \frac{X + Y}{21}$, X = 1, 2, 3
Y = 1, 2

Find $p(X/Y) = \frac{f(x,y)}{f(y)}$

= f1(y) = $\sum_{x=1}^{3} \frac{X + Y}{21}$

= (2 + y)/7

Example: f(x, y) = 4xy, $0 \le x \le 1, 0 \le y \le 1$

Find the marginal p.d.f. of X and hence f(Y/X).

Solution

$f_1(x) = \int_0^1 4xy\,dy$

$= \left.\frac{4^2xy^2}{2}\right|_0^1$

$f_1(x) = 2x,\ 0 \leq x \leq 1$

0, otherwise

$f(Y/X) = \frac{f(x, y)}{f_1(x)} = \frac{4^2xy^2}{2} = 2y$

Exercise: Given $f(x, y) = b(x+2y),\ 1 \leq x \leq 3,$

$0 \leq y \leq 2$

Find (i) b (ii) find the marginal of X (iii) find the marginal of Y

Find the conditional p.d.f. of X given Y